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Research article

A new alpha power type-II Lomax distribution with applications to radiation and materials sciences

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Abstract: This study introduces a novel three-parameter continuous model, termed the alpha power type II Lomax (APIILx) distribution, which is constructed by integrating the alpha power type II-G (APII-G) family with the Lomax distribution. The proposed model demonstrates remarkable flexibility and is capable of capturing a wide variety of distributional shapes. Its key statistical properties are rigorously derived. The model parameters are estimated using eight frequentist estimation methods, and the performance of these estimators is evaluated through extensive Monte Carlo simulations under diverse parameter configurations and sample sizes. The simulation findings confirm the consistency and efficiency of the estimators. To identify the most effective estimation procedure for the APIILx parameters, the estimators are ranked based on both partial and overall ranking criteria. Furthermore, the practical utility of the APIILx distribution is demonstrated through applications to four real-world datasets from medical, biomedical, and engineering sciences. The APIILx model consistently outperforms several well-known competing distributions in modeling radiotherapy, biomedical, and materials science data, highlighting its strong robustness and enhanced adaptability for diverse real-world applications.

Keywords: alpha power type II-G family; Lomax distribution; maximum likelihood estimation; simulation study; engineering and radiation data
Mathematics Subject Classification: 60E05, 62E10, 62F10, 62N05

1. Introduction

The Lomax (Lx) distribution, also known as the Pareto type II distribution [1], was originally proposed as a model for data related to corporate failures. Since its introduction, it has been applied to a wide range of datasets across various disciplines. Several researchers, including [2], have recommended this distribution as a suitable alternative to the exponential distribution, particularly in situations where the data exhibit heavy-tailed behavior.

The Lx distribution serves as a foundational model in versatile applications. Hence, various modified and enhanced versions of the Lx distribution have been proposed in recent literature, for instance, the discrete Poisson-Lx distribution [3], double-Lx distribution [4], gamma-Lx distribution [5], transmuted exponentiated Lx distribution [6], Poisson-Lx distribution [7], exponentiated Lx distribution [8], Weibull-Lx distribution [9], exponential-Lx distribution [10], complementary exponentiated Lx Poisson distribution [11], minimum Lindley-Lx distribution [12], Maxwell Lx distribution [13], alpha power power-Lx distribution [14], extended exponentiated Lx distribution [15] distribution, weighted T-X power Lx distribution [16], new extended Lx distribution [17], exponent sine power Lx distribution [18], Pareto Lx distribution [19], and Topp-Leone exponential-Lx distribution [20], among others.

Mohsin and Kalt [21] introduced the alpha power type II-G (APII-G) family, this manuscript delineates an innovative methodological framework that incorporates an additional variable into the distribution function. This family is fundamentally grounded in the inclusion of a power parameter within the context of the essential continuous distribution.

This manuscript introduced a novel and more flexible three-parameter continuous probability model, termed the alpha power type II Lomax (APIILx) distribution. The APIILx distribution is constructed by inserting the Lx distribution as the baseline model within the APII-G family, thereby integrating an additional shape parameter that substantially enhances the overall flexibility of this family. As a result, the APIILx distribution is capable of capturing a broad range of distributional behaviors: Its hazard rate function (HRF) can exhibit unimodal or decreasing shapes, while its density function accommodates right-skewed and unimodal forms.

Furthermore, the flexibility and practical utility of the APIILx distribution are demonstrated through the analysis of four real-life datasets drawn from medical, biological, and materials sciences. Across these applications—particularly for radiotherapy and related medical datasets—the APIILx model consistently delivers superior fits compared to several well-known competing distributions.

To further assess the model's performance, we examine how various classical estimation methods behave when applied to the APIILx parameters under different parameter settings and sample sizes. The parameters are estimated using a range of frequentist techniques, supported by extensive Monte Carlo simulations to evaluate the accuracy, consistency, and efficiency of the estimators. Finally, the estimators are ranked using partial and overall ranking criteria to identify the most reliable estimation method for the APIILx distribution.

The remainder of this article is organized as follows. Section 2 introduces the proposed APIILx model and presents graphical illustrations of its density and HRFs. Section 3 derives the key mathematical properties of the distribution. Section 4 discusses several estimation methods applicable to the APIILx parameters, while Section 5 reports the results of extensive simulation studies evaluating the performance of these estimators. Section 6 demonstrates the practical utility of the

APIILx distribution through the analysis of four real-world datasets. Finally, Section 7 provides concluding remarks.

2. The APIILx distribution

Given the considerable importance of the Lx distribution, it has been applied in conjunction with the APTII-G family. The newly formulated distribution is called the APIILx distribution. A random variable X is said to possess the Lx distribution characterized by the parameters θ and λ if its cumulative distribution function (CDF) (for $x > 0$) is expressed as

$$F(x;\theta,\lambda) = 1 - (\lambda + x)^{-\theta}\lambda^{\theta}, \tag{2.1}$$

where $\theta > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter.

The CDF of the APII-G family has the form

$$F(x;\alpha,\boldsymbol{\varphi}) = \frac{[1 + G(x;\boldsymbol{\varphi})]^{\alpha} - 1}{2^{\alpha} - 1}, \quad \alpha > 0. \tag{2.2}$$

The corresponding probability density function (PDF) of the APII-G family reduces to

$$f(x;\alpha,\boldsymbol{\varphi}) = \frac{\alpha[1 + G(x;\boldsymbol{\varphi})]^{\alpha-1}\, g(x;\boldsymbol{\varphi})}{2^{\alpha} - 1}, \quad \alpha > 0, \tag{2.3}$$

where $G(x;\boldsymbol{\varphi})$ and $g(x;\boldsymbol{\varphi})$ denote the CDF and PDF of any baseline distribution, respectively, and $\boldsymbol{\varphi}$ is a vector of parameters.

By inserting the CDF of the Lx distribution into Eq (2.1), we obtain the CDF of the APIILx distribution as follows:

$$F(x;\alpha,\theta,\lambda) = \frac{\left[2 - (\lambda + x)^{-\theta}\lambda^{\theta}\right]^{\alpha} - 1}{2^{\alpha} - 1}, \quad x > 0;\ \alpha,\theta,\lambda > 0. \tag{2.4}$$

The corresponding PDF of Eq (2.4) takes the form

$$f(x;\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda^{\theta}}{2^{\alpha} - 1}\left[2 - (\lambda + x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1}(\lambda + x)^{-\theta-1}, \quad x > 0, \tag{2.5}$$

where α and θ refer to the positive shape parameters, and λ denotes the positive scale parameter.

The survival function (SF) and HRF of the APIILx distribution are given by

$$S(x;\alpha,\theta,\lambda) = \frac{2^{\alpha} - \left[2 - (\lambda + x)^{-\theta}\lambda^{\theta}\right]^{\alpha}}{2^{\alpha} - 1} \tag{2.6}$$

and

$$h(x;\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda^{\theta}\left[2 - (\lambda + x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1}(\lambda + x)^{-\theta-1}}{2^{\alpha} - [2 - (\lambda + x)^{-\theta}\lambda^{\theta}]^{\alpha}}. \tag{2.7}$$

The reversed HRF and cumulative HRF of the APIILx model are defined by

$$r(x;\alpha,\theta,\lambda)=\frac{\alpha\theta\lambda^{\theta}\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1}(\lambda+x)^{-\theta-1}}{\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha}-1} \tag{2.8}$$

and

$$H(x;\alpha,\theta,\lambda)=-\ln\left(\frac{2^{\alpha}-\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha}}{2^{\alpha}-1}\right). \tag{2.9}$$

Figure 1 illustrates the graphical representations of various parameter values selected for the PDF and HRF of the APIILx distribution.

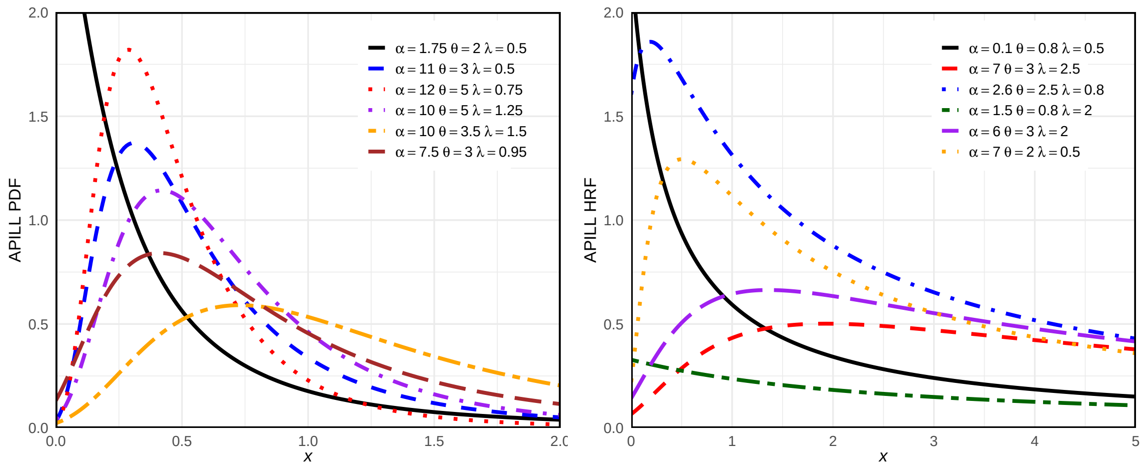

Figure 1. Graphical representations of the PDF and HRF of the APIILx distribution for various combinations of α, θ, and λ.

3. Mathematical properties

In this section, we derive several fundamental mathematical properties of the proposed model, including its distributional characteristics.

3.1. Moments

The rth moment for the APIILx distribution is given by

$$\begin{aligned}\mu_r' = E\left(X^r\right) &= \int_0^{\infty} x^r f(x;\alpha,\theta,\lambda)\,dx, \qquad r=1,2,3,\ldots \\ &= \frac{\alpha\theta\lambda^{\theta}}{2^{\alpha}-1}\int_0^{\infty} x^r\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1}(\lambda+x)^{-\theta-1}\,dx.\end{aligned} \tag{3.1}$$

To solve the integral in Eq (3.1), let $u=\lambda+x$, so that $du=dx$. When $x=0$, $u=\lambda$, and as $x\to\infty$, $u\to\infty$. Substituting, the integral becomes

$$\int_{\lambda}^{\infty}(u-\lambda)^r\left(2-u^{-\theta}\lambda^{\theta}\right)^{\alpha-1}u^{-\theta-1}\,du. \tag{3.2}$$

Next, expand $(u-\lambda)^r$ using the binomial theorem

$$(u-\lambda)^r = \sum_{i=0}^{r} \binom{r}{i} u^{r-i}(-\lambda)^i. \tag{3.3}$$

Substituting into Eq (3.2), we obtain

$$\sum_{i=0}^{r} \binom{r}{i} (-\lambda)^i \int_{\lambda}^{\infty} u^{r-i} \left(2 - u^{-\theta}\lambda^{\theta}\right)^{\alpha-1} u^{-\theta-1}\, du. \tag{3.4}$$

Now, to simplify the integrand in Eq (3.4), let $v = u^{-\theta}\lambda^{\theta}$, then $u = \lambda v^{-1/\theta}$, and

$$du = -\frac{\lambda}{\theta} v^{-1/\theta-1}\, dv.$$

When $u = \lambda$, $v = 1$; and as $u \to \infty$, $v \to 0$. The integral becomes

$$\int_{1}^{0} \left(\lambda v^{-1/\theta}\right)^{r-i-\theta-1} (2-v)^{\alpha-1} \left(-\frac{\lambda}{\theta} v^{-1/\theta-1}\right) dv. \tag{3.5}$$

Combining terms, we get

$$\frac{\lambda^{r-i-\theta}}{\theta} \int_{0}^{1} v^{-\frac{r-i}{\theta}} (2-v)^{\alpha-1}\, dv. \tag{3.6}$$

To evaluate the integral in Eq (3.6), use the substitution $v = 2t$, hence $dv = 2dt$. When $v = 0$, $t = 0$; when $v = 1$, $t = \frac{1}{2}$. The integral becomes

$$\int_{0}^{1} v^{-a}(2-v)^{\lambda}\, dv = 2^{1+\lambda-\alpha} \int_{0}^{1/2} t^{-a}(1-t)^{\lambda}\, dt, \tag{3.7}$$

which is expressed using the incomplete beta function

$$B_x(p,q) = \int_{0}^{x} t^{p-1}(1-t)^{q-1}\, dt. \tag{3.8}$$

Thus,

$$\int_{0}^{1/2} t^{-\alpha}(1-t)^{\beta}\, dt = B_{1/2}(1-\alpha, \beta+1). \tag{3.9}$$

So the result of the integral in Eq (3.6) becomes

$$\int_{0}^{1} v^{-\frac{r-i}{\theta}} (2-v)^{\alpha-1}\, dv = 2^{\alpha-\frac{r-i}{\theta}} B_{1/2}\left(1 - \frac{r-i}{\theta}, \alpha\right). \tag{3.10}$$

Substituting back, the full expression for the moment becomes

$$\sum_{i=0}^{r} \binom{r}{i} (-\lambda)^i \frac{\lambda^{r-i-\theta}}{\theta} 2^{\alpha-\frac{r-i}{\theta}} B_{1/2}\left(1 - \frac{r-i}{\theta}, \alpha\right). \tag{3.11}$$

After some simplifications, the rth moment of the APIILx distribution reduces to

$$\mu_r' = \frac{\alpha\lambda^r}{2^\alpha - 1}\sum_{i=0}^{r}\binom{r}{i}(-1)^i 2^{\alpha-\frac{r-i}{\theta}} B_{1/2}\left(1-\frac{r-i}{\theta},\alpha\right). \tag{3.12}$$

The first and second moments of the APIILx distribution are, respectively, given by

$$E(X) = \frac{\alpha\lambda}{2^\alpha - 1}\left[2^{\alpha-\frac{1}{\theta}} B_{1/2}\left(1-\frac{1}{\theta},\alpha\right) - 2^\alpha B_{1/2}(1,\alpha)\right] \tag{3.13}$$

and

$$E(X^2) = \frac{\alpha\lambda^2}{2^\alpha - 1}\left[2^{\alpha-\frac{2}{\theta}} B_{1/2}\left(1-\frac{2}{\theta},\alpha\right) - 2^{1+\alpha-\frac{1}{\theta}} B_{1/2}\left(1-\frac{1}{\theta},\alpha\right) + 2^\alpha B_{1/2}(1,\alpha)\right]. \tag{3.14}$$

Now, using Eqs (3.13) and (3.14), the variance of the APIILx distribution follows as

$$\begin{aligned}\mathrm{Var}(X) = {} & \frac{\alpha\lambda^2}{2^\alpha - 1}\left[2^{\alpha-\frac{2}{\theta}} B_{1/2}\left(1-\frac{2}{\theta},\alpha\right) \quad - 2^{1+\alpha-\frac{1}{\theta}} B_{1/2}\left(1-\frac{1}{\theta},\alpha\right) + 2^\alpha B_{1/2}(1,\alpha)\right] \\ & - \left(\frac{\alpha\lambda}{2^\alpha - 1}\right)^2\left[2^{\alpha-\frac{1}{\theta}} B_{1/2}\left(1-\frac{1}{\theta},\alpha\right) - 2^\alpha B_{1/2}(1,\alpha)\right]^2.\end{aligned} \tag{3.15}$$

3.2. Moment generating function

The moment generating function (MGF) is defined as

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^\infty e^{tx} f x; \alpha, \theta, \lambda)\, dx \\ &= \frac{\alpha\theta\lambda^\theta}{2^\alpha - 1}\int_0^\infty e^{tx}\left[2-(\lambda+x)^{-\theta}\lambda^\theta\right]^{\alpha-1}(\lambda+x)^{-\theta-1}\, dx. \end{aligned} \tag{3.16}$$

Let $r = -t$, so that $e^{tx} = e^{-rx}$. Define $z(x) = (\lambda + x)^{-\theta}\lambda^\theta$, so $0 < z(x) < 1$. Now, using the binomial expansion given by

$$[2 - z(x)]^{\alpha-1} = \sum_{n=0}^{\infty}\binom{\alpha-1}{n}(-1)^n 2^{\alpha-1-n} z(x)^n,$$

we can write

$$\left[2-(\lambda+x)^{-\theta}\lambda^\theta\right]^{\alpha-1} = \sum_{n=0}^{\infty}\binom{\alpha-1}{n}(-1)^n 2^{\alpha-1-n}\lambda^{\theta n}(\lambda+x)^{-\theta n}. \tag{3.17}$$

Substituting Eq (3.17) into Eq (3.16) gives

$$M_X(t) = \frac{\alpha\theta\lambda^\theta}{2^\alpha - 1}\sum_{n=0}^{\infty}\binom{\alpha-1}{n}(-1)^n 2^{\alpha-1-n}\lambda^{\theta n}\int_0^\infty e^{-rx}(\lambda+x)^{-\theta(n+1)-1}\, dx. \tag{3.18}$$

Now, compute the integral

$$I_n = \int_0^\infty e^{-rx}(\lambda+x)^{-\theta(n+1)-1} dx.$$

Let $u = \lambda + x \Rightarrow du = dx$; when $x = 0$, $u = \lambda$, and $x \to \infty \Rightarrow u \to \infty$. Then, we have

$$I_n = e^{r\lambda}\int_\lambda^\infty e^{-ru} u^{-\theta(n+1)-1} du.$$

Now, by setting $v = ru \Rightarrow dv = r\,du$, $u = v/r$, we obtain

$$I_n = e^{r\lambda} r^{\theta(n+1)} \int_{r\lambda}^{\infty} e^{-v} v^{-\theta(n+1)-1}\, dv.$$

This integral corresponds to the upper incomplete gamma function:

$$\int_a^{\infty} e^{-v} v^{k-1} dv = \Gamma(k, a).$$

Here, $k = -\theta(n + 1)$, so

$$I_n = e^{r\lambda} r^{\theta(n+1)} \Gamma(-\theta(n + 1), r\lambda).$$

Substituting this result back into Eq (3.18), the MGF of the APIILx distribution follows as

$$M_X(t) = \frac{\alpha\theta\lambda^{\theta+\theta n} e^{r\lambda}}{2^{\alpha} - 1} \sum_{n=0}^{\infty} \binom{\alpha - 1}{n} (-1)^n 2^{\alpha-1-n} r^{\theta(n+1)} \Gamma(-\theta(n + 1), r\lambda). \tag{3.19}$$

3.3. Rényi entropy

The Rényi entropy associated with a random variable X serves as a measure of uncertainty and has been widely employed in various scientific disciplines. According to [22], the Rényi entropy of the APIILx distribution is defined (for $w > 0$ and $w \neq 1$) as

$$H(w) = \frac{1}{1 - w} \log \int_0^{\infty} \left\{ \frac{\alpha\theta\lambda^{\theta}}{2^{\alpha} - 1} \left[2 - (\lambda + x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1} (\lambda + x)^{-\theta-1} \right\}^{w} dx. \tag{3.20}$$

Let I denote the integral inside Eq (3.20). Factoring out constants, we obtain

$$I = \frac{\alpha^w \theta^w \lambda^{\theta w}}{(2^{\alpha} - 1)^w} \int_0^{\infty} \left[2 - \left(1 + \frac{x}{\lambda}\right)^{-\theta}\right]^{w(\alpha-1)} (\lambda + x)^{-w(\theta+1)} dx. \tag{3.21}$$

We apply the binomial expansion to the term

$$\left[2 - \left(1 + \frac{x}{\lambda}\right)^{-\theta}\right]^{w(\alpha-1)} = \sum_{k=0}^{\infty} \binom{w(\alpha - 1)}{k} 2^{w(\alpha-1)-k} \left[-\left(1 + \frac{x}{\lambda}\right)^{-\theta}\right]^{k}. \tag{3.22}$$

This expansion is valid for expressions of the form

$$[2 - z(x)]^{\gamma} = \sum_{n=0}^{\infty} \binom{\gamma}{n} (-1)^n 2^{\gamma-n} z(x)^n,$$

where $z(x) = \left(\frac{\lambda}{\lambda+x}\right)^{\theta} \in (0, 1]$ for all $x > 0$. The convergence is guaranteed since the support of the APIILx distribution is $x > 0$, ensuring $|z(x)| < 1$.

Substituting the expansion into the integral, we obtain

$$I = \frac{\alpha^w \theta^w \lambda^{-1}}{(2^{\alpha} - 1)^w} \sum_{k=0}^{\infty} \binom{w(\alpha - 1)}{k} (-1)^k 2^{w(\alpha-1)-k} \int_0^{\infty} \left(1 + \frac{x}{\lambda}\right)^{-\theta k - w(\theta+1)} dx. \tag{3.23}$$

To evaluate the inner integral, apply the substitution $u = 1 + \frac{x}{\lambda} \Rightarrow dx = \lambda du$, with limits $x = 0 \Rightarrow u = 1$ and $x \to \infty \Rightarrow u \to \infty$. Then, we can write

$$\int_0^\infty \left(1 + \frac{x}{\lambda}\right)^{-s} dx = \lambda \int_1^\infty u^{-s} du = \frac{\lambda}{s-1} \quad \text{for } s > 1.$$

Letting $s = \theta k + w(\theta + 1)$, we get

$$\int_0^\infty \left(1 + \frac{x}{\lambda}\right)^{-(\theta k + w(\theta+1))} dx = \frac{\lambda}{\theta k + w(\theta + 1) - 1}. \tag{3.24}$$

Substituting Eq (3.24) into Eq (3.23), the integral becomes

$$I = \frac{\alpha^w \theta^w}{(2^\alpha - 1)^w} \sum_{k=0}^{\infty} \binom{w(\alpha-1)}{k} \frac{(-1)^k 2^{w(\alpha-1)-k}}{\theta k + w(\theta+1) - 1}. \tag{3.25}$$

Finally, substituting Eq (3.25) into Eq (3.20), the Rényi entropy is obtained as

$$\begin{aligned} H(w) &= \frac{1}{1-w} \log\left(\frac{\alpha^w \theta^w}{(2^\alpha - 1)^w} \sum_{k=0}^{\infty} \binom{w(\alpha-1)}{k} \frac{(-1)^k 2^{w(\alpha-1)-k}}{\theta k + w(\theta+1) - 1} \right) \\ &= \frac{1}{1-w} \left[w \log\left(\frac{\alpha\theta}{2^\alpha - 1} \right) + \log\left(\sum_{k=0}^{\infty} \binom{w(\alpha-1)}{k} \frac{(-1)^k 2^{w(\alpha-1)-k}}{\theta k + w(\theta+1) - 1} \right) \right]. \end{aligned} \tag{3.26}$$

3.4. Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the APIILx distribution. The order statistics are denoted by $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$, where $X_{(r)}$ represents the rth order statistic in the sample.

