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Research article

Statistical reliability modeling via Entropy measures: Characterization and testing of log-symmetric lifetime distributions

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Abstract: Statistical reliability modeling of lifetime data routinely encounters distributions that are asymmetric on the original scale yet exhibit fundamental structural balance after logarithmic transformation. Addressing this intrinsic feature of reliability and survival data, we developed a unified entropy-based reliability modeling framework for the characterization and testing of log-symmetry in continuous lifetime distributions. The proposed methodology was built upon distributional transformations induced by linear consecutive k-out-of-n reliability systems, which serve as structured mechanisms for probing how informational balance is preserved or disrupted under reliability-driven system behavior. Within this framework, Shannon entropy, Rényi entropy, and Kerridge inaccuracy were integrated to derive explicit and tractable characterization results that uniquely identified log-symmetric lifetime distributions through intrinsic information-theoretic relationships. These results led naturally to a nonparametric, computationally efficient statistical test for log-symmetry that avoided restrictive modeling assumptions and was well suited for practical reliability analysis. Comprehensive Monte Carlo simulations demonstrated that the proposed test achieved accurate type I error control and competitive power against a wide range of alternatives. Applications to real lifetime datasets from reliability settings further confirmed the effectiveness of the approach. Overall, the study demonstrated how entropy-based inference, when embedded within reliability system modeling, provides a rigorous and practically relevant framework for statistical analysis of lifetime distributions, thereby contributing theoretical insight and applied methodology to modern reliability modeling.

Keywords: Shannon entropy; Rényi entropy; Kerridge inaccuracy; statistical reliability modeling; information-theoretic inference; log-symmetric lifetime distributions; entropy-based testing; consecutive k-out-of-n reliability systems; nonparametric characterization

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1. Introduction

Entropy is a fundamental concept that links information theory with statistical inference, providing a rigorous quantitative framework for assessing uncertainty, structural balance, and distributional organization in probability models. Since its original introduction as a measure of uncertainty, entropy has evolved into a versatile analytical tool with broad influence across statistics, reliability theory, and survival analysis. In statistical reliability modeling of lifetime data, observations are inherently nonnegative and frequently exhibit substantial skewness, making the assumption of symmetry on the original scale unrealistic in most practical settings. Nevertheless, many widely used lifetime distributions exhibit symmetry after logarithmic transformation, a structural property commonly referred to as log-symmetry. From theoretical and applied perspectives, assessing log-symmetry is therefore more meaningful than testing symmetry on the original lifetime scale, particularly in reliability and survival analysis, where logarithmic representations naturally arise.

This observation motivates the use of information-theoretic measures such as Shannon entropy, Rényi entropy, and Kerridge inaccuracy to examine log-symmetry in lifetime distributions. These measures provide intrinsic descriptions of distributional balance and uncertainty that do not depend on moment conditions or restrictive parametric assumptions. Within this setting, log-symmetry serves as a unifying structural principle that encompasses several important models in reliability and survival analysis. As a result, entropy-based methods provide a coherent framework for assessing symmetry properties on the logarithmic scale in lifetime modeling. Related characterizations of symmetry have been obtained through order statistics [1]. Information properties of coherent systems have also been studied in reliability settings [2]. Extropy measures based on order statistics have been used to analyze symmetric distributions [3]. More recent works have developed entropy-based procedures for symmetry testing [4,5].

A deeper perspective emerges when entropy is examined through structured distributional transformations induced by reliability systems. In particular, consecutive k-out-of-n systems provide an analytically tractable mechanism for generating such transformations by systematically reshaping the underlying lifetime distribution. These transformations selectively emphasize specific regions of the distribution, thereby inducing interpretable structural changes that can be analyzed from an information-theoretic viewpoint. Within this framework, entropy functionals respond to reliability-induced modifications in a manner that reveals whether the intrinsic informational balance associated with log-symmetry is preserved. This interaction establishes a natural connection between entropy measures and system structure, offering a principled approach for characterization and inference. Moreover, the transformation induced by consecutive k-out-of-n systems enhances sensitivity to deviations from log-symmetry, enabling entropy-based methods to detect distributional imbalances that may remain less visible on the original scale. Information-theoretic reliability analysis has been developed for linear consecutive k-out-of-n:F systems [6]. Extropy analysis has also been studied for consecutive k-out-of-n:G systems, with applications in reliability and exponentiality testing [7].

Log-symmetry has long attracted considerable attention in statistical research, particularly in the analysis of positive lifetime data. Classical contributions have mostly focused on nonparametric estimation and distributional characterization based on order statistics, record values, and related stochastic structures. Furthermore, entropy-based measures have assumed an increasingly important role in distributional analysis, uncertainty quantification, and statistical inference, while consecutive k-out-of-n reliability systems have been extensively studied for their structural and informational properties. Despite these substantial developments, these research directions have largely evolved independently. In particular, although entropy-related aspects of consecutive reliability systems, including cumulative residual entropy and Shannon differential entropy, have been investigated in the literature, their systematic use for the characterization and testing of log-symmetry has not been established within a unified inferential framework.

In contrast to entropy-based approaches to symmetry, which typically rely on direct distributional properties or order-based arguments, we adopt a fundamentally different perspective by exploiting reliability system-induced transformations to derive new characterization results and develop a corresponding nonparametric inference procedure for log-symmetry.

Motivated by this gap, we develop a unified entropy-based framework for the characterization and nonparametric testing of log-symmetry in continuous lifetime distributions within a statistical reliability modeling context. The proposed approach integrates Shannon entropy, Rényi entropy, and Kerridge inaccuracy with the transformation structure induced by consecutive k-out-of-n systems, leading to explicit and analytically tractable conditions that characterize log-symmetry through intrinsic information-theoretic properties of transformed system lifetimes.

Building on this framework, a novel nonparametric test for log-symmetry is constructed based on entropy estimators computed from transformed samples. The resulting procedure is computationally efficient, free from restrictive parametric assumptions, and applicable to a broad class of lifetime distributions arising in reliability and survival analysis. The proposed framework is particularly well suited to modern reliability modeling, where lifetime distributions are often asymmetric on the original scale, while log-symmetry provides a more natural, interpretable, and practically meaningful structural representation.

The major contributions of this paper can be summarized as follows: First, we develop a unified entropy-based framework that integrates Shannon entropy, Rényi entropy, and Kerridge inaccuracy for the analysis of log-symmetric lifetime distributions, thereby establishing a novel link between information-theoretic measures and reliability system modeling. Second, we demonstrate that consecutive k-out-of-n systems induce structured and analytically tractable transformations that enable a nontrivial characterization of log-symmetry through entropy functionals of transformed system lifetimes. Third, we construct a novel nonparametric test for log-symmetry based on entropy estimators, providing a flexible, model-free, and computationally efficient inferential procedure. Finally, the effectiveness of the proposed framework is demonstrated through comprehensive simulation studies and real-data applications.

The remainder of the paper is organized as follows: In Section 2, we introduce the necessary definitions and preliminary results. In Sections 3 and 4, we present the major theoretical contributions, including entropy-based characterizations of log-symmetry in the context of consecutive k-out-of-n reliability systems. In Section 5, we develop the proposed nonparametric test, establish its statistical properties, and evaluate its finite-sample performance through Monte Carlo simulations and real-data applications. In Section 6, we conclude with a summary of the major findings and outline directions for future research.

For notational simplicity, we write $g^a(x)$ in place of $[g(x)]^a$ for any given function g. Throughout the paper, all expectations and integrals are assumed to exist whenever they appear.

2. Preliminaries

In this section, we introduce the fundamental concepts and analytical tools required for the developments in subsequent sections. Throughout the theoretical analysis, we assume that the underlying lifetime random variable X is absolutely continuous with density function *f*(*x*) and support on (0, ∞). All entropy-related integrals are assumed to exist and be finite, and the transformations induced by consecutive k-out-of-n systems are assumed to preserve absolute continuity and integrability under the stated conditions.

A non-negative random variable X is said to be log-symmetric about $\mu \in \mathbb{R}$ if the transformed variable $Y = \log X$ is symmetric about μ. Formally, if F_Y denotes the cumulative distribution function of Y, then $F_Y(\mu - y) + F_Y(\mu + y) = 1,$ for all $y \in \mathbb{R}$.

This definition implies that μ is the center of symmetry of *Y*, typically coinciding with its median. As a simple illustration, if X follows a lognormal distribution with parameters μ and σ^2, then $Y = \log X \sim N(\mu, \sigma^2)$, which is symmetric about μ. Hence, the lognormal distribution is log-symmetric. In the subsequent developments, parameter μ serves as a central reference point for the entropy-based analysis. While alternative location measures may be considered, the formulation remains directly aligned with the symmetry structure of the logarithmic transformation and is naturally compatible with the entropy-based functionals employed in this study.

Under the log-symmetry condition, a random sample $Y_1, Y_2, \ldots, Y_n$ drawn from the distribution of the transformed variable inherits the same structural balance. Classical examples of log-symmetric distributions include the log-normal, log-logistic, and log-Laplace families, each featuring a central location parameter around which probability of the transformed data is distributed evenly. Despite their fundamental role in statistical modeling, establishing analytical conditions that reliably capture log-symmetry may require additional structural insight. In this work, such insight arises from the behavior of entropy-based quantities under specific distributional transformations. Consecutive k-out-of-*n* systems offer a natural mechanism for generating these transformations. Because their lifetimes depend on the joint behavior of consecutive components, these systems induce systematic modifications of the parent distribution. Such modifications are particularly informative for assessing informational balance and detecting departures from log-symmetry. The concepts summarized in this section lay the groundwork for the entropy-based characterization results and log-symmetry testing procedures developed later. Let X be a nonnegative continuous random variable with probability density function (pdf) $f(x)$ and cdf $F(x)$. The Shannon differential entropy of X is defined by

$$\mathcal{S}(X) = -\int_0^\infty f(x) \log f(x)\, dx = -\int_0^1 \log f\big(F^{-1}(u)\big) du\,, \tag{1}$$

where "log(·)" denotes the natural logarithm. In this formula, $F^{-1}(u) = \inf\{x; F(x) \geq u\}$ is the quantile function with U being a uniform random variable on (0, 1) (see [8]).

This representation highlights the role of the quantile transformation in connecting the distribution of *X* to the structure of its entropy. A natural extension of Shannon entropy is obtained by considering the entropy of order α, commonly referred to as Rényi entropy. For $\alpha > 0, \alpha \neq 1$, the Rényi entropy of *X* is defined as

$$\mathcal{S}_\alpha(X) = \varepsilon(\alpha) \log \int_0^\infty f^\alpha(x)\, dx = \varepsilon(\alpha) \log \int_0^1 f^{\alpha-1}\big(F^{-1}(u)\big) du, \tag{2}$$

where $\varepsilon(\alpha) = 1/(1-\alpha)$. Shannon entropy is recovered as a limiting case of Rényi entropy, i.e.,

$\mathcal{H}(X) = \lim_{\alpha \to 1} \mathcal{H}_\alpha(X)$. Rényi entropy provides a natural and powerful extension of Shannon's formulation by introducing a tuning parameter that governs the relative emphasis placed on different regions of a distribution. This additional degree of flexibility enables Rényi entropy to reveal structural features that may remain hidden under the Shannon entropy, particularly in contexts where sensitivity to tail behavior or localized variation is essential. For this reason, it has become a central tool in modern information-theoretic analysis, supporting the development of characterization results, inference procedures, and model assessment techniques [9].

The Rényi entropy provides a unifying framework that connects information, uncertainty, and structural balance. Its applications in physics, communication theory and signal processing show that it can describe not only randomness, but also organization and equilibrium within complex systems. Comprehensive accounts of its foundations and evolution may be found in the classical references [10,11], the survey in [12], and later contributions that expanded its analytical and practical scope.

