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Research article

Behavior of solutions of higher-order difference equations

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Abstract: This paper examined a new class of difference equations, focusing on the stability, boundedness, and periodicity of solutions. Special cases admitting explicit solutions wrere also considered. Numerical examples were provided to confirm the theoretical results and illustrate the consistency between analytical predictions and computational observations. These findings contributed to a deeper understanding of the qualitative behavior of discrete-time dynamical systems and offered insight into the dynamics of higher-order difference equations.

Keywords: difference equation; stability; boundedness; solution of difference equation
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1. Introduction

Higher order difference equations are essential tools used for modeling discrete time dynamics. They emerge in wide areas of application, including population models, which are represented by difference equations

$$s_{n+1} = as_n (1 - s_n).$$

In economics, higher-order equations, such as

$$s_{n+1} = as_n + bs_{n-1} + \alpha,$$

are used to represent cyclical solutions in markets and macroeconomic systems. The study of these models helps scientists understand many applications. Numerous authors have studied difference equations in [1–3].

Alotaibi et al. [4] studied the qualitative behavior of a rational difference equation

$$s_{n+1} = \frac{\delta s_{n-m}+\beta s_n}{\gamma+\alpha s_{n-k}s_{n-l}s_{n-1}(s_{n-k}+s_{n-l}s_{n-1})}.$$

Jin et al. [5] investigated the global attractivity of a difference equation in the following form

$$s_{n+1} = \frac{\delta+\beta s_n}{\gamma+\alpha s_{n-k}+s_{n-l}}.$$

Elsayed et al. [6] examined the boundedness and qualitative behavior of a recursive sequence

$$s_{l+1} = as_{n-1} + \frac{bs_{n-4}s_{n-1}}{cs_{n-4}+ds_{n-2}}.$$

Ogu et al. [7] studied the solution behavior of a difference equation

$$s_{l+1} = \frac{as_{l-17}}{1+s_{l-2}s_{l-5}s_{l-8}s_{l-11}s_{l-14}s_{l-17}}.$$

In addition, several researchers have studied systems of difference equations [8–10]. Touafek et al. [11] investigated explicit solutions and analyzed the periodicity and asymptotic properties of a second-order difference equation

$$s_{n+1} = c + as_n + \frac{bs_nx_{n-1}}{x_n - d - ay_{n-1}},$$
$$s_{n+1} = d + es_n + \frac{bx_ns_{n-1}}{s_n - c - as_{n-1}}.$$

This investigation focuses on the behavioral analysis of higher-order difference equations

$$s_{n+1} = \frac{\delta s_{n-6}s_{n-4}s_{n-2}s_n}{\gamma+s_{n-5}s_{n-3}s_{n-1}(\beta+\alpha s_{n-6}s_{n-4}s_{n-2}s_n)}, \qquad n = 0,1,2,..., \text{ where } \delta,\gamma,\beta,\alpha \in R, \tag{1.1}$$

such as stability, boundedness, and the emphasis on periodic solutions, as well as the derivation of specific cases that admit explicit solutions. The study includes numerical examples that support the theoretical results.

2. Behavior of the solutions of the equation

In this section, we study the behavior of solutions of the difference equation (1.1), where β, δ, γ α and initial values are positive real numbers, focusing on boundedness, local stability, and global attractivity.

2.1. Boundedness of solutions

This subsection studies the boundedness of solutions of the difference equation (1.1).

Theorem 1. All solutions of the difference equation are bounded. Moreover, if $\frac{\delta}{\gamma} < 1$, the solutions are bounded above b^{k+1}, where $b = \max\{s_{-6}, s_{-5}, ..., s_0\}$.

Proof. From the difference equation (1.1)

$$s_{n+1} = \frac{\delta s_{n-6}s_{n-4}s_{n-2}s_n}{\gamma+s_{n-5}s_{n-3}s_{n-1}(\beta+\alpha s_{n-6}s_{n-4}s_{n-2}s_n)}$$

$$
\begin{aligned}
&< \tfrac{\delta s_{n-6}s_{n-4}s_{n-2}s_n}{\gamma}\\
&< s_{n-6}s_{n-4}s_{n-2}s_n \quad \text{if } \tfrac{\delta}{\gamma} < 1\\
&\leq b^{k+1},
\end{aligned}
$$

where $b = \max\{s_{-6}, s_{-5}, ..., s_0\}$. By contradiction, let $\{s_{2n}\}$ be unbounded

$$
\begin{aligned}
\infty &= \lim_{n\to\infty} s_{2n+2} = \lim_{n\to\infty} \tfrac{\delta s_{2n-5}s_{2n-3}s_{2n-1}s_{2n+1}}{\gamma+s_{2n-4}s_{2n-2}s_{2n}(\beta+\alpha s_{2n-5}s_{2n-3}s_{2n-1}s_{2n+1})}\\
&< \lim_{n\to\infty} \tfrac{\delta s_{2n-5}s_{2n-3}s_{2n-1}s_{2n+1}}{\alpha s_{2n-4}s_{2n-2}s_{2n}s_{2n-5}s_{2n-3}s_{2n-1}s_{2n+1}}\\
&= \lim_{n\to\infty} \tfrac{\delta}{\alpha_{2n-4}s_{2n-2}s_{2n}} = 0.
\end{aligned}
$$

This is a contradiction, which implies that the solution is bounded.