The general PDF of the rth order statistic is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x). \tag{3.27}$$

Substituting the CDF and PDF of the APIILx distribution into the order statistic formula, we obtain

$$\begin{aligned} f_{X_{(r)}}(x; \alpha, \theta, \lambda) = {} & \frac{n!}{(r-1)!(n-r)!} \left\{ \frac{\left[2 - (\lambda + x)^{-\theta} \lambda^\theta\right]^\alpha - 1}{2^\alpha - 1} \right\}^{r-1} \\ & \left\{ 1 - \frac{\left[2 - (\lambda + x)^{-\theta} \lambda^\theta\right]^\alpha - 1}{2^\alpha - 1} \right\}^{n-r} \\ & \frac{\alpha\theta\lambda^\theta}{2^\alpha - 1} \left[2 - (\lambda + x)^{-\theta} \lambda^\theta\right]^{\alpha-1} (\lambda + x)^{-\theta-1}. \end{aligned} \tag{3.28}$$

The minimum order statistic of the APIILx distribution follows (for $r = 1$) as

$$f_{X_{(1)}}(x) = \frac{n\,\alpha\,\theta\,\lambda^\theta \left[2 - (\lambda + x)^{-\theta} \lambda^\theta\right]^{\alpha-1}}{(\lambda + x)^{\theta+1} (2^\alpha - 1)} \left\{ \frac{2^\alpha - \left[2 - (\lambda + x)^{-\theta} \lambda^\theta\right]^\alpha}{2^\alpha - 1} \right\}^{n-1}.$$

The maximum order statistic of the APIILx distribution follows (for $r = n$) as

$$f_{X_{(n)}}(x) = \frac{n\,\alpha\,\theta\,\lambda^{\theta}\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha-1}}{(\lambda+x)^{\theta+1}(2^{\alpha}-1)}\left\{\frac{\left[2-(\lambda+x)^{-\theta}\lambda^{\theta}\right]^{\alpha}-1}{2^{\alpha}-1}\right\}^{n-1}.$$

For the APIILx distribution, the availability of closed-form expressions for both the PDF and CDF enables explicit calculation of order statistic densities, facilitating analytical derivation and numerical computation.

4. Estimation approaches

In this part, we obtain estimates for the APIILx parameters α, θ, and λ using eight frequentist estimation techniques. The approaches considered include least squares (LS), Anderson-Darling (AD), weighted least squares (WLS), maximum likelihood (ML), maximum product of spacings (MPS), Cramér-von Mises (CRVM), right-tail AD (RAD), and percentile (PC) estimators.

The log-likelihood function of the APIILx distribution follows as

$$\begin{aligned}\ell = \ell(\alpha,\theta,\lambda) &= n\ln\alpha + n\ln\theta + n\theta\ln\lambda - n\ln(2^{\alpha}-1)\\ &\quad + (\alpha-1)\sum_{i=1}^{n}\ln\left[2-\left(\frac{\lambda}{\lambda+x_i}\right)^{\theta}\right] - (\theta+1)\sum_{i=1}^{n}\ln(\lambda+x_i).\end{aligned} \tag{4.1}$$

The ML method is one of the most widely used techniques for parameter estimation in statistical modeling (see, e.g., [23–25]). The ML estimators (MLEs) of the APIILx parameters α, θ, and λ are derived by taking the partial derivatives of the log-likelihood function ℓ with respect to each parameter and equating them to zero. This leads to the following system of equations:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - n\frac{2^{\alpha}\ln 2}{2^{\alpha}-1} + \sum_{i=1}^{n}\ln\left[2-\left(\frac{\lambda}{\lambda+x_i}\right)^{\theta}\right], \tag{4.2}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n\theta}{\lambda} - (\theta+1)\sum_{i=1}^{n}\frac{1}{\lambda+x_i} - (\alpha-1)\sum_{i=1}^{n}\frac{\theta x_i\lambda^{\theta-1}(\lambda+x_i)^{-1}}{2\,(\lambda+x_i)^{\theta}-\lambda^{\theta}} \tag{4.3}$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + n\ln\lambda - (\alpha-1)\sum_{i=1}^{n}\frac{\lambda^{\theta}\ln\left(\frac{\lambda}{\lambda+x_i}\right)}{2\,(\lambda+x_i)^{\theta}-\lambda^{\theta}} - \sum_{i=1}^{n}\ln(\lambda+x_i). \tag{4.4}$$

Setting the partial derivatives equal to zero yields a system of nonlinear equations, which has no closed-form solutions. Therefore, numerical optimization methods such as the Newton-Raphson method or iterative algorithms are required to estimate the parameters.

The LS estimators (LSEs) for $\widehat{\alpha}_{LSE}$, $\widehat{\theta}_{LSE}$, and $\widehat{\lambda}_{LSE}$ of the unknown parameters α, θ, and λ of the APIILx distribution can be obtained by minimizing the following function:

$$S(\alpha,\theta,\lambda) = \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2,$$

with respect to α, θ, and λ. Similarly, they can be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left[\frac{\left(2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right)^{\alpha}-1}{2^{\alpha}-1}-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,\lambda\right)=0,\quad s=1,2,3,$$

where

$$\begin{aligned}\Delta_1(x_{(i)}|\alpha,\theta,\lambda)=\frac{\partial}{\partial\alpha}F\left(x_{(i)}|\alpha,\theta,\lambda\right)&=\frac{\left[2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right]^{\alpha}\ln\left(2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right)}{2^{\alpha}-1}\\&-\frac{\left\{\left[2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right]^{\alpha}-1\right\}2^{\alpha}\ln 2}{(2^{\alpha}-1)^2},\end{aligned}\tag{4.5}$$

$$\begin{aligned}\Delta_2(x_{(i)}|\alpha,\theta,\lambda)=\frac{\partial}{\partial\theta}F\left(x_{(i)}|\alpha,\theta,\lambda\right)&=\frac{\alpha\,\lambda^{\theta}\left[2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right]^{\alpha-1}}{(2^{\alpha}-1)(\lambda+x_{(i)})^{\theta}}\\&\times\left[\ln(\lambda+x_{(i)})-\ln(\lambda)\right]\end{aligned}\tag{4.6}$$

and

$$\Delta_3(x_{(i)}|\alpha,\theta,\lambda)=\frac{\partial}{\partial\lambda}F\left(x_{(i)}|\alpha,\theta,\lambda\right)=-\frac{\alpha\,\theta\,x_{(i)}\,\lambda^{\theta-1}\left[2-(\lambda+x_{(i)})^{-\theta}\lambda^{\theta}\right]^{\alpha-1}}{(2^{\alpha}-1)(\lambda+x_{(i)})^{\theta+1}}.\tag{4.7}$$

Note that the solution of Δ_s for $s=1,2,3$ can be obtained numerically.

The WLS [26] estimators (WLSEs) of the APIILx parameters can be obtained by minimizing the following equation:

$$W\left(\alpha,\theta,\lambda\right)=\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F\left(x_{(i)}|\alpha,\theta,\lambda\right)-\frac{i}{n+1}\right]^2.$$

Moreover, the WLSEs are obtained by addressing the corresponding set of nonlinear estimating equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F\left(x_{(i)}|\alpha,\theta,\lambda\right)-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,\lambda\right)=0,\quad s=1,2,3,$$

where Δ_s are provided in Eqs (4.5-4.7) for $s=1,2,3$.

The MPS approach, introduced in [27–29], approximates the Kullback-Leibler information measure and is widely considered a strong alternative to the ML method.

Let $D_i(\alpha,\theta,\lambda)=F\left(x_{(i)}|\alpha,\theta,\lambda\right)-F\left(x_{(i-1)}|\alpha,\theta,\lambda\right)$ for $i=1,2,\ldots,n+1$, denote the uniform spacings associated with an ordered sample from the APIILx distribution, where

$$F\left(x_{(0)}|\alpha,\theta,\lambda\right)=0, F\left(x_{(n+1)}|\alpha,\theta,\lambda\right)=1\ \text{ and }\ \sum_{i=1}^{n+1}D_i(\alpha,\theta,\lambda)=1.$$

The MPS estimators (MPSEs) for the APIILx parameters are then obtained by maximizing the geometric mean of these spacings:

$$G\left(\alpha,\theta,\lambda\right)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\theta,\lambda)\right]^{\frac{1}{n+1}},$$

with respect to the parameters α, θ, and λ. In addition, these estimators may be obtained by maximizing the logarithm of the geometric mean of the sample spacings given by

$$H(\alpha,\theta,\lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\theta,\lambda).$$

The MPSEs for the APIILx parameters may alternatively be computed by solving the system of nonlinear equations given by

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\theta,\lambda)}\left[\Delta_s(x_{(i)}|\alpha,\theta,\lambda)-\Delta_s(x_{(i-1)}|\alpha,\theta,\lambda)\right]=0, \quad s=1,2,3.$$

The CRVM estimators (CRVMEs), which belong to the class of minimum distance estimators, generally exhibit lower bias compared to other estimators of this type. They are derived from the discrepancy between the theoretical CDF and the empirical CDF [30].

For the APIILx parameters, the CRVMEs are obtained by minimizing

$$C(\alpha,\theta,\lambda) = \frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right]^2,$$

with respect to the parameters α, θ, and λ. Additionally, the CRVMEs can be determined by solving the corresponding system of nonlinear equations:

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,\lambda\right)=0, \quad s=1,2,3.$$

The AD statistic, or AD estimators (ADEs), represent another class of minimum distance estimators. The ADEs for the APIILx parameters are derived by minimizing

$$A(\alpha,\theta,\lambda) = -n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)}|\alpha,\theta,\lambda\right)+\log S\left(x_{(i)}|\alpha,\theta,\lambda\right)\right].$$

Alternatively, the ADEs can be computed by solving the associated system of nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s\left(x_{(i)}|\alpha,\theta,\lambda\right)}{F\left(x_{(i)}|\alpha,\theta,\lambda\right)}-\frac{\Delta_j\left(x_{(n+1-i)}|\alpha,\theta,\lambda\right)}{S\left(x_{(n+1-i)}|\alpha,\theta,\lambda\right)}\right]=0, \quad s=1,2,3.$$

The RAD estimators (RADEs) of the APIILx parameters are obtained by minimizing

$$R(\alpha,\theta,\lambda) = \frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)}|\alpha,\theta,\lambda\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{(n+1-i)}|\alpha,\theta,\lambda\right).$$

The RADEs can also be obtained by solving the nonlinear equations

$$-2\sum_{i=1}^{n}\Delta_s\left(x_{(i)}|\alpha,\theta,\lambda\right)+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s\left(x_{(n+1-i)}|\alpha,\theta,\lambda\right)}{S\left(x_{(n+1-i)}|\alpha,\theta,\lambda\right)}=0, \quad s=1,2,3.$$

Consider the unbiased estimator of $F\left(x_{(i)} \mid \alpha, \theta, \lambda\right)$ given by $u_i = \dfrac{i}{n+1}$. The PC estimators (PCEs) of the APIILx parameters are obtained by minimizing the following function with respect to α, θ, and λ:

$$P(\alpha, \theta, \lambda) = \sum_{i=1}^{n} \left[x_{(i)} - \lambda \left(\left\{ 2 - \left[1 + (2^{\alpha} - 1)\, u_i \right]^{\frac{1}{\alpha}} \right\}^{-1/\theta} - 1 \right) \right]^2 .$$

Due to the nonlinear structure and analytical complexity of the APIILx model, deriving closed-form theoretical properties such as unbiasedness, consistency, and asymptotic distributions for all considered estimators is mathematically intractable. Therefore, the properties of the estimators are investigated through an extensive Monte Carlo simulation study, where their finite-sample performance is evaluated in terms of bias and mean squared error under different parameter settings and sample sizes.

5. Simulation analysis

In this section, a comprehensive evaluation is conducted regarding the effectiveness of estimation approaches in estimating the three parameters of the APIILx distribution. The simulation framework incorporates a systematic arrangement of processes, wherein $N = 5000$ iterations of different samples of sizes ($n = 20, 50, 100, 300, 500$) are produced from the APIILx distribution employing the inversion technique for eight distinct selected sets of parameters.

The simulation study was conducted using the `R` software (`R` Core Team, 2023, version 4.3.10), and it compares the estimates from eight estimation methods in terms of the average absolute biases (AB) (AB $= \frac{1}{N} \sum_{i=1}^{N} |\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}|$), the average mean squared errors (MSE) (MSE $= \frac{1}{N} \sum_{i=1}^{N} (\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi})^2$), and the average mean relative errors (MRE) (MRE $= \frac{1}{N} \sum_{i=1}^{N} |\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}|/\boldsymbol{\psi}$), where $\boldsymbol{\psi} = (\alpha, \theta, \lambda)^{\intercal}$.

For each generated sample and parameter setting, the parameters of the APIILx distribution were estimated using eight different methods, namely MLE, WLSE, CRVME, LSE, MPSE, ADE, PCE, and RADE. The corresponding MSE and MRE of these estimates were then evaluated. Tables 1–8 summarize the simulation findings, reporting the AB, MSE, and MRE for all eight estimation techniques. Additionally, these tables display the relative ranking of each estimator in every row, where the superscripts denote the rank indicators and $\sum$ Ranks represent the partial sum of ranks for each column corresponding to a specific sample size n. Table 9 compiles both the partial and overall ranking outcomes for all considered estimators.