Let X and Y be two continuous random variables with cdf's F and G, respectively. Suppose F represents the true distribution underlying the observed data, while G denotes a model or reference distribution specified by the experimenter. If f and g denote their corresponding probability density functions, the Kerridge inaccuracy measure associated with (X,Y) is defined as

$$\mathcal{K}(X,Y) = -\int_0^\infty f(x) \log g(x)\, dx, \tag{3}$$

which quantifies the degree of mismatch between the true distribution and the proposed distribution. This measure quantifies the degree to which the model distribution G deviates from the true distribution F. In the special case where $g = f$, the Kerridge inaccuracy measure simply becomes the Shannon differential entropy of X, i.e.,

$$\mathcal{K}(X,X) = \mathcal{H}(X),$$

thereby emphasizing that entropy serves as the baseline measure when the model perfectly matches the underlying distribution.

In reliability engineering, consecutive k-out-of-n systems constitute a fundamental class of models that combine analytical tractability with substantial practical relevance. In a linear consecutive k-out-of-n:G configuration, the system operates successfully whenever at least k adjacent components, assumed to be independent and identically distributed, function properly. Its dual form, the consecutive k-out-of-n:F system, fails once k adjacent components experience simultaneous failure. The well-known series and parallel arrangements arise as boundary cases: The 1-out-of-n:F system (equivalently, n-out-of-n:G) corresponds to a classical series structure, while the n-out-of-n:F system (equivalently, 1-out-of-n:G) represents a parallel arrangement. These models provide realistic representations of systems whose performance depends on the integrity of consecutive segments, such as communication links, transportation corridors, and pipeline networks. Their reliability properties have been studied extensively in the literature. Reliability of linear consecutive k-out-of-n:F systems with common-mode forced outages has been studied in [13]. General treatments of reliability modeling and consecutive-k systems have been provided in [14,15]. Conditional lifetimes of consecutive k-out-of-n systems have been investigated in [16]. Their sequential dependence structure induces lifetime transformations that interact naturally with information-based measures, enabling entropy to capture subtle variations in distributional shape. This feature makes consecutive systems particularly well suited to this study, where entropy is employed to reveal log-symmetry-related characteristics. Of particular importance are linear consecutive k-out-of-n:F and k-out-of-n:G systems satisfying the condition $2k \geq n$, which ensures that the induced transformations preserve essential structural balance. This condition forms a key analytical foundation for the entropy-based characterizations developed in the sections that follow. The condition $2k \geq n$ plays a structural role in the analysis, as it ensures that the system configuration preserves a form of balance in the induced transformation, which is essential for linking the behavior of entropy functionals to the log-symmetry property of the underlying distribution. When this condition is satisfied, the transformation exhibits monotone and stable behavior that enables a clear characterization of log-symmetry through the proposed entropy measures.

For a system of n i.i.d. components whose lifetimes follow a common distribution function $F(x)$, the lifetime of a consecutive k-out-of-n: G structure is denoted by $T_{k|n:G}$, and the lifetime of its dual k-out-of-n: F structure is denoted by $T_{k|n:F}$. In both configurations, the system reliability admits a closed-form representation. As shown in [17], the corresponding survival functions are given by

$$S_{k|n:G}(x) = (n-k+1)S^k(x) - (n-k)S^{k+1}(x), \qquad x > 0, \tag{4}$$

$$F_{k|n:F}(x) = (n-k+1)F^k(x) - (n-k)F^{k+1}(x), \qquad x > 0, \tag{5}$$

where $S(x) = P(X > x)$ denotes the component survival function.

These expressions highlight how the system lifetime is governed by the joint behavior of consecutive components and how the underlying distribution changes the reliability of the overall structure. Symmetric distributions have played a central role in statistical inference. Estimation of a symmetric distribution function in multistage ranked set sampling has been studied in [18]. Symmetric distributions based on information-measure properties of order statistics have been characterized in [19]. Symmetric continuous distributions based on δ-records and spacings have been characterized in [20]. Building on this, we adopt an information-theoretic viewpoint in which entropy measures are used to examine structural attributes of distributions and, in particular, log-symmetry property. When combined with the transformation properties of consecutive k-out-of-n systems, these measures provide a rigorous analytical framework for identifying log-symmetry and quantifying departures from it.

Next, we present a pivotal lemma from [21] that provides a fundamental analytical tool for the study of symmetric distributions. Let $\mathcal{C}_e$ denote the class of all continuous probability density functions f, with support S_X, that satisfy

$$f(F^{-1}(u)) = f(F^{-1}(1-u)), \text{ for almost all } u \in (0,1).$$

Lemma 2.1. Let X be a continuous random variable with cumulative distribution function F and probability density function f, supported on $[0,\infty)$. Suppose $f \in \mathcal{C}_e$. Then, there exists a constant μ such that $F(\mu - x) = 1 - F(\mu + x)$, for all $x > 0$.

This result provides a key foundational element for the characterizations developed in subsequent sections. A second result that is most useful is Müntz–Szász theorem, cited in [21], which offers an essential approximation tool and plays a significant role in several of the proofs that follow.

Lemma 2.2. Let $\psi(x)$ be an integrable function on the interval (a,b). If $\int_a^b x^{n_j}\psi(x)dx = 0$ for all $j \geq 1$, where $\{n_j\}_{j\geq 1}$ is a strictly increasing sequence of positive integers with $\sum_{j=1}^{\infty} 1/n_j = \infty$, then $\psi(x) = 0$ almost everywhere on (a,b).

This result is known as Müntz–Szász theorem. It establishes the completeness of sequences of monomials of the form $\{x^{n_1}, x^{n_2}, \ldots; 1 \leq n_1 < n_2 < \cdots\}$, and provides a powerful uniqueness principle for functions characterized through generalized moment conditions. If an integrable function possesses vanishing weighted moments along a sufficiently rich sequence of polynomial exponents, then the function must be identically zero almost everywhere on its domain. A generalization of the Müntz-Szász theorem to functions of the form $\{\phi^{n_j}(x), n_j \geq 1\}$, where $\phi(x)$ is absolutely continuous and monotonic on (a,b), is cited in [22–24].

Let $X_1, X_2, \ldots, X_n$ be n i.i.d. continuous nonnegative random variables with cumulative distribution function F, probability density function $f(x)$, and support S_X. Denote the corresponding order statistics by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. For a population that is symmetric about zero, the *r*-th order statistic and the negative of the $(n-r+1)$-th order statistic share the same distribution; that is, $X_{r:n} \stackrel{d}{=} -X_{n-r+1:n}$, where $\stackrel{d}{=}$ denotes equality in distribution. We extend this property to consecutive systems through the following lemma. Consider a linear consecutive (*n*–*r*)-out-of-*n*:F system and a corresponding $(n-r)$-out-of-n:G system for all $n \geq 2r$, where

$$r \in \mathcal{D} = \left\{0,1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\},$$

and $\lfloor \cdot \rfloor$ denotes the floor function. To begin with, we recall that the probability density functions of these linear consecutive systems with lifetimes $T_{n-r|n:G}$ and $T_{n-r|n:F}$ are represented by

$$f_{n-r|n:G}(x) = f(x)\left[a_{r,n} S^{n-r-1}(x) - b_{r,n} S^{n-r}(x)\right], \qquad x > 0, \tag{6}$$
$$f_{n-r|n:F}(x) = f(x)\left[a_{r,n} F^{n-r-1}(x) - b_{r,n} F^{n-r}(x)\right], \qquad x > 0, \tag{7}$$

where $a_{r,n} = (n-r)(r+1)$ and $b_{r,n} = (n-r+1)r$, for all $n \geq 2r$, such that $r \in \mathcal{D}$.

Lemma 2.3. Suppose $X_1, X_2, \ldots, X_n$ are n i.i.d. continuous random variables with cumulative distribution function $F(x)$, probability density function $f(x)$, and support S_X. If $f \in \mathcal{C}_e$, then, for all $n \geq 2r$, $-T_{n-r|n:G} \stackrel{d}{=} T_{n-r|n:F}$, where $r \in \mathcal{D}$.

Proof. Without loss of generality, we can assume that $F(x) = 1 - F(-x)$ for all x in its support S_X, for a symmetric distribution with $\mu = 0$. Recalling Eqs (4) and (5), we have

$$\begin{aligned} F_{-T_{n-r|n:G}}(x) &= P(-T_{n-r|n:G} \leq x) = P(T_{n-r|n:G} > -x) \\ &= (r+1)[S(-x)]^{n-r} - r[S(-x)]^{n-r+1} \\ &= (r+1)[1-F(-x)]^{n-r} - r[1-F(-x)]^{n-r+1} \\ &= (r+1)[F(x)]^{n-r} - r[F(x)]^{n-r+1} = F_{T_{n-r|n:F}}(x), \end{aligned}$$

thus completing the proof of the lemma.

Suppose now the component lifetimes are transformed into independent and identically distributed uniform random variables $U_j = F(X_j)$ on $[0,1]$, for $j = 1, \cdots, n$. In the case where $r \in \mathcal{D}$, the pdfs of $U_{n-r|n:G} = F(T_{n-r|n:G})$ and $U_{n-r|n:F} = F(T_{n-r|n:F})$ are given by

$$\rho_{n-r|n:G}(u) = a_{r,n}(1-u)^{n-r-1} - b_{r,n}(1-u)^{n-r}, \qquad \text{for } 0 < u < 1, \tag{8}$$
$$\rho_{n-r|n:F}(u) = a_{r,n} u^{n-r-1} - b_{r,n} u^{n-r}, \qquad \text{for } 0 < u < 1. \tag{9}$$

The following lemma demonstrates that $U_{n-r|n:G}$ is identically distributed to the random variable $1 - U_{n-r|n:F}$.

Lemma 2.4. Suppose $U_1, U_2, \ldots, U_n$ are n i.i.d. continuous random variables uniformly distributed on the interval $[0,1]$. For all $n \geq 2r$, we have $U_{n-r|n:G} \stackrel{d}{=} 1 - U_{n-r|n:F}$, where $r \in \mathcal{D}$.

Proof. As $\rho_{n-r|n:G}(u) = \rho_{n-r|n:F}(1-u)$, for all $0 < u < 1$, we get

$$\begin{aligned} F_{U_{n-r|n:G}}(u) &= P(U_{n-r|n:G} \leq u) \\ &= \int_0^u \rho_{n-r|n:G}(x)dx = \int_0^u \rho_{n-r|n:F}(1-x)dx \\ &= \int_{1-u}^1 \rho_{n-r|n:F}(z)dz = (U_{n-r|n:F} \geq 1-u) \\ &= P(1-U_{n-r|n:F} \leq u) = F_{1-U_{n-r|n:F}}(u), \end{aligned}$$

for all $r \in \mathcal{D}$, and this completes the proof of the lemma.

3. Entropy-based characterization of log-symmetry

In this section, we develop an entropy-based framework for characterizing log-symmetry in lifetime distributions through consecutive k-out-of-n systems. Unlike generic transformations, consecutive k-out-of-n systems induce structured reliability-driven distortions that systematically amplify departures from log-symmetry, thereby enabling entropy measures to detect asymmetry patterns that are not directly observable on the original scale. Since the lifetimes of the consecutive (n–r)-out-of-n systems can be written as transformations of the uniform variables $U_{n-r|n:G}$ and $U_{n-r|n:F}$, namely,

$$T_{n-r|n:G} = F^{-1}(U_{n-r|n:G}) \text{ and } T_{n-r|n:F} = F^{-1}(U_{n-r|n:F}),$$

the corresponding Jacobians of these transformations are given by $1/f(t)$. Using identity (1), together with the transformation rule for Shannon differential entropy and the analytical results in [21], we obtain a representation for the entropy of a (n–r)-out-of-n:G system as

$$\mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(U_{n-r|n:G}) - E\left[\log f\left(F^{-1}(U_{n-r|n:G})\right)\right], \tag{10}$$

for all $n \geq 2r$, where $\mathcal{S}(U_{n-r|n:G})$ denotes the Shannon differential entropy of the transformed system under i.i.d. uniform components on $[0,1]$. Applying the same reasoning to the transformation $T_{n-r|n:F} = F^{-1}(U_{n-r|n:F})$, we have

$$\mathcal{S}(T_{n-r|n:F}) = \mathcal{S}(U_{n-r|n:F}) - E\left[\log f\left(F^{-1}(U_{n-r|n:F})\right)\right] \text{ for all } n \geq 2r. \tag{11}$$

The following lemma shows that the entropies of $U_{n-r|n:G}$ and $U_{n-r|n:F}$ are identical.

