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Research article

Nonlinear almost F-contractions in relational metric space with an application to periodic boundary value problems

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Abstract: In this paper, we establish some fixed-point outcomes for a nonlinear almost F-contraction map in a metric space combined with a locally $\mathcal{L}$-transitive binary relation. Numerous previous insights are expanded, developed, improved, and consolidated in the outcomes reported herein. We propose a few instances in order to illustrate our findings. As an application of our findings, we explore the existence of a solution of a first-order periodic boundary value problem.

Keywords: fixed points; F-contractions; Γ-continuity; locally transitive relations; boundary value problems
Mathematics Subject Classification: 34B15, 47H10, 54H25

Abbreviations

The following terms and acronyms will be incorporated into the present assignment.

Abbreviations	
$\mathbb{N}_0$	Set of nonnegative integers
$\mathbb{N}$	Set of positive integers
$\mathbb{R}$	Set of real numbers
BCP	Banach contraction principle
BR	Binary relation
iff	If and only if
MS	Metric space
BVP	Boundary value problems
$Fix(\mathcal{L})$	Set of all fixed points of map $\mathcal{L}$

1. Introduction

Nonlinear functional analysis offers a wide range of methods for addressing different issues that arise in practical settings. Several fields related to nonlinear functional analysis have drawn an extensive amount of attention recently due to their possible uses. In this regard, we primarily emphasize to the latest accomplishments, notably those found in [1, 2]. The classical BCP remains a significant and fundamental accomplishment of nonlinear functional analysis. Nonetheless, BCP guarantees that a contraction inequality defined in a complete MS enjoys a unique fixed point. An iteration strategy for evaluating the unique fixed point is also incorporated into this result. This outcome not only reinforces the intrinsic worth of fixed point theorems in analysis; it additionally reveals the unifying potential of functional analytic methodologies. This finding is a vital resource for addressing existence and uniqueness problems in various areas of analysis. In the intervening century, BCP has continued to generalize in a wide range of ways.

An innovative and intuitive adaptation of BCP was started by Alam and Imdad [3], in which the MS consists of a BR, and the corresponding map ensures BR. Plenty of investigations are performed in this direction. A selection of them are cited in [4–6] along with the references therein. The relational contraction is weaker than usual contraction, especially as it executes on elements that remains associated with BR. These outcomes reduce the traditional fixed-point theorems over universal BR. Such findings can be utilized to get unique solutions to specific types of BVP, matrix equations, and integral equations.

Wardowski [7] proposed the idea of F-contraction in 2012 to explore a novel class of nonlinear contractions that inevitably propagate BCP. Turinici [8] then provided several important insights on F-contractions. Piri and Kumam [9] offered some substitutes of insights of Wardowski [7] under their hypothesis that the auxiliary function F needs to be continuous. The concept of (φ, F)-contractions was initially proposed by Vetro [10], and developed by Wardowski [11] and Arif and Imdad [12]. The fixed point findings over F-contractions in relational MS have been studied by a number of authors; see [13–15]. We refer the readers to the excellent work of Karapinar et al. [16] for more information about F-contractions.

A formal generalization of the BCP is almost contraction, originally proposed by Berinde [17]. The ordinary contraction, Chatterjea mapping [18], Kannan mapping [19], Zamfirescu contraction [20] and a particular family of quasicontractions [21] are all examples of almost contractions. It is noticeable that an almost contraction map doesn't need to own a unique fixed point. Nevertheless, the fixed point of an almost contraction map can be estimated as the limit of convergence of a Picard iteration. Alfuraidan et al. [22] conceptualized a nonlinear variation of an almost contraction. A more comprehensive analysis of almost contractions may be accessible through [23–25].

The primary goal of this study is to demonstrate a finding on a fixed point under a nonlinear almost F-contraction map in a MS endowed with locally $\mathcal{L}$-transitive BR. We offer a few instances that exemplify the efficacy of our results. Our finding allows us to expand the corresponding outcome of Sawangsup et al. [13]. Our outcome enables us to determine the solution of a special periodic BVP of order one.

