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Research article

Note on some asymptotic relations for Bloch functions and the distance from a Bloch function to the little Bloch space

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Abstract: We prove, among other results, that if f is an analytic function on the open unit disc in the complex plane belonging to the Bloch space $\mathcal{B}$, then the asymptotic relation holds

$$\operatorname{dist}(f, \mathcal{B}_0) \asymp \limsup_{|z|\to 1-0}(1 - |z|^2)^l |f^{(l)}(z)|,$$

for each $l \in \mathbb{N}$, where

$$\operatorname{dist}(f, \mathcal{B}_0) = \inf_{g\in\mathcal{B}_0} \|f - g\|_{\mathcal{B}}$$

and $\mathcal{B}_0$ is the little Bloch space, extending a result in the literature concerning the case $l = 1$. In passing, we also obtain the following asymptotic relation

$$\operatorname{dist}(f, \mathcal{B}_0) \asymp \limsup_{r\to 1-0} \|f - f_r\|_{\mathcal{B}}.$$

Keywords: Bloch space; little Bloch space; open unit disc; asymptotic equivalence; Cauchy's estimate
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1. Introduction and preliminaries

First, we present the notations used in this note. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_k = \{n \in \mathbb{N}_0 : n \geq k\}$ for some $k \in \mathbb{N}_0$, $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $C(\mathbb{D})$ be the family of all continuous functions on $\mathbb{D}$, $H(\mathbb{D})$ be the class of all analytic/holomorphic functions on $\mathbb{D}$ (for some basics on such functions see, for example, [1]), and let

$$M_\infty(f, r) = \max_{|z|\le r} |f(z)|,$$

where $f \in H(\mathbb{D})$ and $r \in [0, 1)$.

Throughout this note, by C we denote some positive numbers, which may vary from one place to another. The notation $a \lesssim b$ (respectively, $a \gtrsim b$) means that there is a positive number C such that the inequality $a \le Cb$ holds (respectively, $a \ge Cb$), which is independent from all the essential variables and functions appearing in a concrete situation. If the following two asymptotic relations hold: $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that the quantities a and b are asymptotically equivalent.

The Bloch space $\mathcal{B}(\mathbb{D}) = \mathcal{B}$ consists of all $f \in H(\mathbb{D})$ satisfying the condition

$$\|f\|_{\mathcal{B}} = |f(0)| + b_{\mathcal{B}}(f) < +\infty,$$

where

$$b_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)|.$$

The quantity $\|\cdot\|_{\mathcal{B}}$ is a norm on the space, and with the norm it becomes a Banach space.

The little Bloch space $\mathcal{B}_0(\mathbb{D}) = \mathcal{B}_0$ consists of all $f \in H(\mathbb{D})$ satisfying the condition

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0,$$

and is a closed subspace of $\mathcal{B}$.

For these two spaces, some related spaces of analytic functions, as well as some linear operators on them, see, for example, [2, 5, 8–10] and the related references cited therein. The spaces frequently appear during various investigations of analytic functions, and since their definitions are relatively simple and suitable for algebraic calculations and estimations, there are a lot of papers dealing with them.

In [6, Lemma 2.6] it is claimed that the following asymptotic relations were proved in [2], which we incorporate in the following result.

Theorem 1. *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, $0 \le k < n$, and $f \in \mathcal{B}$. Then, the following asymptotic relations hold*

$$\operatorname{dist}(f, \mathcal{B}_0) \asymp \limsup_{|z|\to 1-0}(1 - |z|^2)|f'(z)|, \tag{1.1}$$

$$\operatorname{dist}(f, \mathcal{B}_0) \asymp \limsup_{|z|\to 1-0}(1 - |z|^2)^{n-k}|f^{(n-k)}(z)|, \tag{1.2}$$

$$\operatorname{dist}(f, \mathcal{B}_0) \asymp \limsup_{r\to 1-0} \|f - f_r\|_{\mathcal{B}}, \tag{1.3}$$

where

$$\operatorname{dist}(f, \mathcal{B}_0) = \inf_{g\in\mathcal{B}_0} \|f - g\|_{\mathcal{B}}.$$

However, a detailed inspection of the results proved and quoted in [2] shows that only the asymptotic relation in (1.1) was proved therein, which was done by using some sufficiently separated sequences so that they become interpolating sequences (see [2, Proposition 10]). If $k = n-1$, then the asymptotic relation in (1.2) becomes the one in (1.1). Because of this, it is of some interest the case when $k < n-1$. The asymptotic relation (1.2) in the case $k < n-1$ could be a known result, but we have not managed to find a specific reference for it, up to the moment. Beside this, the asymptotic relation (1.2) is not so obvious nor it easily follows from well-known asymptotic relations in the literature, one of which is the theme of the following well-known result (see, for instance, [9]).