Table 1. Simulation results for $\boldsymbol{\psi} = (\alpha = 3.5, \theta = 4, \lambda = 0.5)^\intercal$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.23041 $^{\{2\}}$	2.61442 $^{\{5\}}$	2.31791 $^{\{4\}}$	2.64465 $^{\{6\}}$	1.98554 $^{\{1\}}$	3.10332 $^{\{8\}}$	2.23233 $^{\{3\}}$	2.69567 $^{\{7\}}$
		$\hat{\theta}$	6.12882 $^{\{8\}}$	2.56601 $^{\{5\}}$	2.49767 $^{\{4\}}$	2.71031 $^{\{6\}}$	2.20831 $^{\{1\}}$	2.86224 $^{\{7\}}$	2.38253 $^{\{2\}}$	2.46722 $^{\{3\}}$
		$\hat{\lambda}$	1.29839 $^{\{8\}}$	0.46629 $^{\{4\}}$	0.45037 $^{\{3\}}$	0.49280 $^{\{6\}}$	0.38245 $^{\{1\}}$	0.49747 $^{\{7\}}$	0.43803 $^{\{2\}}$	0.46992 $^{\{5\}}$
	MSE	$\hat{\alpha}$	4.97474 $^{\{2\}}$	6.83521 $^{\{5\}}$	5.37269 $^{\{4\}}$	6.99418 $^{\{6\}}$	3.94235 $^{\{1\}}$	9.63062 $^{\{8\}}$	4.98330 $^{\{3\}}$	7.26663 $^{\{7\}}$
		$\hat{\theta}$	37.56333 $^{\{8\}}$	6.58442 $^{\{5\}}$	6.23838 $^{\{4\}}$	7.34579 $^{\{6\}}$	4.87665 $^{\{1\}}$	8.19239 $^{\{7\}}$	5.67644 $^{\{2\}}$	6.08717 $^{\{3\}}$
		$\hat{\lambda}$	1.68585 $^{\{8\}}$	0.21743 $^{\{4\}}$	0.20283 $^{\{3\}}$	0.24285 $^{\{6\}}$	0.14627 $^{\{1\}}$	0.24748 $^{\{7\}}$	0.19187 $^{\{2\}}$	0.22083 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.63726 $^{\{2\}}$	0.74698 $^{\{5\}}$	0.66226 $^{\{4\}}$	0.75561 $^{\{6\}}$	0.56730 $^{\{1\}}$	0.88666 $^{\{8\}}$	0.63781 $^{\{3\}}$	0.77019 $^{\{7\}}$
		$\hat{\theta}$	1.53221 $^{\{8\}}$	0.64150 $^{\{5\}}$	0.62442 $^{\{4\}}$	0.67758 $^{\{6\}}$	0.55208 $^{\{1\}}$	0.71556 $^{\{7\}}$	0.59563 $^{\{2\}}$	0.61680 $^{\{3\}}$
		$\hat{\lambda}$	2.59678 $^{\{8\}}$	0.93259 $^{\{4\}}$	0.90073 $^{\{3\}}$	0.98560 $^{\{6\}}$	0.76490 $^{\{1\}}$	0.99494 $^{\{7\}}$	0.87605 $^{\{2\}}$	0.93984 $^{\{5\}}$
		$\sum Ranks$	54 $^{\{6.5\}}$	42 $^{\{4\}}$	33 $^{\{3\}}$	54 $^{\{6.5\}}$	9 $^{\{1\}}$	66 $^{\{8\}}$	21 $^{\{2\}}$	45 $^{\{5\}}$
50	AB	$\hat{\alpha}$	1.45201 $^{\{2\}}$	1.67040 $^{\{6\}}$	1.45224 $^{\{3\}}$	1.67158 $^{\{7\}}$	1.35798 $^{\{1\}}$	2.70044 $^{\{8\}}$	1.45568 $^{\{4\}}$	1.65412 $^{\{5\}}$
		$\hat{\theta}$	2.07736 $^{\{5\}}$	2.26445 $^{\{6\}}$	2.01075 $^{\{4\}}$	2.38081 $^{\{7\}}$	1.56812 $^{\{1\}}$	2.44350 $^{\{8\}}$	1.96835 $^{\{3\}}$	1.96713 $^{\{2\}}$
		$\hat{\lambda}$	0.39855 $^{\{5\}}$	0.41209 $^{\{6\}}$	0.37164 $^{\{3\}}$	0.42827 $^{\{7\}}$	0.28149 $^{\{1\}}$	0.47633 $^{\{8\}}$	0.36644 $^{\{2\}}$	0.38047 $^{\{4\}}$
	MSE	$\hat{\alpha}$	2.10834 $^{\{2\}}$	2.79024 $^{\{6\}}$	2.10899 $^{\{3\}}$	2.79419 $^{\{7\}}$	1.84411 $^{\{1\}}$	7.29237 $^{\{8\}}$	2.11902 $^{\{4\}}$	2.73610 $^{\{5\}}$
		$\hat{\theta}$	4.31543 $^{\{5\}}$	5.12771 $^{\{6\}}$	4.04311 $^{\{4\}}$	5.66825 $^{\{7\}}$	2.45901 $^{\{1\}}$	5.97067 $^{\{8\}}$	3.87442 $^{\{3\}}$	3.86960 $^{\{2\}}$
		$\hat{\lambda}$	0.15884 $^{\{5\}}$	0.16982 $^{\{6\}}$	0.13811 $^{\{3\}}$	0.18342 $^{\{7\}}$	0.07924 $^{\{1\}}$	0.22689 $^{\{8\}}$	0.13428 $^{\{2\}}$	0.14476 $^{\{4\}}$
	MRE	$\hat{\alpha}$	0.41486 $^{\{2\}}$	0.47726 $^{\{6\}}$	0.41492 $^{\{3\}}$	0.47760 $^{\{7\}}$	0.38799 $^{\{1\}}$	0.77155 $^{\{8\}}$	0.41591 $^{\{4\}}$	0.47260 $^{\{5\}}$
		$\hat{\theta}$	0.51934 $^{\{5\}}$	0.56611 $^{\{6\}}$	0.50269 $^{\{4\}}$	0.59520 $^{\{7\}}$	0.39203 $^{\{1\}}$	0.61087 $^{\{8\}}$	0.49209 $^{\{3\}}$	0.49178 $^{\{2\}}$
		$\hat{\lambda}$	0.79709 $^{\{5\}}$	0.82419 $^{\{6\}}$	0.74327 $^{\{3\}}$	0.85654 $^{\{7\}}$	0.56298 $^{\{1\}}$	0.95267 $^{\{8\}}$	0.73287 $^{\{2\}}$	0.76094 $^{\{4\}}$
		$\sum Ranks$	36 $^{\{5\}}$	54 $^{\{6\}}$	30 $^{\{3\}}$	63 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	27 $^{\{2\}}$	33 $^{\{4\}}$
100	AB	$\hat{\alpha}$	1.07727 $^{\{3\}}$	1.18569 $^{\{5\}}$	1.07007 $^{\{2\}}$	1.21698 $^{\{7\}}$	0.91996 $^{\{1\}}$	2.32128 $^{\{8\}}$	1.11033 $^{\{4\}}$	1.19264 $^{\{6\}}$
		$\hat{\theta}$	1.53092 $^{\{2\}}$	1.88123 $^{\{6\}}$	1.56837 $^{\{5\}}$	1.90857 $^{\{7\}}$	0.95301 $^{\{1\}}$	1.99203 $^{\{8\}}$	1.55885 $^{\{4\}}$	1.53585 $^{\{3\}}$
		$\hat{\lambda}$	0.29186 $^{\{2\}}$	0.34586 $^{\{6\}}$	0.29546 $^{\{4\}}$	0.34991 $^{\{7\}}$	0.12949 $^{\{1\}}$	0.42119 $^{\{8\}}$	0.29250 $^{\{3\}}$	0.30016 $^{\{5\}}$
	MSE	$\hat{\alpha}$	1.16051 $^{\{3\}}$	1.40586 $^{\{5\}}$	1.14505 $^{\{2\}}$	1.48104 $^{\{7\}}$	0.84632 $^{\{1\}}$	5.38833 $^{\{8\}}$	1.23284 $^{\{4\}}$	1.42240 $^{\{6\}}$
		$\hat{\theta}$	2.34370 $^{\{2\}}$	3.53903 $^{\{6\}}$	2.45977 $^{\{5\}}$	3.64263 $^{\{7\}}$	0.90823 $^{\{1\}}$	3.96817 $^{\{8\}}$	2.43001 $^{\{4\}}$	2.35882 $^{\{3\}}$
		$\hat{\lambda}$	0.08518 $^{\{2\}}$	0.11962 $^{\{6\}}$	0.08730 $^{\{4\}}$	0.12244 $^{\{7\}}$	0.01677 $^{\{1\}}$	0.17740 $^{\{8\}}$	0.08556 $^{\{3\}}$	0.09010 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.30779 $^{\{3\}}$	0.33877 $^{\{5\}}$	0.30573 $^{\{2\}}$	0.34771 $^{\{7\}}$	0.26284 $^{\{1\}}$	0.66322 $^{\{8\}}$	0.31724 $^{\{4\}}$	0.34075 $^{\{6\}}$
		$\hat{\theta}$	0.38273 $^{\{2\}}$	0.47031 $^{\{6\}}$	0.39209 $^{\{5\}}$	0.47714 $^{\{7\}}$	0.23825 $^{\{1\}}$	0.49801 $^{\{8\}}$	0.38971 $^{\{4\}}$	0.38396 $^{\{3\}}$
		$\hat{\lambda}$	0.58372 $^{\{2\}}$	0.69173 $^{\{6\}}$	0.59092 $^{\{4\}}$	0.69982 $^{\{7\}}$	0.25899 $^{\{1\}}$	0.84238 $^{\{8\}}$	0.58501 $^{\{3\}}$	0.60033 $^{\{5\}}$
		$\sum Ranks$	21 $^{\{2\}}$	51 $^{\{6\}}$	33 $^{\{3.5\}}$	63 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	33 $^{\{3.5\}}$	42 $^{\{5\}}$
300	AB	$\hat{\alpha}$	0.61258 $^{\{2\}}$	0.73210 $^{\{7\}}$	0.62949 $^{\{4\}}$	0.72191 $^{\{6\}}$	0.48877 $^{\{1\}}$	1.86755 $^{\{8\}}$	0.61895 $^{\{3\}}$	0.69267 $^{\{5\}}$
		$\hat{\theta}$	0.85206 $^{\{2\}}$	1.23476 $^{\{6\}}$	0.95672 $^{\{4\}}$	1.25761 $^{\{7\}}$	0.30635 $^{\{1\}}$	1.49305 $^{\{8\}}$	0.97387 $^{\{5\}}$	0.95170 $^{\{3\}}$
		$\hat{\lambda}$	0.16673 $^{\{2\}}$	0.23170 $^{\{6\}}$	0.18426 $^{\{3\}}$	0.23517 $^{\{7\}}$	0.03542 $^{\{1\}}$	0.33142 $^{\{8\}}$	0.18707 $^{\{4\}}$	0.18802 $^{\{5\}}$
	MSE	$\hat{\alpha}$	0.37525 $^{\{2\}}$	0.53597 $^{\{7\}}$	0.39626 $^{\{4\}}$	0.52116 $^{\{6\}}$	0.23890 $^{\{1\}}$	3.48775 $^{\{8\}}$	0.38310 $^{\{3\}}$	0.47979 $^{\{5\}}$
		$\hat{\theta}$	0.72600 $^{\{2\}}$	1.52464 $^{\{6\}}$	0.91531 $^{\{4\}}$	1.58159 $^{\{7\}}$	0.09385 $^{\{1\}}$	2.22920 $^{\{8\}}$	0.94843 $^{\{5\}}$	0.90574 $^{\{3\}}$
		$\hat{\lambda}$	0.02780 $^{\{2\}}$	0.05368 $^{\{6\}}$	0.03395 $^{\{3\}}$	0.05530 $^{\{7\}}$	0.00125 $^{\{1\}}$	0.10984 $^{\{8\}}$	0.03500 $^{\{4\}}$	0.03535 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.17502 $^{\{2\}}$	0.20917 $^{\{7\}}$	0.17985 $^{\{4\}}$	0.20626 $^{\{6\}}$	0.13965 $^{\{1\}}$	0.53359 $^{\{8\}}$	0.17684 $^{\{3\}}$	0.19791 $^{\{5\}}$
		$\hat{\theta}$	0.21301 $^{\{2\}}$	0.30869 $^{\{6\}}$	0.23918 $^{\{4\}}$	0.31440 $^{\{7\}}$	0.07659 $^{\{1\}}$	0.37326 $^{\{8\}}$	0.24347 $^{\{5\}}$	0.23793 $^{\{3\}}$
		$\hat{\lambda}$	0.33346 $^{\{2\}}$	0.46339 $^{\{6\}}$	0.36851 $^{\{3\}}$	0.47034 $^{\{7\}}$	0.07084 $^{\{1\}}$	0.66284 $^{\{8\}}$	0.37414 $^{\{4\}}$	0.37603 $^{\{5\}}$
		$\sum Ranks$	18 $^{\{2\}}$	57 $^{\{6\}}$	33 $^{\{3\}}$	60 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	36 $^{\{4\}}$	39 $^{\{5\}}$
500	AB	$\hat{\alpha}$	0.46742 $^{\{2\}}$	0.55051 $^{\{6\}}$	0.47109 $^{\{3\}}$	0.57009 $^{\{7\}}$	0.30512 $^{\{1\}}$	1.53915 $^{\{8\}}$	0.48215 $^{\{4\}}$	0.52617 $^{\{5\}}$
		$\hat{\theta}$	0.65934 $^{\{2\}}$	0.97301 $^{\{6\}}$	0.71573 $^{\{3\}}$	0.98176 $^{\{7\}}$	0.12827 $^{\{1\}}$	1.23279 $^{\{8\}}$	0.76277 $^{\{5\}}$	0.74560 $^{\{4\}}$
		$\hat{\lambda}$	0.12983 $^{\{2\}}$	0.18263 $^{\{6\}}$	0.14228 $^{\{3\}}$	0.18623 $^{\{7\}}$	0.02274 $^{\{1\}}$	0.28114 $^{\{8\}}$	0.14674 $^{\{4\}}$	0.14864 $^{\{5\}}$
	MSE	$\hat{\alpha}$	0.21848 $^{\{2\}}$	0.30306 $^{\{6\}}$	0.22192 $^{\{3\}}$	0.32500 $^{\{7\}}$	0.09310 $^{\{1\}}$	2.36897 $^{\{8\}}$	0.23247 $^{\{4\}}$	0.27686 $^{\{5\}}$
		$\hat{\theta}$	0.43472 $^{\{2\}}$	0.94676 $^{\{6\}}$	0.51228 $^{\{3\}}$	0.96386 $^{\{7\}}$	0.01645 $^{\{1\}}$	1.51977 $^{\{8\}}$	0.58182 $^{\{5\}}$	0.55592 $^{\{4\}}$
		$\hat{\lambda}$	0.01686 $^{\{2\}}$	0.03335 $^{\{6\}}$	0.02024 $^{\{3\}}$	0.03468 $^{\{7\}}$	0.00052 $^{\{1\}}$	0.07904 $^{\{8\}}$	0.02153 $^{\{4\}}$	0.02209 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.13355 $^{\{2\}}$	0.15729 $^{\{6\}}$	0.13460 $^{\{3\}}$	0.16288 $^{\{7\}}$	0.08718 $^{\{1\}}$	0.43976 $^{\{8\}}$	0.13776 $^{\{4\}}$	0.15034 $^{\{5\}}$
		$\hat{\theta}$	0.16483 $^{\{2\}}$	0.24325 $^{\{6\}}$	0.17893 $^{\{3\}}$	0.24544 $^{\{7\}}$	0.03207 $^{\{1\}}$	0.30820 $^{\{8\}}$	0.19069 $^{\{5\}}$	0.18640 $^{\{4\}}$
		$\hat{\lambda}$	0.25966 $^{\{2\}}$	0.36525 $^{\{6\}}$	0.28456 $^{\{3\}}$	0.37247 $^{\{7\}}$	0.04548 $^{\{1\}}$	0.56229 $^{\{8\}}$	0.29348 $^{\{4\}}$	0.29727 $^{\{5\}}$
		$\sum Ranks$	18 $^{\{2\}}$	54 $^{\{6\}}$	27 $^{\{3\}}$	63 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	39 $^{\{4\}}$	42 $^{\{5\}}$

Table 2. Simulation results for $\underline{\psi} = (\alpha = 0.75, \theta = 0.5, \lambda = 0.67)^\intercal$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.28583 {3}	2.71888 {7}	2.26663 {2}	2.67912 {6}	1.95168 {1}	3.38538 {8}	2.36908 {4}	2.59682 {5}
		$\hat{\theta}$	6.06086 {8}	2.55705 {5}	2.49278 {4}	2.67500 {6}	2.20879 {1}	2.86148 {7}	2.37683 {2}	2.49272 {3}
		$\hat{\lambda}$	3.45336 {8}	1.42126 {4}	1.33099 {3}	1.46607 {6}	1.09822 {1}	1.49156 {7}	1.31100 {2}	1.43537 {5}
	MSE	$\hat{\alpha}$	5.22500 {3}	7.39232 {7}	5.13762 {2}	7.17767 {6}	3.80907 {1}	11.46081 {8}	5.61256 {4}	6.74346 {5}
		$\hat{\theta}$	36.73469 {8}	6.53851 {5}	6.21394 {4}	7.15560 {6}	4.87874 {1}	8.18808 {7}	5.64932 {2}	6.21367 {3}
		$\hat{\lambda}$	11.92568 {8}	2.01997 {4}	1.77153 {3}	2.14935 {6}	1.20608 {1}	2.22475 {7}	1.71872 {2}	2.06029 {5}
	MRE	$\hat{\alpha}$	0.65309 {4}	0.77682 {7}	0.64761 {3}	0.76546 {6}	0.55762 {2}	0.96725 {8}	0.67688 {5}	0.74195 {1}
		$\hat{\theta}$	1.51522 {8}	0.63926 {5}	0.62319 {4}	0.66875 {6}	0.55220 {2}	0.71537 {7}	0.59421 {3}	0.62318 {1}
		$\hat{\lambda}$	2.30224 {8}	0.94750 {5}	0.88733 {4}	0.97738 {6}	0.73214 {2}	0.99437 {7}	0.87400 {3}	0.95691 {1}
		$\sum$ *Ranks*	58 {7}	49 {5}	29 {3.5}	54 {6}	12 {1}	66 {8}	27 {2}	29 {3.5}
50	AB	$\hat{\alpha}$	1.45066 {2}	1.62968 {5}	1.49877 {3}	1.64744 {6}	1.30027 {1}	2.68720 {8}	1.51098 {4}	1.69403 {7}
		$\hat{\theta}$	2.21862 {5}	2.32436 {6}	2.06202 {4}	2.41397 {8}	1.50354 {1}	2.41085 {7}	1.98866 {3}	1.95051 {2}
		$\hat{\lambda}$	1.24950 {5}	1.26704 {6}	1.14762 {4}	1.29059 {7}	0.75898 {1}	1.43026 {8}	1.12365 {2}	1.13400 {3}
	MSE	$\hat{\alpha}$	2.10443 {2}	2.65585 {5}	2.24633 {3}	2.71405 {6}	1.69071 {1}	7.22107 {8}	2.28307 {4}	2.86972 {7}
		$\hat{\theta}$	4.92229 {5}	5.40264 {6}	4.25193 {4}	5.82725 {8}	2.26065 {1}	5.81219 {7}	3.95475 {3}	3.80447 {2}
		$\hat{\lambda}$	1.56126 {5}	1.60539 {6}	1.31703 {4}	1.66563 {7}	0.57605 {1}	2.04565 {8}	1.26259 {2}	1.28597 {3}
	MRE	$\hat{\alpha}$	0.41448 {2}	0.46562 {5}	0.42822 {3}	0.47070 {6}	0.37151 {1}	0.76777 {8}	0.43171 {4}	0.48399 {7}
		$\hat{\theta}$	0.55466 {4}	0.58109 {5}	0.51550 {3}	0.60349 {7}	0.37589 {1}	0.60271 {6}	0.49716 {2}	0.48763 {8}
		$\hat{\lambda}$	0.83300 {4}	0.84469 {5}	0.76508 {3}	0.86039 {6}	0.50598 {1}	0.95351 {7}	0.74910 {2}	0.75600 {8}
		$\sum$ *Ranks*	34 {4}	49 {6}	31 {3}	61 {7}	9 {1}	67 {8}	26 {2}	47 {5}
100	AB	$\hat{\alpha}$	1.07020 {2}	1.24319 {7}	1.07388 {3}	1.23319 {6}	0.93656 {1}	2.35236 {8}	1.09367 {4}	1.19648 {5}
		$\hat{\theta}$	1.60012 {4}	1.89530 {6}	1.62735 {5}	1.98184 {7}	0.82120 {1}	2.01957 {8}	1.57894 {3}	1.52551 {2}
		$\hat{\lambda}$	0.91438 {5}	1.04914 {6}	0.90953 {4}	1.08659 {7}	0.24318 {1}	1.23628 {8}	0.89035 {2}	0.90136 {3}
	MSE	$\hat{\alpha}$	1.14532 {2}	1.54552 {7}	1.15321 {3}	1.52076 {6}	0.87714 {1}	5.53360 {8}	1.19611 {4}	1.43155 {5}
		$\hat{\theta}$	2.56038 {4}	3.59216 {6}	2.64827 {5}	3.92767 {7}	0.67437 {1}	4.07865 {8}	2.49305 {3}	2.32720 {2}
		$\hat{\lambda}$	0.83609 {5}	1.10070 {6}	0.82725 {4}	1.18067 {7}	0.05914 {1}	1.52838 {8}	0.79273 {2}	0.81245 {3}
	MRE	$\hat{\alpha}$	0.30577 {2}	0.35520 {6}	0.30682 {3}	0.35234 {5}	0.26759 {1}	0.67210 {8}	0.31248 {4}	0.34185 {7}
		$\hat{\theta}$	0.40003 {3}	0.47382 {5}	0.40684 {4}	0.49546 {7}	0.20530 {1}	0.50489 {8}	0.39473 {2}	0.38138 {6}
		$\hat{\lambda}$	0.60959 {4}	0.69943 {5}	0.60636 {3}	0.72439 {6}	0.16212 {1}	0.82418 {8}	0.59357 {2}	0.60091 {7}
		$\sum$ *Ranks*	31 {3}	54 {6}	34 {4}	58 {7}	9 {1}	72 {8}	26 {2}	40 {5}
300	AB	$\hat{\alpha}$	0.60134 {2}	0.73455 {6}	0.62976 {3}	0.74897 {7}	0.46271 {1}	1.82811 {8}	0.64430 {4}	0.70344 {5}
		$\hat{\theta}$	0.87534 {2}	1.30034 {7}	0.96502 {4}	1.27938 {6}	0.27722 {1}	1.47036 {8}	1.01155 {5}	0.95861 {3}
		$\hat{\lambda}$	0.51263 {2}	0.72409 {7}	0.55967 {3}	0.72291 {6}	0.08207 {1}	0.97039 {8}	0.58227 {5}	0.58016 {4}
	MSE	$\hat{\alpha}$	0.36161 {2}	0.53956 {6}	0.39659 {3}	0.56096 {7}	0.21410 {1}	3.34200 {8}	0.41512 {4}	0.49483 {5}
		$\hat{\theta}$	0.76622 {2}	1.69089 {7}	0.93127 {4}	1.63681 {6}	0.07685 {1}	2.16197 {8}	1.02323 {5}	0.91893 {3}
		$\hat{\lambda}$	0.26279 {2}	0.52431 {7}	0.31323 {3}	0.52259 {6}	0.00673 {1}	0.94165 {8}	0.33904 {5}	0.33659 {4}
	MRE	$\hat{\alpha}$	0.17181 {2}	0.20987 {5}	0.17993 {3}	0.21399 {6}	0.13220 {1}	0.52232 {8}	0.18408 {4}	0.20098 {7}
		$\hat{\theta}$	0.21883 {2}	0.32509 {6}	0.24126 {3}	0.31984 {5}	0.06930 {1}	0.36759 {7}	0.25289 {4}	0.23965 {8}
		$\hat{\lambda}$	0.34176 {2}	0.48273 {6}	0.37311 {3}	0.48194 {5}	0.05471 {1}	0.64693 {8}	0.38818 {4}	0.38677 {7}
		$\sum$ *Ranks*	18 {2}	57 {7}	29 {3}	54 {6}	9 {1}	71 {8}	40 {4}	46 {5}
500	AB	$\hat{\alpha}$	0.49125 {3}	0.57009 {6}	0.47240 {2}	0.58221 {7}	0.31046 {1}	1.61838 {8}	0.50766 {4}	0.55629 {5}
		$\hat{\theta}$	0.69195 {2}	1.01887 {7}	0.75144 {3}	1.00680 {6}	0.13354 {1}	1.24585 {8}	0.78129 {5}	0.77337 {4}
		$\hat{\lambda}$	0.41233 {2}	0.56898 {7}	0.43328 {3}	0.56466 {6}	0.05564 {1}	0.85886 {8}	0.45145 {4}	0.45782 {5}
	MSE	$\hat{\alpha}$	0.24133 {3}	0.32501 {6}	0.22316 {2}	0.33897 {7}	0.09639 {1}	2.61917 {8}	0.25771 {4}	0.30946 {5}
		$\hat{\theta}$	0.47879 {2}	1.03810 {7}	0.56466 {3}	1.01364 {6}	0.01783 {1}	1.55213 {8}	0.61041 {5}	0.59810 {4}
		$\hat{\lambda}$	0.17002 {2}	0.32374 {7}	0.18773 {3}	0.31884 {6}	0.00310 {1}	0.73764 {8}	0.20380 {4}	0.20960 {5}
	MRE	$\hat{\alpha}$	0.14036 {3}	0.16288 {5}	0.13497 {2}	0.16635 {6}	0.08870 {1}	0.46240 {8}	0.14504 {4}	0.15926 {7}
		$\hat{\theta}$	0.17299 {2}	0.25472 {7}	0.18786 {3}	0.25170 {6}	0.03339 {1}	0.31146 {8}	0.19532 {4}	0.19340 {5}
		$\hat{\lambda}$	0.27489 {2}	0.37932 {6}	0.28885 {3}	0.37644 {5}	0.03709 {1}	0.57258 {8}	0.30096 {4}	0.30518 {7}
		$\sum$ *Ranks*	21 {2}	58 {7}	24 {3}	55 {6}	9 {1}	72 {8}	38 {4}	47 {5}

