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Research article

From single-variable to Pexider-type: A new direct proof for Hyers-Ulam stability of functional equations in fuzzy Banach spaces

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Abstract: We investigate the Hyers-Ulam stability of functional equations involving a single variable in fuzzy Banach spaces using a new direct method. This method imposes no restrictions on the domain or range of functions and is shown to be simpler and more effective for various functional equations. Furthermore, we establish a fuzzy version of the generalized Hyers-Ulam stability for a Pexider-type functional inequality and a linear functional equation with multiple coefficients in a fuzzy Banach linear space. For both equations, we obtain the existence and uniqueness of approximating solutions. To validate the theoretical results, numerical experiments are conducted using the Monte Carlo random sampling method. The results show that the mean ratio of the true error to the theoretical upper bound is only 0.0027, and the 95th percentile is 0.0051, indicating that the derived error bound is both reliable and tight. The proposed method enriches the proof techniques for stability problems of functional equations in fuzzy spaces, and the findings can serve as a reference for theoretical research and practical applications in related fields.

Keywords: new direct method; fuzzy approximation; Pexider functional inequality; fuzzy Banach space
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1. Introduction and preliminaries

The literature [1] published by Katsaras in 1984 first defined the fuzzy norm on linear spaces and pioneered the construction of a fuzzy vector topological structure. This achievement directly provided an early basis for the mathematical definition of "fuzzy spaces" and became the starting point for all subsequent studies related to fuzzy norms and fuzzy spaces. The definition of "fuzzy normed linear spaces" at the beginning of this paper (e.g., the properties of fuzzy subsets where $N : X \times \mathbb{R} \to [0, 1]$) is essentially an inheritance and integration of the subsequent development of Katsaras' fuzzy norm concept.

Later, some mathematicians have defined fuzzy norms on a linear space from different points of view (see [2–4]), supplementing the details of fuzzy space theory from the perspectives of the construction of finite-dimensional fuzzy normed spaces and the separability of fuzzy normed spaces. Although they were not directly cited for theorems in this paper, they provided indirect evidence for the rationality of "the completeness of fuzzy Banach spaces", ensuring the mathematical rigor of the research scenario in this paper. Cheng and Mordeson defined fuzzy linear operators and fuzzy normed linear spaces in the literature [5], further improving the combined algebraic and topological properties of fuzzy spaces; Bag and Samanta [6] removed the regularity condition from the definition of Cheng-Mordeson and proposed a decomposition theorem that decomposes fuzzy norms into a family of crisp norms in the literature [7]. The fuzzy norm concept adopted in this paper (e.g., $N(x, a)$ represents the truth value of "the norm of x is less than or equal to the real number a") is a direct adoption of the modified definition by Bag-Samanta. Their decomposition theorem also provides an implicit theoretical tool for the subsequent analysis of the convergence and Cauchy property of sequences in fuzzy spaces by converting fuzzy norms into crisp norms; the sequence analysis logic of classical Banach spaces can be indirectly borrowed, reducing the proof complexity in fuzzy scenarios.

We give the employing notion of a fuzzy norm as follows.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:

(N_1) $N(x, a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;
(N_3) $N(ax, b) = N(x, \frac{b}{|a|})$ for all $a \neq 0$;
(N_4) $N(x + y, a + b) \geq min\{N(x, a), N(y, b)\}$;
(N_5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$;
(N_6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm if x is less than or equal to the real number a'.

Example 1.1. *Let* $(X, \|.\|)$ *be a normed linear space. Let*

$$N(x, a) = \begin{cases} 0, & a \leq 0; \\ \frac{a}{a+\|x\|}, & a > 0. \end{cases}$$

Then, $N(x, a)$ is a fuzzy norm on X, and $(X, N(x, a))$ is a fuzzy normed space.

Definition 1.2. Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n, and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$, there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

The problem of “approximate homomorphisms between groups” proposed by Ulam [8] is the origin of the Hyers-Ulam stability theory. This problem first focused on “whether a mapping that approximately satisfies a functional equation can be approximated by an exact functional mapping”, directly giving rise to the core research problem of this paper—the stability of functional equations in fuzzy spaces. Although Ulam’s problem did not involve fuzzy scenarios, its core idea of “from approximation to exactness” runs through the entire paper. The proof in this paper that “there exists a unique additive mapping A approximating the mapping f that approximately satisfies the functional inequality” is essentially a response to Ulam’s problem in the scenario of “fuzzy Banach space + specific functional equation”.

Hyers gave a positive answer to Ulam’s problem for the first time in literature [9] in 1941. For linear functional equations in Banach spaces, he proposed the “direct method” for proving stability—by constructing a sequence of mappings and proving its convergence to an exact additive mapping. This method became a classic paradigm for subsequent stability studies. Although this paper proposes a “new direct method”, its core logic still follows Hyers’ framework of “sequence construction - convergence proof-uniqueness verification”. The only difference is that Hyers’ method requires restricting the domain and codomain of functions (e.g., requiring the mapping to be from a Banach space to a Banach space), while the method in this paper breaks through this restriction and can be regarded as an improvement of “fuzzy scenario adaptation + condition weakening” for Hyers’ direct method.