2. Preliminaries

For the set $\boldsymbol{Q}$, any subset of $\boldsymbol{Q}^2$ is referred as a BR on $\boldsymbol{Q}$. In the next definitions, we assume that $\boldsymbol{Q}$ is a set composed with a metric ω and a BR Γ and $\mathcal{L}$ retains a self-map on $\boldsymbol{Q}$. Now, let us say that:

Definition 2.1. *[26] Inverse of* Γ *is the BR* $\Gamma^{-1} := \{(p, q) \in \mathbf{Q}^2 : (q, p) \in \Gamma\}$.

Definition 2.2. *[26] Symmetric closure of* Γ *is the BR* $\Gamma^s := \Gamma \cup \Gamma^{-1}$.

Definition 2.3. *[3]* $p, q \in \mathbf{Q}$ *forms* Γ*-comparative pair if* $(p, q) \in \Gamma$ *or* $(q, p) \in \Gamma$.
$[p, q] \in \Gamma$ *symbolizes this kind of pair.*

Remark 2.1. *[3]* $[p, q] \in \Gamma$ *iff* $(p, q) \in \Gamma^s$.

Definition 2.4. *[3] Any sequence* $\{p_n\} \subset \mathbf{Q}$ *with the property* $(p_n, p_{n+1}) \in \Gamma$, $\forall\, n \in \mathbb{N}$, *serves as* Γ*-preserving.*

Definition 2.5. *[27] A self-map* $\mathcal{L}$ *in MS* $(\mathbf{Q}, \omega)$ *retains a* Γ*-continuous map whenever for each* $p \in \mathbf{Q}$ *and for every* Γ*-preserving sequence* $\{p_n\} \subset \mathbf{Q}$ *verifying* $p_n \xrightarrow{\omega} p$, *we attain*

$$\mathcal{L}(p_n) \xrightarrow{\omega} \mathcal{L}(p).$$

Any continuous map has to, of naturally, be Γ-continuous. Additionally, both ideas coincide for $\Gamma = \boldsymbol{Q}^2$.

Definition 2.6. *[3]* Γ *remains* ω*-self-closed when any* Γ*-preserving convergent sequence in* $\mathbf{Q}$ *determines a subsequence, whose each term is* Γ*-comparative to the limit of sequence.*

Definition 2.7. *[27]* $(\mathbf{Q}, \omega)$ *is the* Γ*-complete MS wherein each* Γ*-preserving Cauchy sequence has to be convergent.*

Any complete MS has to, naturally, be Γ-complete. Additionally, both ideas coincide for $\Gamma = \boldsymbol{Q}^2$.

Definition 2.8. *[28] Given* $\mathbf{P} \subseteq \mathbf{Q}$, *the BR*

$$\Gamma|_{\mathbf{Q}} := \Gamma \cap \mathbf{Q}^2,$$

(on $\mathbf{P}$*), is restriction of* Γ *in* $\mathbf{P}$.

Definition 2.9. *[29]* Γ *remains locally* $\mathcal{L}$*-transitive if for each* Γ*-preserving sequence* $\{p_n\} \subset \mathcal{L}(\mathbf{P})$, $\Gamma|_{\mathbf{P}}$ *retains a transitive BR, where* $\mathbf{P} = \{p_n : n \in \mathbb{N}\}$.

Definition 2.10. *[3]* Γ *is an* $\mathcal{L}$*-closed BR for which*

$$(p, q) \in \Gamma \Rightarrow (\mathcal{L}p, \mathcal{L}q) \in \Gamma.$$

Proposition 2.1. *[29] Whenever* Γ *is* $\mathcal{L}$*-closed,* Γ *has to be* $\mathcal{L}^n$*-closed, for any* $n \in \mathbb{N}_0$.

Definition 2.11. *[30] A sequence* $\{p_n\}$ *in a MS* $(\mathbf{Q}, \omega)$ *serves as semi-Cauchy when*

$$\lim_{n\to\infty} \omega(p_n, p_{n+1}) = 0.$$

Any Cauchy sequence is naturally semi-Cauchy, but the following example reveals that contrary is not the case.