Table 3. Simulation results for $\boldsymbol{\psi} = (\alpha = 3.5, \theta = 2, \lambda = 0.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.16227 {3}	2.49833 {5}	2.15140 {2}	2.68938 {7}	2.01193 {1}	2.89990 {8}	2.28276 {4}	2.51335 {6}
		$\hat{\theta}$	15.12868 {8}	3.38969 {4}	3.43103 {5}	3.56204 {6}	2.49191 {1}	3.75248 {7}	3.31910 {2}	3.33586 {3}
		$\hat{\lambda}$	2.16588 {8}	0.46469 {4}	0.46125 {3}	0.48767 {6}	0.35615 {1}	0.49505 {7}	0.45433 {2}	0.47883 {5}
	MSE	$\hat{\alpha}$	4.67543 {3}	6.24166 {5}	4.62851 {2}	7.23274 {7}	4.04786 {1}	8.40944 {8}	5.21101 {4}	6.31695 {6}
		$\hat{\theta}$	228.87696 {8}	11.49000 {4}	11.77197 {5}	12.68810 {6}	6.20960 {1}	14.08109 {7}	11.01640 {2}	11.12799 {3}
		$\hat{\lambda}$	4.69123 {8}	0.21593 {4}	0.21276 {3}	0.23782 {6}	0.12684 {1}	0.24508 {7}	0.20642 {2}	0.22928 {5}
	MRE	$\hat{\alpha}$	0.61779 {3}	0.71381 {5}	0.61468 {2}	0.76839 {7}	0.57484 {1}	0.82854 {8}	0.65222 {4}	0.71810 {6}
		$\hat{\theta}$	3.02574 {8}	0.67794 {4}	0.68621 {5}	0.71241 {6}	0.49838 {1}	0.75050 {7}	0.66382 {2}	0.66717 {3}
		$\hat{\lambda}$	4.33176 {8}	0.92937 {4}	0.92251 {3}	0.97534 {6}	0.71229 {1}	0.99011 {7}	0.90866 {2}	0.95766 {5}
		$\sum$ *Ranks*	57 {6.5}	39 {4}	30 {3}	57 {6.5}	9 {1}	66 {8}	24 {2}	42 {5}
50	AB	$\hat{\alpha}$	1.33982 {2}	1.50284 {5}	1.40061 {4}	1.52113 {6}	1.25276 {1}	2.31294 {8}	1.38992 {3}	1.52947 {7}
		$\hat{\theta}$	3.25771 {7}	3.20139 {6}	2.89465 {4}	3.29974 {8}	1.36397 {1}	3.20056 {5}	2.81634 {3}	2.66896 {2}
		$\hat{\lambda}$	0.44525 {7}	0.42773 {5}	0.39570 {4}	0.44045 {6}	0.21061 {1}	0.46005 {8}	0.39258 {3}	0.37957 {2}
	MSE	$\hat{\alpha}$	1.79511 {2}	2.25854 {5}	1.96170 {4}	2.31384 {6}	1.56940 {1}	5.34969 {8}	1.93188 {3}	2.33929 {7}
		$\hat{\theta}$	10.61267 {7}	10.24892 {6}	8.37902 {4}	10.88830 {8}	1.86040 {1}	10.24358 {5}	7.93177 {3}	7.12336 {2}
		$\hat{\lambda}$	0.19825 {7}	0.18295 {5}	0.15658 {4}	0.19400 {6}	0.04436 {1}	0.21164 {8}	0.15412 {3}	0.14407 {2}
	MRE	$\hat{\alpha}$	0.38281 {2}	0.42938 {5}	0.40017 {4}	0.43461 {6}	0.35793 {1}	0.66084 {8}	0.39712 {3}	0.43699 {7}
		$\hat{\theta}$	0.65154 {7}	0.64028 {6}	0.57893 {4}	0.65995 {8}	0.27279 {1}	0.64011 {5}	0.56327 {3}	0.53379 {2}
		$\hat{\lambda}$	0.89051 {7}	0.85546 {5}	0.79139 {4}	0.88090 {6}	0.42123 {1}	0.92009 {8}	0.78516 {3}	0.75914 {2}
		$\sum$ *Ranks*	48 {5.5}	48 {5.5}	36 {4}	60 {7}	9 {1}	63 {8}	27 {2}	33 {3}
100	AB	$\hat{\alpha}$	1.01866 {3}	1.11184 {6}	1.00636 {2}	1.11475 {7}	0.87145 {1}	1.92587 {8}	1.02048 {4}	1.10643 {5}
		$\hat{\theta}$	2.27369 {4}	2.75073 {7}	2.32823 {5}	2.75418 {8}	0.48109 {1}	2.74457 {6}	2.27353 {3}	2.24612 {2}
		$\hat{\lambda}$	0.32373 {5}	0.37202 {6}	0.32120 {4}	0.37613 {7}	0.08599 {1}	0.40415 {8}	0.32019 {3}	0.31954 {2}
	MSE	$\hat{\alpha}$	1.03767 {3}	1.23619 {6}	1.01276 {2}	1.24266 {7}	0.75942 {1}	3.70897 {8}	1.04138 {4}	1.22419 {5}
		$\hat{\theta}$	5.16968 {4}	7.56649 {7}	5.42066 {5}	7.58549 {8}	0.23145 {1}	7.53264 {6}	5.16894 {3}	5.04507 {2}
		$\hat{\lambda}$	0.10480 {5}	0.13840 {6}	0.10317 {4}	0.14147 {7}	0.00739 {1}	0.16334 {8}	0.10252 {3}	0.10211 {2}
	MRE	$\hat{\alpha}$	0.29105 {3}	0.31767 {6}	0.28753 {2}	0.31850 {7}	0.24898 {1}	0.55025 {8}	0.29157 {4}	0.31612 {5}
		$\hat{\theta}$	0.45474 {4}	0.55015 {7}	0.46565 {5}	0.55084 {8}	0.09622 {1}	0.54891 {6}	0.45471 {3}	0.44922 {2}
		$\hat{\lambda}$	0.64746 {5}	0.74404 {6}	0.64241 {4}	0.75226 {7}	0.17198 {1}	0.80830 {8}	0.64038 {3}	0.63908 {2}
		$\sum$ *Ranks*	36 {5}	57 {6}	33 {4}	66 {7.5}	9 {1}	66 {7.5}	30 {3}	27 {2}
300	AB	$\hat{\alpha}$	0.60663 {4}	0.71152 {7}	0.58402 {2}	0.70731 {6}	0.46522 {1}	1.45147 {8}	0.60026 {3}	0.67985 {5}
		$\hat{\theta}$	1.32393 {2}	1.89492 {7}	1.42002 {3}	1.85880 {6}	0.14533 {1}	1.97388 {8}	1.45847 {5}	1.43712 {4}
		$\hat{\lambda}$	0.19186 {2}	0.26033 {7}	0.19859 {3}	0.25708 {6}	0.02752 {1}	0.30997 {8}	0.20382 {5}	0.20320 {4}
	MSE	$\hat{\alpha}$	0.36800 {4}	0.50626 {7}	0.34108 {2}	0.50029 {6}	0.21643 {1}	2.10677 {8}	0.36031 {3}	0.46219 {5}
		$\hat{\theta}$	1.75279 {2}	3.59072 {7}	2.01645 {3}	3.45514 {6}	0.02112 {1}	3.89619 {8}	2.12714 {5}	2.06533 {4}
		$\hat{\lambda}$	0.03681 {2}	0.06777 {7}	0.03944 {3}	0.06609 {6}	0.00076 {1}	0.09608 {8}	0.04154 {5}	0.04129 {4}
	MRE	$\hat{\alpha}$	0.17332 {4}	0.20329 {7}	0.16686 {2}	0.20209 {6}	0.13292 {1}	0.41471 {8}	0.17150 {3}	0.19424 {5}
		$\hat{\theta}$	0.26479 {2}	0.37898 {7}	0.28400 {3}	0.37176 {6}	0.02907 {1}	0.39478 {8}	0.29169 {5}	0.28742 {4}
		$\hat{\lambda}$	0.38372 {2}	0.52066 {7}	0.39719 {3}	0.51416 {6}	0.05504 {1}	0.61995 {8}	0.40764 {5}	0.40639 {4}
		$\sum$ *Ranks*	24 {2.5}	63 {7}	24 {2.5}	54 {6}	9 {1}	72 {8}	39 {4.5}	39 {4.5}
500	AB	$\hat{\alpha}$	0.44711 {2}	0.53954 {6}	0.46013 {3}	0.54919 {7}	0.32015 {1}	1.24331 {8}	0.47272 {4}	0.51592 {5}
		$\hat{\theta}$	1.00936 {2}	1.50587 {7}	1.13587 {3}	1.50364 {6}	0.04375 {1}	1.65508 {8}	1.17072 {5}	1.14122 {4}
		$\hat{\lambda}$	0.14801 {2}	0.20959 {7}	0.16142 {3}	0.20946 {6}	0.01907 {1}	0.26383 {8}	0.16762 {5}	0.16221 {4}
	MSE	$\hat{\alpha}$	0.19991 {2}	0.29111 {6}	0.21172 {3}	0.30161 {7}	0.10249 {1}	1.54582 {8}	0.22346 {4}	0.26617 {5}
		$\hat{\theta}$	1.01881 {2}	2.26763 {7}	1.29020 {3}	2.26093 {6}	0.00191 {1}	2.73930 {8}	1.37059 {5}	1.30238 {4}
		$\hat{\lambda}$	0.02191 {2}	0.04393 {7}	0.02606 {3}	0.04387 {6}	0.00036 {1}	0.06961 {8}	0.02810 {5}	0.02631 {4}
	MRE	$\hat{\alpha}$	0.12775 {2}	0.15416 {6}	0.13147 {3}	0.15691 {7}	0.09147 {1}	0.35523 {8}	0.13506 {4}	0.14741 {5}
		$\hat{\theta}$	0.20187 {2}	0.30117 {7}	0.22717 {3}	0.30073 {6}	0.00875 {1}	0.33102 {8}	0.23414 {5}	0.22824 {4}
		$\hat{\lambda}$	0.29602 {2}	0.41918 {7}	0.32284 {3}	0.41893 {6}	0.03813 {1}	0.52766 {8}	0.33525 {5}	0.32443 {4}
		$\sum$ *Ranks*	18 {2}	60 {7}	27 {3}	57 {6}	9 {1}	72 {8}	42 {5}	39 {4}

Table 4. Simulation results for $\boldsymbol{\psi} = (\alpha = 3.5, \theta = 5, \lambda = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.17899 {2}	2.51497 {6}	2.20617 {3}	2.55274 {7}	2.01407 {1}	3.11865 {8}	2.26435 {4}	2.47083 {5}
		$\hat{\theta}$	18.55938 {8}	3.41141 {5}	3.38462 {3}	3.61383 {6}	2.33455 {1}	3.77831 {7}	3.31334 {2}	3.38556 {4}
		$\hat{\lambda}$	8.62519 {8}	1.38049 {4}	1.37068 {3}	1.47425 {6}	0.97487 {1}	1.49018 {7}	1.36243 {2}	1.44287 {5}
	MSE	$\hat{\alpha}$	4.74799 {2}	6.32508 {6}	4.86717 {3}	6.51646 {7}	4.05647 {1}	9.72599 {8}	5.12728 {4}	6.10499 {5}
		$\hat{\theta}$	344.51114 {8}	11.63773 {5}	11.45564 {3}	13.05978 {6}	5.45015 {1}	14.27565 {7}	10.97823 {2}	11.46202 {4}
		$\hat{\lambda}$	74.39400 {8}	1.90575 {4}	1.87877 {3}	2.17341 {6}	0.95037 {1}	2.22063 {7}	1.85622 {2}	2.08188 {5}
	MRE	$\hat{\alpha}$	0.62257 {2}	0.71856 {6}	0.63033 {3}	0.72935 {7}	0.57545 {1}	0.89104 {8}	0.64696 {4}	0.70595 {5}
		$\hat{\theta}$	3.71188 {8}	0.68228 {5}	0.67692 {3}	0.72277 {6}	0.46691 {1}	0.75566 {7}	0.66267 {2}	0.67711 {4}
		$\hat{\lambda}$	5.75013 {8}	0.92033 {4}	0.91379 {3}	0.98283 {6}	0.64991 {1}	0.99345 {7}	0.90829 {2}	0.96191 {5}
		$\sum$ *Ranks*	54 {6}	45 {5}	27 {3}	57 {7}	9 {1}	66 {8}	24 {2}	42 {4}
50	AB	$\hat{\alpha}$	1.32927 {2}	1.57436 {7}	1.41475 {3}	1.53006 {5}	1.23804 {1}	2.36189 {8}	1.44501 {4}	1.55244 {6}
		$\hat{\theta}$	3.32158 {8}	3.23097 {6}	2.91337 {4}	3.27551 {7}	1.13358 {1}	3.17206 {5}	2.75144 {2}	2.77251 {3}
		$\hat{\lambda}$	1.35850 {7}	1.30759 {5}	1.19039 {4}	1.32285 {6}	0.48874 {1}	1.36346 {8}	1.14735 {2}	1.18465 {3}
	MSE	$\hat{\alpha}$	1.76695 {2}	2.47860 {7}	2.00152 {3}	2.34109 {5}	1.53274 {1}	5.57851 {8}	2.08805 {4}	2.41007 {6}
		$\hat{\theta}$	11.03288 {8}	10.43915 {6}	8.48773 {4}	10.72900 {7}	1.28501 {1}	10.06197 {5}	7.57043 {2}	7.68680 {3}
		$\hat{\lambda}$	1.84553 {7}	1.70978 {5}	1.41704 {4}	1.74993 {6}	0.23887 {1}	1.85901 {8}	1.31641 {2}	1.40341 {3}
	MRE	$\hat{\alpha}$	0.37979 {2}	0.44982 {7}	0.40421 {3}	0.43716 {5}	0.35373 {1}	0.67482 {8}	0.41286 {4}	0.44355 {6}
		$\hat{\theta}$	0.66432 {8}	0.64619 {6}	0.58267 {4}	0.65510 {7}	0.22672 {1}	0.63441 {5}	0.55029 {2}	0.55450 {3}
		$\hat{\lambda}$	0.90567 {7}	0.87172 {5}	0.79360 {4}	0.88190 {6}	0.32583 {1}	0.90897 {8}	0.76490 {2}	0.78977 {3}
		$\sum$ *Ranks*	51 {5}	54 {6.5}	33 {3}	54 {6.5}	9 {1}	63 {8}	24 {2}	36 {4}
100	AB	$\hat{\alpha}$	0.99946 {2}	1.11572 {5}	1.02873 {3}	1.16026 {7}	0.86345 {1}	1.89009 {8}	1.03571 {4}	1.13219 {6}
		$\hat{\theta}$	2.30111 {4}	2.78317 {8}	2.35566 {5}	2.77287 {7}	0.48075 {1}	2.67401 {6}	2.28338 {3}	2.24111 {2}
		$\hat{\lambda}$	0.96778 {4}	1.12732 {6}	0.97167 {5}	1.13080 {7}	0.21276 {1}	1.16832 {8}	0.95079 {2}	0.95307 {3}
	MSE	$\hat{\alpha}$	0.99892 {2}	1.24484 {5}	1.05828 {3}	1.34621 {7}	0.74555 {1}	3.57246 {8}	1.07270 {4}	1.28185 {6}
		$\hat{\theta}$	5.29513 {4}	7.74604 {8}	5.54915 {5}	7.68880 {7}	0.23112 {1}	7.15032 {6}	5.21381 {3}	5.02257 {2}
		$\hat{\lambda}$	0.93659 {4}	1.27085 {6}	0.94414 {5}	1.27871 {7}	0.04527 {1}	1.36497 {8}	0.90400 {2}	0.90835 {3}
	MRE	$\hat{\alpha}$	0.28556 {2}	0.31878 {5}	0.29392 {3}	0.33150 {7}	0.24670 {1}	0.54003 {8}	0.29592 {4}	0.32348 {6}
		$\hat{\theta}$	0.46022 {4}	0.55663 {8}	0.47113 {5}	0.55457 {7}	0.09615 {1}	0.53480 {6}	0.45668 {3}	0.44822 {2}
		$\hat{\lambda}$	0.64518 {4}	0.75155 {6}	0.64778 {5}	0.75387 {7}	0.14184 {1}	0.77888 {8}	0.63386 {2}	0.63538 {3}
		$\sum$ *Ranks*	30 {3}	57 {6}	39 {5}	63 {7}	9 {1}	66 {8}	27 {2}	33 {4}
300	AB	$\hat{\alpha}$	0.59122 {2}	0.71466 {7}	0.60821 {3}	0.70206 {6}	0.45283 {1}	1.46312 {8}	0.61559 {4}	0.68110 {5}
		$\hat{\theta}$	1.34262 {2}	1.89537 {7}	1.45326 {4}	1.87397 {6}	0.16060 {1}	1.93171 {8}	1.48693 {5}	1.43967 {3}
		$\hat{\lambda}$	0.57240 {2}	0.78133 {7}	0.61281 {3}	0.77909 {6}	0.07044 {1}	0.91042 {8}	0.63201 {5}	0.61622 {4}
	MSE	$\hat{\alpha}$	0.34954 {2}	0.51073 {7}	0.36992 {3}	0.49289 {6}	0.20506 {1}	2.14072 {8}	0.37895 {4}	0.46389 {5}
		$\hat{\theta}$	1.80262 {2}	3.59245 {7}	2.11196 {4}	3.51176 {6}	0.02579 {1}	3.73151 {8}	2.21097 {5}	2.07265 {3}
		$\hat{\lambda}$	0.32764 {2}	0.61048 {7}	0.37554 {3}	0.60697 {6}	0.00496 {1}	0.82886 {8}	0.39944 {5}	0.37973 {4}
	MRE	$\hat{\alpha}$	0.16892 {2}	0.20419 {7}	0.17378 {3}	0.20059 {6}	0.12938 {1}	0.41803 {8}	0.17588 {4}	0.19460 {5}
		$\hat{\theta}$	0.26852 {2}	0.37907 {7}	0.29065 {4}	0.37479 {6}	0.03212 {1}	0.38634 {8}	0.29739 {5}	0.28793 {3}
		$\hat{\lambda}$	0.38160 {2}	0.52089 {7}	0.40854 {3}	0.51939 {6}	0.04696 {1}	0.60694 {8}	0.42134 {5}	0.41081 {4}
		$\sum$ *Ranks*	18 {2}	63 {7}	30 {3}	54 {6}	9 {1}	72 {8}	42 {5}	36 {4}
500	AB	$\hat{\alpha}$	0.45612 {3}	0.53315 {6}	0.45547 {2}	0.56337 {7}	0.31456 {1}	1.28576 {8}	0.48587 {4}	0.53157 {5}
		$\hat{\theta}$	1.00960 {2}	1.47401 {6}	1.14613 {3}	1.57444 {7}	0.05107 {1}	1.67874 {8}	1.19015 {5}	1.15026 {4}
		$\hat{\lambda}$	0.44225 {2}	0.61236 {6}	0.49298 {3}	0.66172 {7}	0.05015 {1}	0.81159 {8}	0.51309 {5}	0.50023 {4}
	MSE	$\hat{\alpha}$	0.20805 {3}	0.28424 {6}	0.20745 {2}	0.31739 {7}	0.09895 {1}	1.65317 {8}	0.23607 {4}	0.28257 {5}
		$\hat{\theta}$	1.01929 {2}	2.17270 {6}	1.31362 {3}	2.47887 {7}	0.00261 {1}	2.81817 {8}	1.41646 {5}	1.32311 {4}
		$\hat{\lambda}$	0.19558 {2}	0.37498 {6}	0.24303 {3}	0.43787 {7}	0.00251 {1}	0.65867 {8}	0.26326 {5}	0.25023 {4}
	MRE	$\hat{\alpha}$	0.13032 {3}	0.15233 {6}	0.13013 {2}	0.16096 {7}	0.08987 {1}	0.36736 {8}	0.13882 {4}	0.15188 {5}
		$\hat{\theta}$	0.20192 {2}	0.29480 {6}	0.22923 {3}	0.31489 {7}	0.01021 {1}	0.33575 {8}	0.23803 {5}	0.23005 {4}
		$\hat{\lambda}$	0.29483 {2}	0.40824 {6}	0.32865 {3}	0.44115 {7}	0.03343 {1}	0.54106 {8}	0.34206 {5}	0.33349 {4}
		$\sum$ *Ranks*	21 {2}	54 {6}	24 {3}	63 {7}	9 {1}	72 {8}	42 {5}	39 {4}