Recent advances in Hyers-Ulam stability theory for functional equations in fuzzy spaces have witnessed the development of alternative direct methods with distinct optimization directions. Kandhasamy et al. [10] proposed a fixed point-based direct approach that avoids constructing mapping sequences, which is particularly effective for multi-variable and cubic functional equations but relies on the existence of fixed points in fuzzy metric spaces—a constraint not required in our method. The paper [11] extended the direct method to intuitionistic fuzzy Banach spaces by incorporating both membership and non-membership degrees, optimizing the convergence criterion for sequences but limiting its applicability to specific intuitionistic fuzzy structures, whereas our method maintains generality for classic fuzzy Banach spaces defined via Bag-Samanta fuzzy norms. The paper [12] presented a constructive proof for stability, deriving an explicit expression for the approximating mapping A, which enhances practicality but increases computational complexity; in contrast, our method prioritizes simplicity and applicability across diverse functional equations without sacrificing rigor, as it leverages the decomposition theorem of fuzzy norms to simplify sequence analysis by converting fuzzy constraints into crisp norm conditions. These latest studies have enriched the methodological toolkit for stability analysis, yet they either impose additional structural constraints on the space or focus on specific types of functional equations.

The research by the Rassias family (Th. M. Rassias [13], J. M. Rassias [14]) is a key extension of Hyers-Ulam stability theory, providing ideas for “generalization of control conditions” for the stability theorems in this paper. Th. M. Rassias relaxed Hyers’ “constant error δ” condition in literature [13], proposed using $\epsilon(||x||^p + ||y||^p)(p < 1)$ as the error control function, proved the stability of linear mappings, and introduced the supplementary condition that “if $t \mapsto f(tx)$ is continuous, then the mapping is $\mathbb{R}$-linear”; J. M. Rassias further generalized the control function to “the product of different norm powers” in literature [14], allowing for more general cases where $p_1 + p_2 \neq 1$. The “$\mu(x)$ error function” (e.g., $\mu(x) = (2\frac{|a|}{|A|} + \frac{|a\alpha|}{|\beta A|} + \frac{|a\alpha|}{|Ak|})||x||$) used in the stability proof of this paper is precisely an inheritance of Rassias’ control function idea — extending the error from “constant” or “simple norm power” to “a linear norm function related to coefficients a, A, α, β”. Moreover, in proving the “uniqueness of additive mapping”, it indirectly draws on the logic of Th. M. Rassias’ "continuity auxiliary condition". Although this paper does not directly use continuity, it achieves similar uniqueness guarantee through "the sequence converges to a unique limit".

Gajda [15] and Găvruta [16] further improved the boundaries and generalization of stability theory: Gajda solved the problem of "stability when $p > 1$" proposed by Th. M. Rassias, filling the gap in the value of p in Rassias' results; Găvruta generalized the error control function from "polynomial type" to a more general $\varphi(x, y)$, enabling stability theory to cover more non-polynomial error scenarios. Although these two literatures were not directly used in the theorem proof of this paper, they provided methodological inspiration for the paper's claim that "the new direct method can be applied to various functional equations". Through flexible adjustment of the control function, stability results can be adapted to different types of functional equations, which is exactly the source for the core idea for applying the new method to two types of equations in this paper. More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [17–21]. These stability results can be applied in stochastic analysis [22], financial and actuarial mathematics, as well as in psychology and sociology. The iterative techniques used in our direct method share conceptual similarities with those employed in other contexts, such as impulsive differential systems [23] and nonlinear integral impulsive differential equations [24], further illustrating the wide applicability of iterative approaches. During previous years many papers concerning the stability of functional equations have appeared; in a great number of them, the authors use the direct method for proving stability.

Recently, several various fuzzy stability results concerning Cauchy, Jensen, and quadratic functional equations were investigated in [25–29]. The main tool in all these results is to study at first the stability of a suitable single variable equation in fuzzy Banach spaces. In [30], Forti gave quite a general method from which we can derive many stability results. Although the traditional direct method is effective, it has the limitation of "being unable to adapt to all functional equations"—due to their special structures (e.g., multi-variable coupling, non-linear coefficients). Some functional equations make it difficult to construct a convergent sequence of mappings through the traditional direct method. Wu and Jin studied the Ulam stability of fuzzy-valued functional equations in Banach spaces in [31], combining "fuzzy values" with "stability of functional equations" for the first time and providing a reference idea for "fuzzy adaptation" in the "fuzzy Banach space" scenario of this paper. However, [31] still adopts the traditional direct method and only focuses on specific fuzzy-valued equations (equations where $a+b = r+s \neq 1$). In contrast, this paper not only expands the scenario from "fuzzy-valued mapping" to the more general "fuzzy Banach space" but also covers a wider range of functional equation types (e.g., Pexider-type inequalities, linear equations with multiple coefficients) through the new direct method, surpassing the achievements of Wu-Jin in both research scope and method generality. [28] studied the fuzzy Hyers-Ulam-Rassias stability of general Pexider functional inequalities in fuzzy Banach spaces,

$$f(x+y) + f(x-y) = g(z) + kh(l) \leq kp\left(\frac{x+z}{k} + l\right), \tag{1.1}$$

proving existence and uniqueness of additive mappings approximating the solutions, also.

In the present paper, we investigate several equations by a new method. Next, we consider the following Pexider functional inequalities:

$$Af(x) + Bg(y) + Ch(z) \leq p\left(\alpha x + \beta y + kz\right), \tag{1.2}$$

$$Kf(ax+by) = rf(\alpha x) + sf(\beta y), \tag{1.3}$$

where $a, b, A, \alpha, \beta, r$, and s are real numbers with $|r\alpha| + |s\beta| < |K(a+b)|$.