Lemma 3.1. Suppose that $U_1, \ldots, U_n$ be i.i.d. continuous random variables uniformly distributed on the interval $[0,1]$. Then, we have $\mathcal{S}(U_{n-r|n:G}) = \mathcal{S}(U_{n-r|n:F})$, for all $n \geq 2r$, where $r \in \mathcal{D}$.

Proof. Given that $\rho_{n-r|n:G}(u) = \rho_{n-r|n:F}(1-u)$, for all $0 < u < 1$, and $r \in \mathcal{D}$, and recalling (1), (8), and (9), we can write

$$\begin{aligned} \mathcal{S}(U_{n-r|n:G}) &= -\int_0^1 \rho_{n-r|n:G}(u) \log \rho_{n-r|n:G}(u)du \\ &= -\int_0^1 \rho_{n-r|n:F}(1-u) \log \rho_{n-r|n:F}(1-u)du \\ &= -\int_0^1 \rho_{n-r|n:F}(z) \log \rho_{n-r|n:F}(z)du \\ &= \mathcal{S}(U_{n-r|n:F}), \end{aligned}$$

where the third equality follows from the substitution $z = 1 - u$. Hence the lemma.
We now present the following example to illustrate the results.
Example 3.1. Let us consider a random variable X with pdf $f(x) = bx^{b-1}$ and cdf $F_X(x) = x^b$, for $0 < x < 1$ and $b > 0$. Then, $f(F^{-1}(u)) = bu^{\frac{b-1}{b}}$, for $0 < u < 1$. As $H(U_{n-r|n:G}) = H(U_{n-r|n:F})$, for all $r \in \mathcal{D}$, due to Lemma 3.1, by (10) and (11), we have

$$\begin{aligned}
\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) &= E\left[\log f\left(F^{-1}(U_{n-r|n:F})\right)\right] - E\left[\log f\left(F^{-1}(U_{n-r|n:G})\right)\right] \\
&= E\left[\log U_{n-r|n:F}^{\frac{b-1}{b}}\right] - E\left[\log U_{n-r|n:G}^{\frac{b-1}{b}}\right] \\
&= \frac{b-1}{b}\left(E[\log U_{n-r|n:F}] - E[\log U_{n-r|n:G}]\right) \\
&= \frac{b-1}{b}\left(E[\log U_{n-r|n:F}] - E[\log(1 - U_{n-r|n:F})]\right), \qquad (12)
\end{aligned}$$

where the fourth equality is derived from the observation that $U_{n-r|n:G} \overset{d}{=} 1 - U_{n-r|n:F}$, as noted in Lemma 2.4. It is not difficult to verify that

$$E[\log U_{n-r|n:F}] = (r+1)[\psi(n-r) - \psi(n-r+1)] - r[\psi(n-r+1) - \psi(n-r+2)],$$

and

$$E[\log(1 - U_{n-r|n:F})] = (r+1)[\psi(1) - \psi(n-r+1)] - r[\psi(1) - \psi(n-r+2)],$$

where $\psi(\cdot)$ is the digamma function. After some simplifications using the properties of the digamma function, representation (12) can be rewritten as

$$\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) = \frac{b-1}{b}\left[\psi(n-r) - \frac{r}{n-r} - \psi(1)\right], \qquad \text{for all } r \in \mathcal{D}. \quad (13)$$

The expression inside the brackets in (13) is nonzero except when $n = 1$ $(r = 0)$, $\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) = 0$ if $b = 1$ (i.e., F is symmetric). Thus, for all $r \in \mathcal{D}$, the sign of $\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F})$ is determined by b, being positive and negative.
It is worth noting that Example 3.1 shows that the difference $\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F})$ may take positive, negative, or zero values, depending on the shape of the parent distribution. The following result establishes that, for all $n \geq 2r$, this difference is identically zero under symmetry.
Theorem 3.1. Let $X_1, \ldots, X_n$ be n i.i.d. continuous random variables with cumulative distribution function F. Then F is symmetric if and only if, for a fixed $r \in \mathcal{D}, \mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(T_{n-r|n:F})$, for all $n \geq 2r$.
Proof. We first observe that Lemma 3.1 implies $H(U_{n-r|n:G}) = H(U_{n-r|n:F})$, for all $n \geq 2r$. Assuming, without loss of generality, that the symmetric distribution F is centered at $\mu = 0$, it follows that

$$
\begin{aligned}
E\left[\log f\left(F^{-1}(U_{n-r|n:G})\right)\right] &= \int_0^1 \rho_{n-r|n:G}(u)\log f(F^{-1}(u))du \\
&= \int_0^1 \rho_{n-r|n:F}(1-u)\log f(F^{-1}(u))du \\
&= \int_0^1 \rho_{n-r|n:F}(z)\log f(F^{-1}(1-z))dz, \text{ taking } z=1-u \\
&= \int_0^1 \rho_{n-r|n:F}(z)\log f\big(F^{-1}(z)\big)dz = E\left[\log f\left(F^{-1}(U_{n-r|n:F})\right)\right],
\end{aligned}
$$

where the last equality follows from the observation that F is symmetric, thereby completing the necessity part of the proof. For the sufficiency part, we assume that $\mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(T_{n-r|n:F})$, for all $n \geq 2r$. Then, from (10) and (11) and noting that $H(U_{n-r|n:G}) = H(U_{n-r|n:F})$, for all $n \geq 2r$, we have

$$
\begin{aligned}
0 &= \mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) \\
&= E\left[\log f\left(F^{-1}(U_{n-r|n:F})\right)\right] - E\left[\log f\left(F^{-1}(U_{n-r|n:G})\right)\right], \qquad (14)
\end{aligned}
$$

for all $n \geq 2r$. Utilizing the fact that $U_{n-r|n:G} \overset{d}{=} 1 - U_{n-r|n:F}$, due to Lemma 2.4, from Eq (14), we obtain

$$E\left[\log f\left(F^{-1}(U_{n-r|n:F})\right)\right] - E\left[\log f\left(F^{-1}(1-U_{n-r|n:F})\right)\right] = 0.$$

Consequently, we get

$$\int_0^1 \phi_{r,n}(u)\big[\log f\big(F^{-1}(u)\big) - \log f\big(F^{-1}(1-u)\big)\big]u^{n-2r}du = 0,$$

where $\phi_{r,n}(u) = a_{r,n}u^{r-1} - b_{r,n}u^r$, for $0 < u < 1$. By applying Lemma 2.2 to the function

$$\psi(u) = \phi_{r,n}(u)[\log f(F^{-1}(u)) - \log f(F^{-1}(1-u))],$$

and noting that $\{u^{n-2r}, n \geq 2r\}$ is a complete sequence, we conclude that $f \in \mathcal{C}_e$. The proof of the theorem then gets completed using Lemma 2.1.

Remark 3.1. Under the assumptions of Theorem 3.1, if $\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) = d$, for all $n \geq 2r$, where d is a constant independent of n, then, following the proof of Theorem 3.1, we obtain

$$F(a(b-x)) + F(b+x) = 1, \qquad (15)$$

for all $x \in R$, where $a = e^d > 0$ and b are constants. When $a = 1, F$ is symmetric; when $a \neq 1$, (15) defines a family of distributions that deviates from symmetry.

A natural question arises at this point: If the equality $\mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(T_{n-r|n:F})$ holds only for a particular choice of *r* and a single value of *n*, does this necessarily imply that *F* is symmetric? To address this question, let $\mathcal{C}$ denote the collection of all continuous probability density functions f with support S_X, such that $f(F^{-1}(u)) \geq (\text{ or } \leq) f(F^{-1}(1-u))$ for all $u \in (0,1/2)$. We begin first by examining the case $i = 0$.

Theorem 3.2. Under the assumptions of Theorem 3.1, the following statements are equivalent:

(i) X has a symmetric distribution;

(ii) $f \in \mathcal{C}$ and $\mathcal{S}(T_{n|n:G}) = \mathcal{S}(T_{n|n:F})$, for fixed $n \geq 1$.

Proof. The demonstration that (i) implies (ii) is straightforward. To establish the converse, assume that there exists a fixed $n \geq 1$, such that $\mathcal{S}(T_{n|n:G}) = \mathcal{S}(T_{n|n:F})$. This leads to

$$\int_0^1 (1-u)^{n-1} \log f\big(F^{-1}(u)\big) du = \int_0^1 u^{n-1} \log f\big(F^{-1}(u)\big) du.$$

Upon using the substitution $z = 1 - u$, it follows that

$$\int_0^1 z^{n-1} [\log f(F^{-1}(z)) - \log f(F^{-1}(1-z))] dz = 0,$$

or equivalently

$$\int_0^{1/2} [z^{n-1} - (1-z)^{n-1}] \log \frac{f\big(F^{-1}(z)\big)}{f\big(F^{-1}(1-z)\big)} dz = 0.$$

Given that $n \geq 1$, we have $0 < z^{n-1} < (1-z)^{n-1} < 1$, over $z \in (0,1/2)$. Consequently, the term within the brackets of the integrand in the preceding equation becomes negative. Thus, for almost all $z \in (0,1/2)$, it holds that $f(F^{-1}(z)) = f(F^{-1}(1-z)$, as $f \in \mathcal{C}$ by assumption. Thus, the proof of the theorem gets finalized by applying Lemma 2.1.

The following theorem provides an analogue of the previous result, now for all $r \in D \setminus 0 = \{1, \ldots, \lfloor n/2 \rfloor\}$.

Theorem 3.3. Under the assumptions of Theorem 3.1, the following are equivalent:

(i) X has a symmetric distribution;

(ii) $f \in \mathcal{C}$ and $\mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(T_{n-r|n:F})$, for some n $(n \geq 2r)$ and a fixed $r \in D \setminus 0$.

