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Research article

Enhanced estimators for the multi-stress strength reliability using an advanced progressive hybrid censoring scheme

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Abstract: This article addresses the challenge of estimating the stress-strength reliability $\vartheta = P[Q < W < Y]$ under the scenario where strength (W) is in between upper (Y) and lower (Q) conditions. The study assumed that these variables were independent and followed the exponentiated Pareto distribution (EPD) with a common unknown shape parameter. The primary goal was to enhance statistical inference for this reliability metric by employing an adaptive progressive Type II hybrid censoring scheme (APTII-HCS), a method that ensures the collection of a fixed number of failures while effectively reducing experimental time. Using the maximum likelihood method, we obtained point estimates for the system reliability. Subsequently, the delta method was applied to establish the asymptotic confidence intervals for these estimates. Furthermore, Bayesian estimates were produced under both symmetric (squared error) and asymmetric (linear exponential) loss functions using Markov chain Monte Carlo procedures, which established reliable highest posterior density credible intervals. Extensive simulations were used to assess the effectiveness of Bayesian and classical statistical inference techniques. We conclude that the suggested estimators for the EPD model's reliability metrics and unknown parameters perform well according to several measures of accuracy. The importance of these findings lies in the successful integration of the APTII-HCS framework with the multi-stress-strength model, a combination not previously explored in existing literature. Lastly, three real-world data sets were analyzed to validate the proposed methods. This study provides a robust, flexible methodology for practitioners in engineering, medicine, and environmental science, where maintaining system variables within precise bounds is essential for ensuring durability and safety.

Keywords: exponentiated Pareto distribution; Bayesian estimation; Metropolis-Hastings; linear exponential loss function; credible intervals
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1. Introduction

In reliability and life-testing tests, censoring is a common problem where failure times of specific units are not actually measured. This usually occurs as a result of experimental restrictions or uncontrollable causes that result in data loss. The increased frequency of production cycles and improved quality and durability of contemporary items make effective reliability testing essential. Numerous censoring approaches have been developed by statisticians to solve the issues of partial data. The censoring plans used in reliability studies fall into two general categories: single-stage (including Type-I, Type-II, and hybrid) and multi-stage systems. Progressive Type-II (PTII) censoring is a particularly well-liked multistage technique (see Balakrishnan [1]). Using this approach, n units are subjected to testing, and a predetermined number m of items are set to fail according to a specified progressive censoring plan $R_1, R_2, \ldots, R_m$. When the ith failure occurs, denoted as $W_{i:m:n}$, R_i for $i = 1, 2, \ldots, m-1$, a random selection of surviving units is removed from the test. Upon the occurrence of the final failure, $W_{m:m:n}$, all remaining units are eliminated. This procedure continues until the mth failure, at which point all remaining units are censored. For more overview, see [2–4].

The progressive Type-I hybird censoring scheme (PTI-HCS) was established by Kundu and Joarder [5], which shares the same schematic representation as the PTII scheme. References [6, 7] investigated reliability under progressive hybrid censoring for the generalized exponential and Kavya-Manoharan Kumaraswamy distributions, respectively. Furthermore, the estimation of parameters for the inverse Weibull and inverted exponentiated Rayleigh distributions under progressive hybrid censoring was examined by [8,9].For a broader overview of these methodologies and their applications, see [10, 11].

However, in this case, the test is ended at $T^{\bullet} = \min(W_{m:m:n}, T)$, where $W_{m:m:n}$ is the mth failure and $T \in (0, \infty)$ is a predefined time point. The PTI-HCS's disadvantage is that its relatively small effective sample size may be random, making the statistical inference process either inapplicable or inefficient. By evaluating the data under the presumption of an exponential distribution, Ng et al. [12] developed an adaptive progressive Type II-HCS (APTII-HCS) to improve statistical analysis efficiency and decrease the overall test duration. In APTII-HCS, the number of failures m is predetermined, and the test time is permitted to extend over time (T), which is an optimum total test. In this situation, the progressive censoring scheme $R_1, R_2, \ldots, R_m$ is given; however, during the experiment, some of the R_i values may vary accordingly. If $W_{m:m:n} < T$, the test will end at $W_{m:m:n}$ and the conventional PTII censored will be used. Otherwise, if $W_{K:m:n} < T < W_{K+1:m:n}$, with $K + 1 < m$ and $W_{K+1:m:n}$ is the Kth failure time preceding time T, then we will not exclude any items from the experiment by setting $R_{K+1}, R_{K+2}, ., R_{m-1} = 0$ and at the time of the mth failure, all the remaining surviving items are removed, i.e., $R_m = n - m - \sum_{j=1}^{K} R_j$. This scheme's primary benefits are that it guarantees that we will obtain the effective number of failures (m) and it speeds up the experiment when its duration exceeds the predefined time T. Suppose that $w_{1:m:n} < \cdots < w_{K:m:n} < T < w_{K+1:m:n} < w_{m:m:n}$ is an APTII-HCS observed from a continuous population with probability density function (PDF) $f(w)$ and cumulative distribution function (CDF) $F(w)$, and the likelihood function can be expressed as follows:

$$L = C \prod_{j=1}^{m} g(w_{j:m:n}) \prod_{j=1}^{K} [1 - G(w_{j:m:n})]^{R_j} [1 - G(w_{m:m:n})]^{R_m}, \tag{1.1}$$

where C is a constant.

The stress-strength (SS) model determines how long a component with X strength will last under Y stress. Accordingly, if stress (Y) exceeds strength (X), a component cannot survive. When considering the mechanical dependability of this model, the device reliability is measured by the value $\lambda = P(Y < X)$. Therefore, although the quantity λ is known as SS reliability, it extends its applications beyond merely assessing actual SS reliable. Numerous disciplines, including life testing, reliability, the medical and social sciences, economics, psychology, and many more, have adopted the use of the metric λ. In the book by Kotz et al. [13], the theory and applications of SS reliability in industrial and economic systems are reviewed in great detail. The impact of outliers on the reliability of SS models has been rigorously investigated by [14–16]. In a related vein, ref. [17] addressed the reliability of SS model under a partially accelerated life test, specifically assuming a Lomax distribution. References [18, 19] discussed the inference of SS reliability based on adaptive progressive hybrid censoring, drawing from the Chen and inverse Weibull distributions. For recent studies, refer to [20, 21].

However, in many practical situations, a system must operate within a safe interval rather than merely exceeding a single stress limit. For instance, in medical diagnostics, blood pressure must remain between systolic and diastolic bounds; in engineering, electronic components may fail if voltage deviates above or below specified thresholds. This leads to the multi-stress-strength reliability measure $\vartheta = P[Q < W < Y]$, where W stands for strength and Q and Y for the two stresses. In the literature, estimating $\vartheta = P[Q < W < Y]$ for various stress and strength distributions has received a lot of attention lately. Ivshin [22] investigated the estimate using independent samples. The minimum variance unbiased and maximum likelihood estimators for $\vartheta = P[Q < W < Y]$, were investigated by refs. [23, 24], assuming Q, Y, and W are independent random variables with normal, exponential, or uniform distributions. The non-parametric inference of ϑ was covered by ref. [25]. Inference of ϑ for n-standy system was handled by ref. [26] using a Monte-Carlo simulation technique. The estimation of ϑ was studied by ref. [27] when strength and stresses are independent variables with a Kumaraswamy distribution. Estimation of ϑ under an assumption of an inverse Rayleigh distribution for the strength and stress random variables was discussed by refs. [28, 29]. Reference [30] examined the estimation of the SS reliability ϑ in the case of three independent Kumaraswamy distributions. References [31, 32] explored shrinkage estimation and sampling strategies for stress-strength models, assuming all components follow Lomax distributions. Reference [33] used PTII censored data to estimate ϑ in the case of three independent Weibull random variables. Using the upper record value, refs. [34] explored reliability estimation of ϑ for Burr III model. Inference of the SS reliability ϑ was examined by ref. [35] using the Burr-XII distribution under PTII censoring from both Bayesian and frequentist viewpoints. Bayesian and non-Bayesian estimation methods of ϑ for the length-biased exponential model were examined by ref. [36] under the progressive failure censoring scheme. Reference [37] provided the classical and Bayesian inferences ϑ based on the geometric distribution situation. Using progressive Type-II censoring with random removal, ref. [38] discussed the statistical inference for ϑ based on the inverse Weibull distribution. Reference [39] explored the estimation of ϑ for the unit exponentiated half logistic distribution, utilizing classical, Bayesian, and bootstrapping methods, and demonstrated these approaches through real-world data applications. Reference [40] developed Bayesian and non-Bayesian inference for ϑ using the inverted Kumaraswamy distribution with outliers, and demonstrated the approach with real data. For more recent studies, the reader can refer to [41, 42].

Although the APTII-HCS and the SS model $\vartheta = P[Q < W < Y]$ are highly applicable, no studies have yet explored the intersection of these two frameworks. This study uses the APTII-HCS approach to estimate the reliability of $\vartheta = P[Q < W < Y]$, where strength W is limited between stresses Q and Y. It is noteworthy that the APTII-HCS has never been applied to this specific reliability metric before. The exponentiated Pareto distribution, recognized for its significance and popularity, is employed to model the stress and strength variables with a common unknown shape parameter, as detailed in Section 2. The main contributions of this paper are:

- Producing enhanced estimators for the multi-stress-strength reliability parameter $\vartheta = P[Q < W < Y]$, under the EPD with a common unknown shape parameter.
- Employing the APTII-HCS to improve inference efficiency and reduce experimental time.
- Deriving the maximum likelihood (ML) estimators and approximate confidence intervals (AS-CIs) of $\vartheta = P[Q < W < Y]$ and estimated parameters.
- Providing the Bayesian estimator of ϑ under both symmetric (squared error loss function, SELOF) and asymmetric (linear exponential loss function, LIEXLOF).
- Constructing highest posterior density credible intervals of $\vartheta = P[Q < W < Y]$ and estimated parameters.
- Validating the proposed methods through extensive simulations and real-data applications, demonstrating their practical effectiveness in reliability analysis.

The following is a summary of the remainder of the paper: Section 2 explains the model and its SS reliability formula. The conventional ML and AS-CI estimators of parameters and ϑ are obtained in Section 3. Section 4 proposes a Bayes point estimate with a reasonable highest posterior density (HPD) credible interval. Section 5 offers a numerical evaluation of the simulation study's developed outcomes. The examination of a real-world data set that depicts the transformer insulation at three voltage levels is given in Section 6. Section 7 reports the succinct remarks along with our findings and suggestions.

2. Model specification and its reliability

A range of parametric probability distributions are used by researchers in lifetime analysis and reliability theory to accurately model lifespan data. It has been discovered that the exponentiated Pareto distribution (EPD) is a very helpful model for lifespan assessments. The EPD was introduced by Gupta et al. [43] as a straightforward but powerful extension of the basic Pareto distribution by including a shape parameter. Economics, finance, stock market volatility, environmental science, and insurance risk are just a few of the fields that have used the EPD. The PDF, CDF, and the survival function (SF) of the EPD are represented by:

$$f(w) = \alpha \upsilon_1 (1+w)^{-\alpha-1} \left[1-(1+w)^{-\alpha}\right]^{\upsilon_1 - 1}; \qquad w, \upsilon_1, \alpha > 0, \tag{2.1}$$

$$F(w) = \left[1-(1+w)^{-\alpha}\right]^{\upsilon_1}; \qquad w, \upsilon_1, \alpha > 0, \tag{2.2}$$

and

$$\bar{F}(w) = 1 - \left[1-(1+w)^{-\alpha}\right]^{\upsilon_1}; \qquad w, \upsilon_1, \alpha > 0, \tag{2.3}$$

where α and υ_1 are the shape parameters. For its density function, the EPD may capture both heavier and lighter tails. The EPD is perfect for evaluating a range of lifespan data since its hazard rate function can take on many shapes, such as an upside-down bathtub shape. Due to its flexibility, several studies in various domains were examined by several authors. For instance, parameter estimators of the EPD based on different sample sizes were discussed by ref. [44]. A mixture distribution based on the exponential distribution and EPD was developed by ref. [45]. The Bayes and non-Bayes estimators of EPD were investigated under different censoring schemes by ref. [46]. Later, ref. [47] discussed life testing and reliability. Nooghabi [48] looked at parameter estimators for the EPD, considering the impact of outliers. Panahi [49] investigated the statistical inference for the unknown parameters of the EPD based on a progressive first-failure censoring scheme. Khamnei et al. [50] employed the ranked set sampling approach to estimate the EPD. Both Bayesian and classical estimates of the multi-component SS model for the EPD continue to be a topic of interest, see [51].