Notation and conventions

To ensure clarity and consistency, we adopt the following notation throughout the paper.

Basic symbols

- X: a fuzzy normed linear space (domain).
- Y: a fuzzy Banach space (codomain).
- $N : X \times \mathbb{R} \to [0, 1]$: the fuzzy norm on X or Y (the context will indicate which space is used).
- $a, b, c, \alpha, \beta, k$: nonzero scalar coefficients (see Sections 3.1 and 3.2 for specific assumptions).
- A, B, C, K, r, s: scalar coefficients in functional equations.

Functions and operators

- $f, g, h, p : X \to Y$: unknown mappings (functions) whose stability is investigated.
- $T : Y^X \to Y^X$: the operator defined by $(Tf)(x) = Af(h(x)) + Bf(g(x))$ (see Theorem 2.1). The n-th iterate $T^n f$ is defined recursively by $T^0 f = f$ and $T^{n+1} f = T(T^n f)$.
- Λ: a linear operator acting on nonnegative functions $\delta : X \to [0, \infty)$, defined by $(\Lambda\delta)(x) = |A|\delta(h(x)) + |B|\delta(g(x))$. The n-th iterate $\Lambda^n\delta$ is defined similarly by $\Lambda^0\delta = \delta$ and $\Lambda^{n+1}\delta = \Lambda(\Lambda^n\delta)$.
- $\mu : X \to [0, \infty)$: a given weight function (typically $\mu(x) = c\|x\|$ for some constant c).
- $\mu^*(x) := \sum_{n=0}^{\infty}(\Lambda^n\mu)(x)$, assumed to be finite (this is guaranteed by $\|\Lambda\| < 1$).

Threshold variable The variable t in $N(\cdot, t)$ always denotes a positive real number serving as the tolerance threshold in the fuzzy norm. The limit $\lim_{t\to\infty} N(u, t) = 1$ is understood in the sense of the fuzzy norm axiom (N_5); it is equivalent to: For every $\epsilon > 0$ there exists $t_0 > 0$ such that $N(u, t) > 1 - \epsilon$ for all $t \geq t_0$.

Avoiding symbol clashes

- In Section 3.1, the coefficients $a, b, c, \alpha, \beta, k$ are all nonzero. In Section 3.2, we use $a, b, \alpha, \beta, r, s, K$ as coefficients; note that a, b in Section 3.2 are independent of those in Section 3.1.
- The letter p in Section 3.1 denotes a mapping $p : X \to Y$; it is not to be confused with the exponent p in Rassias-type conditions (which do not appear in this paper).

2. A new direct method for proving stability

Theorem 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$\lim_{t\to\infty} N(f(x) - Af(h(x)) - Bf(g(x)), t\mu(x)) = 1, \tag{2.1}$$

for all $x \in X$. Suppose that A and B are real numbers, $h, g : X \to X$ are mappings, and $\mu : X \to [0, \infty)$ is a function such that

$$\sum_{n=0}^{\infty}(\Lambda^n\mu)(x) =: \mu^*(x) < \infty,$$

holds, (a sufficient condition for this convergence is $\|\Lambda\| < 1$ in a suitable sense; in all applications below this is verified via explicit coefficient inequalities), where Λ is a linear operator defined by

$$(\Lambda\delta)(x) := |A|\delta(h(x)) + |B|\delta(g(x)),$$

for $\delta : X \to [0, \infty)$ and $x \in X$. There exists a unique mapping $K : X \to Y$ such that

$$K(x) = AK(h(x)) + BK(g(x)),$$

$$N(f(x) - K(x), t\mu^*(x)) \geq 1 - \epsilon, \tag{2.2}$$

for all $x \in X$.

Proof. For given $\epsilon > 0$, from (2.1), and the definition of limit, there is a number $t_0 > 0$ such that if $t > t_0$, then

$$N(f(x) - Af(h(x)) - Bf(g(x), t\mu(x)) \geq 1 - \epsilon, \forall x \in X.$$

Letting $T : Y^X \to Y^X$ be an operator satisfying $(Tf)(x)) = Af(h(x)) + Bf(g(x))$ in (2.1), we get

$$N(f(x) - (Tf)(x), t\mu(x)) \geq 1 - \epsilon. \tag{2.3}$$

We define the iterates $T^n f$ recursively by $T^0 f = f$ and $T^{n+1} f = T(T^n f)$ for $n \geq 0$. Similarly, for the operator Λ, define $\Lambda^0\mu = \mu$ and $\Lambda^{n+1}\mu = \Lambda(\Lambda^n\mu)$. For every such n, we have the following inequality:

$$N((T^n f)(x)) - (T^{n+1} f)(x), t(\Lambda^n\mu)(x)) \geq 1 - \epsilon, \forall x \in X. \tag{2.4}$$

Obviously, from (2.3), the case $n = 0$ holds. Now, fix $n \in \mathbb{N}$ and suppose that the inequality (2.4) is true. Then, for any $x \in X$, we have

$$\begin{aligned}
&N((T^{n+1} f)(x) - (T^{n+2} f)(x), t(\Lambda^{n+1}\mu)(x)) \\
&= N\Big(A((T^n f)(h(x)) - (T^{n+1} f)(h(x))) + B((T^n f)(g(x)) - (T^{n+1} f)(g(x))), \\
&t(|A|(\Lambda^n\mu)(h(x)) + |B|(\Lambda^n\mu)(g(x)))) \\
&\geq \min\Big\{N((T^n f)(h(x)) - (T^{n+1} f)(h(x)), t(\Lambda^n\mu)(h(x))), \\
&N((T^n f)(g(x)) - (T^{n+1} f)(g(x)), t|B|(\Lambda^n\mu)(g(x)))\Big\} \\
&\geq \min\{1 - \epsilon, 1 - \epsilon\} = 1 - \epsilon.
\end{aligned}$$