Proof. As the demonstration of (i) and (ii) is straightforward, we show that (i) is implied by (ii). Assume that there exists some $n \geq 2$ and a fixed $r \in D \setminus 0$ such that $\mathcal{S}(T_{n-r|n:G}) = \mathcal{S}(T_{n-r|n:F})$. Then, from (14), and observing that $\rho_{n-r|n:G}(u) = \rho_{n-r|n:F}(1-u)$, it can be seen that:

$$\begin{aligned}
\int_0^1 \rho_{n-r|n:G}(u) \log f(F^{-1}(u)) du &= \int_0^1 \rho_{n-r|n:F}(u) \log f\big(F^{-1}(u)\big) du, \\
\int_0^1 \rho_{n-r|n:F}(1-u) \log f(F^{-1}(u)) du &= \int_0^1 \rho_{n-r|n:F}(u) \log f\big(F^{-1}(u)\big) du, \\
\int_0^1 \rho_{n-r|n:F}(z) \log f(F^{-1}(1-z)) du &= \int_0^1 \rho_{n-r|n:F}(z) \log f\big(F^{-1}(z)\big) dz,
\end{aligned}$$

where the last equality follows from the substitution $z = 1 - u$. Thus, for a fixed $r \in D \setminus 0$, we have

$$\int_0^1 \rho_{n-r|n:F}(z) \big[\log f\big(F^{-1}(z)\big) - \log f\big(F^{-1}(1-z)\big)\big] dz = 0,$$

which can be rewritten as

$$\int_0^{1/2} [\rho_{n-r|n:F}(z) - \rho_{n-r|n:F}(1-z)] \log \frac{f\big(F^{-1}(z)\big)}{f\big(F^{-1}(1-z)\big)} dz = 0. \tag{16}$$

We now demonstrate that the function

$$\rho_{n-r|n:F}(z) = a_{r,n} z^{n-r-1} - b_{r,n} z^{n-r}$$

is increasing for $z \in (0,1/2)$ and $r \in D \setminus 0$. Its derivative is

$$\rho'_{n-r|n:F}(z) = (n-r)z^{n-r-2}[(n-r-1)(r+1) - r(n-r+1)z].$$

For this identity, we have $\rho'_{n-r|n:F}(z) > 0$ for all $z < z_r$, where $z_r = \frac{(n-r-1)(r+1)}{i(n-r+1)}$ and $1 \leq r \leq \lfloor n/2 \rfloor$. As z_r is decreasing in r, $\rho_{n-r|n:F}(z)$ is increasing for $z < z_{\lfloor n/2 \rfloor}$. For even n, $z_{\lfloor n/2 \rfloor} = 1 - \frac{2}{n}$, and for odd $n, z_{\lfloor n/2 \rfloor} = 1 - \frac{2}{n+3}$. In both cases, $z_{\lfloor n/2 \rfloor} > 0.5$. Thus, $\rho_{n-r|n:F}(z)$ is increasing over $z \in (0,1/2)$ for $1 \leq r \leq \lfloor n/2 \rfloor$. Consequently, the term inside the brackets of the integrand in (20) is negative, implying $f(F^{-1}(z)) = f(F^{-1}(1-z))$ for almost all $z \in (0,1/2)$, as $f \in \mathcal{C}$. The proof of the theorem then gets completed using Lemma 2.1.

Remark 3.2. It is worth noting that the class $\mathcal{C}$ is nonempty and, in fact, encompasses several prominent distributional families. Examples include the Gumbel, power, Pareto, and standard normal distributions, as documented in [20].

The following examples illustrate the implications of the theorem and demonstrate how the characterization operates in concrete settings. For additional entropy-based characterizations and further illustrative examples, the reader is referred to [20].

Example 3.2. Following Example 3.1, we have $f(F^{-1}(u)) = bu^{\frac{b-1}{b}}$ and $f(F^{-1}(1-u)) = b(1-u)^{\frac{b-1}{b}}$, for $0 < u < 1$. For $u \in (0,1/2), f(F^{-1}(u)) \leq f(F^{-1}(1-u))$ if $b > 1$ or $b < 1$, and $f(F^{-1}(u)) = f(F^{-1}(1-u))$ if $b = 1$. Therefore, the power distribution belongs to class $\mathcal{C}$. Now, considering Theorem 3.3, we have

$$\begin{aligned}
\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) &= \int_0^{1/2} [\rho_{n-r|n:F}(u) - \rho_{n-r|n:F}(1-u)] \log \frac{f(F^{-1}(u))}{f(F^{-1}(1-u))} du \\
&= \int_0^{1/2} [\rho_{n-r|n:F}(u) - \rho_{n-r|n:F}(1-u)] \log \left(\frac{u}{1-u}\right)^{\frac{b-1}{b}} du \\
&= \frac{b-1}{b} \int_0^{1/2} [\rho_{n-r|n:F}(u) - \rho_{n-r|n:F}(1-u)] \log \left(\frac{u}{1-u}\right) du \\
&= \frac{b-1}{b} \Theta_{r,n},
\end{aligned}$$

where $\Theta_{r,n} = \int_0^{1/2} [\rho_{n-r|n:F}(u) - \rho_{n-r|n:F}(1-u)] \log \left(\frac{u}{1-u}\right) du$, for some n $(n \geq 2r)$ and a fixed $r \in D \setminus \{0\}$. As $\chi_{r,n} > 0$, $\mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) = 0$ when $b = 1$, indicating that the power distribution is symmetric for $b = 1$.

Example 3.3. For a Pareto distribution with probability density function $f(x) = bx^{-b-1}$ and cumulative distribution function $F(x) = 1 - x^{-b}$, for $x > 1$ and $b > 0$, we obtain $f(F^{-1}(u)) = b(1-u)^{\frac{b+1}{b}}$ and $f(F^{-1}(1-u)) = bu^{\frac{b+1}{b}}$, for $0 < u < 1$. As $f(F^{-1}(u)) > f(F^{-1}(1-u))$, for $u \in (0,1/2)$, the Pareto distribution belongs to class $\mathcal{C}$. Consequently, by Theorem 3.3, we have

$$\begin{aligned}\mathcal{S}(T_{n-r|n:G})-\mathcal{S}(T_{n-r|n:F}) &= \int_0^{1/2}[\rho_{n-r|n:F}(u)-\rho_{n-r|n:F}(1-u)]\log\frac{f(F^{-1}(u))}{f(F^{-1}(1-u))}du\\ &= \int_0^{1/2}[\rho_{n-r|n:F}(u)-\rho_{n-r|n:F}(1-u)]\log\left(\frac{1-u}{u}\right)^{\frac{b+1}{b}}du\\ &= \frac{b+1}{b}\int_0^{1/2}[\rho_{n-r|n:F}(u)-\rho_{n-r|n:F}(1-u)]\log\left(\frac{1-u}{u}\right)du\\ &= \frac{b+1}{b}\Theta_{r,n},\end{aligned}$$

where $\Theta_{r,n}=\int_0^{1/2}[\rho_{n-r|n:F}(u)-\rho_{n-r|n:F}(1-u)]\log\left(\frac{1-u}{u}\right)du$, for some n $(n\geq 2r)$ and a fixed $r\in D\setminus\{0\}$. Given that $\Theta_{r,n}>0$ and $b>0$, it follows that $\mathcal{S}(T_{n-r|n:G})>\mathcal{S}(T_{n-r|n:F})$ for all $b>0$, which is a consequence of Pareto distribution's asymmetry.

Analogous to the Kerridge inaccuracy measure in (3) for comparing the densities f and g, a corresponding inaccuracy measure can be defined between the parent distribution and the distribution of the (n−r)-out-of-n:G system. Specifically, we define

$$\mathcal{K}(T_{n-r|n:G},X)=-\int_0^{\infty}f_{n-r|n:G}(x)\log f(x)\,dx=-\int_0^1\rho_{n-r|n:G}(u)\log f\left(F^{-1}(u)\right)du, \quad (17)$$

for $n\geq 2r$ and $r\in\mathcal{D}$. A result similar to the one established in the preceding theorem also holds for $\mathcal{K}(T_{n-r|n:F},X)$.

The following theorem delivers a rigorous characterization of symmetric distributions by leveraging the Kerridge inaccuracy measure within the structure of consecutive k-out-of-n systems.

Theorem 3.4. Suppose the assumptions of Theorem 3.1 hold. Then, F is symmetric if and only if, for a fixed $r,\mathcal{K}(T_{n-r|n:G},X)=\mathcal{K}(T_{n-r|n:F},X)$, for all $n\geq 2r$, where $r\in\mathcal{D}$.

Proof. The result follows by applying arguments similar to those used in the proof of Theorem 3.1, with the necessary modifications to accommodate the Kerridge inaccuracy functional. In the same spirit as Theorem 4.3, an analogous characterization can be derived for the inaccuracy measure associated with consecutive k-out-of-n systems.

Theorem 3.5. Under the assumptions of Theorem 3.1, the following are equivalent:

(i) X has a symmetric distribution;

(ii) $f\in\mathcal{C}$ and $\mathcal{K}(T_{n-r|n:G})=\mathcal{K}(T_{n-r|n:F})$, for some n $(n\geq 2)$ and a fixed i, where $r\in\mathcal{D}$.

4. Log-symmetry analysis via Rényi entropy

The results in this section build upon the theoretical framework established in Section 3 and require similar regularity conditions on the underlying distribution, including continuity and integrability assumptions. The transition from Shannon entropy to its Rényi generalization offers a natural and powerful extension for analyzing distributional symmetry. Now by applying [25, Theorem 1], we obtain the following representation for the Rényi entropy of consecutive $(n-r)$-out-of-n:G system:

$$\mathcal{S}_\alpha(T_{n-r|n:G})=\epsilon(\alpha)\log\int_0^1\rho_{n-r|n:G}^{\alpha}(u)f^{\alpha-1}\left(F^{-1}(u)\right)du, \quad (18)$$

where $\alpha>0,\alpha\neq 1$ and for all $n\geq 2r$.

The same result also holds for consecutive $(n-r)$-out-of-n:F systems as

$$\mathcal{S}_\alpha(T_{n-r|n:F}) = \epsilon(\alpha)\log\int_0^1 \rho_{n-r|n:F}^{\alpha}(u) f^{\alpha-1}\big(F^{-1}(u)\big)du. \tag{19}$$

The following theorem shows that the equality of Rényi entropies of $T_{n-r|n:G}$ and $T_{n-r|n:F}$ provides distinctive characterization of symmetric distributions.

Theorem 4.1. Let $X_1,\dots,X_n$ be i.i.d. continuous random variables with cdf F. Then, F is symmetric if and only if, for fixed values of α and $r \in \mathcal{D}, \mathcal{S}_\alpha(T_{n-r|n:G}) = \mathcal{S}_\alpha(T_{n-r|n:F})$, for all $n \geq 2r$.

Proof. For a symmetric distribution centered at μ (without loss of generality, take $\mu = 0$), (18) and (19) imply that

$$\begin{aligned}
\mathcal{S}_\alpha(T_{n-r|n:G}) = {} & \epsilon(\alpha)\log\int_0^1 \rho_{n-r|n:G}^{\alpha}(u) f^{\alpha-1}\big(F^{-1}(u)\big)du \\
& = \epsilon(\alpha)\log\int_0^1 \rho_{n-r|n:F}^{\alpha}(1-u) f^{\alpha-1}\big(F^{-1}(u)\big)du \\
& = \epsilon(\alpha)\log\int_0^1 \rho_{n-r|n:F}^{\alpha}(z) f^{\alpha-1}\big(F^{-1}(1-z)\big)dz, \text{ taking } z = 1-u \\
& = \epsilon(\alpha)\log\int_0^1 \rho_{n-r|n:F}^{\alpha}(z) f^{\alpha-1}\big(F^{-1}(z)\big)du \quad = \mathcal{S}_\alpha(T_{n-r|n:F}),
\end{aligned}$$

where the final equality follows based on the symmetry of F. This completes the proof of the necessity part. For the sufficiency part, let us assume $\mathcal{S}_\alpha(T_{n-r|n:G}) = \mathcal{S}_\alpha(T_{n-r|n:F})$, for all $n \geq 2r$ and fixed α and $r \in \mathcal{D}$. Then, using the Rényi entropies of consecutive $(n-r)$-out-of-n:G and $(n-r)$-out-of-n:F systems, we have

$$\begin{aligned}
\int_0^1 \rho_{n-r|n:F}^{\alpha}(u) f^{\alpha-1}(F^{-1}(u))du &= \int_0^1 \rho_{n-r|n:G}^{\alpha}(u) f^{\alpha-1}(F^{-1}(u))du \\
&= \int_0^1 \rho_{n-r|n:F}^{\alpha}(1-u) f^{\alpha-1}(F^{-1}(u))du \\
&= \int_0^1 \rho_{n-r|n:F}^{\alpha}(z) f^{\alpha-1}\big(F^{-1}(1-z)\big)dz,
\end{aligned}$$

where the second equality follows from the identity $\rho_{n-r|n:G}(u) = \rho_{n-r|n:F}(1-u)$, for $0 < u < 1$ and $n \geq 2r$, and the last equality follows from the substitution $z = 1-u$. Therefore, we have

$$\int_0^1 \rho_{n-r|n:F}^{\alpha}(z)[f^{\alpha-1}(F^{-1}(z)) - f^{\alpha-1}(F^{-1}(1-z))]dz = 0,$$

for a fixed α and $r \in \mathcal{D}$. Equation

$$\int_0^1 \phi_{r,n}^{\alpha}(z)\big[f^{\alpha-1}\big(F^{-1}(z)\big) - f^{\alpha-1}\big(F^{-1}(1-z)\big)\big] z^{(n-2r)\alpha} du = 0$$

holds for a fixed α and $r \in \mathcal{D}$, where $\phi_{r,n}(z) = a_{r,n}z^{r-1} - b_{r,n}z^r$ for $0 < z < 1$. Applying Lemma 2.2 with $\psi(z) = \phi_{r,n}^{\alpha}(z)[f^{\alpha-1}(F^{-1}(z)) - f^{\alpha-1}(F^{-1}(1-z))]$ and the complete sequence $\{(u^{n-2r})\alpha, n \geq 2r\}$, we conclude that $f \in \mathcal{C}_e$. This completes the proof of the theorem using Lemma 2.1.