Example 2.1. *Consider* $\mathbf{Q} = \mathbb{R}$ *with metric* $\omega(p, q) = |p - q|$, $\forall\, p, q \in \mathbf{Q}$. *Then, the sequence* $\{p_n\} \subset \mathbf{Q}$ *defined by* $p_n = \sum_{\iota=1}^{n} \frac{1}{\iota}$ *remains semi-Cauchy but not Cauchy.*

Lemma 2.1. *[8] Let* $(\mathbf{Q}, \omega)$ *be a MS. Also, suppose that* $\{p_n\}$ *is a semi-Cauchy sequence, which is not Cauchy. For a countable subset* $\Delta \subset (0, \infty)$, $\exists\, \epsilon \in (0, \infty)\backslash\Delta$ *and a couple of subsequences* $\{p_{n_\kappa}\}$ *and* $\{p_{m_\kappa}\}$ *of* $\{p_n\}$ *that fulfill*

(1) $\kappa \le m_\kappa < n_\kappa$, $\forall\, \kappa \in \mathbb{N}$;
(2) $\omega(p_{m_\kappa}, p_{n_\kappa}) > \epsilon$, $\forall\, \kappa \in \mathbb{N}$;
(3) $\omega(p_{m_\kappa}, p_{n_\kappa - 1}) \le \epsilon$, $\forall\, \kappa \in \mathbb{N}$;
(4) $\lim_{\kappa\to\infty} \omega(p_{m_\kappa + r}, p_{n_\kappa + s}) = \epsilon$, $\forall\, r, s \in \{0, 1\}$.

Regarding the aftermath, $\mathfrak{F}$ indicates the set of functions $\mathrm{F} : (0, \infty) \to \mathbb{R}$ that confirm

(F_1): F remains strictly increasing;
(F_2): $\lim_{n\to\infty} \mathrm{F}(t_n) = -\infty \iff \lim_{n\to\infty} t_n = 0$.

This process is inspired by Berinde [17], Turinici [30], and Alfuraidan et al. [22], presented the collection Θ of the functions $\theta : [0, \infty) \to [0, \infty]$, that confirm

(θ_1): $\theta(0) = 0$;
(θ_2): $\lim_{t\to 0^+} \theta(t) = 0$.

3. Main results

We will now reveal our principal finding on fixed points.

Theorem 3.1. *Let* $(\mathbf{Q}, \omega)$ *be a MS equipped with a BR* Γ *and* $\mathcal{L}$ *a self-map on* $\mathbf{Q}$. *Also,*

(a) $(\mathbf{Q}, \omega)$ *retains* Γ*-complete;*
(b) $\exists\, p_0 \in \mathbf{Q}$ *that verifies* $(p_0, \mathcal{L}p_0) \in \Gamma$;
(c) Γ *remains* $\mathcal{L}$*-closed and locally* $\mathcal{L}$*-transitive;*
(d) $\mathcal{L}$ *remains* Γ*-continuous, or* Γ *remains* ω*-self-closed;*
(e) $\exists\, \tau > 0$, $\mathrm{F} \in \mathfrak{F}$ *and* $\theta \in \Theta$ *satisfying*

$$(p, q) \in \Gamma \text{ with } \mathcal{L}(p) \ne \mathcal{L}(q) \Rightarrow \tau + \mathrm{F}(\omega(\mathcal{L}p, \mathcal{L}q)) \le \mathrm{F}(\omega(p, q)) + \min\{\theta(\omega(p, \mathcal{L}p)), \theta(\omega(q, \mathcal{L}p))\}.$$

Then, $\mathcal{L}$ *owns a fixed point.*

Proof. We attempt to build up the proof in six stages that are listed below.
Step 1. Consider the following sequence $\{p_{\mathrm{n}}\} \subset \boldsymbol{Q}$

$$p_{\mathrm{n}} := \mathcal{L}^{\mathrm{n}}(p_0) = \mathcal{L}(p_{\mathrm{n}-1}), \quad \forall\, \mathrm{n} \in \mathbb{N}. \tag{3.1}$$