The expression of $\vartheta = P[Q < W < Y]$ is derived where the strength W with the EPD (α, υ_1), the stress Q with the EPD (α, υ_2), and the stress Y with the EPD (α, υ_3) are independent random variables. The SS model's reliability formula has the following structure:

$$\vartheta = P[Q < W < Y] = \textstyle\int_{-\infty}^{\infty} G_Q(w)\,\bar{H}_Y(w)\,dF_W(w)\,, \tag{2.4}$$

where $\bar{H}_Y(w)$ is the SF of stress Y at w, and $G_Q(w)$ is the CDF of the stress Q at w. Consequently, based on Eq (2.4), we obtain the formula of ϑ as follows:

$$\vartheta = \alpha\upsilon_1 \textstyle\int_0^{\infty} \left[1 - \left[1 - (1+w)^{-\alpha}\right]^{\upsilon_3}\right](1+w)^{-\alpha-1}\left[1 - (1+w)^{-\alpha}\right]^{\upsilon_1+\upsilon_2-1} dw.$$

Then, after some simplification, we have

$$\vartheta = \frac{\upsilon_1}{(\upsilon_1+\upsilon_2)} - \frac{\upsilon_1}{(\upsilon_1+\upsilon_2+\upsilon_3)} = \frac{\upsilon_1\upsilon_3}{(\upsilon_1+\upsilon_2)(\upsilon_1+\upsilon_2+\upsilon_3)}. \tag{2.5}$$

3. Maximum likelihood estimation

The ML estimators and AS-CIs of parameters and SS reliability $\vartheta = P[Q < W < Y]$ are produced in this section under the assumption that the stresses (Q, Y) have the EPD with parameters (α, υ_2) and (α, υ_3), respectively, while the strength W has the EPD with parameters (α, υ_1) and they are independent random variables. At first, the ML estimator of the parameter vector $\Omega = (\upsilon_1, \upsilon_2, \upsilon_3, \alpha)^T$ for the EPD based on the APTII-HCS is determined.

Suppose that the APTII-HCS represented by $w_{1:n_1} < \cdots < w_{d_1:m_1:n_1} < T_1 < w_{d_1+1:m_1:n_1} < \cdots < w_{m_1:m_1:n_1}$ is drawn from strength W with PDF (2.1), CDF (2.2), and SF (2.3) of the EPD (α, υ_1). Similarly, suppose that APTII-HCS $q_{1:n_2} < \cdots < q_{d_2:m_2:n_2} < T_2 < q_{d_2+1:m_2:n_2} < \cdots < q_{m_2:m_2:n_2}$ is drawn from lower stress Q with the PDF, CDF, and SF of the EPD (α, υ_2). Finally, let $y_{1:n_3} < \cdots < y_{d_3:m_3:n_3} < T_3 < y_{d_3+1:m_3:n_3} < \cdots < y_{m_3:m_3:n_3}$ APTII-HCS be drawn from upper stress Y with the PDF, CDF, and SF of the EPD (α, υ_3). Under these assumptions, the likelihood function of the unknown parameters based on Eq (1.1) can be expressed as:

$$\begin{aligned} l(\Omega) \propto (\alpha\upsilon_1)^{m_1} \prod_{j_1=1}^{m_1} (1+w_{j_1})^{-\alpha-1} \left[B_{j_1}(\alpha)\right]^{\upsilon_1-1} \prod_{j_1=1}^{K_1} \left[B_{j_1}(\alpha)\right]^{\upsilon_1 R_1 j_1} \left[B_{m_1}(\alpha)\right]^{R_1 m_1} \\ (\alpha\upsilon_2)^{m_2} \prod_{j_2=1}^{m_2} (1+q_{j_2})^{-\alpha-1} \left[B_{j_2}(\alpha)\right]^{\upsilon_2-1} \prod_{j_2=1}^{K_2} \left[B_{j_2}(\alpha)\right]^{\upsilon_2 R_2 j_2} \left[B_{m_2}(\alpha)\right]^{R_2 m_2} \\ (\alpha\upsilon_3)^{m_3} \prod_{j_3=1}^{m_3} (1+y_{j_3})^{-\alpha-1} \left[B_{j_3}(\alpha)\right]^{\upsilon_3-1} \prod_{j_3=1}^{K_3} \left[B_{j_3}(\alpha)\right]^{\upsilon_3 R_3 j_3} \left[B_{m_3}(\alpha)\right]^{R_3 m_3}, \end{aligned} \tag{3.1}$$

where $B_{j_1}(\alpha) = \left[1-\left(1+w_{j_1}\right)^{-\alpha}\right]$, $B_{j_2}(\alpha) = \left[1-\left(1+q_{j_2}\right)^{-\alpha}\right]$, $B_{j_3}(\alpha) = \left[1-\left(1+y_{j_3}\right)^{-\alpha}\right]$, $B_{m_1}(\alpha) = \left[1-(1+w_{m_1})^{-\alpha}\right]$, $B_{m_2}(\alpha) = \left[1-(1+q_{m_2})^{-\alpha}\right]$, $B_{m_3}(\alpha) = \left[1-(1+y_{m_3})^{-\alpha}\right]$ are written in simplified form. The logarithm of Eq (3.1) is given by:

$$\begin{aligned} \ln l^* = {} & (m_1+m_2+m_3)\ln(\alpha) + m_1 \ln\upsilon_1 + m_2 \ln\upsilon_2 + m_3\ln\upsilon_3 + R_1 m_1 \ln(B_{m_1}(\alpha)) \\ & - (\alpha+1)\left[\sum_{j_1=1}^{m_1} \ln(1+w_{j_1}) + \sum_{j_2=1}^{m_2} \ln(1+q_{j_2}) + \sum_{j_3=1}^{m_3} \ln(1+y_{j_3})\right] + R_3 m_3 \ln(B_{m_3}(\alpha)) \\ & + (\upsilon_3-1)\sum_{j_3=1}^{m_3} \ln(B_{j_3}(\alpha)) + \sum_{j_1=1}^{K_1} \upsilon_1 R_1 j_1 \ln(B_{j_1}(\alpha)) + \sum_{j_2=1}^{K_2} \upsilon_2 R_2 j_2 \ln(B_{j_2}(\alpha)) + \sum_{j_3=1}^{K_3} \upsilon_3 R_3 j_3 \ln(B_{j_3}(\alpha)) \\ & + (\upsilon_1-1)\sum_{j_1=1}^{m_1} \ln(B_{j_1}(\alpha)) + (\upsilon_2-1)\sum_{j_2=1}^{m_2} \ln(B_{j_2}(\alpha)) + R_2 m_2 \ln(B_{m_2}(\alpha)). \end{aligned} \tag{3.2}$$

Differentiating Eq (3.2) with respect to $\upsilon_1, \upsilon_2, \upsilon_3$, and α and setting the resulting equations to zero yields the ML estimators $\hat{\upsilon}_1$, $\hat{\upsilon}_2$, $\hat{\upsilon}_3$, and $\hat{\alpha}$.

$$\frac{\partial \ln l^*}{\partial \upsilon_1} = \frac{m_1}{\upsilon_1} + \sum_{j_1=1}^{K_1} R_1 j_1 \ln(B_{j_1}(\alpha)) = 0, \tag{3.3}$$

$$\frac{\partial \ln l^*}{\partial \upsilon_2} = \frac{m_2}{\upsilon_2} + \sum_{j_2=1}^{K_2} R_2 j_2 \ln(B_{j_2}(\alpha)) = 0, \tag{3.4}$$

$$\frac{\partial \ln l^*}{\partial \upsilon_3} = \frac{m_3}{\upsilon_3} + \sum_{j_2=1}^{K_3} R_3 j_3 \ln(B_{j_3}(\alpha)) = 0, \tag{3.5}$$

and

$$\begin{aligned}\frac{\partial \ln l^*}{\partial \alpha} &= \frac{(m_1+m_2+m_3)}{\alpha} - \left[\sum_{j_1=1}^{m_1} \ln(1+w_{j_1}) + \sum_{j_2=1}^{m_2} \ln(1+q_{j_2}) + \sum_{j_3=1}^{m_3} \ln(1+y_{j_3})\right] \\ &+ \frac{R_2 m_2 B'_{m_2}(\alpha)}{B_{m_2}(\alpha)} + \frac{R_3 m_3 B'_{m_3}(\alpha)}{B_{m_3}(\alpha)} + \sum_{j_1=1}^{m_1} \frac{(\upsilon_1-1)B'_{j_1}(\alpha)}{B_{j_1}(\alpha)} + \sum_{j_2=1}^{m_2} \frac{(\upsilon_2-1)B'_{j_2}(\alpha)}{B_{j_2}(\alpha)} \\ &+ \frac{R_1 m_1 B'_{m_1}(\alpha)}{B_{m_1}(\alpha)} + \sum_{j_3=1}^{m_3} \frac{(\upsilon_3-1)B'_{j_3}(\alpha)}{B_{j_3}(\alpha)} + \sum_{j_1=1}^{K_1} \frac{\upsilon_1 R_1 j_1 B'_{j_1}(\alpha)}{B_{j_1}(\alpha)} + \sum_{j_2=1}^{K_2} \frac{\upsilon_2 R_2 j_2 B'_{j_2}(\alpha)}{B_{j_2}(\alpha)} \\ &+ \sum_{j_3=1}^{K_3} \frac{\upsilon_3 R_3 j_3 B'_{j_3}(\alpha)}{B_{j_3}(\alpha)} = 0,\end{aligned} \tag{3.6}$$

where

$$B'_{j_1}(\alpha) = \frac{\partial}{\partial \alpha}\left[1-\left(1+w_{j_1}\right)^{-\alpha}\right] = \left(1+w_{j_1}\right)^{-\alpha} \ln\left(1+w_{j_1}\right), \tag{3.7}$$

$$B'_{j_2}(\alpha) = \frac{\partial}{\partial \alpha}\left[1-\left(1+q_{j_2}\right)^{-\alpha}\right] = \left(1+q_{j_2}\right)^{-\alpha} \ln\left(1+q_{j_2}\right), \tag{3.8}$$

$$B'_{j_3}(\alpha) = \frac{\partial}{\partial \alpha}\left[1-\left(1+y_{j_3}\right)^{-\alpha}\right] = \left(1+y_{j_3}\right)^{-\alpha} \ln\left(1+y_{j_3}\right). \tag{3.9}$$

Also note that $B'_{m_h}(\alpha), h = 1,2,3$, has the same expression of Eqs (3.7)–(3.9) by replacing j with m. From Eqs (3.3)–(3.5), ML estimators $\hat{\upsilon}_1$ of υ_1, $\hat{\upsilon}_2$ of υ_2, and $\hat{\upsilon}_3$ of υ_3 are obtained as a function of α as shown below:

$$\hat{\upsilon}_1 = \frac{-m_1}{\sum\limits_{j_1=1}^{K_1} R_1 j_1 \ln(B_{j_1}(\alpha))}, \hat{\upsilon}_2 = \frac{-m_2}{\sum\limits_{j_2=1}^{K_2} R_2 j_2 \ln(B_{j_2}(\alpha))}, \hat{\upsilon}_3 = \frac{-m_3}{\sum\limits_{j_3=1}^{K_3} R_3 j_3 \ln(B_{j_3}(\alpha))}. \tag{3.10}$$

The ML estimator $\hat{\alpha}$ of α is produced by substituting $\hat{\upsilon}_1$, $\hat{\upsilon}_2$, and $\hat{\upsilon}_3$ given in Eq (3.10) into Eq (3.6) and solving numerically by using an iterative technique. The ML estimators were computed in R with the maxLik package using the Newton-Raphson method. Using the invariance property, the ML estimator $\hat{\vartheta}$ is produced after inserting $\hat{\upsilon}_1, \hat{\upsilon}_2$, and $\hat{\upsilon}_3$ into Eq (2.5) as follows:

$$\hat{\vartheta} = \frac{\hat{\upsilon}_1 \hat{\upsilon}_3}{(\hat{\upsilon}_1+\hat{\upsilon}_2)(\hat{\upsilon}_1+\hat{\upsilon}_2+\hat{\upsilon}_3)}.$$

Furthermore, the AS-CI of SS reliability is created. The asymptotic distribution of $\hat{\vartheta}$, which is derived from the asymptotic distribution of $\upsilon_1, \upsilon_2, \upsilon_3$, and α, is used to calculate the AS-CI of ϑ. For large sample sizes, the asymptotic distribution of $\hat{\Omega}$ is normal, with a mean of Ω and a variance-covariance matrix of $I^{-1}(\Omega)$. Consequently, we can estimate $I^{-1}(\Omega)$ using $I^{-1}(\hat{\Omega})$ that is derived from the observed Fisher information matrix:

$$I^{-1}(\hat{\Omega}) = -\begin{bmatrix} \frac{\partial^2 \ln l^*}{\partial \upsilon_1^2} & \frac{\partial^2 \ln l^*}{\partial \upsilon_1 \partial \upsilon_2} & \frac{\partial^2 \ln l^*}{\partial \upsilon_1 \partial \upsilon_3} & \frac{\partial^2 \ln l^*}{\partial \upsilon_1 \partial \alpha} \\ \frac{\partial^2 \ln l^*}{\partial \upsilon_2 \partial \upsilon_1} & \frac{\partial^2 \ln l^*}{\partial \upsilon_2^2} & \frac{\partial^2 \ln l^*}{\partial \upsilon_2 \partial \upsilon_3} & \frac{\partial^2 \ln l^*}{\partial \upsilon_2 \partial \alpha} \\ \frac{\partial^2 \ln l^*}{\partial \upsilon_3 \partial \upsilon_1} & \frac{\partial^2 \ln l^*}{\partial \upsilon_3 \partial \upsilon_2} & \frac{\partial^2 \ln l^*}{\partial \upsilon_3^2} & \frac{\partial^2 \ln l^*}{\partial \upsilon_3 \partial \alpha} \\ \frac{\partial^2 \ln l^*}{\partial \alpha \partial \upsilon_1} & \frac{\partial^2 \ln l^*}{\partial \alpha \partial \upsilon_2} & \frac{\partial^2 \ln l^*}{\partial \alpha \partial \upsilon_3} & \frac{\partial^2 \ln l^*}{\partial \alpha^2} \end{bmatrix}^{-1}_{(\upsilon_1=\hat{\upsilon}_1, \upsilon_2=\hat{\upsilon}_2, \upsilon_3=\hat{\upsilon}_3, \alpha=\hat{\alpha})}.$$

Note that the diagonal of $\hat{I}^{-1}(\hat{\Omega})$ provides the asymptotic variances of ML estimators $\hat{\upsilon}_1, \hat{\upsilon}_2, \hat{\upsilon}_3$, and $\hat{\alpha}$ respectively. Thus, for $0 < \delta < 1$, the $100(1-\delta)\%$ AS-CIs of Ω are given by $\hat{\Omega} \pm z_{\delta/2}\sqrt{\widehat{\text{var}}(\hat{\Omega})}$, where $z_{\delta/2}$ is the upper percent point of the standard normal distribution.