2.2. Local stability

We discuss the local stability of the solutions of the difference equation (1.1) below. The equilibrium point of this equation is

$$\bar{s} = \tfrac{\delta\bar{s}^4}{\gamma+\bar{s}^3\left(\beta+\alpha\bar{s}^4\right)}. \tag{2.1}$$

By cross-multiplication, we obtain

$$\bar{s}\left[\gamma + \bar{s}^3\left(\beta + \alpha\bar{s}^4\right) - \delta\bar{s}^3\right] = 0.$$

Solving the above equation, we find that $\bar{s} = 0$ is an equilibrium point.

Theorem 2. The equilibrium point $\bar{s} = 0$ of the difference equation (1.1) is locally stable.

Proof. Define a function $G : (0,\infty) \to (0,\infty)$ as $G : (0,\infty) \to (0,\infty)$ as

$$G(v_0, v_2, v_4, v_6, v_1, v_3, v_5) = \tfrac{\delta v_0v_2v_4v_6}{\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6)}.$$

Deriving of the partial derivatives of G,

$$
\begin{aligned}
\frac{\partial G}{\partial v_0} &= \tfrac{\delta v_4v_2v_6(\gamma+\beta v_1v_3v_5)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2},\\
\frac{\partial G}{\partial v_2} &= \tfrac{\delta v_0v_4v_6(\gamma+\beta v_1v_3v_5)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2},\\
\frac{\partial G}{\partial v_4} &= \tfrac{\delta v_0v_2v_6(\gamma+\beta v_1v_3v_5)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2},\\
\frac{\partial G}{\partial v_6} &= \tfrac{\delta v_0v_2v_4(\gamma+\beta v_1v_3v_5)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_0v_2))^2},\\
\frac{\partial G}{\partial v_1} &= -\tfrac{\delta v_0v_2v_4v_6v_3v_5(\beta+\alpha v_0v_2v_4v_6)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2},\\
\frac{\partial G}{\partial v_3} &= -\tfrac{\delta v_0v_2v_4v_6v_1v_5(\beta+\alpha v_0v_2v_4v_6)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2},\\
\frac{\partial G}{\partial v_5} &= -\tfrac{\delta v_0v_2v_4v_6v_3v_1(\beta+\alpha v_0v_2v_4v_6)}{(\gamma+v_1v_3v_5(\beta+\alpha v_0v_2v_4v_6))^2}.
\end{aligned}
$$

Evaluate the partial derivatives at the point $\bar{s} = 0$:

$$\frac{\partial G(0,...,0)}{\partial v_0} = 0,$$
$$\frac{\partial G(0,...,0)}{\partial v_2} = 0,$$
$$\frac{\partial G(0,...,0)}{\partial v_4} = 0,$$
$$\frac{\partial G(0,...,0)}{\partial v_6} = 0,$$

$$\frac{\partial G(0,...,0)}{\partial v_1} = 0,$$
$$\frac{\partial G(0,...,0)}{\partial v_3} = 0,$$
$$\frac{\partial G(0,...,0)}{\partial v_5} = 0.$$

The linearized equation around $\bar{s} = 0$ is

$$s_{n+1} - \frac{\partial G(0,...,0)}{\partial v_0} s_n - \frac{\partial G(0,...,0)}{\partial v_1} s_{n-1} - ... - \frac{\partial G(0,...,0)}{\partial v_6} s_{n-6} = 0.$$

From the above equations, we obtain $S_{n+1} = 0$. Hence, the characteristic equation is

$$\gamma^7 = 0.$$

This equation has the root $\gamma = 0$, therefore, the point $\bar{s} = 0$ is locally stable.

2.3. Global attractivity

The subsection investigates the global attractivity character of equilibrium point of difference equation (1.1).

Theorem 3. If difference equation (1.1) admits a unique equilibrium $\bar{s} = 0$, then it is globally attractive if $\delta < \gamma$.

Proof. From Theorem 1 we obtain

$$\mathbf{s}_{n+1} \leq \mathbf{s}_{n-6}\mathbf{s}_{n-4}\mathbf{s}_{n-2}\mathbf{s}_n \quad \text{if } \tfrac{\delta}{\gamma} < \mathbf{1},$$

Since all solutions are positive, we have $s_n > 0$. Let

$$\mathbf{P} = \lim_{\mathbf{n}\to\infty} \sup s_n.$$

Therefore, by taking the limit on both sides of the inequality, we get

$$\mathbf{P} \leq \tfrac{\delta}{\gamma}\mathbf{P}^4.$$

Hence,

$$\mathbf{P}\left(1-\tfrac{\delta}{\gamma}\mathbf{P}^3\right)\leq \mathbf{0}.$$

Since $P \geq 0$, it follows that either

$$\mathbf{P}=\mathbf{0} \text{ or } \tfrac{\gamma}{\delta}\leq \mathbf{P}^3.$$

Since $\frac{\delta}{\gamma}<1$, we have $\frac{\gamma}{\delta}>1$, and hence, $\frac{\delta}{\gamma}\leq P^3$ cannot be satisfied by any bounded solution. Thus,

$$\lim_{\mathbf{n}\to\infty}\sup s_n=0.$$

Consequently, the equilibrium point $\bar{s}=0$ is globally attractive if $\delta<\gamma$.