Lemma 1. *Let $f \in H(\mathbb{D})$ and $l \in \mathbb{N}_2$. Then, $f \in \mathcal{B}$, if and only if,*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^l|f^{(l)}(z)| < +\infty,$$

and the following asymptotic relation holds

$$b_{\mathcal{B}}(f) \asymp \sum_{j=1}^{l-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^l|f^{(l)}(z)|. \tag{1.4}$$

The asymptotic relation (1.3) could also be known, but we have not managed to find a specific reference for it either.

Because of abovementioned facts, it is of some interest to see if the asymptotic relation in (1.2) holds in the case $k < n-1$, and to see if the asymptotic relation (1.3) holds.

The aim of this note is to prove that they really hold. To do this we employ some point evaluation estimates, which are frequently useful in similar situations (see, for instance, [3, 4, 7, 9] and the related references therein), together with some tricks. It should be noticed that, unlike the proof of Proposition 1 in [2], the proof of our main result uses only complex analysis techniques.

2. Main results

Here we formulate and give a detailed proof of the main result in this note. The result gives an affirmative answer to abovementioned questions concerning the asymptotic relations in (1.2) and (1.3), and it also extends Proposition 10 in [2].

As an immediate consequence of the result, it follows that the following asymptotic relation

$$\limsup_{|z|\to 1-0}(1-|z|^2)^l|f^{(l)}(z)| \asymp \limsup_{|z|\to 1-0}(1-|z|^2)|f'(z)|$$

holds for every $l \in \mathbb{N}$ and $f \in \mathcal{B}$, which is related to the asymptotic equivalence result in Lemma 1.

Theorem 2. *Let $f \in \mathcal{B}$ and $l \in \mathbb{N}$. Then, there are two positive constants c_1 and c_2 depending on l, such that*

$$c_1 \limsup_{|z|\to 1-0}(1-|z|^2)^l|f^{(l)}(z)| \le \operatorname{dist}(f, \mathcal{B}_0) \tag{2.1}$$

$$\le \limsup_{r\to 1-0}\|f-f_r\|_{\mathcal{B}} \tag{2.2}$$

$$\le c_2 \limsup_{|z|\to 1-0}(1-|z|^2)^l|f^{(l)}(z)|. \tag{2.3}$$

Proof. If $l = 1$, then, as we have already mentioned, the asymptotic relation in (1.1) was proved in [2].

Assume that $k \in \{2, 3, \dots, l\}$. Employing Cauchy's estimate to the function $f^{(k-1)}$, and using the asymptotic relation

$$1 - |z| \asymp 1 - |w|$$

when

$$|z - w| < \rho(1 - |z|),$$

where ρ is a fixed number in the interval $[0, 1)$, we obtain

$$\begin{aligned} |f^{(k)}(z)| \leq & C \frac{\sup_{|z-w|<\frac{1-|z|}{4}} |f^{(k-1)}(w)|}{1 - |z|} \\ \leq & C \frac{\sup_{|z-w|<\frac{1-|z|}{4}} (1 - |w|)^{k-1} |f^{(k-1)}(w)|}{(1 - |z|)^k}. \end{aligned} \tag{2.4}$$

Let $(z_j)_{j\in\mathbb{N}} \subset \mathbb{D}$ be a sequence such that

$$\lim_{j\to+\infty} (1 - |z_j|^2)^k |f^{(k)}(z_j)| = \limsup_{|z|\to1-0} (1 - |z|^2)^k |f^{(k)}(z)|. \tag{2.5}$$

Without loss of generality, we may assume that the following sequence of real numbers

$$r_j := |z_j|, \quad j \in \mathbb{N},$$

is strictly increasing.