Additionally, estimating the variances of each estimator is required in order to construct the AS-CI for the SS reliability ϑ. A popular method for approximating these variances is the delta method (Greene [52]). The delta method, under specific regularity criteria, suggested that the distribution of ϑ can be approximated by a normal distribution with mean ϑ and variance $\text{var}(\hat{\vartheta}) = [\Lambda\,]^T[\hat{I}^{-1}(\hat{\Omega})][\Lambda]_{\Omega=\hat{\Omega}}$, where $\Lambda = \left(\frac{\partial\vartheta}{\partial\upsilon_1}, \frac{\partial\vartheta}{\partial\upsilon_2}, \frac{\partial\vartheta}{\partial\upsilon_3}, \frac{\partial\vartheta}{\partial\alpha}\right)$. Thus, the two-sided AS-CI of $100(1-\delta)\%$ of ϑ can be constructed as follows:

$$\hat{\vartheta} \pm z_{\delta/2}\sqrt{\widehat{\text{var}}(\hat{\vartheta})}.$$

4. Bayesian estimation

The Bayesian method is based on combining the data, represented by the likelihood function, with previous knowledge about the model parameters. A crucial component of statistical literature is the selection of relevant prior information. Due to its maximum entropy property and support for positive values, which is frequently preferred for model parameters, the gamma distribution was selected for this application. The exponential and chi-square distributions are also special cases of the gamma distribution, which is a flexible family. These factors lead to the assumption that all parameters in this study have independent gamma priors as follows:

$$\Pi_h(\upsilon_h) \propto \upsilon_h^{a_h-1}e^{-b_h\upsilon_h},\;\; \upsilon_h, a_h, b_h > 0,\;\; h = 1, 2, 3,$$

$$\Pi_4(\alpha) \propto \alpha^{a_4-1}e^{-b_4\alpha},\;\; a_4, b_4, \alpha > 0,$$

where $a_h, b_h > 0$, $h = 1, 2, 3, 4$, are the hyperparameters. The joint prior PDF of $\Omega = (\upsilon_1, \upsilon_2, \upsilon_3, \alpha)$ is given by:

$$\Pi(\Omega) \propto \upsilon_1^{a_1-1}\upsilon_2^{a_2-1}\upsilon_3^{a_3-1}\alpha^{a_4-1}e^{-(b_1\upsilon_1+b_2\upsilon_2+b_3\upsilon_3+b_4\alpha)}. \tag{4.1}$$

The means and variances of the ML estimators for each j-th sample are computed and equated to the mean and variance of the gamma prior in order to obtain the hyperparameters. The estimates $\hat{\upsilon}_1, \hat{\upsilon}_2, \hat{\upsilon}_3, \hat{\alpha}$ correspond to $\upsilon_1, \upsilon_2, \upsilon_3, \alpha$, respectively. For this prior elicitation method, see [53]. Matching the moments of estimators to the gamma prior moments yields the hyperparameter estimators

$$\hat{a}_i = \frac{\frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{(j)}}{\frac{1}{I-1}\sum_{j=1}^{I}\left(\hat{\Omega}_i^{(j)} - \frac{1}{I}\sum_{k=1}^{I}\hat{\Omega}_i^{(k)}\right)^2}$$

and

$$\hat{b}_i = \frac{\left(\frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{(j)}\right)^2}{\frac{1}{I-1}\sum_{j=1}^{I}\left(\hat{\Omega}_i^{(j)} - \frac{1}{I}\sum_{k=1}^{I}\hat{\Omega}_i^{(k)}\right)^2}.$$

For further discussion of the gamma prior, its hyperparameters, and the posterior distribution, refer to [54–57].

Using the likelihood function (3.1) and the joint prior (4.1), the joint posterior density function is created as mentioned below:

$$\begin{aligned}\Pi^{\bullet}(\Omega\,|data) \propto\, & \upsilon_1^{m_1+a_1-1}\upsilon_2^{m_2+a_2-1}\upsilon_3^{m_3+a_3-1}\alpha^{m_1+m_2+m_3+a_4-1}e^{-(b_1\upsilon_1+b_2\upsilon_2+b_3\upsilon_3+b_4\alpha)}\\ & \prod_{j_1=1}^{m_1}(1+w_{j_1})^{-\alpha-1}\left[B_{j_1}(\alpha)\right]^{\upsilon_1-1}\prod_{j_1=1}^{K_1}\left[B_{j_1}(\alpha)\right]^{\upsilon_1 R_1 j_1}\left[B_{m_1}(\alpha)\right]^{R_1 m_1}\\ & \prod_{j_2=1}^{m_2}(1+q_{j_2})^{-\alpha-1}\left[B_{j_2}(\alpha)\right]^{\upsilon_2-1}\prod_{j_2=1}^{K_2}\left[B_{j_2}(\alpha)\right]^{\upsilon_2 R_2 j_2}\left[B_{m_2}(\alpha)\right]^{R_2 m_2}\\ & \prod_{j_3=1}^{m_3}(1+y_{j_3})^{-\alpha-1}\left[B_{j_3}(\alpha)\right]^{\upsilon_3-1}\prod_{j_3=1}^{K_3}\left[B_{j_3}(\alpha)\right]^{\upsilon_3 R_3 j_3}\left[B_{m_3}(\alpha)\right]^{R_3 m_3}.\end{aligned} \tag{4.2}$$

Based on Eq (4.2), the conditional posterior of the parameters υ_1, υ_2, and υ_3 are derived as follows:

$$\upsilon_1\,|\alpha, data \sim Gamma\left(m_1+a_1,\, b_1+\sum_{j_1=1}^{m_1}\ln(B_{j_1}(\alpha))+\sum_{j_1=1}^{K_1}R_1 j_1\ln(B_{j_1}(\alpha))\right), \tag{4.3}$$

$$\upsilon_2\,|\alpha, data \sim Gamma\left(m_2+a_2,\, b_2+\sum_{j_2=1}^{m_2}\ln(B_{j_2}(\alpha))+\sum_{j_2=1}^{K_2}R_2 j_2\ln(B_{j_2}(\alpha))\right), \tag{4.4}$$

$$\upsilon_3\,|\alpha, data \sim Gamma\left(m_3+a_3,\, b_3+\sum_{j_3=1}^{m_3}\ln(B_{j_3}(\alpha))+\sum_{j_3=1}^{K_3}R_3 j_3\ln(B_{j_3}(\alpha))\right), \tag{4.5}$$

$$\begin{aligned}\Pi_4^{\bullet}(\alpha\,|\upsilon_1,\upsilon_2,\upsilon_3, data) \propto\, & \alpha^{m_1+m_2+m_3+a_4-1}e^{-b_4\alpha}\prod_{j_1=1}^{m_1}(1+w_{j_1})^{-\alpha-1}\left[B_{j_1}(\alpha)\right]^{\upsilon_1-1}\\ & \prod_{j_1=1}^{K_1}\left[B_{j_1}(\alpha)\right]^{\upsilon_1 R_1 j_1}\left[B_{m_1}(\alpha)\right]^{R_1 m_1}\prod_{j_2=1}^{m_2}(1+q_{j_2})^{-\alpha-1}\left[B_{j_2}(\alpha)\right]^{\upsilon_2-1}\\ & \prod_{j_2=1}^{K_2}\left[B_{j_2}(\alpha)\right]^{\upsilon_2 R_2 j_2}\left[B_{m_2}(\alpha)\right]^{R_2 m_2}\prod_{j_3=1}^{m_3}(1+y_{j_3})^{-\alpha-1}\left[B_{j_3}(\alpha)\right]^{\upsilon_3-1}\\ & \prod_{j_3=1}^{K_3}\left[B_{j_3}(\alpha)\right]^{\upsilon_3 R_3 j_3}\left[B_{m_3}(\alpha)\right]^{R_3 m_3}.\end{aligned} \tag{4.6}$$

It is evident that the posterior PDFs of υ_1, υ_2, and υ_3 as provided in Eqs (4.3)–(4.5) possess gamma distributions. Hence, we employ the Gibbs sampling method to directly draw samples from these conditionals. The algorithm is described below (Algorithm 1):

Algorithm 1 Gibbs sampling for $\upsilon_1, \upsilon_2, \upsilon_3$.

Input: Initial values $\upsilon_1^{(0)}, \upsilon_2^{(0)}, \upsilon_3^{(0)}$, fixed value of α, number of iterations M
for $t = 1$ to M **do**
 Compute:

$$\upsilon_1^{(t)} \sim \text{Gamma}\left(m_1 + a_1,\ b_1 + \sum_{j=1}^{m_1} \ln(B_{j1}(\alpha)) + \sum_{j=1}^{K_1} R_{1j} \ln(B_{j1}(\alpha))\right)$$

$$\upsilon_2^{(t)} \sim \text{Gamma}\left(m_2 + a_2,\ b_2 + \sum_{j=1}^{m_2} \ln(B_{j2}(\alpha)) + \sum_{j=1}^{K_2} R_{2j} \ln(B_{j2}(\alpha))\right)$$

$$\upsilon_3^{(t)} \sim \text{Gamma}\left(m_3 + a_3,\ b_3 + \sum_{j=1}^{m_3} \ln(B_{j3}(\alpha)) + \sum_{j=1}^{K_3} R_{3j} \ln(B_{j3}(\alpha))\right)$$

 Where $B_{jh}(\alpha) = 1 - (1 + z_{jh})^{-\alpha}$, and z_{jh} are the observed values from the corresponding sample (w_j, q_j, or y_j)
end for
Output: Posterior samples $\left\{\upsilon_1^{(t)}, \upsilon_2^{(t)}, \upsilon_3^{(t)}\right\}_{t=1}^{M}$

Unlike the parameters υ_1, υ_2, and υ_3, the conditional posterior distribution of the parameter α (see Eq (4.6)) does not follow a standard form. Therefore, we use the Metropolis–Hastings (M-H) algorithm to sample from the posterior distribution of α within the Gibbs sampling framework. The procedure is detailed in Algorithm 2.

Algorithm 2 Metropolis–Hastings sampling for parameter α.

Input: Initial value $\alpha^{(0)}$, current values of $\upsilon_1, \upsilon_2, \upsilon_3$, number of iterations M, proposal variance σ^2
for $t = 1$ to M **do**
 Propose candidate $\alpha^* \sim \mathcal{N}(\alpha^{(t-1)}, \sigma^2)$
 Compute posterior density ratio:

$$r = \frac{\pi(\alpha^* \mid \upsilon_1, \upsilon_2, \upsilon_3, \text{data})}{\pi(\alpha^{(t-1)} \mid \upsilon_1, \upsilon_2, \upsilon_3, \text{data})}$$

 Compute acceptance probability: $A = \min(1, r)$
 Generate $u \sim \text{Uniform}(0, 1)$
 if $u < A$ **then**
  Accept: $\alpha^{(t)} = \alpha^*$
 else
  Reject: $\alpha^{(t)} = \alpha^{(t-1)}$
 end if
end for
Output: Posterior samples $\{\alpha^{(t)}\}_{t=1}^{M}$

4.1. Bayesian point estimation under different loss functions

Let $\{\theta^{(1)}, \ldots, \theta^{(M)}\}$ denote the posterior samples of a generic parameter $\theta \in \{\alpha, \upsilon_1, \upsilon_2, \upsilon_3, \vartheta\}$ obtained from the Markov chain Monte Carlo (MCMC) procedure. The Bayesian estimates of each parameter under different loss functions are computed as follows:

- **Under SELOF**: The Bayes estimate is the posterior mean:

$$\hat{\theta}_{\text{SELOF}} = \frac{1}{M}\sum_{i=1}^{M}\theta^{(i)}.$$

- **Under LIEXLOF**: For a constant $a \neq 0$, the Bayes estimate is given by:

$$\hat{\theta}_{\text{LIEXLOF}} = -\frac{1}{a}\log\left(\frac{1}{M}\sum_{i=1}^{M}e^{-a\theta^{(i)}}\right).$$

  This form allows asymmetric penalization of overestimation and underestimation, making it suitable for decision-making applications.

These estimators are applied to all parameters: α, υ_1, υ_2, υ_3, and ϑ, based on their respective posterior samples from the MCMC simulation.

4.2. Highest posterior density credible intervals

To construct the $100(1-\delta)\%$ HPD credible interval for any parameter $\theta \in \{\alpha, \upsilon_1, \upsilon_2, \upsilon_3, \vartheta\}$, we proceed as follows:

(1) Sort the M posterior samples $\{\theta^{(i)}\}_{i=1}^{M}$ in ascending order.

(2) For all possible intervals of the form $[\theta^{(j)}, \theta^{(j+m)}]$, where $m = \lfloor M(1-\delta)\rfloor$, compute their widths.

(3) Select the interval with the minimum width; it corresponds to the HPD region:

$$\text{HPD}_{(1-\delta)}(\theta) = \left[\theta^{(j^*)}, \theta^{(j^*+m)}\right].$$

This method ensures that the HPD interval contains the highest density region of the posterior distribution for each parameter, offering more efficient and interpretable uncertainty quantification, especially when the posterior distribution is skewed.