2.4. Existence of periodic solutions:

This subsection is devoted to the study of period-4 solutions of Eq (1.1) in the case $\gamma=0$ that is

$$s_{n+1}=\frac{\delta s_{n-6}s_{n-4}s_{n-2}s_n}{s_{n-5}s_{n-3}s_{n-1}(\beta+\alpha s_{n-6}s_{n-4}s_{n-2}s_n)}. \tag{2.2}$$

Theorem 4. Equation (2.2) admits a period-4 solution $q_0,q_1,q_2,q_3,q_0,q_1,...,$ if, and only if, $q_3=\sqrt{\frac{\delta-\beta}{\alpha q_1^2}}$, $q_2=\sqrt{\frac{\delta-\beta}{\alpha q_0^2}}$, initial values satisfy $s_{-6}=s_{-2}=q_0, s_{-5}=s_{-1}=q_1, s_{-4}=s_0=q_2, s_{-3}=q_3, \beta<\delta$, and $\Psi=\Upsilon$, where $\Psi=q_1^2q_3^2, \Upsilon=q_0^2q_2^2$, with $q_0\ q_1\in R\backslash\{0\}$.

Proof. Let Eq (2.2) have a period-4 solution, that is, $q_0,q_1,q_2,q_3,q_0,q_1,...,$ where $s_{-6}=q_0,s_{-5}=q_1,...s_0=q_2$. From Eq (2.2),

$$s_1=q_3=\frac{\delta q_0^2q_2^2}{q_1^2q_3\left(\beta+\alpha q_0^2q_2^2\right)}, \tag{2.3}$$

$$s_2=q_2=\frac{\delta q_1^2q_3^2}{q_0^2q_2\left(\beta+\alpha q_1^2q_3^2\right)}, \tag{2.4}$$

$$s_3=q_1=\frac{\delta q_0^2q_2^2}{q_1q_3^2\left(\beta+\alpha q_0^2q_2^2\right)}, \tag{2.5}$$

$$s_4=q_0=\frac{\delta q_1^2q_3^2}{q_0q_2^2\left(\beta+\alpha q_1^2q_3^2\right)}. \tag{2.6}$$

Let

$$\Psi=q_1^2q_3^2, \Upsilon=q_0^2q_2^2, \tag{2.7}$$

hence,

$$\Upsilon=\frac{\delta\Psi}{\alpha\Psi+\beta}, \Psi=\frac{\delta\Upsilon}{\alpha\Upsilon+\beta}.$$

Simplifying and subtracting, we obtain

$$(\beta+\delta)(\Psi-\Upsilon)=0,$$

hence, $\Psi=\Upsilon$. It follows from Eqs (2.4) and (2.7) that

$$q_2=\sqrt{\frac{\delta-\beta}{\alpha q_0^2}}.$$

Similarly, from Eqs (2.7) and (2.3),

$$q_3 = \sqrt{\frac{\delta - \beta}{\alpha q_1^2}}.$$

Hence, Eq (2.2) has a period-4 solution if

$$q_3 = \sqrt{\frac{\delta - \beta}{\alpha q_1^2}}, q_2 = \sqrt{\frac{\delta - \beta}{\alpha q_0^2}}, \text{ and } \Psi = \Upsilon.$$

2.5. Simulations

Numerical examples are provided to illustrate and confirm the theoretical results of this subsection.

Example 1. Let $\beta = 3, \delta = 1.5, \gamma = 2$, and $\alpha = 1$, with initial conditions $s_{-6} = 10, s_{-5} = 30, s_{-4} = 100, s_{-3} = 40, s_{-2} = 50, s_{-1} = 20, s_0 = 70$ (see Figure 1).

Example 2. Assume that $\beta = 2, \delta = 115, \gamma = 1$, and $\alpha = 2$, with initial conditions $s_{-6} = 1, s_{-5} = 0.3, s_{-4} = 2, s_{-3} = 0.4, s_{-2} = 2, s_{-1} = 0.5, s_0 = 3$ (see Figure 2).

Example 3. Let $\beta = 3, \delta = 15, \gamma = 2$, and $\alpha = 1$, with initial conditions $s_{-6} = 0.1, s_{-5} = 0.4, s_{-4} = 0.2, s_{-3} = 0.3, s_{-2} = 0.2, s_{-1} = 0.1, s_0 = 0.3$ (see Figure 3).

Example 4. Assume that $\beta = 5, \delta = 1.8, \gamma = 2$,and $\alpha = 1$ with initial conditions
IC1: $s_{-6} = 40, s_{-5} = 30, s_{-4} = 10, s_{-3} = 60, s_{-2} = 50, s_{-1} = 80, s_0 = 20$.
IC2: $s_{-6} = 50, s_{-5} = 20, s_{-4} = 10, s_{-3} = 40, s_{-2} = 30, s_{-1} = 70, s_0 = 10$.
IC3: $s_{-6} = 90, s_{-5} = 110, s_{-4} = 70, s_{-3} = 80, s_{-2} = 60, s_{-1} = 100, s_0 = 50$ (see Figure 4).