Then, from (2.4) and (2.5), we have

$$\begin{aligned} \limsup_{|z|\to1-0} (1 - |z|^2)^k |f^{(k)}(z)| = & \lim_{j\to+\infty} (1 - |z_j|^2)^k |f^{(k)}(z_j)| \\ \leq & 2^k \lim_{j\to+\infty} \sup_{r_j - \frac{1-r_j}{4} < |z| < r_j + \frac{1-r_j}{4}} (1 - |z|)^k |f^{(k)}(z)| \\ \leq & C \lim_{j\to+\infty} \sup_{r_j - \frac{1-r_j}{4} < |z| < r_j + \frac{1-r_j}{4}} \sup_{|z-w|<\frac{1-|z|}{4}} (1 - |w|)^{k-1} |f^{(k-1)}(w)| \\ \leq & C \lim_{j\to+\infty} \sup_{r_j - \frac{1-r_j}{4} < |z| < r_j + \frac{1-r_j}{4}} \sup_{||z|-|w||<\frac{1-|z|}{4}} (1 - |w|)^{k-1} |f^{(k-1)}(w)| \\ = & C \lim_{j\to+\infty} \sup_{r_j - \frac{1-r_j}{4} < |z| < r_j + \frac{1-r_j}{4}} \sup_{\frac{5|z|-1}{4} < |w| < \frac{3|z|+1}{4}} (1 - |w|)^{k-1} |f^{(k-1)}(w)| \\ \leq & C \lim_{j\to+\infty} \sup_{r_j - \frac{9(1-r_j)}{16} < |w| < r_j + \frac{7(1-r_j)}{16}} (1 - |w|)^{k-1} |f^{(k-1)}(w)| \\ \leq & C \limsup_{|z|\to1-0} (1 - |z|^2)^{k-1} |f^{(k-1)}(z)|, \end{aligned} \tag{2.6}$$

for any $k \in \{2, 3, \dots, l\}$, where the last inequality in (2.6) holds due to the fact that

$$\lim_{j\to+\infty} \left(r_j - \frac{9(1 - r_j)}{16}\right) = \lim_{j\to+\infty} \left(r_j + \frac{7(1 - r_j)}{16}\right) = 1,$$

and the definition of limit superior of a function.

By combining the inequalities which are obtained from the inequality (2.6) for $k = 2, 3, \dots, l$, it follows that the inequality

$$\limsup_{|z|\to 1-0}(1-|z|^2)^l|f^{(l)}(z)| \le C \limsup_{|z|\to 1-0}(1-|z|^2)|f'(z)|, \tag{2.7}$$

holds. Now combining the relations in (1.1) and (2.7), the asymptotic inequality (2.1) easily follows.

Let

$$\begin{aligned} f(z) = \sum_{j=0}^{\infty} a_j z^j &= \sum_{j=0}^{l-1} a_j z^j + \sum_{j=l}^{\infty} a_j z^j \\ &= p_{l-1}(z) + (f - p_{l-1})(z). \end{aligned}$$

Then, since

$$p_{l-1} \in \mathcal{B}_0,$$

for each $l \in \mathbb{N}$, we have

$$\operatorname{dist}(f, \mathcal{B}_0) = \operatorname{dist}(f - p_{l-1}, \mathcal{B}_0 - p_{l-1}) = \operatorname{dist}(f - p_{l-1}, \mathcal{B}_0). \tag{2.8}$$

Hence, we may assume that

$$f(0) = f'(0) = \cdots = f^{(l-1)}(0) = 0.$$

Now note that the function

$$f_r(z) := f(rz), \quad z \in \mathbb{D},$$

belongs to $\mathcal{B}_0$ for each $r \in (0, 1)$, and since $(f - f_r)(0) = 0$ we have

$$b_{\mathcal{B}}(f - f_r) = \|f - f_r\|_{\mathcal{B}}, \tag{2.9}$$

for every $f \in \mathcal{B}$ and $r \in [0, 1)$.

Hence

$$\operatorname{dist}(f, \mathcal{B}_0) \le \|f - f_r\|_{\mathcal{B}}, \tag{2.10}$$

for each $r \in (0, 1)$.

By taking the limit superior in the inequality (2.10) in variable r, the inequality (2.2) immediately follows.

From (1.4), we have

$$\begin{aligned} \|f - f_r\|_{\mathcal{B}} &\asymp \sum_{j=1}^{l-1} |(f - f_r)^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^l|(f - f_r)^{(l)}(z)| \\ &= \sum_{j=1}^{l-1}(1 - r^j)|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^l|(f - f_r)^{(l)}(z)|, \end{aligned} \tag{2.11}$$

for each $r \in (0, 1)$.