5. Simulation

To study the behavior of the proposed estimators for the model parameter and the corresponding SS reliability of the EPD under the APTII-HCS, a series of Monte Carlo simulations are performed. Four different true values of the EPD parameters, $(\alpha = 2, \upsilon_1 = 0.7, \upsilon_2 = 0.4, \upsilon_3 = 5)$, $(\alpha = 5, \upsilon_1 = 1.3, \upsilon_2 = 1.2, \upsilon_3 = 3)$, $(\alpha = 0.8, \upsilon_1 = 1.6, \upsilon_2 = 0.3, \upsilon_3 = 6)$, and $(\alpha = 5, \upsilon_1 = 3, \upsilon_2 = 1.2, \upsilon_3 = 10)$, are considered. Following the algorithm outlined in [12], 1000 APTII-HCS samples are generated for each variable from EPD(α, υ_1), EPD(α, υ_2), and EPD(α, υ_3), using various combinations of N (the total number of test units for the SS model): 30, 105, and 150, n_i; $i = 1, 2, 3$ (the total number of test units for each SS variable), and m_i (the effective censored sample size for each SS variable).

For each scenario, the ML estimates (MLEs) and Bayesian MCMC estimates of the unknown parameters of the SS model, namely, α, υ_1, υ_2, υ_3, as well as the SS reliability ϑ defined in Eq (2.5), are calculated. Approximate Bayesian estimates are obtained under both symmetric and asymmetric loss functions using the M-H algorithm. To generate APTII-HCS samples from the EPD based on the SS model, follow these steps:

(1) Generate an ordinary PTII sample (X_i, R_i) for $i = 1, 2, \dots, m$ using the algorithm described in [58]:

(a) Generate H independent observations of size m, denoted by $H_1, H_2, \dots, H_m$.

(b) Given the values of n, m, T, and R_i for $i = 1, 2, \dots, m$, compute

$$V_i = \left(H_i + \sum_{j=m-i+1}^{m} R_j\right)^{-1}, \quad i = 1, 2, \dots, m.$$

(c) Define

$$U_i = 1 - V_m V_{m-1} \cdots V_{m-i+1}, \quad i = 1, 2, \dots, m,$$

where $\{U_i\}$ represents a PTII sample of size m from the $U(0, 1)$ distribution.

(d) Invert CDF (2.2) for a given value of d to obtain

$$W_i = F^{-1}(U_i; d), \quad i = 1, 2, \dots, m,$$

thus generating the PTII sample from the EPD(α, υ_1) distribution.

(2) Determine the d-th failure such that $W_d < T_1 < W_{d+1}$, and discard the remaining observations $W_{d+2}, \dots, W_m$.

(3) Generate the first $m - d - 1$ order statistics from the truncated distribution with density

$$\frac{f(w)}{1 - F(w_{d+1})}$$

and sample size $n - d - \sum_{j=1}^{d} R_j - 1$ to obtain $W_{d+2}, \dots, W_m$.

(4) Repeat these steps for each varaible of SS model EPD(α, υ_2) with T_2, and EPD(α, υ_3) with T_3.

This numerical comparison is conducted based on different combinations of (n_i, m_i, T_i), specifically with $n_i = 10$, $n_i = 35$, and $n_i = 50$ for each predetermined time by $Q = 0.7$ and 0.9 as $T_i = \left(1 - Q^{\frac{1}{\upsilon_i}}\right)^{\frac{-1}{\alpha}} - 1;\ i = 1, 2, 3$. The test is terminated once the number of failed units reaches or exceeds a specified value m_i, where the failure information is set as 7, 9, 27, 32, 38, and 45.

To perform the Bayesian computations, 12,000 MCMC samples were generated, with the first 2000 iterations discarded as burn-in to eliminate the effect of the initial value selection. Subsequently, the average BEs of the unknown parameters α, υ_1, υ_2, υ_3, as well as the SS reliability ϑ, were computed based on the SELOF using the remaining 10,000 MCMC samples. The initial values of α, υ_1, υ_2, and υ_3 for running the MCMC sampler were set to their corresponding ML estimates (MLEs). Moreover, for each combination of n and m, different censoring schemes (CSs), denoted by R, were employed to remove surviving units during the lifetime experiment, as follows:

- **CS-1**: $R_{m_i} = n_i - m_i$ and $R_j = 0$ for all $j \neq m_i$.
- **CS-2**: $R_1 = \lfloor \frac{n_i - m_i}{2} \rfloor$, $R_j = 0$ for all $j = [2, m_i - 1]$, and $R_{m_i} = n_i - m_i - \lfloor \frac{n_i - m_i}{2} \rfloor$.

For each test, the average point estimates, along with their corresponding mean squared errors (MSEs), bias, and the average confidence widths (ACWs) of the interval estimates, are computed as follows:

$$\mathrm{MSE}(\hat{\theta}_r) = \frac{1}{G}\sum_{i=1}^{G}\left(\hat{\theta}_r^{(i)} - \theta_r\right)^2, \quad r = 1,2,3,4,5, \mathrm{Bias}(\hat{\theta}_r) = \frac{1}{G}\sum_{i=1}^{G}\left(\hat{\theta}_r^{(i)} - \theta_r\right), \quad r = 1,2,3,4,5,$$

$$\mathrm{ACW}(\hat{\theta}_r) = \frac{1}{G}\sum_{i=1}^{G}\left(\hat{\theta}_r^{(U_i)} - \hat{\theta}_r^{(L_i)}\right), \quad r = 1,2,3,4,5,$$

where G is the number of simulation replicates and $\hat{\theta}$ refers to the MLE or Bayesian estimate (BE) of the parametric function θ. The quantities $(\hat{\theta}_r^{(L)}, \hat{\theta}_r^{(U)})$ represent the lower and upper bounds of the 95% confidence interval.

The average MLEs and BEs obtained for α, υ_1, υ_2, υ_3, as well as the SS reliability ϑ, along with their corresponding MSEs and biases, are computed and presented in Tables 1–4. In each table, the average estimate, MSE, and bias for each unknown parameter under each testing scenario are organized. Additionally, the ACWs of the 95% AS-CI and the ACWs of credible intervals (CCI), denoted as LCCI under a symmetric loss function, while LCCI 2 and LCCI 3 under asymmetric loss functions, along with the coverage probability (CP) for α, υ_1, υ_2, υ_3, as well as the SS reliability ϑ, are evaluated and summarized in Tables 1–4. All numerical analyses were conducted using R software version 4.3.0, utilizing three main packages: "CODA" for Bayesian inference, "maxLik" for obtaining MLEs via the Newton-Raphson method, and "GoFKerne" for inverting the EPD. These packages have been recently recommended in the literature.

When losses are a symmetric the posterior mean (BE-SELOF) is preferable, whereas when over- or underestimation carries unequal costs, the LINEX-based Bayes estimators (BE-LIEXLOF1: $a = 0.5$, BE-LIEXLOF2: $a = 1.5$) achieve lower expected LINEX loss, typically at the expense of a small increase in MSE. A summary of findings are now presented from the simulation study (based on Tables 1–4):

- The proposed estimates for the parameters of the EPD and the SS measure $\vartheta = P[Q < W < Y]$ demonstrated satisfactory performance in terms of bias, MSE, and average ACW.
- As the sample size n_i or the effective sample size m_i increased:
  - The bias and MSE of the estimators decreased.
  - The widths of the AS-CIs (LAS-CIs) and HPD credible intervals (LCCIs) also became narrower, indicating improved estimation precision.
- BEs outperformed classical MLEs in terms of both lower MSE and bias, particularly when informative gamma priors were employed.
- The MCMC implementation using the M-H algorithm provided superior results, especially under the asymmetric loss function (LIEXLOF), compared to the symmetric loss function.
- The HPD credible intervals were consistently narrower than the corresponding AS-CIs, offering more efficient interval estimation.
- Among the Bayesian approaches, the LIEXLOF yielded more conservative and accurate point estimates than the SELOF.
- These findings confirm the robustness and applicability of the proposed APTII-HCS in multi-SS reliability modeling.

Table 1. MLE and BE with different symmetric and asymmetric loss functions for $\alpha = 2, \upsilon_1 = 0.7, \upsilon_2 = 0.4, \upsilon_3 = 5$.