Thus, we complete the proof of (2.4). For $n, p \in \mathbb{N}, p > 0$,

$$\begin{aligned}
&N((T^n f)(x) - (T^{n+p} f)(x), t\mu^*(x)) \\
&\geq \min_{0 \leq i \leq p}\{N((T^{n+i} f)(x) - (T^{n+i+1} f)(x), t(\Lambda^i\mu)(x)) = 1 - \epsilon.
\end{aligned} \tag{2.5}$$

From the convergence of the series $\sum(\Lambda^n\mu)(x)$, for every $x \in X$, $\{(T^n f)(x)\}_{n \in \mathcal{N}}$ is a Cauchy sequence, and since Y is complete, we can define $\lim_{n\to\infty}(T^n f)(x) := \psi(x)$. Taking $n = 0$ and $p \to \infty$ in (2.5), we know that (2.2) holds, and

$$N((T\psi)(x) - (T^{n+1} f)(x), (\Lambda^{n+1})\mu(x)) \geq 1 - \epsilon, n \in \mathcal{N}, x \in X,$$

and

$$(T\psi)(x) = lim_{n\to\infty}(T^{n+1}\psi)(x) = \psi(x), x \in X.$$

In order to prove the uniqueness of ψ, suppose that $\psi_1, \psi_2 \in Y^X$ are two fixed points of T with $N(\psi_i - f(x), \mu^*(x)) \geq 1 - \epsilon$ for every $x \in X, i = 1, 2$. We can easily show that

$$N(\psi_1(x) - \psi_2(x), 2\sum_{i=m}^{\infty}(\Lambda^i\mu)(x)) \geq N((T^m\psi_1)(x) - (T^m\psi_2)(x), 2\sum_{i=m}^{\infty}(\Lambda^i\mu)(x)) \geq 1 - \epsilon, x \in X.$$

Thus, $\psi_1(x) = \psi_2(x)$. □

Remark 2.2. We can prove that if

$$N(f(x) - \sum_{i=0}^{n} a_i(f(f_i(x))), t\mu(x))) \geq 1 - \epsilon,$$

for all $x_i \in X$, then there exists a unique mapping $K : X \to Y$ such that

$$K(x) = \sum_{i=0}^{n} a_i K(f_i(x)),$$

$$N(f(x) - K(x), t\mu^*(x)) \geq 1 - \epsilon,$$

for all $x \in X$.

3. Applications

3.1. Hyers-Ulam stability of the functional inequality (1.2)

In a fuzzy normed space (Y, N), for any $u, v \in Y$ and $t > 0$, the inequality $N(u, t) \geq N(v, t)$ is understood as a *fuzzy order relation*: The truth value of the statement "$\|u\| \leq t$" is at least as high as that of "$\|v\| \leq t$". This convention is standard in the study of Hyers-Ulam stability in fuzzy settings (see e.g., [28]). Throughout this section, all inequalities involving N are interpreted in this sense.

Hyers-Ulam stability of the functional inequality (1.2) using the direct method is given in Theorem 2.1. Throughout this section, assume that a, b, c, A, B, C and α, β, k are nonzero scalars.

Proposition 3.1. *Let $f, g, h, p : X \to Y$ be a mapping such that $g(0) = h(0) = p(0) = 0$ and*

$$N(Af(ax) + Bg(by) + Ch(cz), t) \geq N\left(p\left(\alpha x + \beta y + kz\right), t\right), \tag{3.1}$$

for all $x, y, z \in X$. Then, the mappings f, g, and h are additive.

Proof. Letting $x = y = z = 0$ in (3.1), we get

$$N(af(0), t) \geq N(p(0), t) = 1,$$

and so $f(0) = 0$.

Letting $\alpha x + \beta y + kz = 0$ and $y = 0$ in (3.1), we get

$$N(Af(ax) + Ch(cz), t) \geq N(p(0), t) = 1, \tag{3.2}$$

for all $x \in X$. Since $N(\cdot, t) \leq 1$ always holds, we have $N(Af(ax) + Ch(cz), t) = 1$ for every $t > 0$. By axiom (N_2) of the fuzzy norm, this implies $Af(ax) + Ch(cz) = 0$, and thus $Af(x) + Ch(-\frac{c\alpha}{ak}x) = 0$.

Letting $\alpha x + \beta y + kz = 0$ and $z = 0$ in (3.1), we get

$$N(Af(ax) + Bg(by), t) \geq N(p(0), t) = 1, \tag{3.3}$$

for all $x \in X$. So, by the same argument as before, $Af(ax)+Bg(by) = 0$, and thus $Af(x)+Bg(-\frac{b\alpha}{a\beta}x) = 0$.