Example 5. Let $\beta = 0.5, \delta = 1.8, \gamma = 2$,and $\alpha = 1$, with initial conditions $s_{-6} = -40, s_{-5} = 30, s_{-4} = \sqrt{\frac{\delta-\beta}{\alpha s_{-6}^2}} = 0.0202, s_{-3} = \sqrt{\frac{\delta-\beta}{\alpha s_{-5}^2}} = 0.0269, s_{-2} = -40, s_{-1} = 30, s_0 = 0.0202$ (see Figure 5).

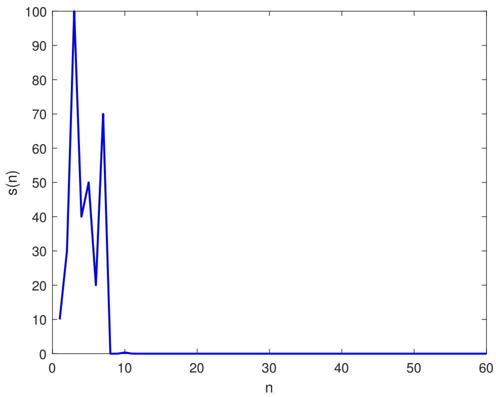

Figure 1. The solution of Eq (1.1) is bounded above by b^{k+1} when $\frac{\delta}{\gamma} < 1$.

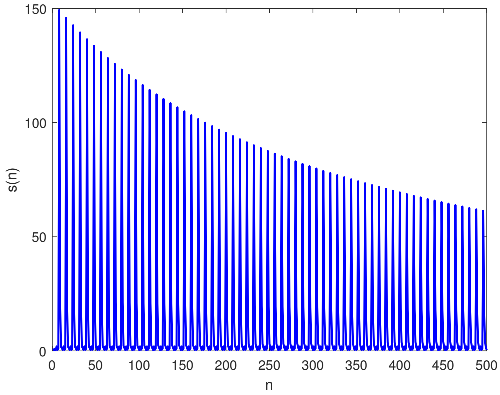

Figure 2. For $\gamma < \delta$, the solution of Eq (1.1) remains bounded.

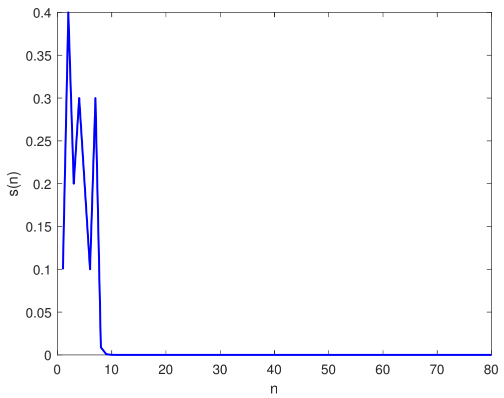

Figure 3. The figure illustrates the conclusion of Theorem 2 regarding the local stability of $\bar{s} = 0$.

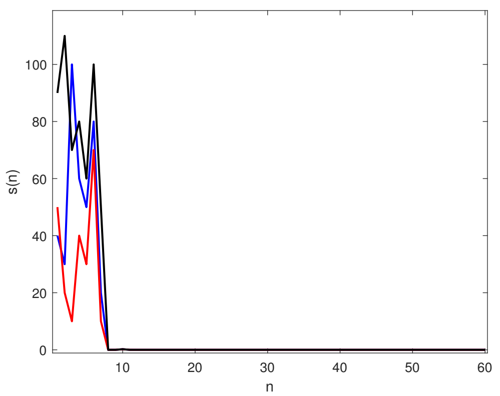

Figure 4. This figure illustrates the global attractivity of $s = 0$ for different initial conditions.

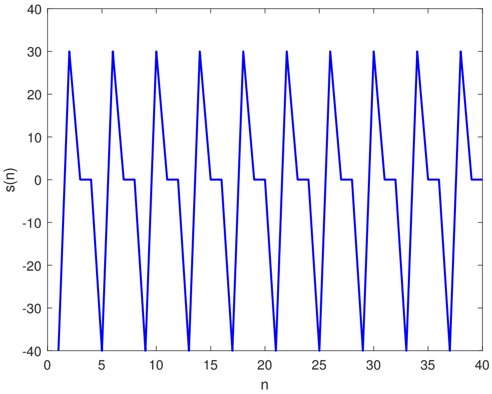

Figure 5. Equation (2.2) admits a periodic solution under the conditions of Theorem 4.

3. On the solution of the difference equation

In this section, our goal is to solve two special cases of the Eq (1.1) where all initial values are nonzero real numbers and $n \in \mathbb{N}_0$.