					MLE		BE SELOF		BE LIEXLOF1		BE LIEXLOF2		AS-CI			HPD	
	n_i	Q	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LAS-CI	CP	LCCI	LCCI 2	LCCI 3
1	10	0.7	7	α	0.2339	0.3562	0.0024	0.0213	-0.0043	0.0213	-0.0079	0.0213	2.1535	95.4%	0.5811	0.5854	0.5802
				υ_1	0.1404	0.1690	0.0033	0.0167	0.0051	0.0166	-0.0034	0.0164	1.5151	95.3%	0.5044	0.5026	0.4987
				υ_2	0.0672	0.0508	0.0092	0.0089	0.0074	0.0087	0.0039	0.0084	0.8437	95.4%	0.3486	0.3489	0.3416
				υ_3	1.5257	12.5818	0.0073	0.0245	-0.0093	0.0244	-0.0061	0.0244	12.5588	95.3%	0.5949	0.5958	0.5887
				ϑ	-0.0086	0.0088	-0.0048	0.0029	-0.0043	0.0029	-0.0035	0.0029	0.3666	95.4%	0.2094	0.2085	0.2068
		0.9	9	α	0.2232	0.3015	-0.0022	0.0124	-0.0038	0.0124	-0.0071	0.0125	1.9675	94.8%	0.4352	0.4345	0.4365
				υ_1	0.1072	0.1102	-0.0032	0.0105	-0.0046	0.0105	-0.0032	0.0105	1.2319	94.8%	0.3824	0.3812	0.3823
				υ_2	0.0668	0.0392	0.0051	0.0066	0.0040	0.0066	0.0017	0.0064	0.7299	94.8%	0.3081	0.3060	0.3028
				υ_3	1.5088	11.5337	-0.0067	0.0125	-0.0084	0.0126	-0.0031	0.0127	11.7740	94.8%	0.4363	0.4357	0.4356
				ϑ	-0.0081	0.0084	-0.0032	0.0023	-0.0029	0.0023	-0.0023	0.0023	0.3589	94.8%	0.1848	0.1860	0.1867
	35	0.7	27	α	0.0645	0.0733	0.0045	0.0132	0.0025	0.0132	-0.0017	0.0131	1.0313	95.7%	0.4486	0.4468	0.4483
				υ_1	0.0304	0.0227	0.0012	0.0080	-0.0005	0.0079	-0.0038	0.0078	0.5791	95.0%	0.3308	0.3301	0.3282
				υ_2	0.0160	0.0069	0.0103	0.0029	0.0094	0.0028	0.0075	0.0027	0.3194	95.7%	0.1934	0.1928	0.1901
				υ_3	0.4879	2.1688	0.0069	0.0227	0.0040	0.0226	-0.0015	0.0225	5.4496	95.0%	0.5645	0.5660	0.5560
				ϑ	-0.0006	0.0027	-0.0062	0.0011	-0.0061	0.0011	-0.0058	0.0011	0.2056	95.7%	0.1212	0.1210	0.1199
		0.9	32	α	0.0489	0.0644	-0.0015	0.0093	-0.0025	0.0093	-0.0015	0.0093	0.9762	94.3%	0.3730	0.3729	0.3712
				υ_1	0.0304	0.0196	0.0010	0.0064	-0.0002	0.0063	-0.0025	0.0063	0.5335	96.0%	0.3051	0.3043	0.3016
				υ_2	0.0156	0.0064	0.0096	0.0026	0.0089	0.0026	0.0075	0.0025	0.3074	94.3%	0.1806	0.1799	0.1778
				υ_3	0.3804	1.7536	-0.0038	0.0114	-0.0035	0.0138	-0.0011	0.0138	4.9747	96.0%	0.4398	0.4392	0.4386
				ϑ	-0.0006	0.0025	-0.0057	0.0010	-0.0056	0.0010	-0.0053	0.0010	0.1970	94.3%	0.1168	0.1164	0.1156
	50	0.7	38	α	0.0454	0.0515	0.0033	0.0127	0.0013	0.0127	-0.0028	0.0126	0.8717	95.3%	0.4268	0.4276	0.4265
				υ_1	0.0260	0.0176	0.0076	0.0070	0.0061	0.0069	0.0031	0.0067	0.5104	95.6%	0.3144	0.3130	0.3087
				υ_2	0.0111	0.0048	0.0096	0.0021	0.0088	0.0021	0.0073	0.0020	0.2672	95.3%	0.1618	0.1614	0.1593
				υ_3	0.3050	1.3941	-0.0006	0.0222	-0.0033	0.0222	-0.0087	0.0222	4.4735	95.6%	0.5645	0.5651	0.5640
				ϑ	-0.0006	0.0020	-0.0048	0.0008	-0.0047	0.0008	-0.0045	0.0008	0.1770	95.3%	0.1073	0.1074	0.1077
		0.9	45	α	0.0367	0.0496	0.0009	0.0086	-0.0004	0.0086	-0.0030	0.0087	0.8613	95.4%	0.3475	0.3480	0.3501
				υ_1	0.0196	0.0156	0.0017	0.0052	0.0007	0.0052	-0.0014	0.0051	0.4832	96.0%	0.2719	0.2715	0.2702
				υ_2	0.0107	0.0041	0.0065	0.0018	0.0059	0.0018	0.0047	0.0017	0.2474	95.4%	0.1566	0.1560	0.1546
				υ_3	0.2824	1.3416	-0.0005	0.0104	-0.0035	0.0137	-0.0070	0.0137	4.4056	96.0%	0.4553	0.4573	0.4592
				ϑ	-0.0005	0.0019	-0.0039	0.0007	-0.0038	0.0007	-0.0036	0.0007	0.1698	95.4%	0.1054	0.1055	0.1052
2	10	0.7	7	α	0.2938	0.4374	-0.0032	0.0213	-0.0046	0.0213	-0.0102	0.0214	2.3237	95.5%	0.5594	0.5545	0.5525
				υ_1	0.1327	0.1383	0.0071	0.0161	-0.0016	0.0160	-0.0062	0.0158	1.3625	95.8%	0.4824	0.4817	0.4779
				υ_2	0.0690	0.0449	0.0064	0.0078	0.0048	0.0077	0.0016	0.0075	0.7854	95.5%	0.3332	0.3324	0.3298
				υ_3	1.8894	14.5921	0.0025	0.0228	-0.0034	0.0227	-0.0060	0.0227	13.0208	95.8%	0.5907	0.5858	0.5850
				ϑ	-0.0020	0.0085	-0.0040	0.0028	-0.0037	0.0028	-0.0029	0.0028	0.3623	95.5%	0.2036	0.2029	0.2027
		0.9	9	α	0.2232	0.2920	-0.0029	0.0111	-0.0044	0.0111	-0.0075	0.0111	1.9302	94.4%	0.4045	0.4050	0.4052
				υ_1	0.1241	0.1396	-0.0034	0.0105	-0.0015	0.0105	-0.0058	0.0105	1.3820	96.2%	0.4024	0.4023	0.3999
				υ_2	0.0603	0.0373	0.0043	0.0059	0.0032	0.0058	0.0010	0.0057	0.7195	94.4%	0.2849	0.2831	0.2793
				υ_3	1.4694	9.5364	0.0023	0.0137	0.0024	0.0136	0.0006	0.0136	10.6524	96.2%	0.4400	0.4399	0.4431
				ϑ	-0.0019	0.0083	-0.0030	0.0021	-0.0027	0.0021	-0.0021	0.0021	0.3571	94.4%	0.1754	0.1752	0.1753
	35	0.7	27	α	0.0595	0.0763	0.0024	0.0141	0.0006	0.0140	-0.0041	0.0140	1.0576	94.8%	0.4672	0.4666	0.4678
				υ_1	0.0276	0.0196	0.0062	0.0078	0.0045	0.0077	0.0012	0.0076	0.5382	94.6%	0.3304	0.3279	0.3263
				υ_2	0.0164	0.0070	0.0116	0.0027	0.0107	0.0027	0.0088	0.0026	0.3216	94.8%	0.1917	0.1898	0.1864
				υ_3	0.4338	2.1660	-0.0041	0.0220	-0.0069	0.0221	-0.0124	0.0224	5.5157	94.6%	0.5563	0.5534	0.5488
				ϑ	-0.0018	0.0026	-0.0062	0.0010	-0.0060	0.0010	-0.0057	0.0010	0.1993	94.8%	0.1159	0.1152	0.1145
		0.9	32	α	0.0458	0.0636	0.0021	0.0100	0.0002	0.0100	-0.0020	0.0100	0.9730	96.0%	0.3951	0.3947	0.3927
				ϑ	0.0277	0.0199	0.0010	0.0062	-0.0005	0.0062	-0.0022	0.0061	0.5420	94.7%	0.2953	0.2945	0.2943
				υ_2	0.0164	0.0067	0.0097	0.0025	0.0090	0.0025	0.0075	0.0024	0.3138	96.0%	0.1865	0.1862	0.1840
				υ_3	0.3698	1.8511	-0.0010	0.0136	-0.0027	0.0136	-0.0061	0.0137	5.1352	94.7%	0.4461	0.4457	0.4473
				ϑ	-0.0016	0.0025	-0.0058	0.0010	-0.0056	0.0010	-0.0053	0.0010	0.1943	96.0%	0.1109	0.1109	0.1182
	50	0.7	38	α	0.0497	0.0556	0.0023	0.0126	0.0004	0.0125	-0.0036	0.0125	0.9038	95.2%	0.4371	0.4375	0.4361
				υ_1	0.0188	0.0136	0.0054	0.0053	0.0040	0.0053	0.0011	0.0052	0.4521	95.8%	0.2771	0.2761	0.2751
				υ_2	0.0135	0.0045	0.0104	0.0017	0.0097	0.0016	0.0083	0.0016	0.2572	95.2%	0.1468	0.1462	0.1449
				υ_3	0.2917	1.4409	-0.0008	0.0211	-0.0036	0.0211	-0.0092	0.0212	4.5667	95.8%	0.5615	0.5622	0.5651
				ϑ	-0.0029	0.0018	-0.0055	0.0006	-0.0054	0.0006	-0.0053	0.0006	0.1655	95.2%	0.0953	0.0952	0.0955
		0.9	45	α	0.0357	0.0448	0.0020	0.0084	0.0003	0.0084	-0.0002	0.0084	0.8178	95.1%	0.3603	0.3602	0.3603
				υ_1	0.0180	0.0130	0.0042	0.0052	0.0004	0.0052	0.0010	0.0051	0.4422	94.0%	0.2689	0.2687	0.2687
				υ_2	0.0098	0.0038	0.0068	0.0017	0.0062	0.0017	0.0050	0.0015	0.2393	95.1%	0.1404	0.1450	0.1439
				υ_3	0.2892	1.2644	0.0007	0.0126	0.0017	0.0125	-0.0016	0.0125	4.2618	94.0%	0.4276	0.4274	0.4245
				ϑ	-0.0009	0.0018	-0.0029	0.0005	-0.0028	0.0005	-0.0025	0.0005	0.1641	95.1%	0.0948	0.0948	0.0947

Table 2. MLE and BE with different symmetric and asymmetric loss functions for $\alpha = 5, \upsilon_1 = 1.3, \upsilon_2 = 1.2, \upsilon_3 = 3$.

					MLE		BE SELOF		BE LIEXLOF1		BE LIEXLOF2		AS-CI		HPD		
	n_i	Q	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LAS-CI	CP	LCCI	LCCI 2	LCCI 3
1	10	0.7	7	α	1.0016	3.9843	0.0095	0.0243	-0.0083	0.0243	-0.0108	0.0243	6.7715	95.3%	0.5962	0.5954	0.6007
				υ_1	0.3914	0.9154	-0.0025	0.0215	-0.0131	0.0215	-0.0131	0.0215	3.4240	95.5%	0.5555	0.5576	0.5551
				υ_2	0.3671	0.8901	-0.0072	0.0205	-0.0198	0.0205	-0.0151	0.0205	3.4086	95.3%	0.5507	0.5535	0.5491
				υ_3	1.6086	11.1417	-0.0025	0.0231	-0.0054	0.0232	-0.0113	0.0233	11.4706	95.5%	0.5801	0.5825	0.5809
				ϑ	0.0063	0.0085	0.0019	0.0007	0.0025	0.0007	0.0018	0.0007	0.3598	95.3%	0.1005	0.1011	0.1018
		0.9	9	α	0.6900	2.2708	-0.0054	0.0132	-0.0070	0.0133	-0.0103	0.0133	5.2542	95.4%	0.4446	0.4448	0.4455
				υ_1	0.3495	0.8330	-0.0018	0.0134	-0.0098	0.0134	-0.0131	0.0135	3.3067	95.0%	0.4571	0.4579	0.4581
				υ_2	0.3480	0.6354	-0.0062	0.0127	-0.0112	0.0127	-0.0144	0.0128	2.8126	95.4%	0.4316	0.4320	0.4353
				υ_3	1.3216	9.2162	0.0019	0.0142	0.0023	0.0141	-0.0014	0.0141	10.7190	95.0%	0.4576	0.4577	0.4569
				ϑ	-0.0014	0.0072	0.0014	0.0004	0.0016	0.0004	0.0018	0.0004	0.3330	95.4%	0.0802	0.0800	0.0807
	35	0.7	27	α	0.2113	0.4951	-0.0027	0.0218	-0.0055	0.0218	-0.0111	0.0219	2.6324	95.1%	0.5740	0.5745	0.5721
				υ_1	0.0945	0.1015	-0.0021	0.0172	-0.0037	0.0171	-0.0085	0.0170	1.1934	95.9%	0.5183	0.5185	0.5158
				υ_2	0.0784	0.0891	0.0013	0.0158	-0.0010	0.0157	-0.0056	0.0156	1.1295	95.1%	0.4814	0.4833	0.4855
				υ_3	0.2747	0.7817	0.0008	0.0214	-0.0021	0.0214	-0.0078	0.0214	3.2960	95.9%	0.5588	0.5573	0.5560
				ϑ	0.0009	0.0022	-0.0002	0.0005	0.0013	0.0005	0.0003	0.0005	0.1836	95.1%	0.0881	0.0884	0.0887
		0.9	32	α	0.1870	0.4179	-0.0008	0.0130	-0.0025	0.0130	-0.0058	0.0130	2.4271	95.4%	0.4528	0.4557	0.4593
				υ_1	0.0810	0.0970	-0.0020	0.0103	-0.0035	0.0103	-0.0069	0.0103	1.1515	95.7%	0.3868	0.3873	0.3881
				υ_2	0.0658	0.0717	-0.0011	0.0106	-0.0008	0.0107	-0.0055	0.0107	1.0181	95.4%	0.3989	0.3980	0.4001
				υ_3	0.2545	0.6724	0.0007	0.0128	-0.0007	0.0128	-0.0042	0.0128	3.0571	95.7%	0.4292	0.4278	0.4272
				ϑ	0.0009	0.0019	0.0001	0.0004	0.0012	0.0004	0.0003	0.0004	0.1722	95.4%	0.0712	0.0714	0.0720
	50	0.7	38	α	0.1433	0.3392	0.0015	0.0215	-0.0012	0.0215	-0.0066	0.0216	2.2139	95.4%	0.5730	0.5666	0.5671
				υ_1	0.0551	0.0644	-0.0034	0.0154	-0.0056	0.0153	-0.0100	0.0152	0.9714	94.8%	0.4721	0.4729	0.4696
				υ_2	0.0650	0.0557	0.0043	0.0138	0.0022	0.0138	-0.0021	0.0137	0.8841	95.4%	0.4672	0.4673	0.4635
				υ_3	0.1906	0.5308	-0.0009	0.0219	-0.0037	0.0220	-0.0094	0.0222	2.7580	94.8%	0.5773	0.5785	0.5712
				ϑ	-0.0007	0.0015	-0.0009	0.0004	-0.0008	0.0004	-0.0005	0.0004	0.1527	95.4%	0.0822	0.0824	0.0830
		0.9	45	α	0.1187	0.2628	0.0014	0.0117	0.0011	0.0116	-0.0008	0.0116	1.9558	95.8%	0.4119	0.4111	0.4118
				υ_1	0.0524	0.0542	0.0016	0.0102	0.0002	0.0101	-0.0027	0.0101	0.8794	94.1%	0.3935	0.3947	0.3973
				υ_2	0.0462	0.0463	-0.0004	0.0095	-0.0017	0.0095	-0.0019	0.0095	0.8246	95.8%	0.3720	0.3716	0.3741
				υ_3	0.1717	0.4224	0.0008	0.0130	-0.0009	0.0129	-0.0019	0.0129	2.4585	94.1%	0.4443	0.4431	0.4397
				ϑ	0.0006	0.0015	0.0003	0.0003	0.0004	0.0003	0.0005	0.0003	0.1506	95.8%	0.0700	0.0704	0.0709
2	10	0.7	7	α	0.8850	3.3489	-0.0039	0.0224	-0.0068	0.0225	-0.0124	0.0226	6.2820	96.3%	0.5930	0.5901	0.5854
				υ_1	0.4007	0.9880	-0.0061	0.0210	-0.0088	0.0210	-0.0143	0.0211	3.5675	96.4%	0.5503	0.5537	0.5535
				υ_2	0.3274	1.0546	-0.0051	0.0196	-0.0070	0.0196	-0.0090	0.0195	3.8174	96.3%	0.5273	0.5292	0.5325
				υ_3	1.5507	12.7813	-0.0067	0.0241	-0.0099	0.0242	-0.0161	0.0244	12.6337	96.4%	0.6107	0.6116	0.6235
				ϑ	0.0106	0.0084	-0.0012	0.0007	-0.0051	0.0007	0.0013	0.0007	0.3567	96.3%	0.1002	0.1001	0.1021
		0.9	9	α	0.6884	2.4394	0.0008	0.0140	-0.0010	0.0140	-0.0044	0.0140	5.4985	95.1%	0.4634	0.4621	0.4633
				υ_1	0.3388	0.8190	-0.0043	0.0123	-0.0060	0.0123	-0.0093	0.0124	3.2912	95.9%	0.4302	0.4315	0.4308
				υ_2	0.2959	0.5223	-0.0048	0.0132	-0.0064	0.0132	-0.0088	0.0133	2.5861	95.1%	0.4523	0.4537	0.4545
				υ_3	1.2805	8.5130	0.0018	0.0147	0.0021	0.0146	-0.0034	0.0146	10.2822	95.9%	0.4786	0.4758	0.4757
				ϑ	0.0033	0.0070	0.0009	0.0004	0.0010	0.0004	0.0012	0.0004	0.3277	95.1%	0.0835	0.0839	0.0844
	35	0.7	27	α	0.2541	0.5471	-0.0041	0.0216	-0.0070	0.0216	-0.0127	0.0216	2.7244	94.5%	0.5728	0.5691	0.5669
				υ_1	0.0899	0.0977	-0.0019	0.0161	-0.0027	0.0161	-0.0074	0.0161	1.1744	95.7%	0.4918	0.4918	0.4902
				υ_2	0.0763	0.0815	0.0071	0.0167	-0.0086	0.0166	-0.0054	0.0166	1.0792	94.5%	0.4972	0.4954	0.4930
				υ_3	0.3449	0.8857	-0.0018	0.0241	-0.0048	0.0241	-0.0108	0.0242	3.4342	95.7%	0.5937	0.5932	0.5894
				ϑ	0.0035	0.0021	-0.0014	0.0005	-0.0018	0.0005	0.0002	0.0006	0.1789	94.5%	0.0897	0.0899	0.0901
		0.9	32	α	0.1891	0.4283	0.0019	0.0142	0.0024	0.0142	-0.0031	0.0142	2.4573	94.5%	0.4722	0.4727	0.4713
				υ_1	0.0801	0.0906	0.0018	0.0111	0.0004	0.0111	-0.0025	0.0110	1.1378	95.3%	0.4191	0.4194	0.4181
				υ_2	0.0724	0.0739	-0.0068	0.0102	-0.0082	0.0102	-0.0041	0.0102	1.0277	94.5%	0.3863	0.3848	0.3834
				υ_3	0.2652	0.6999	0.0013	0.0136	0.0011	0.0136	-0.0023	0.0135	3.1118	95.3%	0.4558	0.4565	0.4551
				ϑ	0.0009	0.0019	0.0011	0.0003	0.0015	0.0003	0.0002	0.0003	0.1714	94.5%	0.0705	0.0707	0.0710
	50	0.7	38	α	0.1470	0.3424	0.0046	0.0221	0.0018	0.0221	-0.0038	0.0220	2.2212	95.4%	0.5784	0.5789	0.5762
				υ_1	0.0518	0.0572	0.0036	0.0141	0.0013	0.0141	-0.0032	0.0140	0.9155	96.0%	0.4595	0.4588	0.4600
				υ_2	0.0598	0.0512	-0.0024	0.0118	-0.0044	0.0118	-0.0083	0.0119	0.8563	95.4%	0.4073	0.4089	0.4097
				υ_3	0.2160	0.5451	-0.0048	0.0221	-0.0076	0.0221	-0.0132	0.0223	2.7688	96.0%	0.5760	0.5730	0.5743
				ϑ	-0.0001	0.0014	0.0004	0.0004	0.0004	0.0004	0.0006	0.0004	0.1487	95.4%	0.0767	0.0771	0.0776
		0.9	45	α	0.1008	0.2848	-0.0013	0.0126	-0.0017	0.0126	-0.0036	0.0126	2.0554	95.4%	0.4375	0.4365	0.4338
				υ_1	0.0512	0.0575	-0.0030	0.0103	-0.0011	0.0103	-0.0027	0.0103	0.9190	96.0%	0.3944	0.3949	0.3960
				υ_2	0.0413	0.0488	-0.0016	0.0094	-0.0037	0.0094	-0.0071	0.0094	0.8510	95.4%	0.3712	0.3721	0.3689
				υ_3	0.1850	0.4336	-0.0045	0.0124	-0.0068	0.0124	-0.0100	0.0125	2.4786	96.0%	0.4308	0.4313	0.4344
				ϑ	0.0001	0.0014	0.0003	0.0003	0.0003	0.0003	0.0005	0.0003	0.1446	95.4%	0.0684	0.0680	0.0689