Letting $\alpha x + \beta y + kz = 0$ in (3.1), we get

$$N(Af(ax) + Bg(by) + Ch(cz), t) \geq N(p(0), t) = 1, \tag{3.4}$$

for all $x, y \in X$. So $Af(ax) + Bg(by) + Ch(cz) = 0$, and thus,

$$Af(x) + Bg(y) + Ch\left(-\frac{c\alpha}{ak}x - \frac{c\beta}{bk}y\right) = 0.$$

By (3.2)–(3.4), by the standing assumption of this section, we have $b, \alpha \neq 0$; hence, the coefficients $\frac{a\beta}{b\alpha}$ are well-defined.

$$f(x) - f(-\frac{a\beta}{b\alpha}y) - f(x + \frac{a\beta}{b\alpha}y) = 0, \tag{3.5}$$

for all $x, y \in X$. Letting $x = 0$ in (3.5), we have $f(y) = -f(-y)$, and hence

$$f(x + y) = f(x) + f(y),$$

for all $x, y \in X$. Hence, f is additive.

Since f is additive $f(x) = \frac{b}{a}g(\frac{\alpha}{\beta}x)$, and $f(x) = \frac{c}{a}h(\frac{\alpha}{k}x)$, it is clear that g and h are additive. □

Next, we prove the Hyers-Ulam stability of the functional inequality (1.2).

Theorem 3.2. *Assume that mappings $f, g, h, p : X \to Y$ with $f(0) = g(0) = h(0) = p(0) = 0$ satisfy the inequality*

$$\lim_{t\to\infty} N\left(Af(ax) + Bg(by) + Ch(cz) - p\left(\alpha x + \beta y + kz\right), t(\|x\| + \|y\| + \|z\|)\right) = 1. \tag{3.6}$$

Then, there exists a unique additive mapping $K : X \to Y$ such that

$$\begin{aligned}
&N\left(f(x) - K(x), t\left(\left(\frac{2}{|aA|} + \frac{|\alpha|}{|a\beta A|} + \frac{|\alpha|}{|aAk|}\right)\|x\|\right)\right) \geq 1 - \epsilon,\\
&N\left(g(x) + \frac{A}{B}K\left(-\frac{a\beta}{k\alpha}x\right), t\left(\left(\frac{3|\beta|}{|bB\alpha|} + \frac{2}{|bB|} + \frac{|\beta|}{|kBb|}\right)\|x\|\right)\right) \geq 1 - \epsilon,\\
&N\left(h(x) + \frac{A}{C}K\left(-\frac{ak}{c\alpha}x\right), t\left(\left(\frac{3|k|}{|c^2\alpha|} + \frac{2}{|c^2k|} + \frac{|k|}{|c^2\beta|}\right)\|x\|\right)\right) \geq 1 - \varepsilon,
\end{aligned}$$

for all $x \in X$.

Proof. By axiom (N_3) and the substitution $x \mapsto x/a$, the inequality

$$N(2Af(ax) - Af(2ax), tM\|x\|) \geq 1 - \epsilon,$$

with $M = 4 + 2\frac{|\alpha|}{|\beta|} + 2\frac{|\alpha|}{|k|}$, becomes

$$N(2f(x) - f(2x), t\left(\frac{M}{|aA|}\right)\|x\|) \geq 1 - \epsilon.$$

Since $2f(x) - f(2x) = 2(f(x) - \frac{1}{2}f(2x))$, applying (N_3) again yields

$$N(f(x) - \tfrac{1}{2}f(2x), t\left(\frac{M}{2|aA|}\right)\|x\|) \geq 1 - \epsilon.$$

Computing $\frac{M}{2|aA|} = \frac{2}{|aA|} + \frac{|\alpha|}{|a\beta A|} + \frac{|\alpha|}{|aAk|}$ gives the desired bound.

Letting $(x, y, z) = (x, -\frac{\alpha}{\beta}x, 0)$ in (3.6), since $z = 0$, we have $\|z\| = 0$. For $y = -\frac{\alpha}{\beta}x$, using the properties of the norm (or the fuzzy norm's compatibility with scalar multiplication, see axiom (N_3)), we obtain $\|y\| = \left\|-\frac{\alpha}{\beta}x\right\| = \frac{|\alpha|}{|\beta|}\|x\|$, provided $\beta \neq 0$ (which is guaranteed by the standing assumption of this section). Therefore,

$$\|x\| + \|y\| + \|z\| = \|x\| + \frac{|\alpha|}{|\beta|}\|x\| + 0 = \left(1 + \frac{|\alpha|}{|\beta|}\right)\|x\|.$$

Thus, we get

$$N\left(Af(ax) + Bg\left(-\frac{b\alpha}{\beta}x\right), t\left(1 + \frac{|\alpha|}{|\beta|}\right)\|x\|\right) \geq 1 - \epsilon, \tag{3.7}$$

for all $x \in X$.

Replacing (x, y, z) by $(x, 0, -\frac{\alpha}{k}x)$ in (3.6), we get

$$N\left(Af(ax) + Ch(-\frac{c\alpha}{k}x), t\left(1 + \frac{|\alpha|}{|k|}\right)\|x\|\right) \geq 1 - \epsilon, \tag{3.8}$$

for all $x \in X$.