3.1. Case 1: Formulations of solution of the equation $s_{n+1} = \frac{s_n s_{n-2} s_{n-4} s_{n-6}}{s_{n-3}(1 - s_n s_{n-2} s_{n-4} s_{n-6})}$

In this case, we investigate the solutions of the difference equation

$$s_{n+1} = \frac{s_n s_{n-2} s_{n-4} s_{n-6}}{s_{n-5} s_{n-3} s_{n-1}(1 - s_n s_{n-2} s_{n-4} s_{n-6})}. \tag{3.1}$$

Theorem 5. For $n = 0, 1, 2, ...,$ the solutions of difference equations (3.1) are expressed as

$$s_{8n+1} = \frac{-s_0 s_{-2} s_{-4} s_{-6} \prod_{i=1}^{n}((8i-1)s_0 s_{-2} s_{-4} s_{-6} - 1)}{s_{-1} s_{-3} s_{-5} \prod_{i=0}^{n}((8i+1)s_0 s_{-2} s_{-4} s_{-6} - 1)}, \quad s_{8n+2} = \frac{-s_{-6} \prod_{i=1}^{n}(8i s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+2)s_0 s_{-2} s_{-4} s_{-6} - 1)},$$

$$s_{8n+3} = \frac{s_{-5} \prod_{i=0}^{n}((8i+1)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+3)s_0 s_{-2} s_{-4} s_{-6} - 1)}, \quad s_{8n+4} = \frac{s_{-4} \prod_{i=0}^{n}((8i+2)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+4)s_0 s_{-2} s_{-4} s_{-6} - 1)},$$

$$s_{8n+5} = \frac{s_{-3} \prod_{i=0}^{n}((8i+3)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+5)s_0 s_{-2} s_{-4} s_{-6} - 1)}, \quad s_{8n+6} = \frac{s_{-2} \prod_{i=0}^{n}((8i+4)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+6)s_0 s_{-2} s_{-4} s_{-6} - 1)},$$

$$s_{8n+7} = \frac{s_{-1} \prod_{i=0}^{n}((8i+5)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+7)s_0 s_{-2} s_{-4} s_{-6} - 1)}, \quad s_{8n+8} = \frac{s_0 \prod_{i=0}^{n}((8i+6)s_0 s_{-2} s_{-4} s_{-6} - 1)}{\prod_{i=0}^{n}((8i+8)s_0 s_{-2} s_{-4} s_{-6} - 1)}.$$

Proof. The result holds for $n = 1$. Assume it is valid for $n - 1$. Then,

$$s_{8n-7} = \frac{-s_0s_{-2}s_{-4}s_{-6}\prod_{i=1}^{n-1}((8i-1)s_0s_{-2}s_{-4}s_{-6}-1)}{s_{-1}s_{-3}s_{-5}\prod_{i=0}^{n-1}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)}, \quad s_{8n-6} = \frac{-s_{-6}\prod_{i=1}^{n-1}(8is_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n-5} = \frac{s_{-5}\prod_{i=0}^{n-1}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)}, \quad s_{8n-4} = \frac{s_{-4}\prod_{i=0}^{n-1}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n-3} = \frac{s_{-3}\prod_{i=0}^{n-1}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)}, \quad s_{8n-2} = \frac{s_{-2}\prod_{i=0}^{n-1}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n-1} = \frac{s_{-1}\prod_{i=0}^{n-1}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+7)s_0s_{-2}s_{-4}s_{-6}-1)}, \quad s_{8n} = \frac{s_0\prod_{i=0}^{n-1}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+8)s_0s_{-2}s_{-4}s_{-6}-1)}.$$

By using the recursive formula of the difference equation, the solution for n follows as

$$s_{8n+1} = \frac{s_{8n-6}s_{8n-4}s_{8n-2}s_{8n}}{(s_{8n-5}s_{8n-3}s_{8n-1})(1-s_{8n-6}s_{8n-4}s_{8n-2}s_{8n})}.$$

After substitution and simplification, we obtain

$$\begin{aligned}
s_{8n+1} &= \frac{\left(\frac{-s_{-6}s_{-4}s_{-2}s_0}{8ns_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{s_{-1}s_{-3}s_{-5}\prod_{i=0}^{n-1}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+7)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_{-6}s_{-4}s_{-2}s_0}{8ns_0s_{-2}s_{-4}s_{-6}-1}\right)\right]} \\
&= \frac{-s_{-6}s_{-4}s_{-2}s_0\prod_{i=0}^{n-1}((8i+7)\,s_0s_{-2}s_{-4}s_{-6}-1)}{\left(s_{-1}s_{-3}s_{-5}\prod_{i=0}^{n-1}((8i+1)\,s_0s_{-2}s_{-4}s_{-6}-1)\right)[8ns_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0]} \\
&= \frac{-s_0s_{-2}s_{-4}s_{-6}\prod_{i=1}^{n}((8i-1)s_0s_{-2}s_{-4}s_{-6}-1)}{s_{-1}s_{-3}s_{-5}\prod_{i=0}^{n}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)},
\end{aligned}$$

$$\begin{aligned}
s_{8n+2} &= \frac{s_{8n-5}s_{8n-3}s_{8n-1}s_{8n+1}}{(s_{8n-4}s_{8n-2}s_{8n})(1-s_{8n-5}s_{8n-3}s_{8n-1}s_{8n+1})} \\
&= \frac{\left(\frac{-s_0s_{-2}s_{-4}s_{-6}}{(8n+1)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{s_0s_{-2}s_{-4}\prod_{i=0}^{n-1}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n-1}((8i+8)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+1)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}
\end{aligned}$$

$$= \frac{-s_{-6}s_{-4}s_{-2}s_0\prod_{i=0}^{n-1}\left((8i+8)\,s_0s_{-2}s_{-4}s_{-6}-1\right)}{\left(s_0s_{-2}s_{-4}\prod_{i=0}^{n-1}\left((8i+2)\,s_0s_{-2}s_{-4}s_{-6}-1\right)\right)\left[(8n+1)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0\right]}$$