The conclusions of Theorem 3.1 carry over to Rényi entropy setting, as shown in the following theorem.

Theorem 4.2. Let $X_1,\dots,X_n$ be i.i.d. continuous random variables with cdf F. Then, the following are equivalent:

(i) X has a symmetric distribution;

(ii) $f \in \mathcal{C}$ and $\mathcal{S}_\alpha(T_{n|n:G}) = \mathcal{S}_\alpha(T_{n|n:F})$, for fixed values of α and $n \geq 1$.
Proof. The implication (i) $\Longrightarrow$ (ii) is clear. Conversely, let us assume that $\mathcal{S}_\alpha(T_{n|n:G}) = \mathcal{S}_\alpha(T_{n|n:F})$, for a fixed value of α and $n \geq 1$. Then,

$$\int_0^1 (1-u)^{(n-1)\alpha} f^{\alpha-1}\big(F^{-1}(u)\big)du = \int_0^1 u^{(n-1)\alpha} f^{\alpha-1}\big(F^{-1}(u)\big)du.$$

The substitution $z = 1-u$ yields

$$\int_0^1 z^{(n-1)\alpha}[f^{\alpha-1}(F^{-1}(z)) - f^{\alpha-1}(F^{-1}(1-z))]dz = 0,$$

which is equivalent to

$$\int_0^{1/2} \left[z^{(n-1)\alpha} - (1-z)^{(n-1)\alpha}\right][f^{\alpha-1}(F^{-1}(z)) - f^{\alpha-1}(F^{-1}(1-z))]dz = 0.$$

As $n \geq 1$ and $\alpha > 0$, we have $(n-1)\alpha > 0$, and so $0 < z^{(n-1)\alpha} < (1-z)^{(n-1)\alpha} < 1$, for $z \in (0,1/2)$ and fixed $\alpha > 0, \alpha \neq 1$. Therefore, the first factor in the integrand is negative. Consequently, $f(F^{-1}(z)) = f(F^{-1}(1-z))$ for almost all $z \in (0,1/2)$, given $f \in \mathcal{C}$. Then, the proof of the theorem gets completed using Lemma 2.1.
The following theorem provides a corresponding extension for all $r \in \{1, \dots, \lfloor n/2 \rfloor\}$.
Theorem 4.3. Suppose the assumptions of Theorem 3.1 hold. Then, the following two statements are equivalent:
(i) X has a symmetric distribution;
(ii) $f \in \mathcal{C}$ and $\mathcal{S}_\alpha(T_{n-r|n:G}) = \mathcal{S}_\alpha(T_{n-r|n:F})$, for some $n, (n \geq 2)$ and fixed values of α and $r \in D \setminus 0$.
Proof. As the demonstration of (i) and (ii) is straightforward, we show that (i) is implied by (ii). Assume there exists some $n \geq 2$ and fixed values of α and $r \in D \setminus 0$, such that $\mathcal{S}_\alpha(T_{n-r|n:G}) = \mathcal{S}_\alpha(T_{n-r|n:F})$. Then, from (14), and observing that $\rho_{n-r|n:G}(u) = \rho_{n-r|n:F}(1-u)$, we have

$$\begin{aligned}
\int_0^1 \rho^\alpha_{n-r|n:G}(u) f^{\alpha-1}(F^{-1}(u))du &= \int_0^1 \rho^\alpha_{n-r|n:F}(u) f^{\alpha-1}\big(F^{-1}(u)\big)du,\\
\int_0^1 \rho^\alpha_{n-r|n:F}(1-u) f^{\alpha-1}(F^{-1}(u))du &= \int_0^1 \rho^\alpha_{n-r|n:F}(u) f^{\alpha-1}\big(F^{-1}(u)\big)du,\\
\int_0^1 \rho^\alpha_{n-r|n:F}(z) f^{\alpha-1}(F^{-1}(1-z))dz &= \int_0^1 \rho^\alpha_{n-r|n:F}(z) f^{\alpha-1}\big(F^{-1}(z)\big)dz,
\end{aligned}$$

where the last equality follows from the substitution $z = 1-u$. Thus, for fixed values of α and $r \in D \setminus 0$, we have

$$\int_0^1 \rho^\alpha_{n-r|n:F}(z)\big[f^{\alpha-1}\big(F^{-1}(u)\big) - f^{\alpha-1}\big(F^{-1}(u)\big)\big]dz = 0,$$

which can be rewritten as

$$\int_0^{1/2} \left[\rho^\alpha_{n-r|n:F}(z) - \rho^\alpha_{n-r|n:F}(1-z)\right][f^{\alpha-1}(F^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))]dz = 0. \tag{20}$$

As given in the proof of Theorem 4.3, we observe that $\rho^\alpha_{n-r|n:F}(z)$ is increasing for $z \in (0,1/2)$ and fixed values of α and $r \in D \setminus 0$. Consequently, the term inside the brackets of the integrand in (20) is negative, implying $f(F^{-1}(z)) = f(F^{-1}(1-z))$ for almost all $z \in (0,1/2)$, as $f \in \mathcal{C}$. Thus, the proof of the theorem gets completed by Lemma 2.1.

5. Entropy-based test for log-symmetry of the distribution

In this section, we introduce an entropy-based framework for testing log-symmetry of the underlying distribution based on the characterization principles established in Theorem 3.1. Given a random sample $Y_1, Y_2, \dots, Y_N$ from an absolutely continuous cumulative distribution function F_Y, with order statistics $Y_{1:N} \leq Y_{2:N} \leq \cdots \leq Y_{N:N}$, we consider the problem of testing log-distributional symmetry about a point μ_Y. Specifically, we test the hypotheses

$$H_0\colon F_Y(\mu_Y + y) = 1 - F_Y(\mu_Y - y), \text{ for all } y > 0, \textit{ vs.}$$
$$H_1\colon \exists\ y > 0 \text{ such that } F_Y(\mu_Y + y) \neq 1 - F_Y(\mu_Y - y),$$

where μ_Y denotes the center of log-symmetry.
Based on (14), we construct the following test statistic for log-symmetry:

$$\Lambda_{r,n} = \mathcal{S}(T_{n-r|n:G}) - \mathcal{S}(T_{n-r|n:F}) = \int_0^1 [\rho_{n-r|n:F}(u) - \rho_{n-r|n:G}(u)] \log f_Y(F_Y^{-1}(u))du$$
$$= \int_0^1 [\rho_{n-r|n:G}(u) - \rho_{n-r|n:F}(u)] \log\left[\frac{dF_Y^{-1}(u)}{du}\right] du, \text{valid for } n \geq 2r. \tag{21}$$

The proposed test statistic is constructed as a difference between entropy measures associated with transformed system lifetimes, and its behavior reflects the direction and magnitude of deviation from log-symmetry. Under log-symmetry, the entropy measures corresponding to the transformed samples are expected to be balanced, leading to values of the test statistic close to zero. In contrast, in the presence of asymmetry, this balance is disrupted. For instance, right-skewed distributions typically induce systematic increases in the entropy of the transformed lifetimes, resulting in positive values of the test statistic, whereas left-skewed behavior may produce negative deviations. More complex forms of asymmetry, such as heavy tails or multimodality, can lead to larger fluctuations in the statistic, reflecting the sensitivity of the entropy-based measure to global distributional features.

To assess the log-symmetry of the distribution F, we estimate $\Lambda_{r,n}$ using a difference operator-based estimator, as proposed by [26], to approximate $dF_Y^{-1}(u)/\mathrm{du}$. This estimator uses the empirical distribution function to estimate the distribution function of the random variable in (21), yielding the following estimator for $\Lambda_{r,n}$

$$\widehat{\Lambda}_{r,n} = \frac{1}{N}\sum_{k=1}^{N} \left(\rho_{n-r|n:G}\left(\frac{k}{N+1}\right) - \rho_{n-r|n:F}\left(\frac{k}{N+1}\right)\right) \log\left(\frac{N(Y_{k+m:N} - Y_{k-m:N})}{2m}\right), \tag{22}$$

for $n \geq 2r$, where $m < N/2$ is the window size. The window parameter plays a smoothing role in the estimation process. Smaller values increase sensitivity to local fluctuations but may introduce higher variability, whereas larger values improve stability at the expense of local responsiveness. In practice, moderate choices of the window parameter provide a balance between bias and variance. When $k - m \leq 1, Y_{k-m:N}$ defaults to $Y_{1:N}$, and when $k + m \geq N, Y_{k+m:N}$ defaults to $Y_{N:N}$. Rejecting the null hypothesis of symmetry occurs for small or large values of $\widehat{\Lambda}_{r,n}$. Consistency is vital for estimators, particularly those for parametric functions. Under standard regularity conditions and for sufficiently large sample sizes, the proposed entropy-based test statistic is expected to exhibit asymptotic normal behavior due to its dependence on empirical entropy estimators. This provides a theoretical justification for its stability and supports its use in large-sample settings.

The next theorem presents a rigorous formulation of the statement given in Eq (22). The structure of its proof parallels the arguments in [26, Theorem 1], with additional methodological insights drawn from [27,28], where similar techniques have been used to establish the consistency of related test statistics.

Theorem 5.1. Let $Y_1, Y_2, \dots, Y_N$ be a random sample of size N from a population with pdf f_Y, cdf F_Y, and finite variance. Then, as $N \to +\infty, m \to +\infty$, and $\frac{m}{N} \to 0, \widehat{\Lambda}_{r,n} \overset{p}{\to} \Lambda_{r,n}$, where $\overset{p}{\to}$ denotes convergence in probability.

A location shift of the random variable Y does not affect the variance of $\widehat{\Lambda}_{,n}$ when estimating $\Lambda_{r,n}$, whereas scale transformations do alter it. This result follows from the arguments in [29].

Theorem 5.2. Assume that $Y_1, Y_2, \dots, Y_N$ is a random sample of size N taken from a population with pdf f_Y and cdf F_Y and $Z_i = aY_i + b, a > 0, b \in R$. Denote the estimators for $\Lambda_{r,n}$ on the basis of Y_i and Z_i by $\widehat{\Lambda}^Y_{r,n}$ and $\widehat{\Lambda}^Z_{r,n}$, respectively. Then, the following properties apply:

(i) $\mathbb{E}(\widehat{\Lambda}^Z_{r,n}) = \mathbb{E}(\widehat{\Lambda}^Y_{r,n}) + \log(a)$,

(ii) $\mathrm{Var}(\widehat{\Lambda}^Z_{r,n}) = \mathrm{Var}(\widehat{\Lambda}^Y_{r,n})$,

Proof. Equation (22) shows that

$$\begin{aligned}
\widehat{\Lambda}^Z_{r,n} &= \frac{1}{N}\sum_{k=1}^{N}\left(\rho_{n-r|n:G}\left(\frac{k}{N+1}\right) - \rho_{n-r|n:F}\left(\frac{k}{N+1}\right)\right)\log\frac{N(Z_{k+m:N} - Z_{k-m:N})}{2m} \\
&= \frac{1}{N}\sum_{k=1}^{N}\left(\rho_{n-r|n:G}\left(\frac{k}{N+1}\right) - \rho_{n-r|n:F}\left(\frac{k}{N+1}\right)\right)\log\frac{Na(Y_{k+m:N} - Y_{k-m:N})}{2m} \\
&= \widehat{\Lambda}^Y_{r,n} + \log(a).
\end{aligned}$$

The proof concludes using the mean and variance of $\widehat{\Lambda}^Z_{r,n} = \widehat{\Lambda}^Y_{r,n} + \log(a)$.