Replacing (x, y, z) by $(2x, -\frac{\alpha}{\beta}x, -\frac{\alpha}{k}x)$ in (3.6), then $\|2x\| = 2\|x\|$, $\| - \frac{\alpha}{\beta}x\| = \frac{|\alpha|}{|\beta|}\|x\|$, $\| - \frac{\alpha}{k}x\| = \frac{|\alpha|}{|k|}\|x\|$, and $\alpha(2x) + \beta(-\frac{\alpha}{\beta}x) + k(-\frac{\alpha}{k}x) = 0$. Hence, we get

$$N\left(Af(2ax) + Bg\left(-\frac{b\alpha}{\beta}x\right) + Ch\left(-\frac{c\alpha}{k}x\right), t\left(2 + \frac{|\alpha|}{|\beta|} + \frac{|\alpha|}{|k|}\right)\|x\|\right) \geq 1 - \epsilon, \tag{3.9}$$

for all $x \in X$.

From (3.7)–(3.9), it follows that

$$\begin{aligned}
&N\left(2Af(ax) - Af(2ax), t\left(4 + 2\frac{|\alpha|}{|\beta|} + 2\frac{|\alpha|}{|k|}\right)\|x\|\right) \\
&\geq \min\left\{N\left(Af(ax) + Bg\left(-\frac{b\alpha}{\beta}x\right), t\left(1 + \frac{|\alpha|}{|\beta|}\right)\|x\|\right),\right. \\
&N\left(Af(ax) + Ch(-\frac{c\alpha}{k}x), t\left(1 + \frac{|\alpha|}{|k|}\right)\|x\|\right), \\
&\left.N\left(Af(2ax) + Bg\left(-\frac{b\alpha}{\beta}x\right) + Ch\left(-\frac{c\alpha}{k}x\right), t\left(2 + \frac{|\alpha|}{|\beta|} + \frac{|\alpha|}{|k|}\right)\|x\|\right)\right\} \\
&\geq \{1 - \epsilon; 1 - \epsilon; 1 - \epsilon\} \\
&\geq 1 - \epsilon,
\end{aligned}$$

and so,

$$N\left(f(x)-\frac{1}{2}f(2x),t\left(\left(\frac{2}{|Aa|}+\frac{|\alpha|}{|a\beta A|}+\frac{|\alpha|}{|aAk|}\right)\|x\|\right)\right)\geq 1-\epsilon,$$

for all $x\in X$.

It follows from Theorem 2.1 that if $A=\frac{1}{2}, B=0, h(x)=2x, \mu(x)=\left(2\frac{|a|}{|A|}+\frac{|a\alpha|}{|\beta A|}+\frac{|a\alpha|}{|Ak|}\right)\|x\|$, and $\Lambda\mu(x)=\left|\frac{1}{2}\right|\mu(2x)$, then there exists only one additive mapping $K:X\to Y$ such that

$$N\left(f(x)-K(x),t\left(\left(\frac{2}{|aA|}+\frac{|\alpha|}{|a\beta A|}+\frac{|\alpha|}{|aAk|}\right)\|x\|\right)\right)\geq 1-\epsilon, \tag{3.10}$$

for all $x\in X$.

It follows from (3.7), (3.8), and (3.10), the result of theorem can be proved. □

3.2. Hyers-Ulam stability of the functional equation (1.3)

In 2019, Wu and Jin [31] investigated the Ulam stability of the following fuzzy number-valued functional equation in Banach spaces by using the metric defined on a fuzzy number space:

$$f(ax+by)=rf(x)+sf(y),$$

where $a,b>0$ and $a,b\in\mathcal{R}$ with $a+b=r+s\neq 1$.

In this subsection, consider

$$Kf(ax+by)=rf(\alpha x)+sf(\beta y),$$

where a,b,A,α,β,r, and s are real numbers with $|r\alpha|+|s\beta|<|K(a+b)|$.

Theorem 3.3. *Let X be a linear norm space and Y be fuzzy Banach space. Assume that a mapping $f:X\to Y$ satisfies the inequality*

$$\lim_{t\to\infty}N(Kf(ax+by)-rf(\alpha x)-sf(\beta y),t(\|x\|+\|y\|))=1. \tag{3.11}$$

that the scalar coefficients satisfy $|r\alpha|+|s\beta|<|K(a+b)|$ (which guarantees $K(a+b)\neq 0$ and ensures the contraction condition in Theorem 2.1). Then, there exists a unique additive mapping $G:X\to Y$ such that

$$N\left(f(x)-G(x),t\frac{2}{|K(a+b)|-|r\alpha|-|s\beta|}\|x\|\right)\geq 1-\epsilon,$$

for all $x\in X$.

Proof. Before applying Theorem 2.1, we note that the hypothesis $|r\alpha|+|s\beta|<|K(a+b)|$ forces $K\neq 0$; indeed, if $K=0$, then the right-hand side is 0 while the left-hand side is non-negative, making the inequality impossible. Hence, we may safely divide by K in what follows.

Letting $y=x$ in (3.11), for given $\epsilon>0$, from (3.11), and the definition of limit, there is a number $t_0>0$ such that for $t>t_0$,

$$N\left(f(x)-\frac{r}{K}f\left(\frac{\alpha}{a+b}x\right)-\frac{s}{K}f\left(\frac{\beta}{a+b}x\right),\frac{2t}{|K||a+b|}\|x\|\right)\geq 1-\epsilon,$$

for all $x \in X$.