$$= \frac{-s_{-6}\prod_{i=1}^{n}(8is_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n+3} = \frac{s_{8n-4}s_{8n-2}s_{8n}s_{8n+2}}{(s_{8n-3}s_{8n-1}s_{8n+1})(1-s_{8n-4}s_{8n-2}s_{8n}s_{8n+2})}$$

$$= \frac{\left(\frac{s_{-5}}{(8n+2)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod_{i=0}^{n-1}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+2)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}$$

$$= \frac{s_{-5}\prod_{i=0}^{n}\left((8i+1)\,s_0s_{-2}s_{-4}s_{-6}-1\right)}{\left(\prod_{i=0}^{n-1}\left((8i+3)\,s_0s_{-2}s_{-4}s_{-6}-1\right)\right)\left[(8n+2)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0\right]}$$

$$= \frac{s_{-5}\prod_{i=0}^{n}((8i+1)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n+4} = \frac{s_{8n-3}s_{8n-1}s_{8n+1}s_{8n+3}}{(s_{8n-2}s_{8n}s_{8n+2})(1-s_{8n-3}s_{8n-1}s_{8n+1}s_{8n+3})}$$

$$= \frac{\left(\frac{s_{-4}}{(8n+3)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod_{i=0}^{n-1}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+3)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}$$

$$= \frac{s_{-4}\prod_{i=0}^{n}\left((8i+2)\,s_0s_{-2}s_{-4}s_{-6}-1\right)}{\left(\prod_{i=0}^{n-1}\left((8i+4)\,s_0s_{-2}s_{-4}s_{-6}-1\right)\right)\left[(8n+3)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0\right]}$$

$$= \frac{s_{-4}\prod_{i=0}^{n}((8i+2)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n+5} = \frac{s_{8n-2}s_{8n}s_{8n+2}s_{8n+4}}{(s_{8n-1}s_{8n+1}s_{8n+3})(1-s_{8n-2}s_{8n}s_{8n+2}s_{8n+4})}$$

$$= \frac{\left(\frac{s_{-3}}{(8n+4)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod\limits_{i=0}^{n}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod\limits_{i=0}^{n}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+4)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}$$

$$= \frac{s_{-3}\prod\limits_{i=0}^{n}((8i+3)\,s_0s_{-2}s_{-4}s_{-6}-1)}{\left(\prod\limits_{i=0}^{n-1}((8i+5)\,s_0s_{-2}s_{-4}s_{-6}-1)\right)[(8n+4)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0]}$$

$$= \frac{s_{-3}\prod\limits_{i=0}^{n}((8i+3)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod\limits_{i=0}^{n}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n+6} = \frac{s_{8n-1}s_{8n+1}s_{8n+3}s_{8n+5}}{(s_{8n}s_{8n+2}s_{8n+4})(1-s_{8n-1}s_{8n+1}s_{8n+3}s_{8n+5})}$$

$$= \frac{\left(\frac{s_{-2}}{(8n+5)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod\limits_{i=0}^{n-1}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod\limits_{i=0}^{n}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+5)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}$$

$$= \frac{s_{-2}\prod\limits_{i=0}^{n}((8i+4)\,s_0s_{-2}s_{-4}s_{-6}-1)}{\left(\prod\limits_{i=0}^{n-1}((8i+6)\,s_0s_{-2}s_{-4}s_{-6}-1)\right)[(8n+5)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0]}$$

$$= \frac{s_{-2}\prod\limits_{i=0}^{n}((8i+4)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod\limits_{i=0}^{n}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$s_{8n+7} = \frac{s_{8n}s_{8n+2}s_{8n+4}s_{8n+6}}{(s_{8n+1}s_{8n+3}s5)(1-s_{8n}s_{8n+2}s_{8n+4}s_{8n+6})}$$

$$= \frac{\left(\frac{s_{-1}}{(8n+6)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod\limits_{i=0}^{n-1}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod\limits_{i=0}^{n}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+6)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]}$$

$$= \frac{s_{-2}\prod\limits_{i=0}^{n}((8i+5)\,s_0s_{-2}s_{-4}s_{-6}-1)}{\left(\prod\limits_{i=0}^{n-1}((8i+7)\,s_0s_{-2}s_{-4}s_{-6}-1)\right)[(8n+6)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0]}$$

$$= \frac{s_{-1}\prod_{i=0}^{n}((8i+5)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+7)s_0s_{-2}s_{-4}s_{-6}-1)},$$

$$\begin{aligned} s_{8n+8} &= \frac{s_{8n+1}s_{8n+3}s_{8n+5}s_{8n+7}}{(s_{8n+2}s_{8n+4}s_{8n+6})(1-s_{8n+1}s_{8n+3}s_{8n+5}s_{8n+7})} \\ &= \frac{\left(\frac{s_0}{(8n+7)s_0s_{-2}s_{-4}s_{-6}-1}\right)}{\left(\frac{\prod_{i=0}^{n-1}((8i+8)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)}\right)\left[1+\left(\frac{s_0s_{-2}s_{-4}s_{-6}}{(8n+7)s_0s_{-2}s_{-4}s_{-6}-1}\right)\right]} \\ &= \frac{s_0\prod_{i=0}^{n}((8i+6)\,s_0s_{-2}s_{-4}s_{-6}-1)}{\left(\prod_{i=0}^{n-1}((8i+8)\,s_0s_{-2}s_{-4}s_{-6}-1)\right)[(8n+7)\,s_0s_{-2}s_{-4}s_{-6}-1+s_{-6}s_{-4}s_{-2}s_0]} \\ &= \frac{s_0\prod_{i=0}^{n}((8i+6)s_0s_{-2}s_{-4}s_{-6}-1)}{\prod_{i=0}^{n}((8i+8)s_0s_{-2}s_{-4}s_{-6}-1)}. \end{aligned}$$