Table 5. Simulation results for $\boldsymbol{\psi} = (\alpha = 3.5, \theta = 2, \lambda = 0.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.73373 $^{\{4\}}$	2.97250 $^{\{5\}}$	2.52905 $^{\{3\}}$	3.06247 $^{\{6\}}$	2.32264 $^{\{1\}}$	3.71695 $^{\{8\}}$	2.43793 $^{\{2\}}$	3.21448 $^{\{7\}}$
		$\hat{\theta}$	2.59089 $^{\{6\}}$	2.56781 $^{\{5\}}$	2.39035 $^{\{4\}}$	2.67259 $^{\{7\}}$	2.29311 $^{\{1\}}$	2.75649 $^{\{8\}}$	2.35143 $^{\{2\}}$	2.36168 $^{\{3\}}$
		$\hat{\lambda}$	0.49403 $^{\{7\}}$	0.48526 $^{\{5\}}$	0.45671 $^{\{3\}}$	0.49359 $^{\{6\}}$	0.40934 $^{\{1\}}$	0.49770 $^{\{8\}}$	0.44486 $^{\{2\}}$	0.48369 $^{\{4\}}$
	MSE	$\hat{\alpha}$	7.47325 $^{\{4\}}$	8.83573 $^{\{5\}}$	6.39608 $^{\{3\}}$	9.37872 $^{\{6\}}$	5.39467 $^{\{1\}}$	13.81570 $^{\{8\}}$	5.94351 $^{\{2\}}$	10.33288 $^{\{7\}}$
		$\hat{\theta}$	6.71269 $^{\{6\}}$	6.59366 $^{\{5\}}$	5.71378 $^{\{4\}}$	7.14274 $^{\{7\}}$	5.25834 $^{\{1\}}$	7.59824 $^{\{8\}}$	5.52923 $^{\{2\}}$	5.57753 $^{\{3\}}$
		$\hat{\lambda}$	0.24407 $^{\{7\}}$	0.23547 $^{\{5\}}$	0.20858 $^{\{3\}}$	0.24363 $^{\{6\}}$	0.16756 $^{\{1\}}$	0.24770 $^{\{8\}}$	0.19790 $^{\{2\}}$	0.23396 $^{\{4\}}$
	MRE	$\hat{\alpha}$	0.45562 $^{\{4\}}$	0.49542 $^{\{5\}}$	0.42151 $^{\{3\}}$	0.51041 $^{\{6\}}$	0.38711 $^{\{1\}}$	0.61949 $^{\{8\}}$	0.40632 $^{\{2\}}$	0.53575 $^{\{7\}}$
		$\hat{\theta}$	0.64772 $^{\{6\}}$	0.64195 $^{\{5\}}$	0.59759 $^{\{4\}}$	0.66815 $^{\{7\}}$	0.57328 $^{\{1\}}$	0.68912 $^{\{8\}}$	0.58786 $^{\{2\}}$	0.59042 $^{\{3\}}$
		$\hat{\lambda}$	0.98807 $^{\{7\}}$	0.97051 $^{\{5\}}$	0.91341 $^{\{3\}}$	0.98719 $^{\{6\}}$	0.81868 $^{\{1\}}$	0.99540 $^{\{8\}}$	0.88972 $^{\{2\}}$	0.96739 $^{\{4\}}$
		$\sum$ *Ranks*	51 $^{\{6\}}$	45 $^{\{5\}}$	30 $^{\{3\}}$	57 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	18 $^{\{2\}}$	42 $^{\{4\}}$
50	AB	$\hat{\alpha}$	1.64995 $^{\{3\}}$	1.93631 $^{\{5\}}$	1.67330 $^{\{4\}}$	1.93746 $^{\{6\}}$	1.48535 $^{\{1\}}$	3.10352 $^{\{8\}}$	1.62976 $^{\{2\}}$	2.07879 $^{\{7\}}$
		$\hat{\theta}$	1.71277 $^{\{2\}}$	2.11620 $^{\{7\}}$	1.80437 $^{\{5\}}$	2.07737 $^{\{6\}}$	1.55230 $^{\{1\}}$	2.27305 $^{\{8\}}$	1.74379 $^{\{4\}}$	1.73347 $^{\{3\}}$
		$\hat{\lambda}$	0.32995 $^{\{2\}}$	0.39914 $^{\{7\}}$	0.34508 $^{\{4\}}$	0.39901 $^{\{6\}}$	0.27729 $^{\{1\}}$	0.47827 $^{\{8\}}$	0.33081 $^{\{3\}}$	0.35006 $^{\{5\}}$
	MSE	$\hat{\alpha}$	2.72233 $^{\{3\}}$	3.74931 $^{\{5\}}$	2.79992 $^{\{4\}}$	3.75377 $^{\{6\}}$	2.20626 $^{\{1\}}$	9.63181 $^{\{8\}}$	2.65612 $^{\{2\}}$	4.32135 $^{\{7\}}$
		$\hat{\theta}$	2.93358 $^{\{2\}}$	4.47829 $^{\{7\}}$	3.25575 $^{\{5\}}$	4.31547 $^{\{6\}}$	2.40964 $^{\{1\}}$	5.16676 $^{\{8\}}$	3.04080 $^{\{4\}}$	3.00491 $^{\{3\}}$
		$\hat{\lambda}$	0.10886 $^{\{2\}}$	0.15932 $^{\{7\}}$	0.11908 $^{\{4\}}$	0.15921 $^{\{6\}}$	0.07689 $^{\{1\}}$	0.22874 $^{\{8\}}$	0.10943 $^{\{3\}}$	0.12254 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.27499 $^{\{3\}}$	0.32272 $^{\{5\}}$	0.27888 $^{\{4\}}$	0.32291 $^{\{6\}}$	0.24756 $^{\{1\}}$	0.51725 $^{\{8\}}$	0.27163 $^{\{2\}}$	0.34646 $^{\{7\}}$
		$\hat{\theta}$	0.42819 $^{\{2\}}$	0.52905 $^{\{7\}}$	0.45109 $^{\{5\}}$	0.51934 $^{\{6\}}$	0.38808 $^{\{1\}}$	0.56826 $^{\{8\}}$	0.43595 $^{\{4\}}$	0.43337 $^{\{3\}}$
		$\hat{\lambda}$	0.65989 $^{\{2\}}$	0.79829 $^{\{7\}}$	0.69016 $^{\{4\}}$	0.79801 $^{\{6\}}$	0.55457 $^{\{1\}}$	0.95653 $^{\{8\}}$	0.66162 $^{\{3\}}$	0.70012 $^{\{5\}}$
		$\sum$ *Ranks*	21 $^{\{2\}}$	57 $^{\{7\}}$	39 $^{\{4\}}$	54 $^{\{6\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	27 $^{\{3\}}$	45 $^{\{5\}}$
100	AB	$\hat{\alpha}$	1.17521 $^{\{2\}}$	1.41304 $^{\{5\}}$	1.23683 $^{\{4\}}$	1.45017 $^{\{6\}}$	0.97728 $^{\{1\}}$	2.77817 $^{\{8\}}$	1.19773 $^{\{3\}}$	1.48414 $^{\{7\}}$
		$\hat{\theta}$	1.20044 $^{\{2\}}$	1.56249 $^{\{6\}}$	1.33156 $^{\{5\}}$	1.59012 $^{\{7\}}$	0.99266 $^{\{1\}}$	1.79033 $^{\{8\}}$	1.26397 $^{\{3\}}$	1.31099 $^{\{4\}}$
		$\hat{\lambda}$	0.23517 $^{\{2\}}$	0.29908 $^{\{6\}}$	0.25487 $^{\{4\}}$	0.30687 $^{\{7\}}$	0.16178 $^{\{1\}}$	0.38849 $^{\{8\}}$	0.24700 $^{\{3\}}$	0.26245 $^{\{5\}}$
	MSE	$\hat{\alpha}$	1.38113 $^{\{2\}}$	1.99668 $^{\{5\}}$	1.52975 $^{\{4\}}$	2.10299 $^{\{6\}}$	0.95508 $^{\{1\}}$	7.71821 $^{\{8\}}$	1.43456 $^{\{3\}}$	2.20266 $^{\{7\}}$
		$\hat{\theta}$	1.44106 $^{\{2\}}$	2.44136 $^{\{6\}}$	1.77306 $^{\{5\}}$	2.52847 $^{\{7\}}$	0.98538 $^{\{1\}}$	3.20528 $^{\{8\}}$	1.59762 $^{\{3\}}$	1.71870 $^{\{4\}}$
		$\hat{\lambda}$	0.05530 $^{\{2\}}$	0.08945 $^{\{6\}}$	0.06496 $^{\{4\}}$	0.09417 $^{\{7\}}$	0.02617 $^{\{1\}}$	0.15092 $^{\{8\}}$	0.06101 $^{\{3\}}$	0.06888 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.19587 $^{\{2\}}$	0.23551 $^{\{5\}}$	0.20614 $^{\{4\}}$	0.24169 $^{\{6\}}$	0.16288 $^{\{1\}}$	0.46303 $^{\{8\}}$	0.19962 $^{\{3\}}$	0.24736 $^{\{7\}}$
		$\hat{\theta}$	0.30011 $^{\{2\}}$	0.39062 $^{\{6\}}$	0.33289 $^{\{5\}}$	0.39753 $^{\{7\}}$	0.24817 $^{\{1\}}$	0.44758 $^{\{8\}}$	0.31599 $^{\{3\}}$	0.32775 $^{\{4\}}$
		$\hat{\lambda}$	0.47034 $^{\{2\}}$	0.59816 $^{\{6\}}$	0.50974 $^{\{4\}}$	0.61375 $^{\{7\}}$	0.32356 $^{\{1\}}$	0.77697 $^{\{8\}}$	0.49400 $^{\{3\}}$	0.52491 $^{\{5\}}$
		$\sum$ *Ranks*	18 $^{\{2\}}$	51 $^{\{6\}}$	39 $^{\{4\}}$	60 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	27 $^{\{3\}}$	48 $^{\{5\}}$
300	AB	$\hat{\alpha}$	0.64187 $^{\{2\}}$	0.83740 $^{\{5\}}$	0.67288 $^{\{4\}}$	0.85844 $^{\{6\}}$	0.47483 $^{\{1\}}$	2.06664 $^{\{8\}}$	0.66357 $^{\{3\}}$	0.88189 $^{\{7\}}$
		$\hat{\theta}$	0.68609 $^{\{2\}}$	0.98530 $^{\{6\}}$	0.77252 $^{\{4\}}$	0.99252 $^{\{7\}}$	0.42898 $^{\{1\}}$	1.25651 $^{\{8\}}$	0.76424 $^{\{3\}}$	0.81159 $^{\{5\}}$
		$\hat{\lambda}$	0.13299 $^{\{2\}}$	0.18795 $^{\{6\}}$	0.14605 $^{\{4\}}$	0.19291 $^{\{7\}}$	0.05135 $^{\{1\}}$	0.29052 $^{\{8\}}$	0.14468 $^{\{3\}}$	0.16593 $^{\{5\}}$
	MSE	$\hat{\alpha}$	0.41199 $^{\{2\}}$	0.70124 $^{\{5\}}$	0.45276 $^{\{4\}}$	0.73691 $^{\{6\}}$	0.22546 $^{\{1\}}$	4.27100 $^{\{8\}}$	0.44032 $^{\{3\}}$	0.77774 $^{\{7\}}$
		$\hat{\theta}$	0.47072 $^{\{2\}}$	0.97081 $^{\{6\}}$	0.59678 $^{\{4\}}$	0.98509 $^{\{7\}}$	0.18403 $^{\{1\}}$	1.57882 $^{\{8\}}$	0.58406 $^{\{3\}}$	0.65868 $^{\{5\}}$
		$\hat{\lambda}$	0.01769 $^{\{2\}}$	0.03533 $^{\{6\}}$	0.02133 $^{\{4\}}$	0.03721 $^{\{7\}}$	0.00264 $^{\{1\}}$	0.08440 $^{\{8\}}$	0.02093 $^{\{3\}}$	0.02753 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.10698 $^{\{2\}}$	0.13957 $^{\{5\}}$	0.11215 $^{\{4\}}$	0.14307 $^{\{6\}}$	0.07914 $^{\{1\}}$	0.34444 $^{\{8\}}$	0.11059 $^{\{3\}}$	0.14698 $^{\{7\}}$
		$\hat{\theta}$	0.17152 $^{\{2\}}$	0.24632 $^{\{6\}}$	0.19313 $^{\{4\}}$	0.24813 $^{\{7\}}$	0.10725 $^{\{1\}}$	0.31413 $^{\{8\}}$	0.19106 $^{\{3\}}$	0.20290 $^{\{5\}}$
		$\hat{\lambda}$	0.26599 $^{\{2\}}$	0.37591 $^{\{6\}}$	0.29210 $^{\{4\}}$	0.38582 $^{\{7\}}$	0.10271 $^{\{1\}}$	0.58105 $^{\{8\}}$	0.28935 $^{\{3\}}$	0.33187 $^{\{5\}}$
		$\sum$ *Ranks*	18 $^{\{2\}}$	51 $^{\{5.5\}}$	36 $^{\{4\}}$	60 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	27 $^{\{3\}}$	51 $^{\{5.5\}}$
500	AB	$\hat{\alpha}$	0.48932 $^{\{2\}}$	0.64703 $^{\{5\}}$	0.50205 $^{\{3\}}$	0.65596 $^{\{7\}}$	0.27101 $^{\{1\}}$	1.83415 $^{\{8\}}$	0.53351 $^{\{4\}}$	0.65374 $^{\{6\}}$
		$\hat{\theta}$	0.52712 $^{\{2\}}$	0.76656 $^{\{6\}}$	0.57398 $^{\{3\}}$	0.79998 $^{\{7\}}$	0.22775 $^{\{1\}}$	1.07699 $^{\{8\}}$	0.60167 $^{\{5\}}$	0.60071 $^{\{4\}}$
		$\hat{\lambda}$	0.10310 $^{\{2\}}$	0.14834 $^{\{6\}}$	0.11109 $^{\{3\}}$	0.15170 $^{\{7\}}$	0.02799 $^{\{1\}}$	0.25302 $^{\{8\}}$	0.11732 $^{\{4\}}$	0.12213 $^{\{5\}}$
	MSE	$\hat{\alpha}$	0.23943 $^{\{2\}}$	0.41865 $^{\{5\}}$	0.25206 $^{\{3\}}$	0.43028 $^{\{7\}}$	0.07345 $^{\{1\}}$	3.36412 $^{\{8\}}$	0.28464 $^{\{4\}}$	0.42737 $^{\{6\}}$
		$\hat{\theta}$	0.27785 $^{\{2\}}$	0.58762 $^{\{6\}}$	0.32945 $^{\{3\}}$	0.63997 $^{\{7\}}$	0.05187 $^{\{1\}}$	1.15991 $^{\{8\}}$	0.36201 $^{\{5\}}$	0.36086 $^{\{4\}}$
		$\hat{\lambda}$	0.01063 $^{\{2\}}$	0.02201 $^{\{6\}}$	0.01234 $^{\{3\}}$	0.02301 $^{\{7\}}$	0.00078 $^{\{1\}}$	0.06402 $^{\{8\}}$	0.01376 $^{\{4\}}$	0.01491 $^{\{5\}}$
	MRE	$\hat{\alpha}$	0.08155 $^{\{2\}}$	0.10784 $^{\{5\}}$	0.08368 $^{\{3\}}$	0.10933 $^{\{7\}}$	0.04517 $^{\{1\}}$	0.30569 $^{\{8\}}$	0.08892 $^{\{4\}}$	0.10896 $^{\{6\}}$
		$\hat{\theta}$	0.13178 $^{\{2\}}$	0.19164 $^{\{6\}}$	0.14350 $^{\{3\}}$	0.20000 $^{\{7\}}$	0.05694 $^{\{1\}}$	0.26925 $^{\{8\}}$	0.15042 $^{\{5\}}$	0.15018 $^{\{4\}}$
		$\hat{\lambda}$	0.20620 $^{\{2\}}$	0.29669 $^{\{6\}}$	0.22219 $^{\{3\}}$	0.30339 $^{\{7\}}$	0.05598 $^{\{1\}}$	0.50603 $^{\{8\}}$	0.23464 $^{\{4\}}$	0.24425 $^{\{5\}}$
		$\sum$ *Ranks*	18 $^{\{2\}}$	51 $^{\{6\}}$	27 $^{\{3\}}$	63 $^{\{7\}}$	9 $^{\{1\}}$	72 $^{\{8\}}$	39 $^{\{4\}}$	45 $^{\{5\}}$

Table 6. Simulation results for $\boldsymbol{\psi} = (\alpha = 6, \theta = 4, \lambda = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.74615 {4}	2.96936 {5}	2.58459 {3}	2.98358 {6}	2.48222 {1}	3.85976 {8}	2.51910 {2}	3.30802 {7}
		$\hat{\theta}$	2.85644 {8}	2.60829 {5}	2.44779 {4}	2.68493 {6}	2.28081 {1}	2.73959 {7}	2.33577 {2}	2.36111 {3}
		$\hat{\lambda}$	1.49708 {8}	1.46921 {5}	1.40456 {3}	1.47879 {6}	1.25421 {1}	1.49221 {7}	1.32707 {2}	1.45614 {4}
	MSE	$\hat{\alpha}$	7.54132 {4}	8.81710 {5}	6.68011 {3}	8.90174 {6}	6.16143 {1}	14.89774 {8}	6.34587 {2}	10.94299 {7}
		$\hat{\theta}$	8.15928 {8}	6.80318 {5}	5.99167 {4}	7.20886 {6}	5.20212 {1}	7.50535 {7}	5.45583 {2}	5.57486 {3}
		$\hat{\lambda}$	2.24126 {8}	2.15859 {5}	1.97279 {3}	2.18683 {6}	1.57305 {1}	2.22670 {7}	1.76111 {2}	2.12034 {4}
	MRE	$\hat{\alpha}$	0.45769 {4}	0.49489 {5}	0.43077 {3}	0.49726 {6}	0.41370 {1}	0.64329 {8}	0.41985 {2}	0.55134 {7}
		$\hat{\theta}$	0.71411 {8}	0.65207 {5}	0.61195 {4}	0.67123 {6}	0.57020 {1}	0.68490 {7}	0.58394 {2}	0.59028 {3}
		$\hat{\lambda}$	0.99806 {8}	0.97948 {5}	0.93637 {3}	0.98586 {6}	0.83614 {1}	0.99481 {7}	0.88471 {2}	0.97076 {4}
		$\sum$ *Ranks*	60 {7}	45 {5}	30 {3}	54 {6}	9 {1}	66 {8}	18 {2}	42 {4}
50	AB	$\hat{\alpha}$	1.68879 {4}	2.00495 {5}	1.68135 {3}	2.01936 {6}	1.45271 {1}	3.28401 {8}	1.66905 {2}	2.06402 {7}
		$\hat{\theta}$	1.84691 {5}	2.09557 {6}	1.81404 {4}	2.11220 {7}	1.45835 {1}	2.26237 {8}	1.71671 {2}	1.78435 {3}
		$\hat{\lambda}$	1.05468 {4}	1.16864 {6}	1.02935 {3}	1.20178 {7}	0.78125 {1}	1.42890 {8}	0.98958 {2}	1.07271 {5}
	MSE	$\hat{\alpha}$	2.85200 {4}	4.01981 {5}	2.82693 {3}	4.07783 {6}	2.11038 {1}	10.78473 {8}	2.78573 {2}	4.26018 {7}
		$\hat{\theta}$	3.41109 {5}	4.39142 {6}	3.29073 {4}	4.46139 {7}	2.12678 {1}	5.11833 {8}	2.94710 {2}	3.18391 {3}
		$\hat{\lambda}$	1.11235 {4}	1.36572 {6}	1.05956 {3}	1.44427 {7}	0.61036 {1}	2.04176 {8}	0.97926 {2}	1.15071 {5}
	MRE	$\hat{\alpha}$	0.28146 {4}	0.33416 {5}	0.28022 {3}	0.33656 {6}	0.24212 {1}	0.54733 {8}	0.27818 {2}	0.34400 {7}
		$\hat{\theta}$	0.46173 {5}	0.52389 {6}	0.45351 {4}	0.52805 {7}	0.36459 {1}	0.56559 {8}	0.42918 {2}	0.44609 {3}
		$\hat{\lambda}$	0.70312 {4}	0.77909 {6}	0.68623 {3}	0.80119 {7}	0.52083 {1}	0.95260 {8}	0.65972 {2}	0.71514 {5}
		$\sum$ *Ranks*	39 {4}	51 {6}	30 {3}	60 {7}	9 {1}	72 {8}	18 {2}	45 {5}
100	AB	$\hat{\alpha}$	1.17935 {2}	1.41101 {5}	1.21039 {3}	1.44516 {6}	0.99292 {1}	2.77076 {8}	1.22533 {4}	1.52947 {7}
		$\hat{\theta}$	1.23834 {2}	1.61168 {6}	1.33736 {4}	1.62872 {7}	0.95513 {1}	1.85055 {8}	1.29697 {3}	1.34587 {5}
		$\hat{\lambda}$	0.72872 {2}	0.92627 {6}	0.75117 {4}	0.92647 {7}	0.42159 {1}	1.17783 {8}	0.74830 {3}	0.81927 {5}
	MSE	$\hat{\alpha}$	1.39086 {2}	1.99095 {5}	1.46505 {3}	2.08850 {6}	0.98588 {1}	7.67713 {8}	1.50143 {4}	2.33928 {7}
		$\hat{\theta}$	1.53349 {2}	2.59750 {6}	1.78853 {4}	2.65274 {7}	0.91227 {1}	3.42452 {8}	1.68213 {3}	1.81137 {5}
		$\hat{\lambda}$	0.53104 {2}	0.85797 {6}	0.56426 {4}	0.85835 {7}	0.17774 {1}	1.38728 {8}	0.55995 {3}	0.67121 {5}
	MRE	$\hat{\alpha}$	0.19656 {2}	0.23517 {5}	0.20173 {3}	0.24086 {6}	0.16549 {1}	0.46179 {8}	0.20422 {4}	0.25491 {7}
		$\hat{\theta}$	0.30959 {2}	0.40292 {6}	0.33434 {4}	0.40718 {7}	0.23878 {1}	0.46264 {8}	0.32424 {3}	0.33647 {5}
		$\hat{\lambda}$	0.48582 {2}	0.61751 {6}	0.50078 {4}	0.61765 {7}	0.28106 {1}	0.78522 {8}	0.49887 {3}	0.54618 {5}
		$\sum$ *Ranks*	18 {2}	51 {5.5}	33 {4}	60 {7}	9 {1}	72 {8}	30 {3}	51 {5.5}
300	AB	$\hat{\alpha}$	0.66286 {2}	0.83484 {5}	0.68428 {3}	0.84246 {6}	0.48298 {1}	2.25030 {8}	0.68955 {4}	0.88489 {7}
		$\hat{\theta}$	0.72378 {2}	0.99977 {6}	0.75743 {3}	1.00797 {7}	0.38779 {1}	1.28578 {8}	0.77196 {4}	0.84526 {5}
		$\hat{\lambda}$	0.42703 {2}	0.57866 {6}	0.43847 {3}	0.58348 {7}	0.12231 {1}	0.88772 {8}	0.44984 {4}	0.50591 {5}
	MSE	$\hat{\alpha}$	0.43939 {2}	0.69696 {5}	0.46824 {3}	0.70974 {6}	0.23327 {1}	5.06387 {8}	0.47548 {4}	0.78303 {7}
		$\hat{\theta}$	0.52386 {2}	0.99955 {6}	0.57370 {3}	1.01601 {7}	0.15038 {1}	1.65322 {8}	0.59592 {4}	0.71447 {5}
		$\hat{\lambda}$	0.18235 {2}	0.33484 {6}	0.19226 {3}	0.34045 {7}	0.01496 {1}	0.78804 {8}	0.20236 {4}	0.25595 {5}
	MRE	$\hat{\alpha}$	0.11048 {2}	0.13914 {5}	0.11405 {3}	0.14041 {6}	0.08050 {1}	0.37505 {8}	0.11493 {4}	0.14748 {7}
		$\hat{\theta}$	0.18095 {2}	0.24994 {6}	0.18936 {3}	0.25199 {7}	0.09695 {1}	0.32144 {8}	0.19299 {4}	0.21132 {5}
		$\hat{\lambda}$	0.28469 {2}	0.38577 {6}	0.29232 {3}	0.38899 {7}	0.08154 {1}	0.59181 {8}	0.29989 {4}	0.33727 {5}
		$\sum$ *Ranks*	18 {2}	51 {5.5}	27 {3}	60 {7}	9 {1}	72 {8}	36 {4}	51 {5.5}
500	AB	$\hat{\alpha}$	0.51451 {3}	0.64765 {6}	0.50442 {2}	0.64692 {5}	0.29565 {1}	1.88823 {8}	0.54886 {4}	0.66467 {7}
		$\hat{\theta}$	0.55179 {2}	0.76539 {6}	0.57252 {3}	0.78934 {7}	0.21867 {1}	1.07146 {8}	0.60323 {4}	0.63483 {5}
		$\hat{\lambda}$	0.32489 {2}	0.44224 {6}	0.32829 {3}	0.45341 {7}	0.07453 {1}	0.75576 {8}	0.34787 {4}	0.38679 {5}
	MSE	$\hat{\alpha}$	0.26472 {3}	0.41945 {6}	0.25444 {2}	0.41850 {5}	0.08741 {1}	3.56540 {8}	0.30125 {4}	0.44178 {7}
		$\hat{\theta}$	0.30447 {2}	0.58582 {6}	0.32778 {3}	0.62306 {7}	0.04782 {1}	1.14802 {8}	0.36388 {4}	0.40301 {5}
		$\hat{\lambda}$	0.10555 {2}	0.19557 {6}	0.10777 {3}	0.20558 {7}	0.00556 {1}	0.57117 {8}	0.12101 {4}	0.14961 {5}
	MRE	$\hat{\alpha}$	0.08575 {3}	0.10794 {6}	0.08407 {2}	0.10782 {5}	0.04928 {1}	0.31470 {8}	0.09148 {4}	0.11078 {7}
		$\hat{\theta}$	0.13795 {2}	0.19135 {6}	0.14313 {3}	0.19734 {7}	0.05467 {1}	0.26786 {8}	0.15081 {4}	0.15871 {5}
		$\hat{\lambda}$	0.21659 {2}	0.29483 {6}	0.21886 {3}	0.30227 {7}	0.04969 {1}	0.50384 {8}	0.23191 {4}	0.25786 {5}
		$\sum$ *Ranks*	21 {2}	54 {6}	24 {3}	57 {7}	9 {1}	72 {8}	36 {4}	51 {5}