Table 3. MLE and BE with different symmetric and asymmetric loss functions for $\alpha = 0.8, \upsilon_1 = 1.6, \upsilon_2 = 0.3, \upsilon_3 = 6$.

					MLE		BE SELOF		BE LIEXLOF1		BE LIEXLOF2		AS-CI		HPD		
	n_i	Q	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LAS-CI	CP	LCCI	LCCI 2	LCCI 3
1	10	0.7	7	α	0.0555	0.0359	0.0035	0.0097	0.0016	0.0096	-0.0021	0.0094	0.7105	95.7%	0.3744	0.3727	0.3685
				υ_1	0.3166	0.8473	0.0010	0.0216	-0.0017	0.0216	-0.0073	0.0216	3.3899	94.8%	0.5613	0.5608	0.5622
				υ_2	0.0386	0.0187	0.0157	0.0057	0.0143	0.0055	0.0115	0.0053	0.5139	95.7%	0.2650	0.2626	0.2592
				υ_3	0.9619	7.4327	0.0022	0.0255	-0.0008	0.0255	-0.0067	0.0257	10.0048	94.8%	0.6112	0.6146	0.6145
				ϑ	-0.0164	0.0048	-0.0060	0.0009	-0.0055	0.0009	-0.0044	0.0009	0.2650	95.7%	0.1130	0.1118	0.1098
		0.9	9	α	0.0566	0.0272	0.0049	0.0073	0.0037	0.0073	0.0014	0.0072	0.6070	94.0%	0.3327	0.3321	0.3319
				υ_1	0.3261	0.7864	-0.0013	0.0132	-0.0029	0.0132	-0.0063	0.0133	3.2342	94.9%	0.4475	0.4469	0.4422
				υ_2	0.0409	0.0182	0.0083	0.0046	0.0073	0.0045	0.0054	0.0044	0.5035	94.0%	0.2507	0.2495	0.2479
				υ_3	0.9003	5.8512	-0.0027	0.0141	-0.0044	0.0141	-0.0077	0.0142	8.8053	94.9%	0.4687	0.4666	0.4624
				ϑ	-0.0159	0.0044	-0.0030	0.0008	-0.0026	0.0008	-0.0018	0.0007	0.2525	94.0%	0.1039	0.1036	0.1029
	35	0.7	27	α	0.0193	0.0084	0.0077	0.0030	0.0067	0.0030	0.0047	0.0030	0.3516	94.2%	0.2070	0.2059	0.2040
				υ_1	0.0844	0.1463	0.0002	0.0182	-0.0022	0.0182	-0.0070	0.0182	1.4632	95.2%	0.5230	0.5242	0.5228
				υ_2	0.0113	0.0039	0.0114	0.0018	0.0108	0.0017	0.0094	0.0017	0.2398	94.2%	0.1379	0.1372	0.1355
				υ_3	0.3178	1.5414	0.0000	0.0219	-0.0026	0.0220	-0.0079	0.0221	4.7069	95.2%	0.5729	0.5716	0.5702
				ϑ	-0.0043	0.0014	-0.0050	0.0003	-0.0048	0.0003	-0.0043	0.0003	0.1463	94.2%	0.0602	0.0598	0.0590
		0.9	32	α	0.0172	0.0074	0.0052	0.0027	0.0045	0.0027	0.0030	0.0027	0.3308	95.5%	0.2008	0.2009	0.1998
				υ_1	0.0956	0.1443	-0.0042	0.0110	-0.0057	0.0110	-0.0088	0.0111	1.4421	96.2%	0.4087	0.4100	0.4107
				υ_2	0.0093	0.0028	0.0073	0.0013	0.0068	0.0013	0.0057	0.0013	0.2053	95.5%	0.1285	0.1283	0.1277
				υ_3	0.2956	1.5088	0.0020	0.0126	0.0004	0.0126	-0.0028	0.0126	4.6759	96.2%	0.4307	0.4292	0.4300
				ϑ	-0.0040	0.0012	-0.0030	0.0002	-0.0028	0.0002	-0.0024	0.0002	0.1351	95.5%	0.0576	0.0575	0.0575
	50	0.7	38	α	0.0060	0.0055	0.0031	0.0022	0.0024	0.0022	0.0008	0.0022	0.2887	94.9%	0.1809	0.1803	0.1792
				υ_1	0.0497	0.0891	0.0028	0.0179	0.0005	0.0179	-0.0040	0.0178	1.1541	96.1%	0.5244	0.5220	0.5185
				υ_2	0.0070	0.0026	0.0088	0.0011	0.0083	0.0011	0.0073	0.0011	0.1985	94.9%	0.1179	0.1174	0.1160
				υ_3	0.1709	1.0316	-0.0017	0.0203	-0.0046	0.0225	-0.0103	0.0226	3.9266	96.1%	0.5750	0.5742	0.5714
				ϑ	-0.0034	0.0010	-0.0041	0.0002	-0.0040	0.0002	-0.0036	0.0002	0.1210	94.9%	0.0538	0.0538	0.0535
		0.9	45	α	0.0094	0.0045	0.0044	0.0018	0.0038	0.0018	0.0026	0.0018	0.2597	94.6%	0.1599	0.1598	0.1600
				υ_1	0.0367	0.0719	0.0052	0.0109	0.0038	0.0113	0.0008	0.0112	1.0419	95.6%	0.4051	0.4028	0.3998
				υ_2	0.0081	0.0023	0.0066	0.0009	0.0063	0.0009	0.0055	0.0009	0.1870	94.6%	0.1116	0.1110	0.1091
				υ_3	0.1681	0.8907	0.0002	0.0127	-0.0014	0.0127	-0.0046	0.0126	3.6422	95.6%	0.4342	0.4366	0.4364
				ϑ	-0.0032	0.0009	-0.0030	0.0002	-0.0029	0.0002	-0.0026	0.0002	0.1153	94.6%	0.0480	0.0479	0.0479
2	10	0.7	7	α	0.0619	0.0374	0.0068	0.0104	0.0050	0.0103	0.0014	0.0101	0.7181	95.0%	0.3824	0.3802	0.3780
				υ_1	0.2897	0.7415	-0.0058	0.0222	-0.0085	0.0222	-0.0140	0.0223	3.1804	94.9%	0.5777	0.5776	0.5773
				υ_2	0.0475	0.0215	0.0176	0.0059	0.0162	0.0058	0.0134	0.0055	0.5439	95.0%	0.2742	0.2729	0.2687
				υ_3	1.0527	6.7881	-0.0040	0.0243	-0.0069	0.0244	-0.0126	0.0246	9.3471	94.9%	0.6037	0.6063	0.6020
				ϑ	-0.0155	0.0046	-0.0069	0.0010	-0.0064	0.0009	-0.0053	0.0009	0.2587	95.0%	0.1142	0.1140	0.1124
		0.9	9	α	0.0589	0.0310	0.0015	0.0075	0.0003	0.0075	-0.0012	0.0074	0.6509	93.9%	0.3227	0.3206	0.3208
				υ_1	0.2731	0.6412	-0.0032	0.0130	-0.0049	0.0131	-0.0083	0.0131	3.3810	96.2%	0.4463	0.4461	0.4436
				υ_2	0.0421	0.0173	0.0073	0.0046	0.0064	0.0045	0.0045	0.0044	0.4886	93.9%	0.2460	0.2454	0.2425
				υ_3	0.9937	5.4475	0.0009	0.0138	-0.0008	0.0138	-0.0042	0.0138	8.2827	96.2%	0.4505	0.4495	0.4482
				ϑ	-0.0125	0.0040	-0.0024	0.0008	-0.0021	0.0008	-0.0013	0.0007	0.2429	93.9%	0.1036	0.1036	0.1033
	35	0.7	27	α	0.0165	0.0085	0.0060	0.0029	0.0050	0.0029	0.0031	0.0028	0.3556	95.0%	0.2025	0.2019	0.1993
				υ_1	0.0641	0.1183	0.0030	0.0192	0.0005	0.0192	-0.0045	0.0192	1.3253	95.1%	0.5526	0.5512	0.5518
				υ_2	0.0130	0.0036	0.0113	0.0015	0.0107	0.0014	0.0095	0.0014	0.2289	95.0%	0.1321	0.1314	0.1304
				υ_3	0.3147	1.6482	0.0008	0.0233	-0.0020	0.0233	-0.0075	0.0234	4.8815	95.1%	0.6055	0.6027	0.5950
				ϑ	-0.0040	0.0012	-0.0051	0.0003	-0.0049	0.0003	-0.0045	0.0003	0.1344	95.0%	0.0578	0.0575	0.0568
		0.9	32	α	0.0161	0.0073	0.0024	0.0029	0.0017	0.0029	0.0003	0.0026	0.3285	95.5%	0.1921	0.1921	0.1821
				υ_1	0.0593	0.1045	0.0011	0.0109	-0.0004	0.0108	-0.0042	0.0108	1.4467	95.6%	0.4160	0.4124	0.4104
				υ_2	0.0089	0.0031	0.0062	0.0014	0.0057	0.0013	0.0047	0.0013	0.2140	95.5%	0.1287	0.1274	0.1249
				υ_3	0.3038	1.6562	-0.0007	0.0132	-0.0019	0.0132	-0.0057	0.0132	4.8706	95.6%	0.4527	0.4524	0.4518
				ϑ	-0.0022	0.0012	-0.0027	0.0002	-0.0025	0.0002	-0.0021	0.0002	0.1338	95.5%	0.0559	0.0554	0.0548
	50	0.7	38	α	0.0135	0.0057	0.0068	0.0022	0.0060	0.0022	0.0045	0.0022	0.2903	95.1%	0.1788	0.1783	0.1776
				υ_1	0.0615	0.0870	0.0013	0.0171	-0.0009	0.0171	-0.0055	0.0170	1.1311	95.9%	0.5202	0.5179	0.5154
				υ_2	0.0078	0.0023	0.0082	0.0009	0.0078	0.0009	0.0068	0.0008	0.1872	95.1%	0.1074	0.1062	0.1044
				υ_3	0.2068	1.0051	-0.0058	0.0227	-0.0086	0.0228	-0.0141	0.0232	3.8474	95.9%	0.5824	0.5789	0.5843
				ϑ	-0.0029	0.0009	-0.0040	0.0002	-0.0038	0.0002	-0.0035	0.0002	0.1152	95.1%	0.0486	0.0483	0.0477
		0.9	45	α	0.0145	0.0051	0.0063	0.0019	0.0057	0.0019	0.0046	0.0019	0.2748	95.8%	0.1683	0.1680	0.1676
				υ_1	0.0598	0.0761	-0.0007	0.0108	-0.0021	0.0108	-0.0049	0.0108	1.0561	95.7%	0.4018	0.4001	0.3986
				υ_2	0.0099	0.0022	0.0074	0.0010	0.0070	0.0010	0.0062	0.0010	0.1817	95.8%	0.1134	0.1127	0.1121
				υ_3	0.2100	0.8673	-0.0015	0.0130	-0.0031	0.0135	-0.0064	0.0136	3.5584	95.7%	0.4436	0.4425	0.4411
				ϑ	-0.0032	0.0007	-0.0033	0.0002	-0.0032	0.0002	-0.0029	0.0002	0.1065	95.8%	0.0492	0.0494	0.0493

Table 4. MLE and BE with different symmetric and asymmetric loss functions for $\alpha = 5, \upsilon_1 = 3, \upsilon_2 = 1.2, \upsilon_3 = 10$.