Example 6. We consider numerical examples to verify the results presented in this subsection for the difference equation (3.1) with the initial conditions $s_{-6} = -4$, $s_{-5} = 1$, $s_{-4} = -6$, $s_{-3} = 3$, $s_{-2} = -4$, $s_{-1} = -1$, and $s_0 = 5$ (see Figure 6).

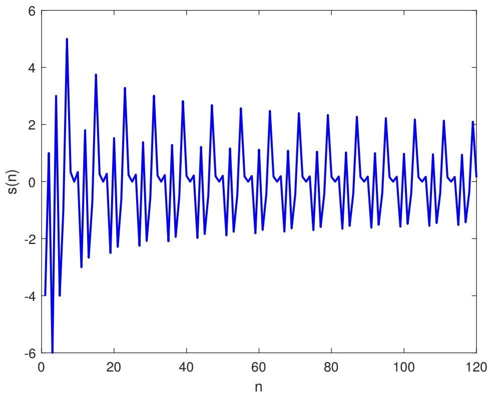

Figure 6. Numerical trajectories of Eq (3.1).

3.2. *Case 2: Formulations of solutions of* $s_{n+1} = \frac{s_ns_{n-2}s_{n-4}s_{n-6}}{s_{n-5}s_{n-3}s_{n-1}(-1-s_ns_{n-2}s_{n-4}s_{n-6})}$

In this case, we investigate the solutions of the equation

$$s_{n+1} = \frac{s_ns_{n-2}s_{n-4}s_{n-6}}{s_{n-5}s_{n-3}s_{n-1}(-1-s_ns_{n-2}s_{n-4}s_{n-6})}. \tag{3.2}$$

Theorem 6. Assume that $\{s_n\}_{n=1}^{\infty}$ is a solution of the difference equations (3.2). For $n = 1, 2, ...,$ the solutions of (3.2) are given by the following equations

$$s_{8n+1} = \frac{-s_0 s_{-2} s_{-4} s_{-6}}{s_{-1} s_{-3} s_{-5}(s_0 s_{-2} s_{-4} s_{-6}+1)}, \qquad s_{8n+2} = s_{-6},$$
$$s_{8n+3} = s_{-5}, \qquad s_{8n+4} = s_{-4},$$
$$s_{8n+5} = s_{-3}, \qquad s_{8n+6} = s_{-2},$$
$$s_{8n+7} = s_{-1}, \qquad s_{8n+8} = s_0.$$

Proof. The result holds if $n = 1$. Assume now that $n > 0$ and assumption holds for $n - 1$,

$$s_{8n-7} = \frac{-s_0 s_{-2} s_{-4} s_{-6}}{s_{-1} s_{-3} s_{-5}(s_0 s_{-2} s_{-4} s_{-6}+1)}, \qquad s_{8n-6} = s_{-6},$$
$$s_{8n-5} = s_{-5}, \qquad s_{8n-4} = s_{-4},$$
$$s_{8n-3} = s_{-3}, \qquad s_{8n-2} = s_{-2},$$
$$s_{8n-1} = s_{-1}, \qquad s_{8n} = s_0.$$