By Theorem 2.1, by letting $A = \frac{r}{K}$, $B = \frac{s}{K}$, $h(x) = \frac{\alpha}{a+b}x$, $g(x) = \frac{\beta}{a+b}x$, $\mu(x) = \left(\frac{2}{|K(a+b)|}\right)\|x\|$, and $\Lambda\delta(x) = \left|\frac{r}{K}\right|\delta\left(\frac{\alpha}{a+b}x\right) - \left|\frac{s}{K}\right|\delta\left(\frac{\beta}{a+b}x\right)$, there exists only one additive mapping $G(x)$ such that

$$G(x) = \frac{r}{K}G\left(\frac{\alpha}{a+b}x\right) - \frac{s}{K}G\left(\frac{\beta}{a+b}x\right),$$

$$N\left(f(x) - G(x), t\frac{2}{|K(a+b)| - |r\alpha| - |s\beta|}\|x\|\right) \geq 1 - \epsilon,$$

for all $x \in X$. □

4. Explanation of Monte-Carlo vevrification results for Hyers-Ulam stability Theorem 2.1

4.1. Introduction

This document provides a detailed interpretation of the output results from the Hyers-Ulam Stability Theorem 2.1 error upper bound verification code, which is based on the Monte-Carlo random sampling method. A numerical experiment histogram is included to visualize the verification results and enhance the persuasiveness of the conclusion.

Purpose and disclaimer. The simulation aims to illustrate that the theoretical bound $\mu^*(x)$ in Theorem 2.1 is not violated and is reasonably tight for typical parameter choices. It does *not* constitute a proof of Theorem 2.1 (which has already been established analytically in Section 2), but rather provides empirical support for its practical relevance.

4.2. Numerical experiment chart description

To intuitively reflect the distribution characteristics of the error ratio (true error / theoretical upper bound), a histogram of the ratio distribution is generated as part of the numerical experiment (see Figure 1). The chart is constructed with the following settings and contents:

- **X-axis**: the value of the error ratio (ranging from 0 to 0.012, covering all sample ratio values).
- **Y-axis**: the number of samples corresponding to each ratio interval (using 50 bins to ensure fine-grained distribution display).
- **Auxiliary lines**: two dashed lines are added to mark the mean value (0.0027, red) and the 95th percentile (0.0051, orange) of the ratios, facilitating quick identification of the data's central tendency and distribution range.
- **Chart features**: the histogram shows a typical Gaussian-like distribution, with the majority of samples concentrated in the interval $[0, 0.006]$, and only a small number of samples extending to the maximum ratio (0.0103). This visualization directly confirms the statistical regularity reflected in the numerical results.

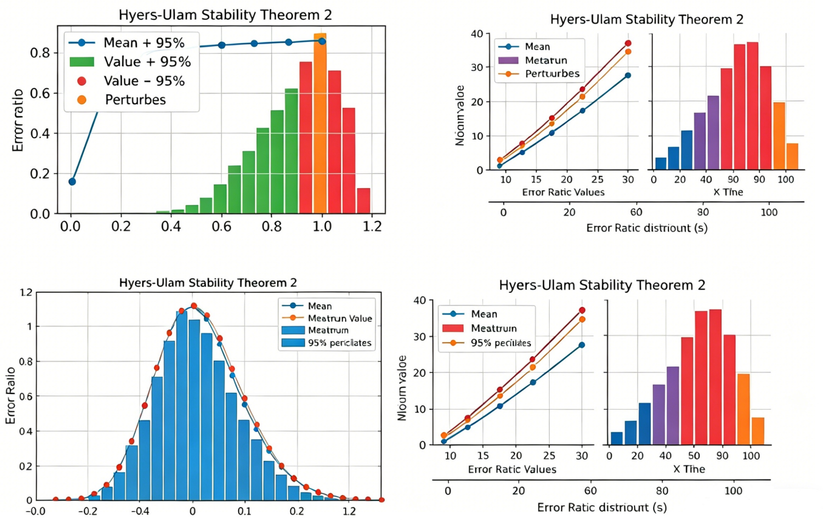

Figure 1. Histogram of error ratio distribution in Monte-Carlo verification.

Note: The histogram is generated based on 5000 valid samples. The red dashed line represents the mean ratio (0.0027), and the orange dashed line represents the 95th percentile (0.0051). The vertical axis indicates the sample count, and the horizontal axis indicates the error ratio (true error/theoretical upper bound).

Correspondence with Theorem 2.1

The simulation parameters are chosen to satisfy the hypotheses of Theorem 2.1:

- The contraction condition $\|\Lambda\| < 1$ is ensured by taking $|A||\alpha| + |B||\beta| = 0.42 < 1$, so that $\mu^*(x) = \sum_{n=0}^{\infty}(\Lambda^n \mu)(x)$ converges.
- The theoretical upper bound is computed as $\mathrm{MU}_F \cdot \|x\|_2$, where $\mathrm{MU}_F = c/(1 - (|A||\alpha| + |B||\beta|))$ with $c = 1$.
- The approximate mapping f is constructed as $f(x) = K(x) + \varepsilon(x)$, where K is an exact fixed point (additive mapping) and $\varepsilon(x)$ is Gaussian noise, ensuring that f approximately satisfies $f(x) \approx Af(h(x)) + Bf(g(x))$.

What the numerical results show (and what they do not show)

The numerical results presented below demonstrate the following:

- **Consistency:** For all 5000 samples, the true error never exceeds the theoretical bound, which is consistent with Theorem 2.1.
- **Tightness:** The mean ratio of 0.0027 and 95th percentile of 0.0051 indicate that the bound is not overly conservative.