To simplify the analysis, we set $r = 1$ and $n = 3$ and employ the sample estimate

$$\widehat{\Lambda}_{1,3} = \frac{1}{N}\sum_{k=1}^{N}\left(\rho_{2|3:G}\left(\frac{k}{N+1}\right) - \rho_{2|3:F}\left(\frac{k}{N+1}\right)\right)\log\left(\frac{N(Y_{k+m:N} - Y_{k-m:N})}{2m}\right), \tag{23}$$

of $\Lambda_{1,3} = \mathcal{S}(T_{2|3:G}) - \mathcal{S}(T_{2|3:F})$ to test for log-symmetry in Y. We reject the log-symmetry hypothesis for extreme values of $\Lambda_{1,3}$. Under the null hypothesis $H_0, \widehat{\Lambda}_{1,3}$ converges to zero as $N \to \infty$. Critical values are calculated using the first case of the generalized lambda distribution (GLD). A random variable X follows a GLD with parameters $\lambda_i, i = 1,2,3,4$, if its quantile function is defined as

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \text{ for } 0 < u < 1.$$

Parameters λ_3 and λ_4 govern skewness and kurtosis, λ_2 influences variance, and λ_1 determines the mean. When $\lambda_3 = \lambda_4$, the distribution is symmetric around λ_1. Nine commonly used GLD cases for evaluating symmetry tests are presented in Table 1, and probability density function shapes can be found in [30, Figure 4]. We reject the null hypothesis at significance level α for sample size N if $|\widehat{\Lambda}_{1,3}| \geq \widehat{\Lambda}_{1,3,(1-\alpha)}$. Because the distribution of $\widehat{\Lambda}_{1,3}$ is complex and sensitive to N and the window parameter m, we use Monte Carlo simulations with 10000 samples to determine critical values. Specifically, we generate samples of sizes $N = 5, 10, 20, 30, 40, 50, 100$ from the null distribution and estimated the $(1-\alpha)$-th quantile.

It is important to note that the estimator underlying the proposed test statistic is constructed via a difference-operator approximation to the derivative of the quantile function, which is inherently sensitive to sample variability, particularly in small samples. In such settings, the estimator may exhibit moderate bias and increased variability due to discretization effects and local fluctuations in the empirical quantile function. This behavior, however, is intrinsic to finite-difference approximations and does not compromise the overall reliability of the proposed testing procedure. Importantly, the entropy-based formulation of the test statistic aggregates information over the distribution, thereby mitigating the impact of local estimation errors and enhancing stability. This is supported by the simulation results, which demonstrate that the test maintains satisfactory performance even for moderate sample sizes. Further methodological refinements, such as the incorporation of smoothing techniques or higher-order approximation schemes, may improve the small-sample properties of the estimator and constitute a natural direction for future research.

To ensure a parsimonious and well-focused simulation design, the power of the proposed test is evaluated using a symmetric GLD under the null hypothesis of log-symmetry. The parameter values used in the simulations are reported in Table 1. This enables us to examine the finite-sample behavior of the proposed entropy-based test under log-symmetry without introducing unnecessary complexity. In particular, it enables a clear assessment of type I error control and stability of the test statistic in a symmetric setting, which is the primary objective under the null hypothesis. The GLD is adopted due to its high flexibility in representing a wide range of distributional shapes, including symmetric, skewed, and moderately heavy-tailed behaviors. This makes it a suitable benchmark for assessing the performance of the proposed test under diverse structural conditions. In other words, Table 1 presents the parameter configurations of the GLD used in the simulation study. Case 1 ($\lambda_3 = \lambda_4$) corresponds to a symmetric distribution and is employed to evaluate the Type I error rate of the proposed test under the null hypothesis of log-symmetry. Cases 2–9 ($\lambda_3 \neq \lambda_4$) are asymmetric distributions with varying degrees of skewness and kurtosis; these are used to assess the power of the test against different alternatives. Among these, Cases 2 and 3 represent near-symmetric alternatives, Cases 4–7 correspond to moderate asymmetry, and Cases 8 and 9 exhibit strong asymmetry.

Table 1. Parameter values of a symmetric GLD used to evaluate the proposed entropy-based test under the null hypothesis of log-symmetry and under alternative settings where α_3 and α_4 represent skewness and kurtosis, respectively.

Case	λ_1	λ_2	λ_3	λ_4	α_3	α_4
1	0.0000	0.1975	0.1349	0.1349	0.0000	3.0000
2	-0.1167	-0.3517	-0.1300	-0.1600	0.8000	11.4000
3	0.0000	-1.0000	-0.1000	-0.1800	2.0000	21.2000
4	3.5865	0.0431	0.0252	0.0940	0.9000	4.2000
5	0.0000	-1.0000	-0.0075	-0.0300	1.5000	7.5000
6	0.0000	1.0000	1.4000	0.2500	0.5000	2.2000
7	0.0000	1.0000	0.0001	0.1000	1.5000	5.8000
8	0.0000	-1.0000	-0.0010	-0.1300	3.1600	23.8000
9	0.0000	-1.0000	-0.0001	-0.1700	3.8800	40.7000

The performance of the proposed test depends on the choice of the window parameter m, as is common in entropy-based estimation procedures. In our implementation, m is selected as a function of the sample size to balance bias and variance. A sensitivity analysis conducted over a range of admissible values of m indicates that the test results remain stable for moderate variations of this parameter. In particular, no significant changes in the empirical power or size of the test are observed within this range. This suggests that the proposed procedure is not overly sensitive to the specific choice of m, provided that it is selected within a reasonable interval. A more detailed investigation of optimal data-driven selection of m constitutes an interesting direction for future research.

5.1. Comparative study of test power

The proposed entropy-based testing procedure also exhibits robustness to moderate deviations from model assumptions. Since it is based on entropy measures that capture global distributional characteristics rather than relying on specific parametric forms, the test remains stable under mild model misspecification and variability in data structure. Several parameter configurations of the GLD considered in the simulations can approximate heavy-tailed behavior, thereby providing indirect evidence of the performance of the proposed test under such conditions. This flexibility also supports the robustness of the proposed procedure, as it enables the test to be evaluated under a wide range of distributional shapes, including settings that deviate from standard modeling assumptions. We note that for the choice $m = [N/2] - 1$, the computation of $Y_{k+m:N} - Y_{k-m:N}$ in (22) involves indices that may fall outside the range $1, \dots, N$ when k is small or large. In such cases, following standard practice in entropy-based testing, we set $Y_{k-m:N} = Y_{1:N}$ whenever $k - m \leq 1$, and $Y_{k+m:N} = Y_{1:N}$ whenever $k + m \geq N$. This boundary adjustment ensures that the estimator remains well-defined. It is worth noting that the choice $m = [N/2] - 1$ is determined following a preliminary simulation study that evaluates the empirical size and power of the test for $m = 2, 3, \dots, [N/2] - 1$ across sample sizes $N = 20, 30, 50, 100$. Values close to $[N/2] - 1$ stabilize the test while preserving higher power. Based on these findings, we adopt $m = [N/2] - 1$ for all simulations.

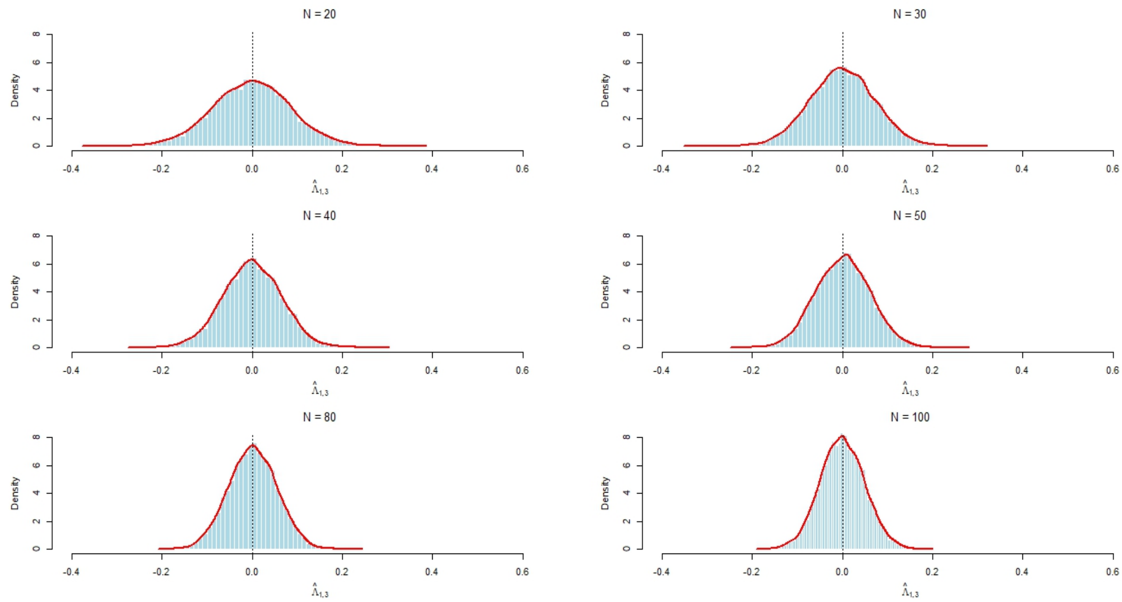

Figure 1. Histograms of $\widehat{\Lambda}_{1,3}$ under the null hypothesis (symmetric GLD, Case 1) for $N = 20, 30, 40, 50, 80, 100$ with $m = [N/2] - 1$ with 10,000 replications.

To further illustrate the finite-sample behavior of the proposed test statistic under the null hypothesis, Figure 1 displays histograms of $\widehat{\Lambda}_{1,3}$ based on 10,000 Monte Carlo replications for sample sizes $N = 20, 30, 40, 50, 80, 100$, using the symmetric GLD (Case 1 in Table 1) and window size $m = [N/2] - 1$. From the figure, for all sample sizes, the distribution of $\widehat{\Lambda}_{1,3}$ is centered approximately at zero, which is consistent with the null hypothesis of symmetry. Specifically, as N increases, the distribution becomes progressively more concentrated around zero and increasingly symmetric, indicating consistency of the estimator. This experimental setup enables a comprehensive sensitivity assessment of the test and ensures a robust evaluation across a range of distributional shapes. For sample sizes of 20, 30, 50, and 100, a total of 10,000 Monte Carlo replications are generated for each configuration. Power is reported as the proportion of rejections of the null hypothesis of log-symmetry at the 5% significance level. The empirical power is evaluated under a broad collection of alternatives drawn from the generalized lambda family, together with ten widely used symmetry tests for comparison. Power estimates are computed for the nine reference distributions listed in Table 1, using window sizes defined as $m = [N/2] - 1$. Critical values for the statistic $\widehat{\Lambda}_{1,3}$ at the 5% and 1% significance levels are given in Tables 2 and 3, respectively. These values provide the required thresholds for implementing the test across the sample sizes examined. The resulting values appear in Table 4. We investigate the performance of the proposed log-symmetry test based on the statistic $\widehat{\Lambda}_{1,3}$ by benchmarking it against the established procedures summarized in Table 5. Under a symmetric distribution (Case 1 in Table 1), all procedures exhibit rejection rates close to the nominal 0.05 level, confirming their correct behavior under the null. In the asymmetric settings (Cases 2 and 3, which represent near-symmetric alternatives), the proposed test, apart from $\widehat{D}_2$, achieves consistently strong performance and frequently surpasses the competing methods. These findings indicate that the proposed procedure is well-suited for practical applications where sensitivity to mild departures from symmetry is essential.