					MLE		BE SELOF		BE LIEXLOF1		BE LIEXLOF2		AS-CI		HPD		
	n_i	Q	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LAS-CI	CP	LCCI	LCCI 2	LCCI 3
1	10	0.7	7	α	0.5626	1.5886	0.0014	0.0230	-0.0014	0.0229	-0.0072	0.0229	4.4235	95.3%	0.5877	0.5873	0.5896
				υ_1	1.0220	5.9112	-0.0140	0.0262	-0.0169	0.0263	-0.0228	0.0267	8.6521	94.4%	0.6560	0.6540	0.6561
				υ_2	0.2867	0.6420	0.0078	0.0213	0.0051	0.0213	-0.0004	0.0212	2.9345	95.3%	0.5549	0.5557	0.5571
				υ_3	1.4290	4.4433	-0.0036	0.0232	-0.0065	0.0233	-0.0123	0.0234	8.2588	94.4%	0.6104	0.6082	0.5975
				ϑ	0.0082	0.0084	-0.0010	0.0005	-0.0006	0.0005	0.0001	0.0005	0.3589	95.3%	0.0864	0.0868	0.0873
		0.9	9	α	0.4781	1.2700	-0.0071	0.0142	-0.0088	0.0143	-0.0123	0.0144	4.0023	94.9%	0.4462	0.4443	0.4449
				υ_1	0.9183	4.7329	0.0004	0.0141	-0.0014	0.0141	-0.0048	0.0141	7.7350	95.4%	0.4437	0.4435	0.4435
				υ_2	0.2909	0.5401	-0.0005	0.0131	-0.0022	0.0131	-0.0055	0.0130	2.6469	94.9%	0.4516	0.4513	0.4522
				υ_3	1.0979	3.9027	-0.0007	0.0144	-0.0025	0.0144	-0.0061	0.0145	7.5824	95.4%	0.4614	0.4611	0.4625
				ϑ	-0.0071	0.0073	0.0004	0.0003	0.0006	0.0003	0.0010	0.0003	0.3338	94.9%	0.0715	0.0710	0.0705
	35	0.7	27	α	0.1625	0.3301	-0.0087	0.0209	-0.0114	0.0209	-0.0167	0.0210	2.1614	94.1%	0.5697	0.5713	0.5714
				υ_1	0.2717	0.6562	0.0002	0.0230	-0.0026	0.0231	-0.0083	0.0233	2.9930	95.1%	0.5842	0.5854	0.5817
				υ_2	0.0670	0.0753	-0.0050	0.0155	-0.0073	0.0155	-0.0118	0.0155	1.0435	94.1%	0.4761	0.4754	0.4771
				υ_3	0.9229	3.0113	0.0021	0.0225	-0.0009	0.0225	-0.0069	0.0225	6.7085	95.1%	0.5831	0.5823	0.5740
				ϑ	0.0016	0.0021	0.0011	0.0004	0.0014	0.0004	0.0019	0.0004	0.1803	94.1%	0.0756	0.0757	0.0764
		0.9	32	α	0.1604	0.3218	0.0073	0.0123	0.0057	0.0123	0.0024	0.0122	2.1144	94.3%	0.4290	0.4286	0.4264
				υ_1	0.2771	0.6857	0.0018	0.0122	0.0001	0.0122	-0.0032	0.0122	3.0605	96.0%	0.4117	0.4116	0.4099
				υ_2	0.0880	0.0806	-0.0012	0.0104	-0.0026	0.0104	-0.0053	0.0104	1.0589	94.3%	0.3856	0.3873	0.3877
				υ_3	0.9026	2.9240	-0.0014	0.0145	-0.0032	0.0145	-0.0067	0.0146	6.3860	96.0%	0.4721	0.4753	0.4768
				ϑ	-0.0009	0.0023	0.0004	0.0003	0.0006	0.0003	0.0009	0.0003	0.1888	94.3%	0.0614	0.0617	0.0623
	50	0.7	38	α	0.1289	0.2665	0.0033	0.0198	0.0007	0.0198	-0.0043	0.0197	1.9606	95.8%	0.5393	0.5384	0.5364
				υ_1	0.1998	0.5044	-0.0046	0.0215	-0.0074	0.0216	-0.0128	0.0217	2.6728	95.3%	0.5532	0.5510	0.5497
				υ_2	0.0475	0.0534	0.0008	0.0129	-0.0013	0.0128	-0.0055	0.0128	0.8872	95.8%	0.4480	0.4450	0.4407
				υ_3	0.7955	1.2332	0.0062	0.0222	0.0033	0.0222	-0.0025	0.0222	5.6121	95.3%	0.5983	0.5999	0.5939
				ϑ	0.0024	0.0018	0.0001	0.0003	0.0004	0.0003	0.0009	0.0003	0.1648	95.8%	0.0706	0.0706	0.0705
		0.9	45	α	0.1056	0.2140	-0.0028	0.0117	-0.0043	0.0117	-0.0073	0.0118	1.7666	94.9%	0.4317	0.4320	0.4310
				υ_1	0.1466	0.4092	-0.0023	0.0132	-0.0039	0.0132	-0.0072	0.0133	2.4421	95.4%	0.4578	0.4575	0.4598
				υ_2	0.0438	0.0436	-0.0017	0.0097	-0.0031	0.0097	-0.0059	0.0097	0.8003	94.9%	0.3774	0.3783	0.3783
				υ_3	0.7550	1.1554	0.0031	0.0147	0.0013	0.0147	-0.0022	0.0147	4.0645	95.4%	0.4812	0.4802	0.4786
				ϑ	0.0001	0.0015	0.0005	0.0002	0.0007	0.0002	0.0010	0.0002	0.1523	94.9%	0.0587	0.0587	0.0590
2	10	0.7	7	α	0.5721	1.6218	-0.0017	0.0233	-0.0036	0.0233	-0.0094	0.0234	4.4623	96.0%	0.6042	0.6061	0.6063
				υ_1	1.0335	5.2442	-0.0051	0.0234	-0.0081	0.0235	-0.0140	0.0238	7.0148	95.2%	0.5907	0.5905	0.5926
				υ_2	0.2593	0.5935	0.0019	0.0220	-0.0049	0.0220	-0.0066	0.0220	2.8451	96.0%	0.5589	0.5612	0.5622
				υ_3	1.3220	6.6618	0.0026	0.0230	-0.0028	0.0229	-0.0059	0.0229	8.8825	95.2%	0.6049	0.6035	0.5977
				ϑ	0.0083	0.0074	0.0008	0.0006	0.0011	0.0006	0.0013	0.0006	0.3382	96.0%	0.0892	0.0894	0.0901
		0.9	9	α	0.5365	1.2409	0.0011	0.0137	-0.0006	0.0138	-0.0041	0.0138	3.8289	95.2%	0.4361	0.4371	0.4424
				υ_1	1.0150	4.3383	-0.0040	0.0183	-0.0056	0.0163	-0.0089	0.0153	7.0014	95.4%	0.4495	0.4498	0.4495
				υ_2	0.2478	0.4443	-0.0015	0.0134	-0.0031	0.0134	-0.0065	0.0135	2.4266	95.2%	0.4561	0.4562	0.4539
				υ_3	1.2835	5.0577	-0.0023	0.0138	-0.0018	0.0138	-0.0054	0.0138	5.9404	95.4%	0.4579	0.4580	0.4594
				ϑ	0.0065	0.0071	0.0005	0.0003	0.0007	0.0003	0.0012	0.0003	0.3301	95.2%	0.0722	0.0719	0.0719
	35	0.7	27	α	0.1559	0.3442	0.0044	0.0203	0.0017	0.0203	-0.0037	0.0203	2.2182	95.3%	0.5614	0.5609	0.5556
				υ_1	0.2586	0.6639	0.0033	0.0227	0.0048	0.0216	-0.0052	0.0210	3.0304	96.0%	0.5548	0.5550	0.5508
				υ_2	0.0746	0.0729	-0.0024	0.0154	-0.0047	0.0154	-0.0093	0.0154	1.0176	95.3%	0.4694	0.4699	0.4712
				υ_3	1.2011	3.9290	0.0020	0.0226	-0.0010	0.0227	-0.0070	0.0229	5.8170	96.0%	0.5882	0.5891	0.5883
				ϑ	0.0001	0.0023	0.0008	0.0004	0.0010	0.0004	0.0016	0.0004	0.1880	95.3%	0.0743	0.0745	0.0753
		0.9	32	α	0.1266	0.2828	0.0014	0.0123	-0.0002	0.0123	-0.0034	0.0123	2.0256	95.9%	0.4202	0.4205	0.4193
				υ_1	0.1959	0.5400	-0.0030	0.0150	-0.0045	0.0150	-0.0049	0.0151	2.7779	96.0%	0.4698	0.4696	0.4712
				υ_2	0.0611	0.0610	0.0006	0.0114	-0.0009	0.0114	-0.0039	0.0114	0.9388	95.9%	0.4054	0.4032	0.4028
				υ_3	1.0415	3.1031	-0.0014	0.0139	-0.0009	0.0139	-0.0065	0.0140	5.7779	96.0%	0.4517	0.4516	0.4521
				ϑ	-0.0001	0.0022	0.0001	0.0003	0.0003	0.0003	0.0006	0.0003	0.1834	95.9%	0.0629	0.0627	0.0630
	50	0.7	38	α	0.1220	0.2522	-0.0024	0.0198	-0.0050	0.0198	-0.0102	0.0199	1.9107	95.5%	0.5474	0.5514	0.5536
				υ_1	0.1737	0.4494	0.0006	0.0220	-0.0023	0.0219	-0.0080	0.0219	2.5393	96.1%	0.5884	0.5873	0.5817
				υ_2	0.0454	0.0476	-0.0018	0.0131	-0.0038	0.0130	-0.0080	0.0130	0.8373	95.5%	0.4390	0.4385	0.4399
				υ_3	0.9930	2.8300	0.0042	0.0217	0.0012	0.0210	-0.0048	0.0207	4.9842	96.1%	0.5954	0.5954	0.5959
				ϑ	0.0028	0.0017	0.0006	0.0003	0.0008	0.0003	0.0013	0.0003	0.1608	95.5%	0.0691	0.0692	0.0691
		0.9	45	α	0.0999	0.2256	0.0002	0.0111	-0.0013	0.0111	-0.0043	0.0111	1.8210	95.1%	0.4082	0.4073	0.4090
				υ_1	0.1354	0.3671	0.0005	0.0134	0.0021	0.0133	-0.0004	0.0133	2.3163	96.2%	0.4538	0.4528	0.4522
				υ_2	0.0412	0.0451	-0.0013	0.0097	-0.0030	0.0097	-0.0075	0.0097	0.8167	95.1%	0.3736	0.3719	0.3708
				υ_3	0.8082	1.2677	0.0022	0.0130	0.0005	0.0130	-0.0028	0.0130	4.6223	96.2%	0.4392	0.4367	0.4355
				ϑ	0.0008	0.0016	0.0005	0.0002	0.0006	0.0002	0.0010	0.0002	0.1556	95.1%	0.0592	0.0590	0.0589

6. Application of real data

This section analyzes a real-world data set to illustrate the application of the methods discussed earlier. The same data set is utilized within a multi-SS framework to showcase the progressive first-failure censoring techniques based on the EPD. Data from a stress-strength life test on transformer insulation, detailed in Chapter 3 of Nelson (Nelson [59]), was analyzed. The experiment involved three voltage levels: 35.4, 42.4, and 46.7 kV, with a standard voltage of 14.4 kV as a reference. The observed data sets at 42.4 kV are: 0.6, 13.4, 15.2, 19.9, 25.0, 30.2, 32.8, 44.4, 50.2, and 56.2. At 35.4 kV, the recorded values are: 40.1, 59.4, 71.2, 166.5, 204.7, 229.7, 308.3, and 537.9. Meanwhile, for 46.7 kV, the observations are: 3.1, 8.3, 8.9, 9.0, 13.6, 14.9, 16.1, 16.9, 21.3, and 48.1. The aim was to estimate the parameters of the EPD using the APTII-HCS under the multi-SS model. Additionally, we computed the reliability of the system, represented by the probability $\vartheta = P[Q < W < Y]$.