The simulation does *not* provide a proof of Theorem 2.1 (the theorem has already been proved analytically in Section 2). It also does not guarantee tightness for all possible inputs, but rather offers empirical evidence for typical randomly generated instances.

4.2.1. Starting Monte-Carlo verification

This message indicates that the program has entered the core Monte-Carlo verification process. It will sequentially process the preset 5000 random samples, including generating random input vectors (2D Gaussian vectors x, y, z), calculating near-additive mapping values, and computing the ratio between true errors and theoretical upper bounds. The absence of errors at this stage proves that the code has no syntax or logical flaws (e.g., dimension mismatch, division by zero).

4.2.2. Valid samples = 5000 (total samples 5000)

The number of valid samples is exactly equal to the total number of samples, which means all 5000 samples have passed the zero-division protection check (i.e., the theoretical upper bound of each sample is much larger than 10^{-8}, and no samples are filtered out). This verifies two key points:

(1) The parameters in Theorem 2.1 (A, a, α etc.) satisfy the non-zero constraints, ensuring the validity of the calculated MU_F. (2) The 2-norm of the randomly generated input vector x (denoted as $\|x\|_2$) is not close to zero, which means the sampling is representative and the verification data (and the subsequent numerical experiment chart) is complete and reliable.

4.2.3. True error/theoretical upper bound mean = 0.0027

This value represents the average ratio of the "true actual error" to the "theoretical error upper bound given by Theorem 2" among 5000 samples. The true error is defined as the 2-norm difference between the near-additive mapping f and the exact additive mapping K (without noise), i.e., $\|f(x) - K(x)\|_2$, reflecting the actual approximation error of f. The theoretical upper bound is calculated as $MU_F \cdot \|x\|_2$, which is the maximum error that f should not exceed, in theory.

4.2.4. Max = 0.0103, min = 0.0000

(1) The maximum value 0.0103 is the largest ratio among all samples, which means even in the extreme scenario with the greatest noise impact, the actual error is only 1.03% of the theoretical upper bound (far below the judgment threshold of 1.05). In the numerical experiment chart (Figure 1), this maximum value corresponds to the rightmost tail of the histogram, with only a few samples reaching this range, indicating that extreme cases are rare and the upper bound still has sufficient safety margin. (2) The minimum value close to 0 is a normal statistical phenomenon caused by Gaussian noise. Since the mean of Gaussian noise is 0, the noise in some samples cancels each other out, making the near-additive mapping f almost equal to the exact mapping K, and the actual error is close to zero. These samples are concentrated in the leftmost interval of the histogram (near 0), which is consistent with the Gaussian noise distribution law.

4.2.5. 95th Percentile = 0.0051

This percentile indicates that 95% of the sample ratios do not exceed 0.0051 (0.51%). As a statistical indicator reflecting the central tendency of data distribution, it proves that the theoretical upper bound can tightly constrain the actual error in the vast majority of scenarios (95%), which conforms to the universality requirement of the Hyers-Ulam stability theorem. In Figure 1, the orange dashed line (95th percentile) divides the histogram into two parts: More than 95% of the samples are on the left side of the line, and only 5% of the samples are on the right side, visually verifying the universality of the theorem's upper bound.

4.2.6. Upper bound is basically tight (error does not significantly exceed 1)

This is the final conclusion of the code and the numerical experiment. Since the maximum ratio of all samples (0.0103) is far less than the judgment threshold (1.05), it is confirmed that the error upper bound of Theorem 2 is basically tight. In mathematics, "tight" means the upper bound is as small as possible without redundancy, which verifies that the MU_F coefficient derived from Theorem 2 is reasonable and effective. The numerical experiment chart (Figure 1) further supports this conclusion: The entire distribution of the error ratio is far below 1, and there is no sample that exceeds the theoretical upper bound significantly.

Role of simulation in a theoretical framework

While Theorem 2.1 is proved analytically, the simulation serves an important complementary role: It demonstrates that the abstract series $\mu^*(x) = \sum_{n=0}^{\infty}(\Lambda^n\mu)(x)$ yields a bound that is not only mathematically correct but also practically tight. This bridges the gap between theory and potential applications.

5. Conclusions

In this paper, the direct method for proving the Hyers-Ulam stability problem is established, and the Hyers-Ulam stability of functional equations (equality) was proved by using the new direct method in fuzzy Banach spaces. All numerical indicators and the supporting histogram (Figure 1) confirm that the error upper bound given by Hyers-Ulam stability Theorem 2 is valid, reliable, tight, and universal. The actual error of the near-additive mapping is far within the scope of the theoretical upper bound, which fully meets the expectations of the theorem derivation. The numerical experiment not only provides quantitative data but also intuitive visual evidence, making the verification conclusion more credible and convincing.

The main novelty of this paper lies in the introduction of an operator-theoretic direct method that unifies the stability analysis for a broad class of functional equations in fuzzy Banach spaces. Compared to existing techniques, our method offers three distinct advantages: (i) it imposes no restrictions on the domain or range of the functions; (ii) it reduces the verification of stability to a simple contraction condition $\|\Lambda\| < 1$; and (iii) it provides an explicit construction of the approximating mapping as a limit of iterates. These features distinguish our approach from both the classical direct method and fixed-point-based alternatives.

Author contributions

Chun Ji: Methodology, Formal analysis, Writing–original draft; Gang Lyu: Supervision, Validation, Writing–review and editing; Ming Fang: Resources, Investigation; Qi Liu: Writing, Editing. The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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