Table 2. Critical values of the $\widehat{\Lambda}_{1,3}$ statistic at $\alpha = 0.05$ level of significance.

m	N = 5	N = 10	N = 20	N = 30	N = 40	N = 50	N = 100
2	0.337480	0.332817	0.248638	0.194601	0.166579	0.146355	0.102773
3		0.288437	0.231631	0.192653	0.166169	0.143266	0.100325
4		0.238387	0.222997	0.185786	0.162502	0.145227	0.098050
5		0.179560	0.216528	0.185453	0.163816	0.141576	0.099719
6			0.203262	0.182095	0.159725	0.142282	0.098907
7			0.193798	0.184174	0.160965	0.142214	0.099150
8			0.180016	0.175320	0.158022	0.144272	0.100689
9			0.168054	0.171633	0.160028	0.147753	0.101133
10			0.146390	0.164778	0.153992	0.143982	0.100275
11				0.163605	0.154932	0.144048	0.101943

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m	N = 5	N = 10	N = 20	N = 30	N = 40	N = 50	N = 100
12				0.152948	0.152271	0.141767	0.102676
13				0.149072	0.151492	0.141645	0.102728
14				0.140064	0.146209	0.140619	0.105070
15				0.130124	0.142142	0.143189	0.107068
16					0.141217	0.139802	0.106648
17					0.137039	0.137556	0.105520
18					0.132326	0.135803	0.108570
19					0.128398	0.134512	0.108861
20					0.120920	0.131519	0.108157
21						0.129385	0.108696
22						0.125503	0.108736
23						0.121454	0.110790
24						0.118439	0.108556
25						0.116704	0.109283
26							0.109636
27							0.111259
28							0.110474
29							0.109429
30							0.112262

Table 3. Critical values of the $\widehat{\Lambda}_{1,3}$ statistic at $\alpha = 0.01$.

m	N = 5	N = 10	N = 20	N = 30	N = 40	N = 50	N = 100
2	0.335374	0.338007	0.245691	0.198773	0.166696	0.146442	0.102372
3		0.283134	0.236263	0.192829	0.165036	0.143716	0.100402
4		0.239196	0.222015	0.184328	0.163091	0.142606	0.097955
5		0.187968	0.213615	0.188665	0.162909	0.143705	0.098133
6			0.203479	0.183678	0.159274	0.142400	0.099036

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m	N = 5	N = 10	N = 20	N = 30	N = 40	N = 50	N = 100
7			0.192638	0.177518	0.158648	0.145767	0.099942
8			0.180675	0.174479	0.158714	0.140362	0.100679
9			0.168623	0.171073	0.158424	0.144113	0.099099
10			0.148344	0.165699	0.157388	0.143698	0.102542
11				0.160330	0.154764	0.141731	0.102240
12				0.155286	0.152198	0.142970	0.102704
13				0.150216	0.151789	0.140423	0.103384
14				0.140014	0.145087	0.143775	0.104386
15				0.131211	0.144018	0.139572	0.105299
16					0.142116	0.137785	0.105654
17					0.135995	0.140932	0.106949
18					0.131338	0.135932	0.107688
19					0.124631	0.132821	0.107809
20					0.121384	0.131132	0.106712
21						0.127555	0.108246
22						0.123258	0.109114
23						0.121674	0.108227
24						0.115953	0.110048
25						0.115765	0.111397
26							0.110145
27							0.111035
28							0.109858
29							0.110339
30							0.110239

Table 4 provides a comparative summary of the empirical powers of the proposed test and the benchmark procedures against GLD alternatives, computed at the 5% significance level**.**

Table 4. Empirical powers for the GLD with $\alpha = 0.05$.

Cas	N	R	RB	W	T	C_6	$M_{0.2}$	$M_{0.6}$	$L^{\star}_{n,0.8}$	RM	J_6	$\widehat{D}_2$	$\widehat{\Lambda}_{1,3}$
1	20	0.04	0.05	0.04	0.04	0.04	0.04	0.04	0.05	0.04	0.05	0.04	0.04
	30	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04
	50	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.04	0.05	0.04
	10	0.05	0.04	0.05	0.04	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.05
2	20	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.05	0.07	0.09	0.20
	30	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.06	0.06	0.06	0.09	0.27
	50	0.05	0.05	0.05	0.06	0.07	0.06	0.06	0.06	0.07	0.06	0.11	0.33
	10	0.05	0.05	0.05	0.07	0.09	0.05	0.06	0.06	0.10	0.08	0.15	0.40
3	20	0.06	0.07	0.05	0.07	0.08	0.07	0.08	0.08	0.11	0.11	0.60	0.32
	30	0.07	0.07	0.06	0.09	0.11	0.09	0.10	0.12	0.15	0.12	0.66	0.41
	50	0.08	0.09	0.06	0.13	0.20	0.12	0.15	0.14	0.25	0.20	0.77	0.54
	10	0.11	0.10	0.08	0.22	0.36	0.16	0.21	0.22	0.48	0.35	0.90	0.69
4	20	0.09	0.10	0.06	0.10	0.11	0.12	0.14	0.13	0.18	0.17	0.01	0.25
	30	0.11	0.12	0.07	0.14	0.21	0.16	0.19	0.22	0.28	0.24	0.02	0.40
	50	0.14	0.15	0.08	0.20	0.42	0.23	0.30	0.30	0.49	0.44	0.03	0.57
	10	0.21	0.20	0.12	0.38	0.75	0.39	0.52	0.57	0.81	0.75	0.13	0.76
5	20	0.11	0.13	0.06	0.13	0.15	0.15	0.19	0.16	0.25	0.23	0.98	0.43
	30	0.15	0.16	0.08	0.19	0.30	0.22	0.28	0.33	0.40	0.34	0.99	0.62
	50	0.19	0.21	0.10	0.28	0.58	0.33	0.43	0.45	0.66	0.60	0.99	0.79
	10	0.32	0.31	0.16	0.52	0.89	0.55	0.69	0.76	0.93	0.88	1.00	0.94
6	20	0.20	0.23	0.07	0.16	0.25	0.33	0.39	0.26	0.42	0.46	0.80	0.26
	30	0.30	0.33	0.09	0.23	0.60	0.54	0.67	0.64	0.65	0.71	0.92	0.38
	50	0.49	0.52	0.12	0.36	0.95	0.81	0.92	0.90	0.89	0.97	0.99	0.60
	10	0.78	0.78	0.19	0.63	1.00	0.98	0.99	1.00	0.99	1.00	1.00	0.90

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Cas	N	R	RB	W	T	C_6	$M_{0.2}$	$M_{0.6}$	$L^{\star}_{n,0.8}$	RM	J_6	$\widehat{D}_2$	$\widehat{\Lambda}_{1,3}$
7	20	0.31	0.35	0.09	0.28	0.42	0.49	0.57	0.33	0.59	0.64	0.99	0.76
	30	0.45	0.49	0.12	0.39	0.79	0.73	0.82	0.82	0.85	0.86	1.00	0.93
	50	0.68	0.70	0.18	0.60	0.99	0.93	0.97	0.97	0.98	0.99	1.00	0.99
	10	0.92	0.92	0.35	0.88	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00
8	20	0.37	0.42	0.10	0.33	0.49	0.58	0.65	0.36	0.66	0.71	0.99	0.90
	30	0.53	0.57	0.15	0.48	0.86	0.81	0.87	0.87	0.91	0.91	1.00	0.98
	50	0.76	0.78	0.23	0.69	0.99	0.96	0.98	0.99	0.99	0.99	1.00	0.99
	10	0.96	0.96	0.42	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	20	0.39	0.45	0.11	0.35	0.53	0.61	0.69	0.35	0.69	0.75	1.00	0.91
	30	0.58	0.61	0.15	0.49	0.87	0.84	0.90	0.89	0.92	0.92	1.00	0.98
	50	0.80	0.82	0.24	0.72	0.99	0.97	0.99	0.99	0.99	0.99	1.00	0.99
	10	0.98	0.98	0.44	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 5. Symmetry tests.

Test	1	2	3	4	5	6	7	8	9	10
Reference	[31]	[32]	[33]	[34]	[35]	[36]	[37]	[38]	[39]	[40]
Notation	R	RB	W	T	C_6	M_p	$L^{\star}_{n,0.8}$	RM	J_6	$\widehat{D}_2$

5.2. Illustrative data analyses

In this section, we illustrate the practical performance of the proposed entropy-based test for log-distributional symmetry through the analysis of real datasets. The primary objective of these applications is not to validate or compare specific parametric models, but rather to assess whether the underlying lifetime distributions exhibit log-symmetry, as characterized in the preceding sections.

To this end, three real datasets, studied in the literature, demonstrate the applicability of the proposed test in realistic settings. For each dataset, the hypothesis of log-symmetry is evaluated at the 5% nominal significance level using the proposed entropy-based test statistic. The selected datasets are widely used in the reliability and survival analysis literature and encompass real-world scenarios involving lifetime data with varying distributional characteristics. This diversity provides a meaningful and representative testbed for assessing the practical effectiveness of the proposed methodology under conditions commonly encountered in reliability applications.

Example 5.1. We analyze a real dataset originally reported in [30], consisting of active repair times (in hours) for an airborne communication transceiver:

Dataset: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Lifetime data of this type are typically right-skewed on the original scale, which motivates the investigation of symmetry after logarithmic transformation. Applying the proposed entropy-based test to the original observations yields a test statistic value of $\widehat{\Lambda}_{1,3}$=0.5075, with an associated p-value approximately equal to zero, providing strong evidence against symmetry on the original scale. This behavior is also evident from Figure 2 (left panel), which shows pronounced asymmetry in the empirical distribution. To assess log-distributional symmetry, we apply the proposed test to the transformed data Y=log(X).

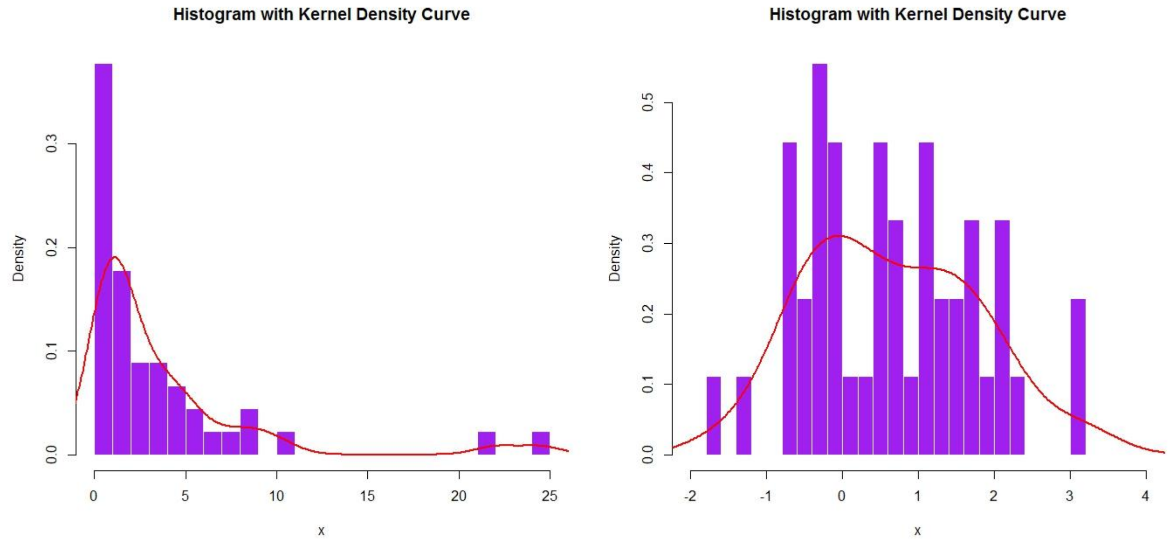

Figure 2. Histogram with kernel density estimates of the observed lifetime data (left panel) and the logarithmically transformed data (right panel).