Table 7. Simulation results for $\boldsymbol{\psi} = (\alpha = 6, \theta = 5, \lambda = 0.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.49813 {4}	2.86461 {5}	2.41795 {2}	2.96884 {6}	2.27267 {1}	3.32406 {8}	2.43195 {3}	3.06804 {7}
		$\hat{\theta}$	3.64095 {7}	3.52691 {5}	3.32422 {4}	3.57660 {6}	2.98483 {1}	3.67066 {8}	3.21474 {2}	3.26045 {3}
		$\hat{\lambda}$	0.49815 {8}	0.49033 {5}	0.46787 {3}	0.49250 {6}	0.38822 {1}	0.49783 {7}	0.45222 {2}	0.48331 {4}
	MSE	$\hat{\alpha}$	6.24067 {4}	8.20599 {5}	5.84646 {2}	8.81400 {6}	5.16502 {1}	11.04935 {8}	5.91438 {3}	9.41289 {7}
		$\hat{\theta}$	13.25653 {7}	12.43907 {5}	11.05045 {4}	12.79207 {6}	8.90921 {1}	13.47378 {8}	10.33457 {2}	10.63056 {3}
		$\hat{\lambda}$	0.24816 {8}	0.24042 {5}	0.21890 {3}	0.24256 {6}	0.15072 {1}	0.24784 {7}	0.20450 {2}	0.23359 {4}
	MRE	$\hat{\alpha}$	0.41636 {4}	0.47743 {5}	0.40299 {2}	0.49481 {6}	0.37878 {1}	0.55401 {8}	0.40532 {3}	0.51134 {7}
		$\hat{\theta}$	0.72819 {7}	0.70538 {5}	0.66484 {4}	0.71532 {6}	0.59697 {1}	0.73413 {8}	0.64295 {2}	0.65209 {3}
		$\hat{\lambda}$	0.99631 {8}	0.98065 {5}	0.93574 {3}	0.98500 {6}	0.77644 {1}	0.99566 {7}	0.90444 {2}	0.96663 {4}
		$\sum$ *Ranks*	57 {7}	45 {5}	27 {3}	54 {6}	9 {1}	69 {8}	21 {2}	42 {4}
50	AB	$\hat{\alpha}$	1.52038 {2}	1.80322 {6}	1.58940 {3}	1.77512 {5}	1.41776 {1}	2.70950 {8}	1.58941 {4}	1.94883 {7}
		$\hat{\theta}$	2.47436 {4}	2.90111 {6}	2.51109 {5}	2.97022 {7}	1.86741 {1}	3.03600 {8}	2.40775 {2}	2.46824 {3}
		$\hat{\lambda}$	0.35262 {4}	0.40206 {6}	0.35136 {3}	0.41672 {7}	0.24445 {1}	0.45873 {8}	0.33690 {2}	0.36634 {5}
	MSE	$\hat{\alpha}$	2.31155 {2}	3.25159 {6}	2.52618 {3}	3.15106 {5}	2.01003 {1}	7.34138 {8}	2.52624 {4}	3.79795 {7}
		$\hat{\theta}$	6.12246 {4}	8.41644 {6}	6.30556 {5}	8.82219 {7}	3.48722 {1}	9.21732 {8}	5.79728 {2}	6.09219 {3}
		$\hat{\lambda}$	0.12434 {4}	0.16165 {6}	0.12346 {3}	0.17366 {7}	0.05976 {1}	0.21044 {8}	0.11350 {2}	0.13421 {5}
	MRE	$\hat{\alpha}$	0.25340 {2}	0.30054 {6}	0.26490 {3.5}	0.29585 {5}	0.23629 {1}	0.45158 {8}	0.26490 {3.5}	0.32481 {7}
		$\hat{\theta}$	0.49487 {4}	0.58022 {6}	0.50222 {5}	0.59404 {7}	0.37348 {1}	0.60720 {8}	0.48155 {2}	0.49365 {3}
		$\hat{\lambda}$	0.70524 {4}	0.80411 {6}	0.70273 {3}	0.83344 {7}	0.48891 {1}	0.91747 {8}	0.67379 {2}	0.73268 {5}
		$\sum$ *Ranks*	30 {3}	54 {6}	33.5 {4}	57 {7}	9 {1}	72 {8}	23.5 {2}	45 {5}
100	AB	$\hat{\alpha}$	1.09367 {2}	1.31871 {5}	1.13554 {3}	1.35709 {6}	0.99003 {1}	2.33480 {8}	1.14178 {4}	1.38330 {7}
		$\hat{\theta}$	1.76970 {2}	2.28643 {6}	1.92307 {5}	2.31373 {7}	0.79718 {1}	2.47683 {8}	1.85688 {3}	1.90790 {4}
		$\hat{\lambda}$	0.25600 {2}	0.32069 {6}	0.27259 {4}	0.32852 {7}	0.07535 {1}	0.38041 {8}	0.26584 {3}	0.28136 {5}
	MSE	$\hat{\alpha}$	1.19612 {2}	1.73900 {5}	1.28944 {3}	1.84169 {6}	0.98016 {1}	5.45130 {8}	1.30365 {4}	1.91352 {7}
		$\hat{\theta}$	3.13184 {2}	5.22777 {6}	3.69821 {5}	5.35334 {7}	0.63549 {1}	6.13469 {8}	3.44799 {3}	3.64009 {4}
		$\hat{\lambda}$	0.06554 {2}	0.10284 {6}	0.07431 {4}	0.10793 {7}	0.00568 {1}	0.14471 {8}	0.07067 {3}	0.07916 {5}
	MRE	$\hat{\alpha}$	0.18228 {2}	0.21979 {5}	0.18926 {3}	0.22618 {6}	0.16500 {1}	0.38913 {8}	0.19030 {4}	0.23055 {7}
		$\hat{\theta}$	0.35394 {2}	0.45729 {6}	0.38461 {5}	0.46275 {7}	0.15944 {1}	0.49537 {8}	0.37138 {3}	0.38158 {4}
		$\hat{\lambda}$	0.51201 {2}	0.64138 {6}	0.54518 {4}	0.65705 {7}	0.15070 {1}	0.76082 {8}	0.53169 {3}	0.56272 {5}
		$\sum$ *Ranks*	18 {2}	51 {6}	36 {4}	60 {7}	9 {1}	72 {8}	30 {3}	48 {5}
300	AB	$\hat{\alpha}$	0.61960 {2}	0.81733 {6}	0.68613 {4}	0.80281 {5}	0.54558 {1}	1.70525 {8}	0.68421 {3}	0.85567 {7}
		$\hat{\theta}$	1.00118 {2}	1.51241 {7}	1.14625 {3}	1.46053 {6}	0.24227 {1}	1.67728 {8}	1.15881 {4}	1.24827 {5}
		$\hat{\lambda}$	0.14668 {2}	0.21366 {7}	0.16247 {3}	0.20614 {6}	0.02674 {1}	0.27092 {8}	0.16701 {4}	0.18404 {5}
	MSE	$\hat{\alpha}$	0.38390 {2}	0.66802 {6}	0.47077 {4}	0.64451 {5}	0.29766 {1}	2.90787 {8}	0.46815 {3}	0.73217 {7}
		$\hat{\theta}$	1.00237 {2}	2.28738 {7}	1.31388 {3}	2.13314 {6}	0.05870 {1}	2.81326 {8}	1.34283 {4}	1.55818 {5}
		$\hat{\lambda}$	0.02152 {2}	0.04565 {7}	0.02640 {3}	0.04249 {6}	0.00072 {1}	0.07340 {8}	0.02789 {4}	0.03387 {5}
	MRE	$\hat{\alpha}$	0.10327 {2}	0.13622 {6}	0.11435 {4}	0.13380 {5}	0.09093 {1}	0.28421 {8}	0.11404 {3}	0.14261 {7}
		$\hat{\theta}$	0.20024 {2}	0.30248 {7}	0.22925 {3}	0.29211 {6}	0.04845 {1}	0.33546 {8}	0.23176 {4}	0.24965 {5}
		$\hat{\lambda}$	0.29337 {2}	0.42732 {7}	0.32494 {3}	0.41228 {6}	0.05348 {1}	0.54184 {8}	0.33402 {4}	0.36807 {5}
		$\sum$ *Ranks*	18 {2}	60 {7}	30 {3}	51 {5.5}	9 {1}	72 {8}	33 {4}	51 {5.5}
500	AB	$\hat{\alpha}$	0.47011 {2}	0.61670 {5}	0.53092 {4}	0.63872 {6}	0.26940 {1}	1.50141 {8}	0.49516 {3}	0.65309 {7}
		$\hat{\theta}$	0.76136 {2}	1.19046 {6}	0.86057 {3}	1.20148 {7}	0.01982 {1}	1.47984 {8}	0.88457 {4}	0.94869 {5}
		$\hat{\lambda}$	0.11055 {2}	0.16862 {6}	0.12368 {3}	0.17286 {7}	0.01777 {1}	0.24321 {8}	0.12758 {4}	0.13868 {5}
	MSE	$\hat{\alpha}$	0.22100 {2}	0.38032 {5}	0.28187 {4}	0.40796 {6}	0.07258 {1}	2.25424 {8}	0.24518 {3}	0.42653 {7}
		$\hat{\theta}$	0.57966 {2}	1.41719 {6}	0.74057 {3}	1.44356 {7}	0.00039 {1}	2.18992 {8}	0.78246 {4}	0.90001 {5}
		$\hat{\lambda}$	0.01222 {2}	0.02843 {6}	0.01530 {3}	0.02988 {7}	0.00032 {1}	0.05915 {8}	0.01628 {4}	0.01923 {5}
	MRE	$\hat{\alpha}$	0.07835 {2}	0.10278 {5}	0.08849 {4}	0.10645 {6}	0.04490 {1}	0.25024 {8}	0.08253 {3}	0.10885 {7}
		$\hat{\theta}$	0.15227 {2}	0.23809 {6}	0.17211 {3}	0.24030 {7}	0.00396 {1}	0.29597 {8}	0.17691 {4}	0.18974 {5}
		$\hat{\lambda}$	0.22111 {2}	0.33724 {6}	0.24737 {3}	0.34571 {7}	0.03554 {1}	0.48642 {8}	0.25517 {4}	0.27736 {5}
		$\sum$ *Ranks*	18 {2}	51 {5.5}	30 {3}	60 {7}	9 {1}	72 {8}	33 {4}	51 {5.5}

Table 8. Simulation results for $\boldsymbol{\psi} = (\alpha = 6, \theta = 5, \lambda = 1.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	AB	$\hat{\alpha}$	2.61999 {4}	2.82251 {5}	2.48410 {3}	2.85330 {6}	2.26688 {1}	3.50327 {8}	2.35654 {2}	2.97282 {7}
		$\hat{\theta}$	3.74173 {8}	3.54374 {5}	3.33897 {4}	3.62347 {6}	2.92277 {1}	3.65115 {7}	3.24955 {3}	3.24467 {2}
		$\hat{\lambda}$	1.49565 {8}	1.47036 {5}	1.41411 {3}	1.48181 {6}	1.13437 {1}	1.49225 {7}	1.35944 {2}	1.44991 {4}
	MSE	$\hat{\alpha}$	6.86433 {4}	7.96658 {5}	6.17074 {3}	8.14135 {6}	5.13872 {1}	12.27293 {8}	5.55328 {2}	8.83769 {7}
		$\hat{\theta}$	14.00052 {8}	12.55811 {5}	11.14872 {4}	13.12953 {6}	8.54256 {1}	13.33091 {7}	10.55960 {3}	10.52789 {2}
		$\hat{\lambda}$	2.23698 {8}	2.16197 {5}	1.99970 {3}	2.19578 {6}	1.28679 {1}	2.22680 {7}	1.84809 {2}	2.10225 {4}
	MRE	$\hat{\alpha}$	0.43666 {4}	0.47042 {5}	0.41402 {3}	0.47555 {6}	0.37781 {1}	0.58388 {8}	0.39276 {2}	0.49547 {7}
		$\hat{\theta}$	0.74835 {8}	0.70875 {5}	0.66779 {4}	0.72469 {6}	0.58455 {1}	0.73023 {7}	0.64991 {3}	0.64893 {2}
		$\hat{\lambda}$	0.99710 {8}	0.98024 {5}	0.94274 {3}	0.98788 {6}	0.75624 {1}	0.99483 {7}	0.90630 {2}	0.96661 {4}
		$\sum$ *Ranks*	60 {7}	45 {5}	30 {3}	54 {6}	9 {1}	66 {8}	21 {2}	39 {4}
50	AB	$\hat{\alpha}$	1.56054 {2}	1.82748 {5}	1.59365 {4}	1.86539 {6}	1.37476 {1}	2.71280 {8}	1.57761 {3}	1.92814 {7}
		$\hat{\theta}$	2.61895 {5}	2.96687 {6}	2.51602 {3}	3.04707 {8}	1.69436 {1}	2.98826 {7}	2.37081 {2}	2.51989 {4}
		$\hat{\lambda}$	1.11818 {5}	1.23609 {6}	1.06580 {3}	1.28117 {7}	0.64847 {1}	1.34690 {8}	1.01861 {2}	1.10998 {4}
	MSE	$\hat{\alpha}$	2.43528 {2}	3.33968 {5}	2.53971 {4}	3.47967 {6}	1.88998 {1}	7.35929 {8}	2.48885 {3}	3.71772 {7}
		$\hat{\theta}$	6.85892 {5}	8.80234 {6}	6.33034 {3}	9.28463 {8}	2.87084 {1}	8.92971 {7}	5.62076 {2}	6.34986 {4}
		$\hat{\lambda}$	1.25033 {5}	1.52793 {6}	1.13594 {3}	1.64140 {7}	0.42051 {1}	1.81413 {8}	1.03756 {2}	1.23206 {4}
	MRE	$\hat{\alpha}$	0.26009 {2}	0.30458 {5}	0.26561 {4}	0.31090 {6}	0.22913 {1}	0.45213 {8}	0.26293 {3}	0.32136 {7}
		$\hat{\theta}$	0.52379 {5}	0.59337 {6}	0.50320 {3}	0.60941 {8}	0.33887 {1}	0.59765 {7}	0.47416 {2}	0.50398 {4}
		$\hat{\lambda}$	0.74545 {5}	0.82406 {6}	0.71054 {3}	0.85411 {7}	0.43231 {1}	0.89793 {8}	0.67907 {2}	0.73999 {4}
		$\sum$ *Ranks*	36 {4}	51 {6}	30 {3}	63 {7}	9 {1}	69 {8}	21 {2}	45 {5}
100	AB	$\hat{\alpha}$	1.11982 {2}	1.34593 {5}	1.17110 {3}	1.35612 {6}	0.98287 {1}	2.26681 {8}	1.17776 {4}	1.40914 {7}
		$\hat{\theta}$	1.81068 {2}	2.33323 {6}	1.98724 {5}	2.36244 {7}	0.74405 {1}	2.48267 {8}	1.85934 {3}	1.93357 {4}
		$\hat{\lambda}$	0.77594 {2}	0.99001 {7}	0.83612 {4}	0.98924 {6}	0.19627 {1}	1.14053 {8}	0.79601 {3}	0.85360 {5}
	MSE	$\hat{\alpha}$	1.25399 {2}	1.81153 {5}	1.37148 {3}	1.83907 {6}	0.96604 {1}	5.13841 {8}	1.38711 {4}	1.98568 {7}
		$\hat{\theta}$	3.27856 {2}	5.44398 {6}	3.94911 {5}	5.58112 {7}	0.55361 {1}	6.16367 {8}	3.45716 {3}	3.73868 {4}
		$\hat{\lambda}$	0.60208 {2}	0.98012 {7}	0.69910 {4}	0.97859 {6}	0.03852 {1}	1.30081 {8}	0.63364 {3}	0.72864 {5}
	MRE	$\hat{\alpha}$	0.18664 {2}	0.22432 {5}	0.19518 {3}	0.22602 {6}	0.16381 {1}	0.37780 {8}	0.19629 {4}	0.23486 {7}
		$\hat{\theta}$	0.36214 {2}	0.46665 {6}	0.39745 {5}	0.47249 {7}	0.14881 {1}	0.49653 {8}	0.37187 {3}	0.38671 {4}
		$\hat{\lambda}$	0.51729 {2}	0.66001 {7}	0.55741 {4}	0.65949 {6}	0.13085 {1}	0.76035 {8}	0.53067 {3}	0.56907 {5}
		$\sum$ *Ranks*	18 {2}	54 {6}	36 {4}	57 {7}	9 {1}	72 {8}	30 {3}	48 {5}
300	AB	$\hat{\alpha}$	0.64405 {2}	0.83062 {5}	0.68265 {4}	0.83758 {7}	0.52213 {1}	1.74536 {8}	0.67819 {3}	0.83690 {6}
		$\hat{\theta}$	1.05719 {2}	1.51245 {7}	1.16071 {3}	1.50241 {6}	0.23129 {1}	1.75185 {8}	1.17074 {4}	1.20606 {5}
		$\hat{\lambda}$	0.46096 {2}	0.64712 {7}	0.50048 {4}	0.64309 {6}	0.07228 {1}	0.84469 {8}	0.49961 {3}	0.54027 {5}
	MSE	$\hat{\alpha}$	0.41480 {2}	0.68994 {5}	0.46601 {4}	0.70154 {7}	0.27262 {1}	3.04630 {8}	0.45994 {3}	0.70040 {6}
		$\hat{\theta}$	1.11766 {2}	2.28749 {7}	1.34726 {3}	2.25725 {6}	0.05350 {1}	3.06896 {8}	1.37064 {4}	1.45458 {5}
		$\hat{\lambda}$	0.21248 {2}	0.41876 {7}	0.25048 {4}	0.41356 {6}	0.00522 {1}	0.71350 {8}	0.24961 {3}	0.29189 {5}
	MRE	$\hat{\alpha}$	0.10734 {2}	0.13844 {5}	0.11377 {4}	0.13960 {7}	0.08702 {1}	0.29089 {8}	0.11303 {3}	0.13948 {6}
		$\hat{\theta}$	0.21144 {2}	0.30249 {7}	0.23214 {3}	0.30048 {6}	0.04626 {1}	0.35037 {8}	0.23415 {4}	0.24121 {5}
		$\hat{\lambda}$	0.30731 {2}	0.43141 {7}	0.33366 {4}	0.42873 {6}	0.04818 {1}	0.56313 {8}	0.33307 {3}	0.36018 {5}
		$\sum$ *Ranks*	18 {2}	57 {6.5}	33 {4}	57 {6.5}	9 {1}	72 {8}	30 {3}	48 {5}
500	AB	$\hat{\alpha}$	0.49225 {2}	0.62698 {5}	0.49622 {3}	0.63979 {7}	0.31655 {1}	1.47611 {8}	0.53751 {4}	0.63948 {6}
		$\hat{\theta}$	0.80039 {2}	1.15190 {6}	0.86891 {3}	1.20127 {7}	0.04710 {1}	1.43473 {8}	0.92650 {4}	0.96313 {5}
		$\hat{\lambda}$	0.34942 {2}	0.49115 {6}	0.36941 {3}	0.51991 {7}	0.04923 {1}	0.70554 {8}	0.39976 {4}	0.42288 {5}
	MSE	$\hat{\alpha}$	0.24231 {2}	0.39310 {5}	0.24624 {3}	0.40933 {7}	0.10021 {1}	2.17890 {8}	0.28892 {4}	0.40894 {6}
		$\hat{\theta}$	0.64062 {2}	1.32688 {6}	0.75501 {3}	1.44305 {7}	0.00222 {1}	2.05846 {8}	0.85841 {4}	0.92763 {5}
		$\hat{\lambda}$	0.12209 {2}	0.24123 {6}	0.13646 {3}	0.27031 {7}	0.00242 {1}	0.49778 {8}	0.15981 {4}	0.17883 {5}
	MRE	$\hat{\alpha}$	0.08204 {2}	0.10450 {5}	0.08270 {3}	0.10663 {7}	0.05276 {1}	0.24602 {8}	0.08959 {4}	0.10658 {6}
		$\hat{\theta}$	0.16008 {2}	0.23038 {6}	0.17378 {3}	0.24025 {7}	0.00942 {1}	0.28695 {8}	0.18530 {4}	0.19263 {5}
		$\hat{\lambda}$	0.23294 {2}	0.32743 {6}	0.24627 {3}	0.34661 {7}	0.03282 {1}	0.47036 {8}	0.26651 {4}	0.28192 {5}
		$\sum$ *Ranks*	18 {2}	51 {6}	27 {3}	63 {7}	9 {1}	72 {8}	36 {4}	48 {5}

Table 9. Partial and overall ranks of all estimation methods for various combinations of $\boldsymbol{\psi}$.