Taking the limit superior in (2.11) in r, we have

$$\limsup_{r\to 1-0}\|f-f_r\|_{\mathcal{B}} \le C\limsup_{r\to 1-0}\sup_{z\in\mathbb{D}}(1-|z|^2)^l|(f-f_r)^{(l)}(z)|. \tag{2.12}$$

Let $\delta\in(0,1)$ be fixed and

$$r\in\Big(\frac{1+\delta}{2},1\Big). \tag{2.13}$$

Then, by using the fact

$$\sup_{z\in A\cup B}|g(z)|\le \sup_{z\in A}|g(z)|+\sup_{z\in B}|g(z)|,$$

for $g\in C(\mathbb{D})$, where A and B are two disjoint subsets of $\mathbb{D}$, and the monotonicity of the function

$$h_l(t)=(1-t^2)^l$$

on the interval $[0,1)$, it follows that

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^l|(f-f_r)^{(l)}(z)| \le{}& \sup_{|z|\le\delta/r}(1-|z|^2)^l|f^{(l)}(z)-r^lf^{(l)}(rz)|\\ &+\sup_{\delta/r<|z|<1}(1-|z|^2)^l|f^{(l)}(z)-r^lf^{(l)}(rz)|\\ \le{}& \sup_{|z|\le\delta/r}(1-|z|^2)^l|f^{(l)}(z)-f^{(l)}(rz)|\\ &+\sup_{|z|\le\delta/r}(1-|z|^2)^l(1-r^l)|f^{(l)}(rz)|\\ &+2\sup_{\delta<|z|<1}(1-|z|^2)^l|f^{(l)}(z)|. \end{aligned} \tag{2.14}$$

From the assumption (2.13), inequality (2.14), the maximum modulus principle, Lemma 1, and the monotonicity of the function $m(r):=M_\infty(g,r)$ in variable r, for each $g\in H(\mathbb{D})$ ([1]), we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^l|(f-f_r)^{(l)}(z)| \le{}& \sup_{|z|\le\delta/r}(1-|z|^2)^l\Big|\int_r^1 f^{(l+1)}(tz)z\,dt\Big|\\ &+\sup_{|z|\le\delta/r}(1-|z|^2)^l(1-r^l)|f^{(l)}(rz)|\\ &+2\sup_{\delta<|z|<1}(1-|z|^2)^l|f^{(l)}(z)|\\ \le{}& \sup_{|z|\le\delta/r}(1-|z|^2)^l|z|(1-r)M_\infty\Big(f^{(l+1)},\frac{\delta}{r}\Big)\\ &+C(1-r^l)b_{\mathcal{B}}(f)\\ &+2\sup_{\delta<|z|<1}(1-|z|^2)^l|f^{(l)}(z)|\\ \le{}&\frac{2\delta}{1+\delta}(1-r)M_\infty\Big(f^{(l+1)},\frac{2\delta}{1+\delta}\Big)\\ &+C(1-r^l)b_{\mathcal{B}}(f)\\ &+2\sup_{\delta<|z|<1}(1-|z|^2)^l|f^{(l)}(z)|. \end{aligned} \tag{2.15}$$

Taking the limit superior in (2.15) in variable r, and using the assumption $f \in \mathcal{B}$, we obtain

$$\limsup_{r\to 1-0} \sup_{z\in\mathbb{D}} (1-|z|^2)^l |(f-f_r)^{(l)}(z)| \le 2 \sup_{\delta\le |z|<1} (1-|z|^2)^l |f^{(l)}(z)|, \tag{2.16}$$

for each $\delta \in (0,1)$.

Combining (2.12) and (2.16), we obtain

$$\limsup_{r\to 1-0} \|f-f_r\|_{\mathcal{B}} \le C \sup_{\delta\le |z|<1} (1-|z|^2)^l |f^{(l)}(z)|, \tag{2.17}$$

for each $\delta \in (0,1)$.

Taking the limit superior in (2.17) in variable δ, we obtain the inequality (2.3), finishing the proof of the theorem. □

3. Conclusions

We give some asymptotic relations for Bloch functions and the distance from a Bloch function to the little Bloch space on the open unit disk in the complex plane, extending a result in the literature, and show that they can be obtained by using only complex analysis techniques. The relations can be used in dealing with some linear operators on Bloch type spaces of holomorphic functions, whereas the methods and tricks in this note could be employed in related situations.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

This work does not have any conflict of interest.

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