As illustrated in Figure 2 (right panel), the distribution of the transformed observations appears approximately symmetric. This visual assessment is supported by the test result, yielding $\widehat{\Lambda}_{1,3}$=0.0359 with an estimated p-value of 0.5573. Consequently, the null hypothesis of log-symmetry is not rejected for the transformed data.

Example 5.2. We analyze a real dataset reported in [41], representing the number of failures (in thousands of cycles) observed for electrical appliances in a life test. The dataset consists of 50 observations:

Dataset: 0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100.

The empirical distribution of the observed data exhibits noticeable asymmetry on the original scale, as illustrated in Figure 3 (left panel). Applying the proposed entropy-based test to the original observations yields a test statistic value of $\widehat{\Lambda}_{1,3}$=0.1306 with an associated p-value of 0.0340, providing evidence against symmetry on the original scale.

To assess log-distributional symmetry, we apply the proposed test to the logarithmically transformed data Y = log(X). As shown in Figure 3 (right panel), the transformed data also display asymmetric behavior. This observation is confirmed by the test result, which yields $\widehat{\Lambda}_{1,3}$ = 0.2718 with an estimated p-value approximately equal to zero. Consequently, the null hypothesis of log-symmetry is rejected for this dataset.

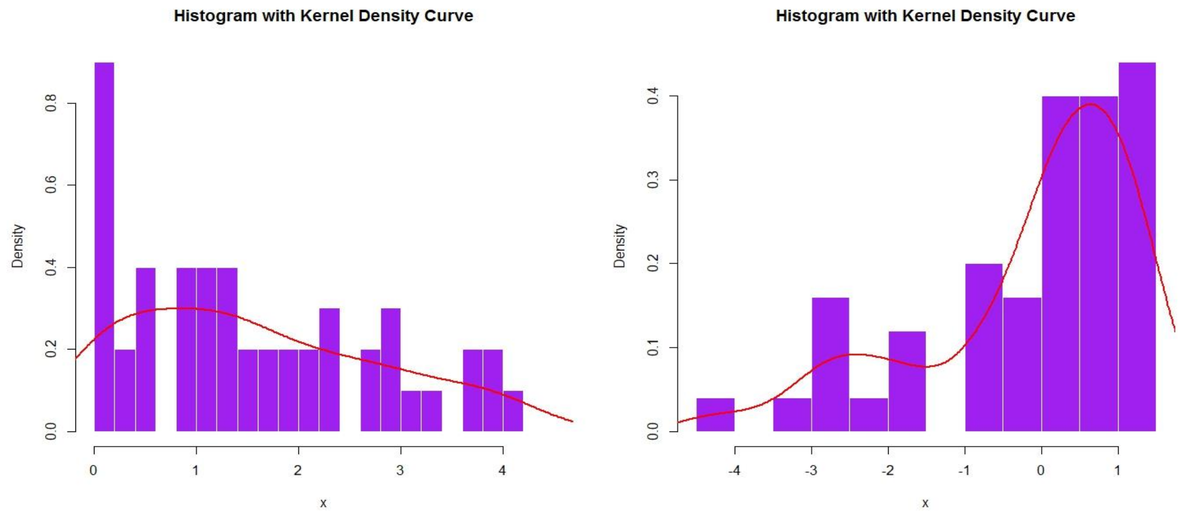

Figure 3. Histogram with kernel density estimates of the observed lifetime data (left panel) and the logarithmically transformed data (right panel).

Example 5.3. We consider a dataset consisting of 29 observations representing the times between successive air conditioning failures in a Boeing 720 airplane. The observed data are as follows:
Dataset: 90, 10, 60, 186, 61, 49, 14, 24, 56, 20, 79, 84, 44, 59, 29, 118, 25, 156, 310, 76, 26, 44, 23, 62, 130, 208, 70, 101, 208.
The empirical distribution of the observed data exhibits pronounced asymmetry on the original scale, as illustrated in Figure 4 (left panel). Applying the proposed entropy-based test to the original observations yields a test statistic value of $\widehat{\Lambda}_{1,3}$=0.3043 with an associated p-value approximately equal to zero, providing strong evidence against symmetry on the original scale. To examine log-distributional symmetry, we apply the proposed test to the logarithmically transformed data Y=log(X).

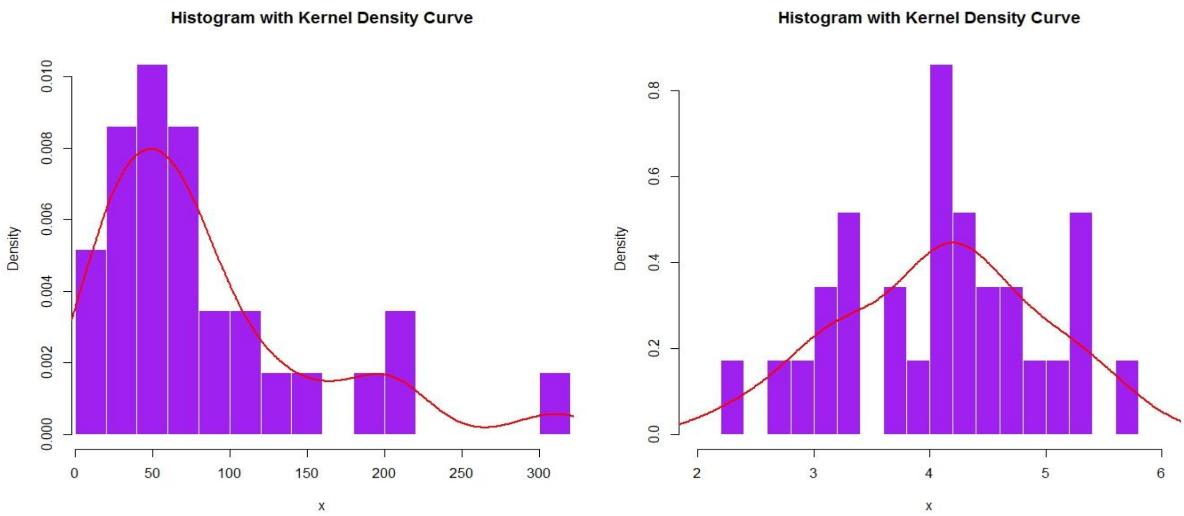

Figure 4. Histogram with kernel density estimates of the observed lifetime data (left panel) and the logarithmically transformed data (right panel).

As shown in Figure 4 (right panel), the transformed data appear approximately symmetric. This visual assessment is supported by the test result, which yields $\widehat{\Lambda}_{1,3}$ = 0.0234 with an estimated p-value of 0.7438. Consequently, the null hypothesis of log-symmetry is not rejected for the transformed data.

For completeness, the conclusions obtained from the proposed test are consistent with those typically suggested by classical graphical and transformation-based diagnostics for symmetry assessments, thereby providing additional support for the validity of the results in practical applications. The empirical findings offer consistent, robust, and compelling evidence regarding the presence or absence of log-symmetry, which agrees with the theoretical properties of the proposed entropy-based test. Although our primary objective of this study is the development of a novel entropy-based inferential framework, the results exhibit strong concordance with established diagnostic approaches.

Moreover, the stability of the findings across datasets reinforces the reliability of the proposed methodology and highlights its effectiveness in capturing essential global distributional characteristics, even in the presence of substantial variability in the underlying data structure. It is important to emphasize that the framework is developed under the standard assumption of independent and identically distributed component lifetimes, a common setting in reliability modeling. When these assumptions are violated, such as in the presence of dependence or heterogeneity among components, the behavior of the system-induced transformation may be altered, which could, in turn, affect the performance of the test. Nevertheless, because the methodology is based on entropy functionals that capture global features of the distribution, it is expected to retain a meaningful degree of robustness under moderate deviations from these assumptions. Extending the framework to explicitly accommodate dependent or heterogeneous structures therefore represents a natural and important direction for future research.

In practical applications, lifetime data may exhibit complex features such as heavy-tailed behavior or multimodality, which can influence distributional functionals. In such settings, the proposed entropy-based characterization remains sensitive to the overall distributional structure and reflects these features through variations in the entropy measures. While this sensitivity enhances the ability of the method to detect departures from log-symmetry, it may also lead to more nuanced interpretations when multiple structural characteristics are present simultaneously. Accordingly, the proposed framework is best viewed as a tool for capturing global distributional imbalance rather than isolating specific sources of asymmetry, and its use is most effective when complemented by additional diagnostic methods. A systematic investigation of its performance under heavy-tailed and multimodal regimes constitutes an important direction for future research.

Finally, in the presence of outliers or contaminated observations, entropy-based estimators may be influenced by extreme values. However, the proposed test relies on entropy functionals applied to transformed system lifetimes, which aggregate information across the distribution. This aggregation mitigates the impact of isolated extreme observations and provides a degree of robustness under moderate contamination. Nevertheless, in cases of severe contamination or pronounced outliers, the performance of the test may be affected, and the results should be interpreted with appropriate caution. The development of robust entropy-based extensions therefore remains an important avenue for future research.

6. Conclusions and future directions

In this paper, we develop a unified information-theoretic framework for the characterization and nonparametric testing of log-distributional symmetry in continuous lifetime distributions within a reliability modeling context. By exploiting the interaction between entropy measures and transformation structures induced by consecutive *k*-out-of-*n* systems, the we establish explicit and analytically tractable characterizations that link intrinsic informational balance to log-symmetry. Through the integrated use of Shannon entropy, Rényi entropy, and Kerridge inaccuracy, the proposed approach provides a principled and fully nonparametric methodology for assessing structural symmetry on the logarithmic scale, which arises naturally in reliability and survival analysis. Building on these characterization results, we develop an entropy-based test for log-symmetry that combines theoretical rigor with computational tractability. Extensive Monte Carlo simulations demonstrate accurate control of the type I error and strong, competitive power across a broad class of alternatives. The real-data applications further confirm the practical relevance of the method and its effectiveness in detecting meaningful departures from log-symmetry in applied settings. Collectively, these findings highlight the strength of entropy-based inference in capturing global structural features of lifetime distributions that may remain undetected by classical moment-based or model-dependent approaches.

Beyond the results, this work emphasizes the broader role of information-theoretic measures as powerful inferential tools in reliability analysis. By embedding entropy within reliability system theory, the proposed framework shows how system-induced transformations can be systematically leveraged to reveal fundamental structural properties of probability distributions in a transparent and interpretable manner. In this sense, the methodology provides a bridge between theoretical characterization and practical statistical inference, contributing to the ongoing integration of information theory and reliability modeling. From a practical perspective, the proposed test offers several advantages over simpler approaches for detecting log-symmetry. In particular, it delivers a quantitative and formally testable criterion within a unified, model-free framework that avoids restrictive parametric assumptions. Unlike purely graphical or transformation-based diagnostics, the method captures global distributional characteristics through entropy functionals, enabling sensitivity to a wide range of departures from log-symmetry. Furthermore, the procedure remains computationally feasible and can be implemented using standard empirical estimators, making it readily applicable in practice.

Several directions for future research arise naturally from this work. These include deriving the asymptotic null distribution of the test statistic and studying its local power properties, as well as developing robust variants capable of handling contamination, heavy-tailed behavior, and outliers. Further extensions to broader symmetry structures, such as elliptical or high-dimensional settings, together with scalable computational implementations, represent promising avenues for advancing the proposed framework.

Author contributions

All authors have worked equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

All datasets analyzed in this study are reproduced in the article and originate from previously published sources cited in the manuscript.

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