$\boldsymbol{\psi}^{\intercal}$	n	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
($\hat{\alpha}$=3.5, $\hat{\theta}$=4, $\hat{\lambda}$=0.5)	20	6.5	4	3	6.5	1	8	2	5
	50	5	6	3	7	1	8	2	4
	100	2	6	3.5	7	1	8	3.5	5
	300	2	6	3	7	1	8	4	5
	500	2	6	3	7	1	8	4	5
($\hat{\alpha}$=3.5, $\hat{\theta}$=4, $\hat{\lambda}$=1.5)	20	7	5	3.5	6	1	8	2	3.5
	50	4	6	3	7	1	8	2	5
	100	3	6	4	7	1	8	2	5
	300	2	7	3	6	1	8	4	5
	500	2	7	3	6	1	8	4	5
($\hat{\alpha}$=3.5, $\hat{\theta}$=5, $\hat{\lambda}$=0.5)	20	6.5	4	3	6.5	1	8	2	5
	50	5.5	5.5	4	7	1	8	2	3
	100	5	6	4	7.5	1	7.5	3	2
	300	2.5	7	2.5	6	1	8	4.5	4.5
	500	2	7	3	6	1	8	5	4
($\hat{\alpha}$=3.5, $\hat{\theta}$=5, $\hat{\lambda}$=1.5)	20	6	5	3	7	1	8	2	4
	50	5	6.5	3	6.5	1	8	2	4
	100	3	6	5	7	1	8	2	4
	300	2	7	3	6	1	8	5	4
	500	2	6	3	7	1	8	5	4
($\hat{\alpha}$=6, $\hat{\theta}$=4, $\hat{\lambda}$=0.5)	20	6	5	3	7	1	8	2	4
	50	2	7	4	6	1	8	3	5
	100	2	6	4	7	1	8	3	5
	300	2	5.5	4	7	1	8	3	5.5
	500	2	6	3	7	1	8	4	5
($\hat{\alpha}$=6, $\hat{\theta}$=4, $\hat{\lambda}$=1.5)	20	7	5	3	6	1	8	2	4
	50	4	6	3	7	1	8	2	5
	100	2	5.5	4	7	1	8	3	5.5
	300	2	5.5	3	7	1	8	4	5.5
	500	2	6	3	7	1	8	4	5
($\hat{\alpha}$=6, $\hat{\theta}$=5, $\hat{\lambda}$=0.5)	20	7	5	3	6	1	8	2	4
	50	3	6	4	7	1	8	2	5
	100	2	6	4	7	1	8	3	5
	300	2	7	3	5.5	1	8	4	5.5
	500	2	5.5	3	7	1	8	4	5.5
($\hat{\alpha}$=6, $\hat{\theta}$=5, $\hat{\lambda}$=1.5)	20	7	5	3	6	1	8	2	4
	50	4	6	3	7	1	8	2	5
	100	2	6	4	7	1	8	3	5
	300	2	6.5	4	6.5	1	8	3	5
	500	2	6	3	7	1	8	4	5
$\sum$ *Ranks*		139	235.5	133.5	267	40	319.5	121	184.5
Overall Rank		4	6	3	7	1	8	2	5

A notable trend in the simulation results is that as the sample size n increases, the values of AB, MSE, and MRE consistently decline across all parameter combinations. This behavior demonstrates that all estimation approaches possess the property of consistency, a key requirement for dependable parameter estimation. Consequently, the estimates obtained using the proposed estimators are credible, showing relatively small AB, MSE, and MRE values in all investigated cases. Hence, the estimated parameters of the APIILx model are shown to closely approximate their true values.

The comparative performance ranking of the estimators, ordered from best to worst based on the overall rank values, is as follows: MPSE, ADE, WLSE, MLE, RADE, LSE, CRVME, and PCE. This ordering was obtained by arranging the $\sum$ Ranks values in ascending order, as summarized in Table 9.

It is important to emphasize that the simulation study primarily evaluates parameter uncertainty arising from finite-sample variability within a frequentist framework. The Monte Carlo design enables empirical approximation of the sampling distribution of each estimator under repeated sampling across different sample sizes and parameter regimes. The performance metrics AB, MSE, and MRE capture complementary aspects of estimation uncertainty, including systematic deviation, dispersion, and relative error. The consistent reduction of these measures as the sample size increases reflects the attenuation of sampling variability and confirms estimator consistency. Although the study includes a ranking of estimators, this ranking is grounded in their relative stability and robustness under repeated sampling. Broader forms of uncertainty, such as predictive uncertainty or robustness to model misspecification, are beyond the immediate scope of this methodological investigation and may be considered in future research.

In conclusion, all classical estimators exhibit satisfactory performance, with MPSEs providing the most accurate results overall. Specifically, the MPSE method achieved the lowest total rank of 40, followed by ADEs with a score of 121. Therefore, based on the overall ranking results, MPSEs and ADEs can be regarded as the most efficient estimation methods for the parameters of the APIILx distribution. This summary, together with the detailed tables, provides both transparency and concise interpretation of estimator performance across diverse scenarios.

6. Four real-life data applications

This section aims to assess the suitability of the APIILx distribution for modeling real-world datasets from different applied fields. The APIILx distribution and other competing models are fitted to four real-life datasets. It should be noted that model superiority claims are made relative to the competing models considered and the observed datasets, and thus are subject to the inherent variability of empirical data and parameter estimation uncertainty.

The first dataset consists of the remission times (in months) of 128 bladder cancer patients, as reported by [31]. The second dataset concerns head and neck cancer (HNC) patients, encompassing a range of malignancies of the mouth, throat, and other regions of the head and neck. It contains survival times (in days) for 44 HNC patients who received radiation therapy and chemotherapy concurrently. The data were originally provided by [32] and later modified by [33], who transformed the values by dividing the original observations by 100 prior to their analysis. The third dataset includes the survival durations (in days) of 72 guinea pigs infected with virulent tubercle bacilli, as documented by [34]. The fourth dataset captures the fatigue fracture life of Kevlar 373/epoxy samples exposed to a constant pressure at 90% of their stress capacity until failure, based on the studies in [35].

The performance of the APIILx distribution is evaluated against several competing models, including the transmuted Weibull-Lomax (TWLx) [36], Weibull-Lomax (WLx) [9], alpha power II exponential (APIIE) [21], Burr X Lomax (BXLx) [37], and Lx distribution. Model performance is evaluated through multiple goodness-of-fit metrics, such as the Akaike information criterion (AIC), consistent AIC (CAIC), Hannan-Quinn criterion (HQIC), Bayesian information criterion (BIC), CRVM statistic (W), AD statistic (A), Kolmogorov-Smirnov statistic (KS), and the corresponding p-value.

The ML estimates of the model parameters, along with their corresponding standard errors (SEs) (in parentheses), and the analytical fit measures for the four datasets are reported in Tables 10–13. The results indicate that the APIILx distribution consistently attains the lowest values across all goodness-of-fit criteria and the highest KS p-values, demonstrating strong performance compared with the other Lx-based models. Visual comparisons of the PDF, CDF, SF, and quantile-quantile (QQ) plots for the APIILx model across the four datasets are provided in Figures 2–5, while probability-probability (PP) plots for the four datasets are presented in Figures 6–9. These graphical assessments confirm that the APIILx distribution closely matches the empirical data and exhibits competitive performance relative to the alternative Lx distributions. All numerical computations were performed using the R software package.

Table 10. ML estimates, corresponding SEs, and goodness-of-fit measures for bladder cancer dataset.

Model	Param.	Estimates	SEs	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	p-value
APIILx	$\hat{\alpha}$	5.3375	(1.4642)	824.7863	824.9799	833.3424	828.2627	0.0138	0.0868	409.3932	0.0284	0.9999491
	$\hat{\theta}$	3.1275	(1.0634)									
	$\hat{\lambda}$	10.7161	(6.3963)									
TWLx	$\hat{\alpha}$	0.1403	(0.3711)	829.6097	830.1015	843.8699	835.4037	0.0235	0.1564	409.8049	0.0380	0.992493
	$\hat{\beta}$	11.2690	(14.4984)									
	$\hat{\lambda}$	0.6697	(0.4678)									
	$\hat{a}$	24.5503	(124.0050)									
	$\hat{b}$	1.4761	(0.2580)									
WLx	$\hat{\alpha}$	0.0710	(0.1273)	828.0774	828.4026	839.4855	832.7126	0.0295	0.1950	410.0387	0.0413	0.9808894
	$\hat{\beta}$	7.7416	(7.3047)									
	$\hat{a}$	79.4402	(244.2565)									
	$\hat{b}$	1.5097	(0.2835)									
BXLx	$\hat{\alpha}$	0.2982	(0.0511)	828.2918	828.4854	836.8479	831.7682	0.05439	0.3550	411.1459	0.0477	0.9327219
	$\hat{\beta}$	1.0201	(0.6641)									
	$\hat{\theta}$	0.9338	(0.2499)									
APIIE	$\hat{\alpha}$	1.5817	(0.8829)	832.2763	832.3723	837.9804	834.5939	0.1363	0.8124	414.1382	0.0696	0.5642341
	$\hat{\lambda}$	8.4982	(1.3834)									
Lx	$\hat{\alpha}$	13.9384	(15.3868)	831.6658	831.7618	837.3698	833.9834	0.0806	0.4873	413.8329	0.0966	0.1826948
	$\hat{\lambda}$	121.0225	(142.7235)									

Table 11. ML estimates, corresponding SEs, and goodness-of-fit measures for radiotherapy dataset.

Model	Param.	Estimates	SEs	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	p-value
APIILx	$\hat{\alpha}$	15.9981	(14.2799)	152.3397	152.9550	157.6233	154.2881	0.0170	0.1456	73.1698	0.0585	0.9965949
	$\hat{\theta}$	1.4869	(0.4641)									
	$\hat{\lambda}$	0.2877	(0.3851)									
TWLx	$\hat{\alpha}$	0.0549	(0.0641)	157.1352	158.7569	165.9412	160.3826	0.0329	0.2223	73.5676	0.0758	0.9500433
	$\hat{\theta}$	0.2046	(0.3591)									
	$\hat{\lambda}$	0.5831	(0.5035)									
	$\hat{\beta}$	128.8881	(473.3134)									
	$\hat{\gamma}$	2.6393	(1.2091)									
WLx	$\hat{\alpha}$	0.0641	(0.0640)	156.0418	157.0945	163.0866	158.6397	0.0463	0.2962	74.0209	0.0896	0.8499335
	$\hat{\theta}$	0.2362	(0.3648)									
	$\hat{\lambda}$	79.6733	(223.1180)									
	$\hat{\gamma}$	2.3351	(0.9431)									
BXLx	$\hat{\alpha}$	0.1604	(0.0117)	152.4368	153.0522	157.7204	154.3852	0.0283	0.1913	73.2184	0.0751	0.9535145
	$\hat{\theta}$	0.0058	(0.0027)									
	$\hat{\lambda}$	4.5417	(1.3547)									
APIIE	$\hat{\alpha}$	0.0100	(1.9427)	160.1199	160.4199	163.6423	161.4189	0.1418	0.8424	78.0600	0.1288	0.437145
	$\hat{\theta}$	2.7460	(1.1558)									
Lx	$\hat{\alpha}$	4.7714	(3.6966)	158.3623	158.6623	161.8847	159.6613	0.0981	0.5947	77.1812	0.1214	0.511618
	$\hat{\theta}$	8.5675	(7.8715)									

Table 12. ML estimates, corresponding SEs, and goodness-of-fit measures for guinea pigs dataset.

Model	Param.	Estimates	SEs	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	p-value
APIILx	$\hat{\alpha}$	10.8273	(2.8861)	192.1945	192.5475	199.0245	194.9136	0.0701	0.4210	93.0972	0.0765	0.7930629
	$\hat{\theta}$	22.8020	(47.0589)									
	$\hat{\lambda}$	16.3030	(36.9920)									
TWLx	$\hat{\alpha}$	0.1480	(0.2647)	197.6776	198.5867	209.609	202.2093	0.0884	0.5392	0.0890	0.0380	0.6175842
	$\hat{\beta}$	1.3901	(2.2362)									
	$\hat{\lambda}$	0.5761	(0.4252)									
	$\hat{a}$	171.5830	(1071.7518)									
	$\hat{b}$	2.7795	(0.9267)									
WLx	$\hat{\alpha}$	0.1619	(0.1814)	196.5941	197.1911	205.7008	200.2195	0.1038	0.6312	94.29706	0.0951	0.5321822
	$\hat{\beta}$	0.8687	(1.3132)									
	$\hat{a}$	83.8436	(341.3955)									
	$\hat{b}$	2.8240	(1.0022)									
BXLx	$\hat{\alpha}$	0.4928	(0.1406)	194.3719	194.7249	201.2019	197.091	0.0934	0.5791	94.1859	0.0897	0.6078425
	$\hat{\beta}$	0.4960	(0.3610)									
	$\hat{\theta}$	1.6041	(0.5725)									
APIIE	$\hat{\alpha}$	9.8970	(1.9122)	190.4381	190.612	194.9914	192.2508	0.0835	0.4884	93.2190	0.0803	0.7408592
	$\hat{\lambda}$	0.7900	(0.0822)									
Lx	$\hat{\alpha}$	11466.91	(1369.1358)	230.0782	230.2522	234.6316	231.8909	0.0967	0.5977	113.0391	0.2945	0.0000075
	$\hat{\lambda}$	20275.38	(431.1545)									

Table 13. ML estimates, corresponding SEs, and goodness-of-fit measures for fatigue fracture dataset.

Model	Param.	Estimates	SEs	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	p-value
APIILx	$\hat{\alpha}$	5.9097	(1.5971)	247.2356	247.569	254.2278	250.0301	0.0720	0.4161	120.6178	0.0837	0.6312499
	$\hat{\theta}$	9.5198	(9.3595)									
	$\hat{\lambda}$	9.1063	(10.8814)									
TWLx	$\hat{\alpha}$	0.1543	(0.8013)	253.1011	253.9583	264.7548	257.7585	0.0964	0.5742	121.5506	0.0874	0.5773
	$\hat{\beta}$	10.0501	(22.1734)									
	$\hat{\lambda}$	0.6984	(0.3803)									
	$\hat{a}$	153.9790	(1494.1574)									
	$\hat{b}$	1.5602	(0.2745)									
WLx	$\hat{\alpha}$	0.0804	(0.3124)	251.7014	252.2647	261.0243	255.4273	0.1060	0.6290	121.8507	0.0928	0.5001
	$\hat{\beta}$	4.9893	(6.8558)									
	$\hat{a}$	272.4669	(1850.4055)									
	$\hat{b}$	1.5713	(0.2951)									
BXLx	$\hat{\alpha}$	0.4755	(0.1154)	250.7377	251.0710	257.7299	253.5321	0.1227	0.7261	122.3688	0.1012	0.3920
	$\hat{\beta}$	0.7126	(0.4290)									
	$\hat{\theta}$	0.8158	(0.2005)									
APIIE	$\hat{\alpha}$	4.8073	(1.1205)	246.3845	246.5488	251.0459	248.2474	0.0924	0.5443	121.1922	0.0977	0.4349
	$\hat{\lambda}$	1.2024	(0.1537)									
Lx	$\hat{\alpha}$	3581.424	(739.4566)	258.2364	258.4008	262.8978	260.0993	0.1193	0.7073	127.1182	0.1664	0.0263
	$\hat{\lambda}$	7015.332	(1202.8465)									

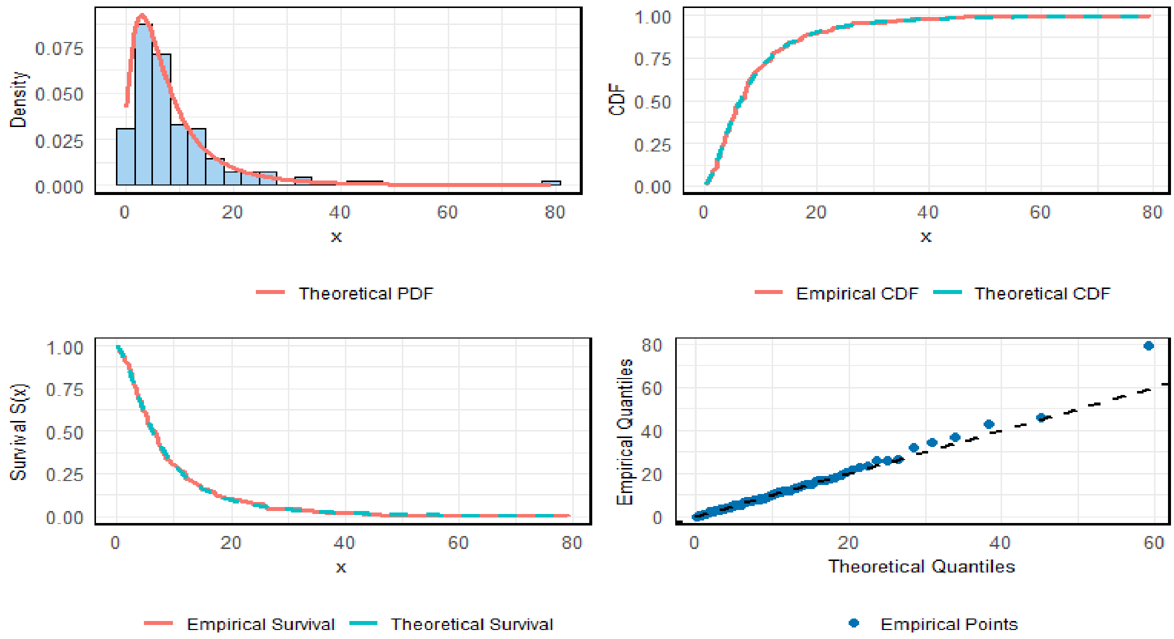

Figure 2. The fitted functions of the APIILx model for bladder cancer dataset.

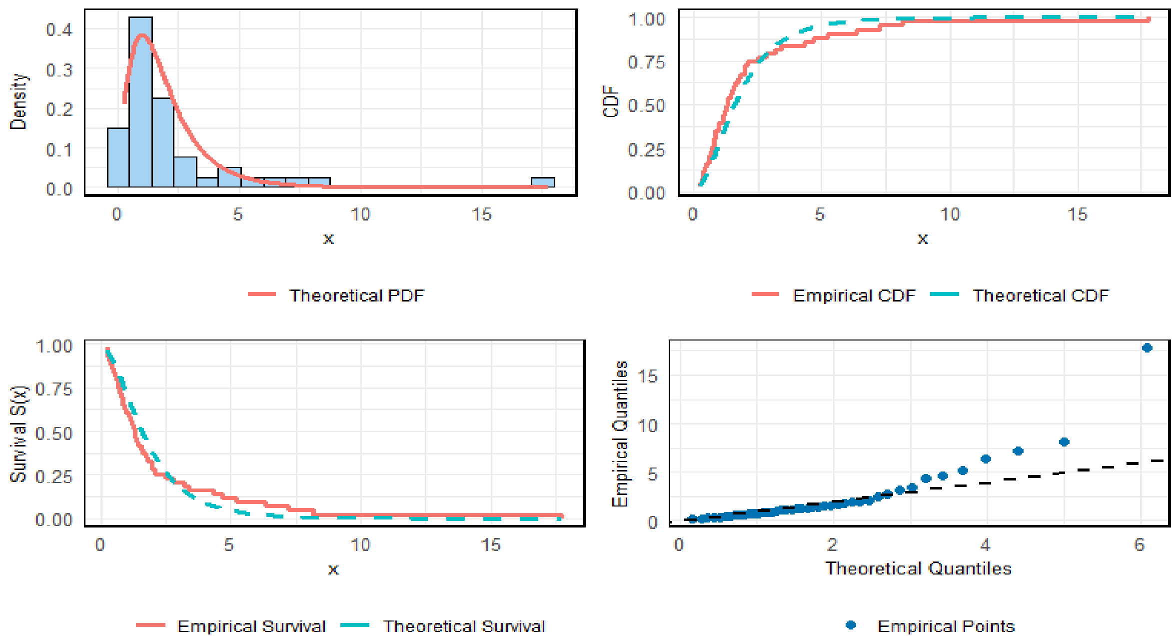

Figure 3. The fitted functions of the APIILx model for radiotherapy dataset.

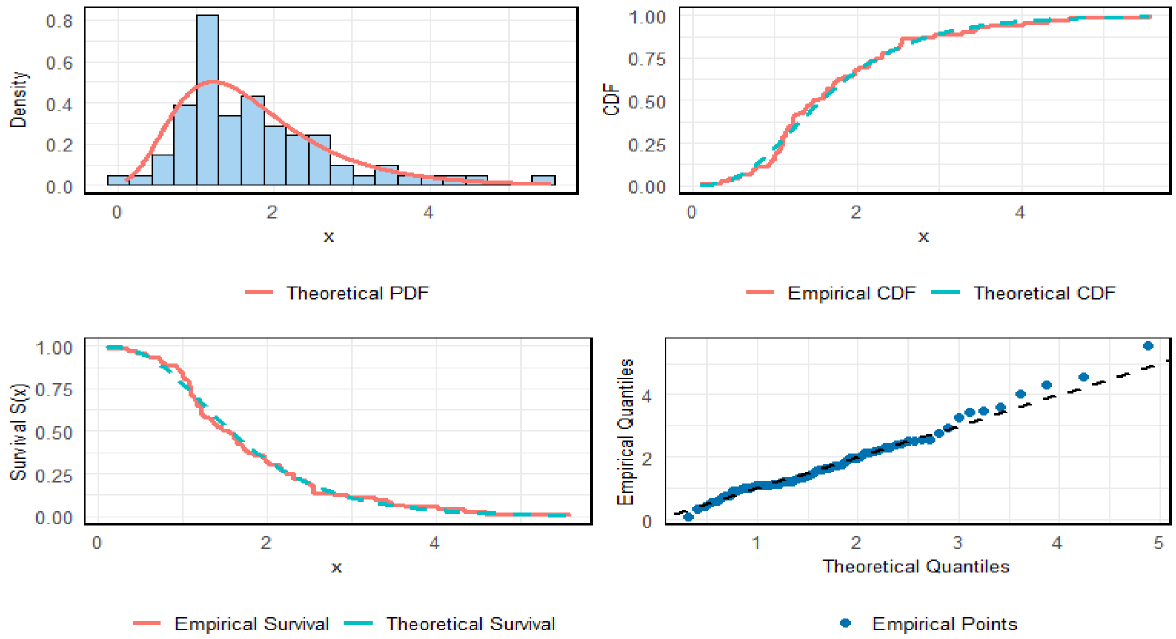

Figure 4. The fitted functions of the APIILx model for guinea pigs dataset.