From the formula difference equation (3.2),

$$\begin{aligned}
s_{8n+1} &= \frac{s_{8n-6} s_{8n-4} s_{8n-2} s_{8n}}{(s_{8n-5} s_{8n-3} s_{8n-1})(-1-s_{8n-6} s_{8n-4} s_{8n-2} s_{8n})}, \\
&= \frac{-s_0 s_{-2} s_{-4} s_{-6}}{s_{-1} s_{-3} s_{-5}(s_0 s_{-2} s_{-4} s_{-6}+1)}, \\
s_{8n+2} &= \frac{s_{8n+1} s_{8n-1} s_{8n-3} s_{8n-5}}{s_{8n-2} s_{8n-4} s_{8n}(-s_{8n+1} s_{8n-1} s_{8n-3} s_{8n-5}-1)} \\
&= \frac{\left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} s_{-5}}{s_{-2} s_{-4} s_0 \left(-1-\left(\frac{s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} s_{-5}\right)} \\
&= s_{-6}, \\
s_{8n+3} &= \frac{s_{8n-4} s_{8n-2} s_{8n} s_{8n+2}}{(s_{8n-3} s_{8n-1} s_{8n+1})(-1-s_{8n-4} s_{8n-2} s_{8n} s_{8n+2})} \\
&= \frac{s_{-2} s_{-4} s_0 s_{-6}}{\left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} \left[-1 - s_{-2} s_{-4} s_0 s_{-6}\right]} \\
&= s_{-5}, \\
s_{8n+4} &= \frac{s_{8n-3} s_{8n-1} s_{8n+1} s_{8n+3}}{(s_{8n-2} s_{8n} s_{8n+2})(1-s_{8n-3} s_{8n-1} s_{8n+1} s_{8n+3})} \\
&= \frac{\left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} s_{-5}}{s_{-2} s_{-6} s_0 \left[-1 - \left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} s_{-5}\right]} \\
&= s_{-4}, \\
s_{8n+5} &= \frac{s_{8n-2} s_{8n} s_{8n+2} s_{8n+4}}{(s_{8n-1} s_{8n+1} s_{8n+3})(1-s_{8n-2} s_{8n} s_{8n+2} s_{8n+4})} \\
&= \frac{s_{-2} s_{-4} s_0 s_{-6}}{\left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-5} \left[-1 - s_{-2} s_{-4} s_0 s_{-6}\right]} \\
&= s_{-3}, \\
s_{8n+6} &= \frac{s_{8n-1} s_{8n+1} s_{8n+3} s_{8n+5}}{(s_{8n} s_{8n+2} s_{8n+4})(1-s_{8n-1} s_{8n+1} s_{8n+3} s_{8n+5})} \\
&= \frac{s_{-1} \left(\frac{-s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-5} s_{-3}}{s_{-2} s_{-6} s_0 \left[-1 - \left(\frac{s_{-6} s_{-4} s_{-2} s_0}{(s_{-5} s_{-3} s_{-1})(+1+s_{-6} s_{-4} s_{-2} s_0)}\right) s_{-1} s_{-3} s_{-5}\right]} \\
&= s_{-2},
\end{aligned}$$

$$
\begin{aligned}
s_{8n+7} &= \frac{s_{8n}s_{8n+2}s_{8n+4}s_{8n+6}}{(s_{8n+1}s_{8n+3}s_{8n+5})(1-s_{8n}s_{8n+2}s_{8n+4}s_{8n+6})} \\
&= \frac{s_{-2}s_{-4}s_0s_{-6}}{\left(\frac{-s_{-6}s_{-4}s_{-2}s_0}{(s_{-5}s_{-3}s_{-1})(+1+s_{-6}s_{-4}s_{-2}s_0)}\right)s_{-3}s_{-5}\left[-1-s_{-2}s_{-4}s_0s_{-6}\right]} \\
&= s_{-1}, \\
s_{8n+8} &= \frac{s_{8n+1}s_{8n+3}s_{8n+5}s_{8n+7}}{(s_{8n+2}s_{8n+4}s_{8n+6})(1-s_{8n+1}s_{8n+3}s_{8n+5}s_{8n+7})} \\
&= \frac{\left(\frac{-s_{-6}s_{-4}s_{-2}s_0}{(s_{-5}s_{-3}s_{-1})(+1+s_{-6}s_{-4}s_{-2}s_0)}\right)s_{-5}s_{-3}s_{-1}}{s_{-6}s_{-4}s_{-2}\left[-1-\left(\frac{s_{-6}s_{-4}s_{-2}s_0}{(s_{-5}s_{-3}s_{-1})(+1+s_{-6}s_{-4}s_{-2}s_0)}\right)s_{-1}s_{-3}s_{-5}\right]} \\
&= s_0.
\end{aligned}
$$

Example 7. We consider numerical examples to verify the results presented in this subsection for the difference equation (3.2) with the initial conditions $s_{-6} = -1$, $s_{-5} = 2, s_{-4} = -3$, $s_{-3} = 4, s_{-2} = -1$, $s_{-1} = 5$, and $s_0 = -2$ (see Figure 7).

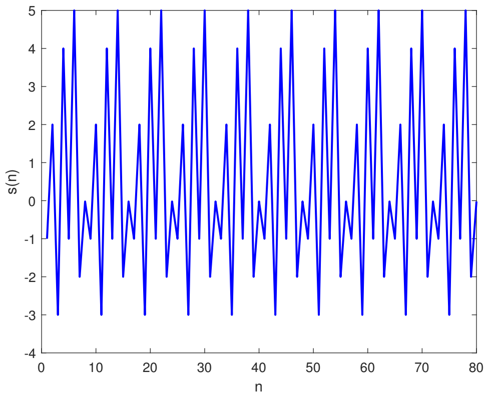

Figure 7. Numerical trajectories of Eq (3.2).

4. Conclusions

In this paper, we discuss some properties of $s_{n+1} = \frac{\delta s_{n-6}s_{n-4}s_{n-2}s_n}{\gamma+s_{n-5}s_{n-3}s_{n-1}(\beta+\alpha s_{n-6}s_{n-4}s_{n-2}s_n)}$ such as boundedness, stability characteristics and solution formula. First, we study the boundedness of solutions of the Eq (1). We prove that the equilibrium point is locally stable and is a global attractor. After that, we derive the conditions for the existence of a period-4 solution. We also obtain the solutions of two special cases of the Eq (1). Finally, we confirm our results by numerical examples.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors confirm that the content of this article is entirely their own work. Artificial intelligence (AI) tools were not used in the mathematical analysis or results. AI tools were used for language editing and proofreading.

Conflict of interest

The authors declare no conflict of interest.

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