Table 5 displays the MLEs along with their standard errors (StEr), as well as several model evaluation metrics, including the Kolmogorov-Smirnov distance (KSD) statistic with its corresponding P-value (PVKS), the Akaike information criterion (AIC), and the Bayesian information criterion (BIC), for the EPD applied to the transformer insulation data. Based on the KS test and its associated P-values, the EPD provides an appropriate fit for each data set.

Table 5. MLE and different measures of fitting of the EPD for a real data set.

	Q	W	Y
α	1.5961	0.6594	0.8166
υ	40.1625	4.4495	52.4555
KSD	0.2136	0.3402	0.1697
PVKS	0.6770	0.1566	0.9356
AIC	77.3061	113.6159	142.7888
BIC	77.9113	114.4117	143.3940

The censoring removal was as follows: R_3 was 0, 0, 0, 0, 0, 0, 0, 2, R_2 was 0, 0, 0, 0, 0, 0, 0, 1, 0, and R_1 was 0, 0, 0, 0, 0, 0, 0, 0, 1. Table 6 presents the MLEs and BEs for the parameters of the EPD based on APTII-HCS, under the multi-SS model, with various items considered within each group. It is observed that the BEs had lower StEr compared to those obtained by MLE. Under the multi-SS model, BEs demonstrated the best performance for parameter estimation in the EPD based on the PTII censoring framework. Furthermore, the reliability $\vartheta = P[Q < W < Y]$, derived from the Bayesian method, was higher than that from the ML approach, confirming the superior performance of the Bayesian approach.

Table 6. MLE and BE with point and interval estimation for a real data set.

	MLE				Bayesian			
	Estimates	StEr	Lower	Upper	Estimates	StEr	Lower	Upper
α	0.8315	0.1224	0.5916	1.0714	0.8394	0.0690	0.7082	0.9794
υ_1	7.6982	3.3149	1.2009	14.1955	7.9053	1.5216	5.1028	10.8882
υ_2	6.7196	2.4385	1.9402	11.4991	6.7883	1.3132	4.3702	9.4079
υ_3	69.9420	22.3410	26.1536	113.7303	70.1088	4.7238	61.0711	79.3961
ϑ	0.4427	0.1224	0.5916	1.0714	0.4448	0.0690	0.4663	0.4272

Convergence diagnostics, including trace plots and histogram density estimates overlaid with a normal curve for the parameters, are illustrated in Figure 1, based on 10,000 MCMC iterations. Figure 1 displays the posterior density plots for α, υ_1, υ_2, and υ_3, each showing an approximately symmetric normal distribution. Figure 2 presents a pairwise scatterplot matrix displaying the posterior samples of α, υ_1, υ_2, υ_3 with density plots along the diagonal and pairwise correlation coefficients in the upper panels. Similarly, Figure 3 demonstrates that the posterior densities for the reliability ϑ also exhibit a symmetric normal distribution. Autocorrelation function (ACF) plots for α, υ_1, υ_2, υ_3, and ϑ are shown in Figure 4.

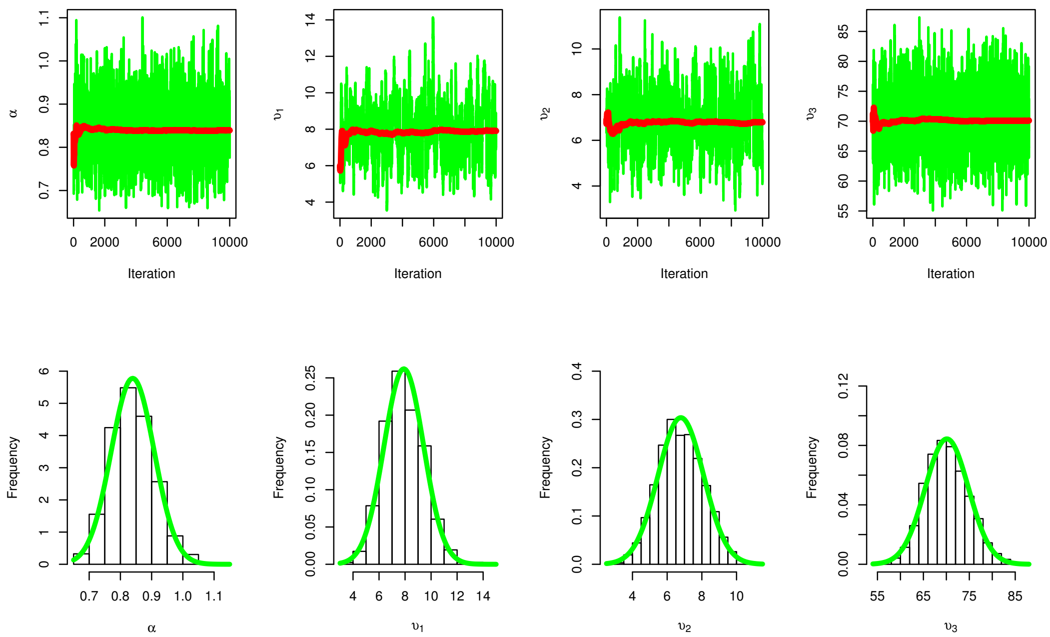

Figure 1. Trace plots of the MCMC results.

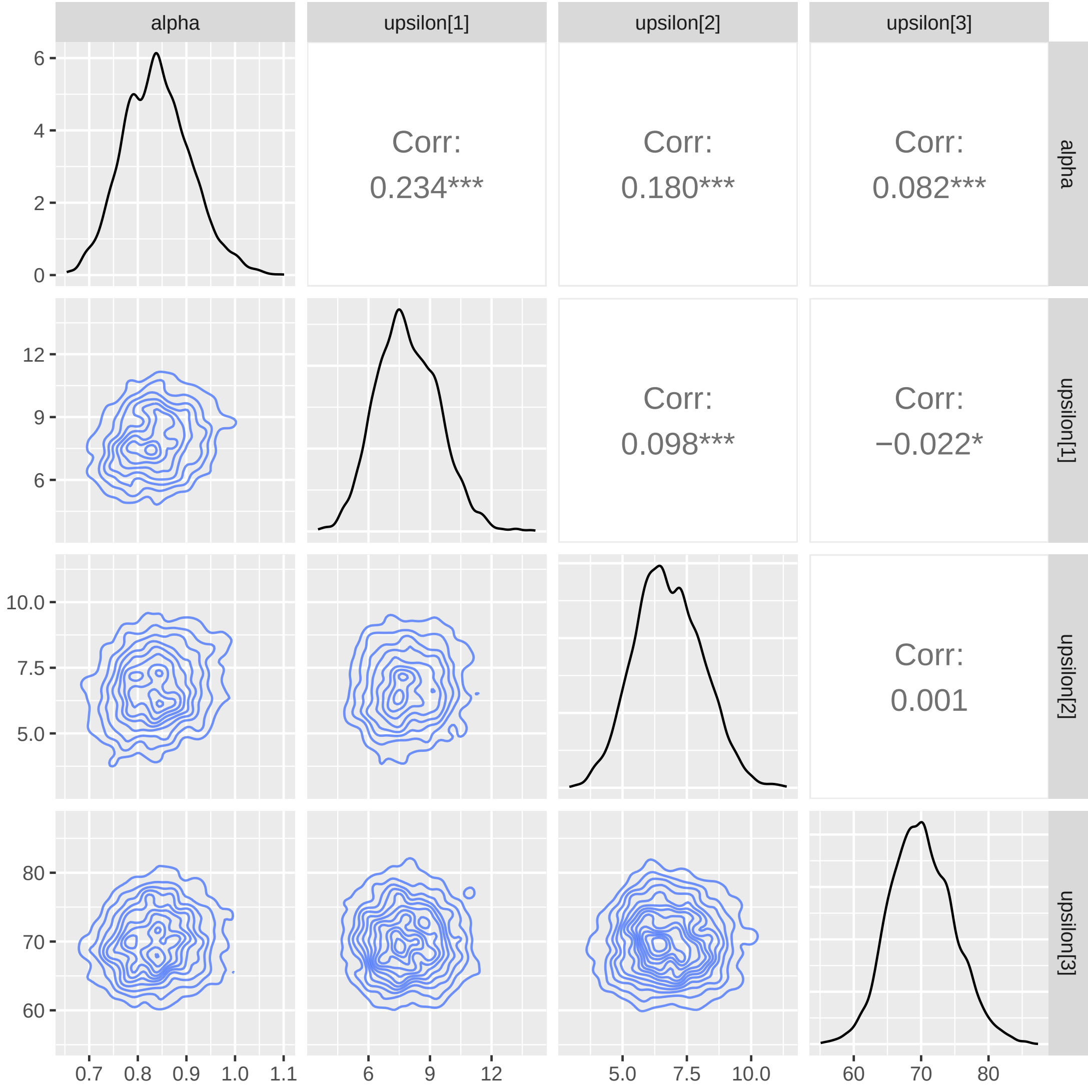

Figure 2. Pairwise scatterplot matrix displaying the posterior samples of α, υ_1, υ_2, υ_3 with density plots along the diagonal and pairwise correlation coefficients in the upper panels.

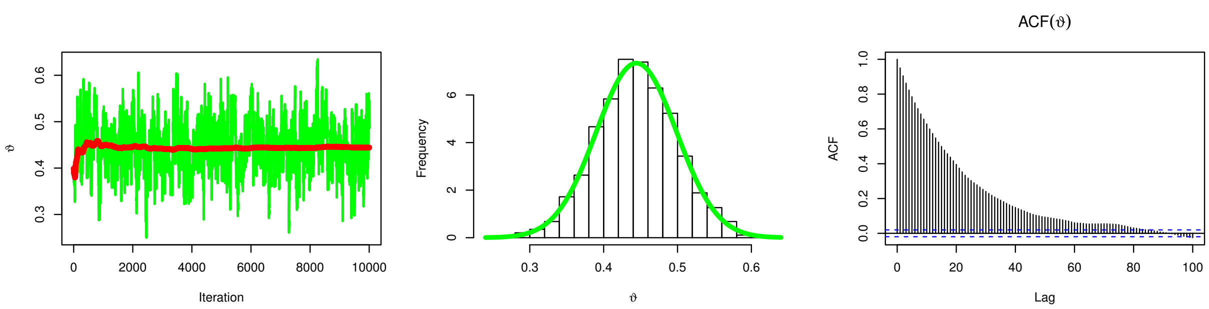

Figure 3. MCMC plots of the reliability stress-strength parameter.

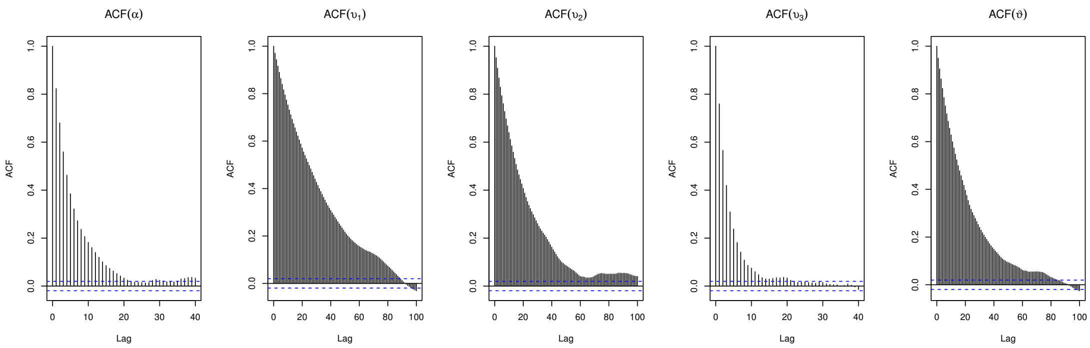

Figure 4. ACF plots of α, υ_1, υ_2, υ_3 and the reliability stress-strength parameter.

7. Conclusions

This paper addresses the estimation of stress-strength reliability $\vartheta = P[Q < W < Y]$, where the strength variable W is constrained between a lower stress Q and an upper stress Y. The study assumes that W, Q, and Y are independent random variables following the EPD with a common unknown shape parameter. Both classical and Bayesian estimation methods are developed under the adaptive progressive APTII-HCS. Maximum likelihood estimates and asymptotic confidence intervals for the system reliability are obtained. Bayesian estimates are derived under symmetric (squared error) and asymmetric (linear exponential) loss functions, and highest posterior density credible intervals are constructed using MCMC procedures. The performance of the proposed estimators is evaluated through extensive simulations, and the results demonstrate satisfactory accuracy across multiple criteria. The practical applicability of the methodology is illustrated using three real data sets.

Despite its efficacy, this study is limited by the assumption that the strength and stress variables follow the EPD with a common unknown shape parameter. While this assumption facilitates analytical tractability, it may restrict the applicability of the proposed methods in more general settings. Future research could extend this work in several directions. First, fully Bayesian approaches employing hierarchical, non-informative, or robust prior distributions could be explored to assess the sensitivity of posterior inferences and enhance objectivity. Second, the APTII-HCS framework could be applied to other reliability metrics or adapted to different stress-strength distributions beyond the EPD. Additional extensions may include the common shape parameter assumption, incorporating dependent stress-strength structures via copulas, or addressing high-dimensional stress scenarios involving multiple lower and upper bounds.

Author contributions

All authors contributed equally to the writing and review of this manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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