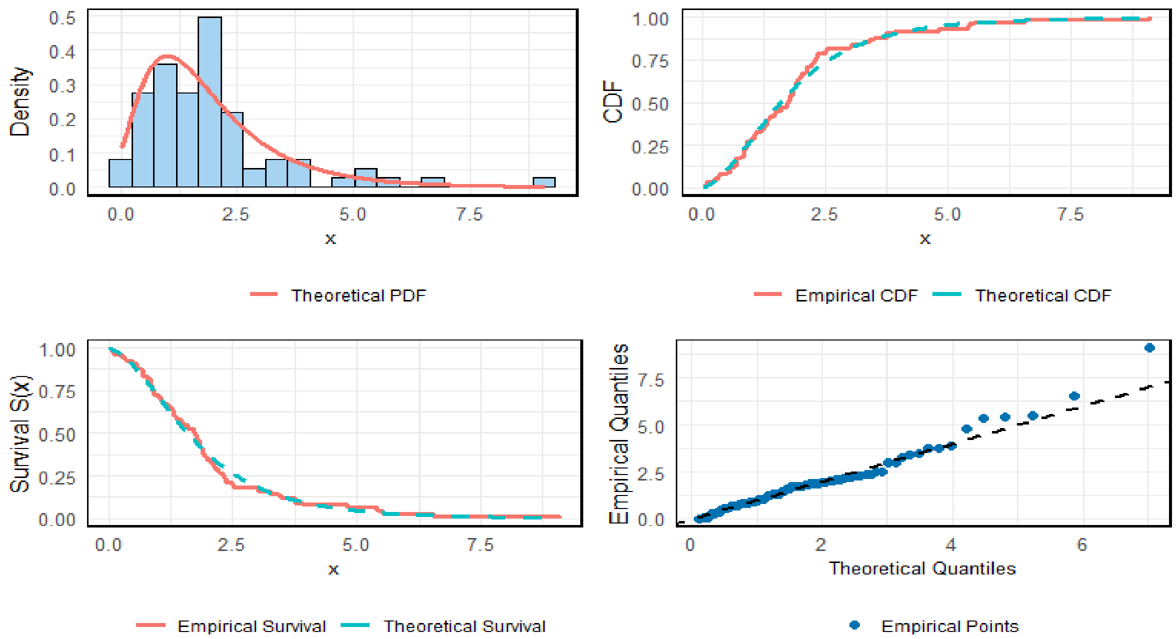

Figure 5. The fitted functions of the APIILx model for fatigue fracture dataset.

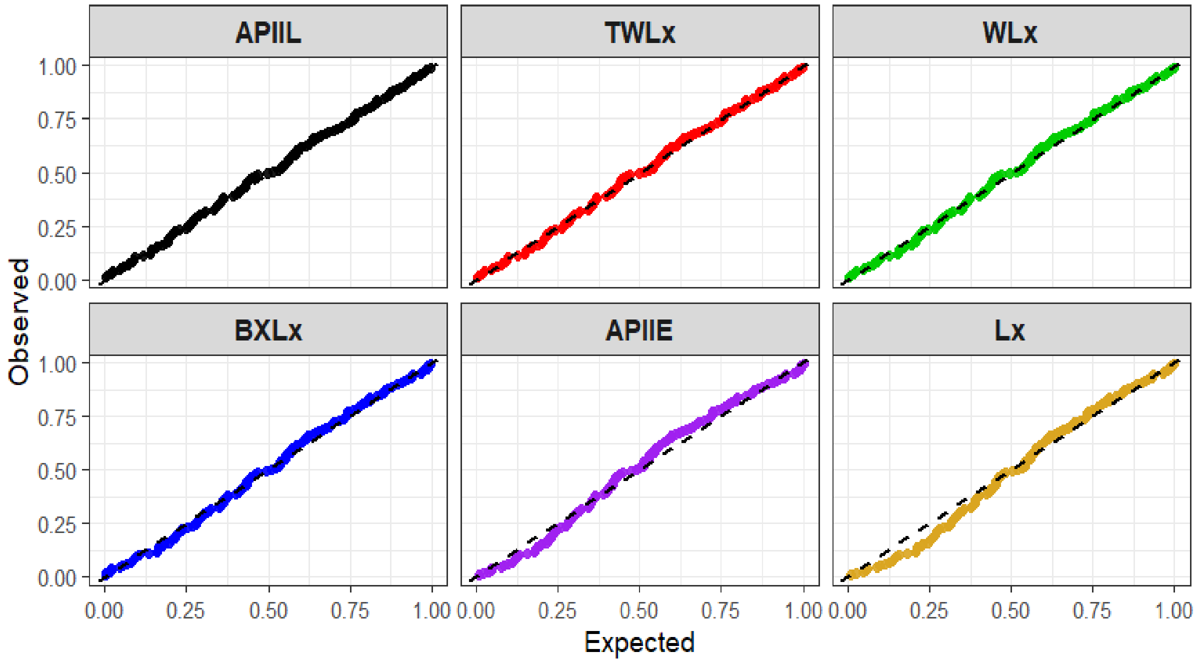

Figure 6. The PP plots of the fitted models for bladder cancer data.

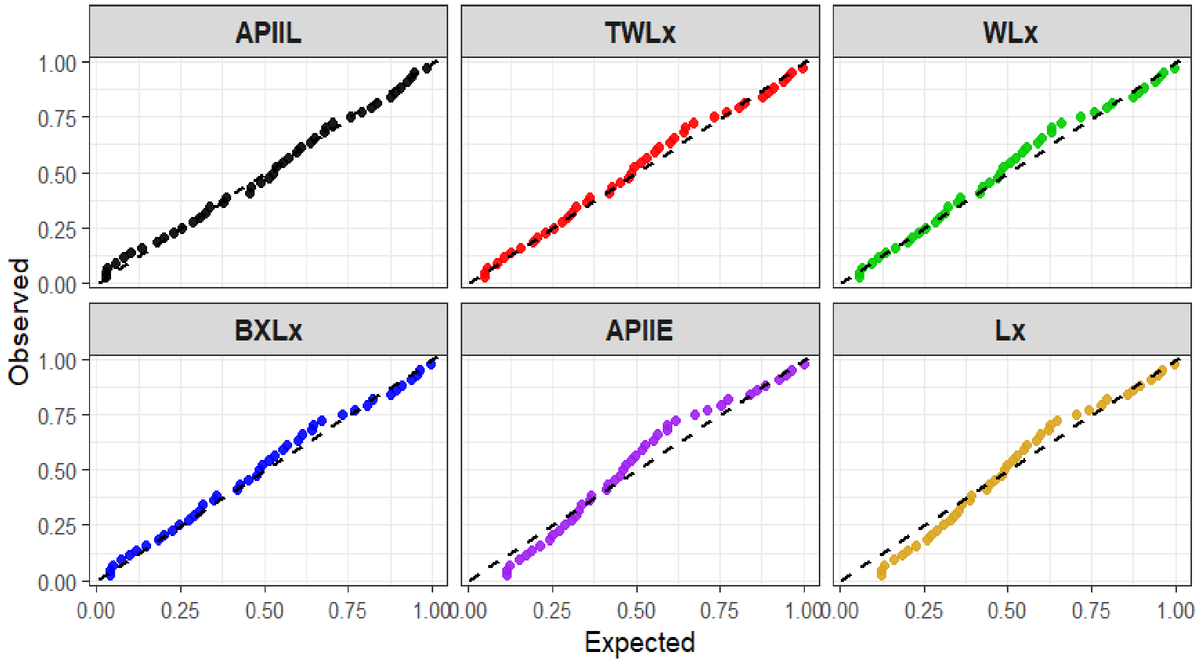

Figure 7. The PP plots of the fitted models for radiotherapy data.

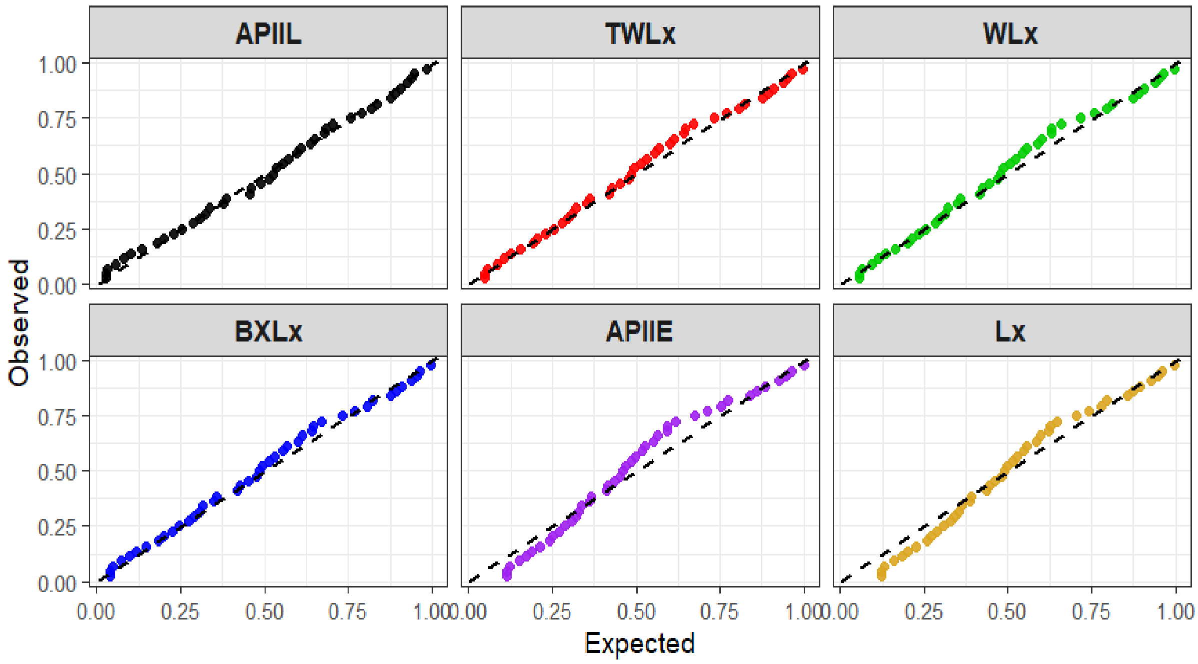

Figure 8. The PP plots of the fitted models for guinea pigs data.

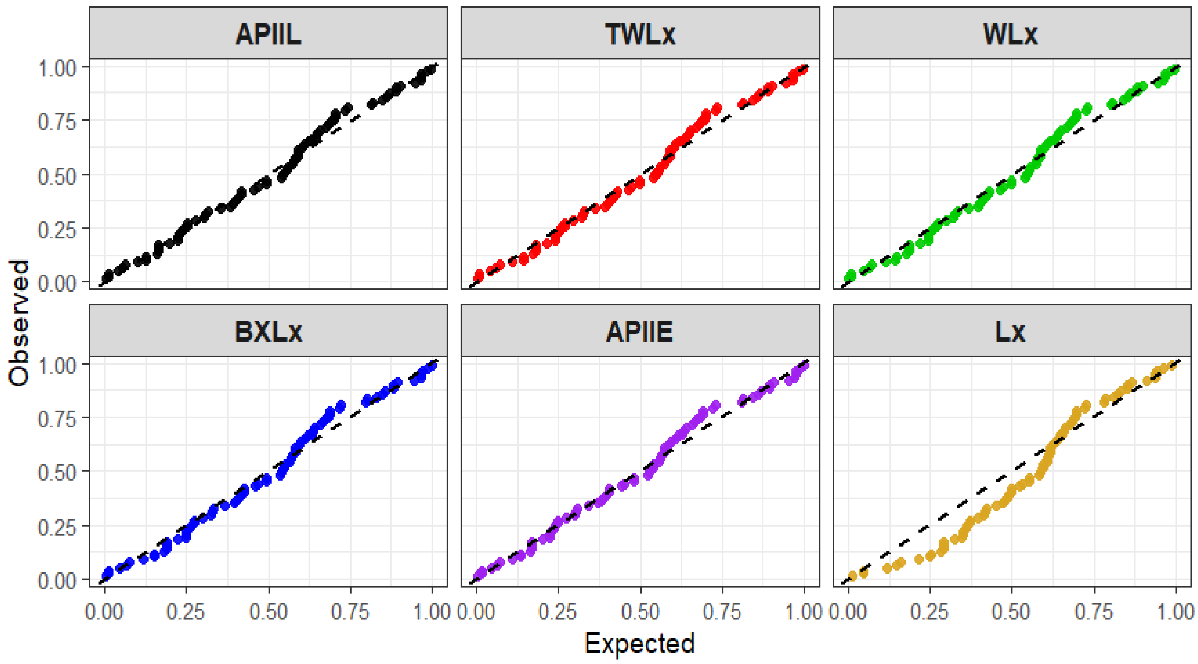

Figure 9. The PP plots of the fitted models for fatigue fracture.

In comparison, the proposed APIILx model provides superior fits to the radiation therapy dataset, outperforming all three competing models examined in these earlier studies. Across all datasets, the APIILx distribution consistently demonstrates superior goodness-of-fit metrics and closer alignment with the empirical data in the PP plots, indicating robust performance across diverse data characteristics. The reported standard errors further illustrate the reliability of the parameter estimates and quantify the inherent estimation uncertainty.

The results of the data analyses provide practical insights for applied users. For instance, the parameter estimates of the APIILx model reflect the underlying risk or failure behavior in each dataset, such as increasing hazard over time in survival studies or material fatigue processes. The superior fit of the APIILx distribution across all datasets indicates that it can reliably capture the data characteristics, offering practitioners a robust tool for prediction and decision-making. These interpretations, combined with the goodness-of-fit metrics and graphical assessments, demonstrate the practical usefulness and flexibility of the APIILx model in diverse applied contexts.

The total time on test (TTT) plots for the four datasets are presented in Figure 10. The TTT plots suggest that the bladder cancer and radiotherapy datasets exhibit unimodal failure rate shapes, whereas the guinea pigs and fatigue fracture datasets appear to follow increasing failure rate patterns. Figure 11 presents the HRF plots of the APIILx model based on the parameter estimates obtained from the four datasets. These HRF plots are consistent with the corresponding TTT plots and further confirm that the APIILx model provides an appropriate fit for all four datasets.

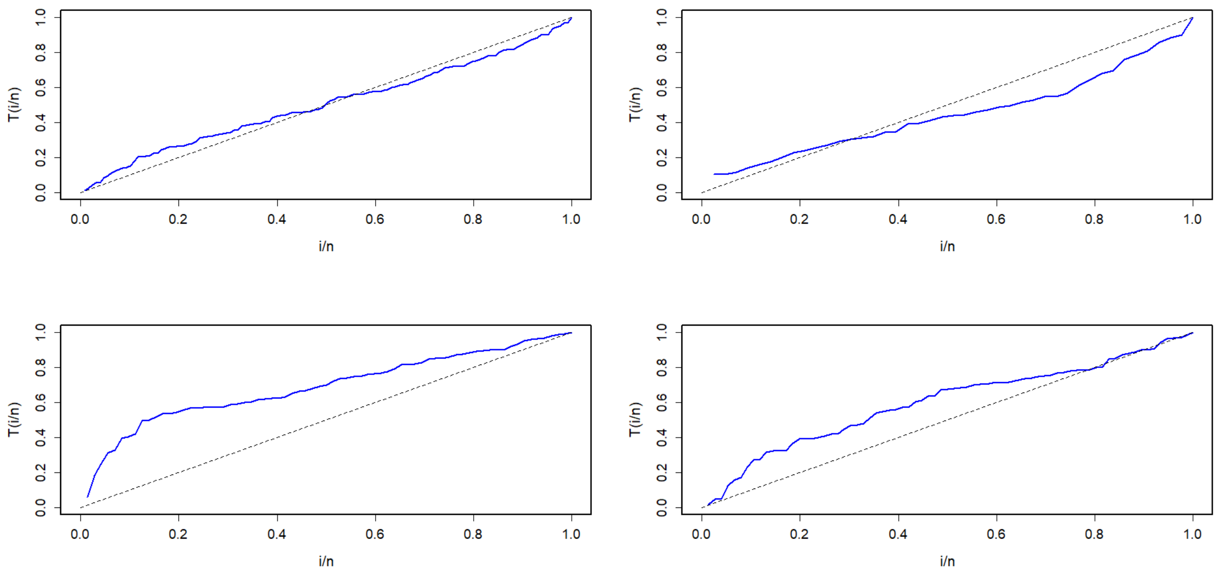

Figure 10. TTT plots for the bladder cancer (top left), radiotherapy (top right), guinea pigs (bottom left), and fatigue fracture (bottom right) datasets.

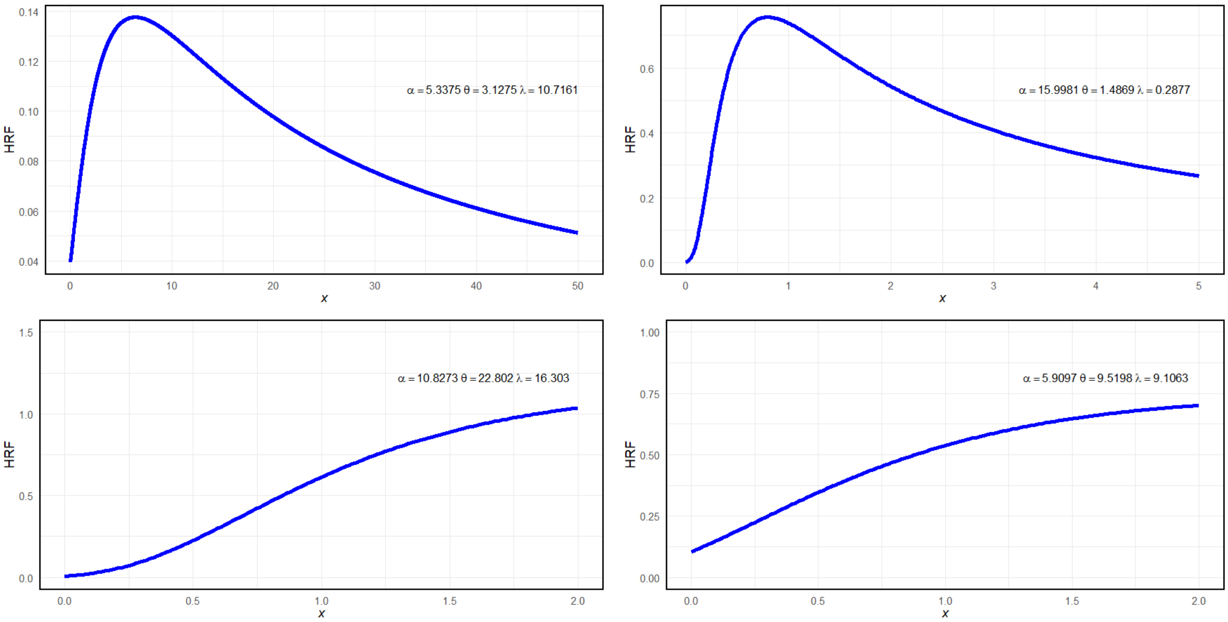

Figure 11. HRF plots for the bladder cancer (top left), radiotherapy (top right), guinea pigs (bottom left), and fatigue fracture (bottom right) datasets.

7. Conclusions

In this paper, a new probability distribution—the APIILx—was developed and investigated as an extension of the Lomax distribution within the APII-G family. The proposed model incorporates three parameters, thereby improving its flexibility in capturing heavy-tailed and skewed data. The key mathematical properties of the APIILx distribution were derived and discussed.

Eight estimation methods—namely, MPS, WLS, ML, LS, AD, CRVM, RAD, and PC estimators—were employed to estimate the model parameters. Extensive simulation studies were carried out to assess and compare the performance of these estimators. The results revealed that the MPS and AD methods consistently outperformed the other approaches, achieving overall scores of 40 and 121, respectively. Based on these findings, we recommend these two methods for estimating the parameters of the APIILx distribution.

Four real-data applications drawn from the medical, biomedical, and materials science (engineering) sectors demonstrated that the APIILx distribution provides a superior fit compared to several well-established competing models, as verified by multiple information criteria and goodness-of-fit statistics. These results highlight the usefulness and effectiveness of the APIILx distribution as a flexible and practical tool for modeling real-world datasets with complex structures.

While the present study focuses on frequentist parameter estimation and finite-sample performance evaluation, several extensions are possible. Future research may explore interval estimation and bootstrap-based uncertainty quantification for the APIILx parameters, as well as sensitivity analyses to assess robustness under model misspecification. Bayesian formulations of the APIILx distribution could also be developed to allow posterior-based uncertainty assessment and predictive inference. In addition, robustness investigations under data contamination or small perturbations may further clarify the stability of model selection conclusions in practical applications. Moreover, the exploration of non-parametric or semi-parametric adaptations of the model could be advantageous.

Author contributions

H. G. K. and A. B. G.: Validation, Formal analysis, Investigation, Resources, Writing-original draft preparation, Writing-review and editing; I. E., A. Z. A. and H. A. M: Conceptualization, Methodology, Software, Writing-original draft preparation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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