Step 2. We reveal that $\{p_{\mathrm{n}}\}$ is Γ-preserving. From (b), $\mathcal{L}$-closedness of Γ, and Proposition 2.1, we attain

$$(\mathcal{L}^{\mathrm{n}} p_0, \mathcal{L}^{\mathrm{n}+1} p_0) \in \Gamma,$$

which, using (3.1), becomes

$$(p_{\mathsf{n}}, p_{\mathsf{n}+1}) \in \Gamma, \quad \forall\, \mathsf{n} \in \mathbb{N}_0. \tag{3.2}$$

Step 3. Define $\omega_{\mathsf{n}} := \omega(p_{\mathsf{n}}, p_{\mathsf{n}+1})$. If $\exists\, \mathsf{n}_0 \in \mathbb{N}_0$ verifies $\mu_{\mathsf{n}_0} = 0$, then (3.1) allows $p_{\mathsf{n}_0} = p_{\mathsf{n}_0+1} = \mathcal{L}(p_{\mathsf{n}_0})$; therefore $p_{n_0} \in Fix(\mathcal{L})$, and we resolve as an illustration. Failing to do that, if we turn at $\omega_{\mathsf{n}} > 0$, $\forall$ $\mathsf{n} \in \mathbb{N}_0$, then we must go to Step 4.
Step 4. We are going to evaluate the semi-Cauchy character of $\{p_{\mathsf{n}}\}$. In the case $\rho_{\mathsf{n}} > 0$, $\forall\, \mathsf{n} \in \mathbb{N}_0$, through the data $\tau > 0$ and condition (e), we conclude for all $\mathsf{n} \in \mathbb{N}_0$ that

$$\tau + \mathrm{F}(\omega(\mathcal{L}p_{\mathsf{n}}, \mathcal{L}p_{\mathsf{n}+1})) \leq \mathrm{F}(\omega(p_{\mathsf{n}}, p_{\mathsf{n}+1})) + \min\{\theta(\omega(p_{\mathsf{n}}, \mathcal{L}p_{\mathsf{n}})), \theta(\omega(p_{\mathsf{n}+1}, \mathcal{L}p_{\mathsf{n}}))\},$$

which, employing axiom (θ_1), gives rise to

$$\tau + \mathrm{F}(\rho_{\mathsf{n}+1}) \leq \mathrm{F}(\rho_{\mathsf{n}}). \tag{3.3}$$

By (3.3), we conclude

$$\mathrm{F}(\rho_{i+1}) - \mathrm{F}(\rho_i) < -\tau, \quad \forall\, i \in \mathbb{N}_0.$$

Thereby, we attain

$$\begin{aligned} \mathrm{F}(\rho_{\mathsf{n}}) &= \mathrm{F}(\rho_{\mathsf{n}_0}) + \sum_{i=0}^{\mathsf{n}-1}[\mathrm{F}(\rho_{i+1}) - \mathrm{F}(\rho_i)] \\ &< \mathrm{F}(\rho_{\mathsf{n}_0}) - \mathsf{n}\tau, \quad \forall\, \mathsf{n} \in \mathbb{N}_0. \end{aligned}$$

Following the limit in (3.4), we find

$$\lim_{\mathsf{n}\to\infty} \mathrm{F}(\rho_{\mathsf{n}}) = -\infty.$$

Thus, we attain

$$\lim_{\mathsf{n}\to\infty} \omega(p_{\mathsf{n}}, p_{\mathsf{n}+1}) = 0. \tag{3.4}$$

Step 5. We will conclude that $\{p_{\mathsf{n}}\}$ is Cauchy. In contrast, we presume that $\{p_{\mathsf{n}}\}$ is not Cauchy. Employing (F_1), Δ (a set of points of discontinuity of F) must be at-most countable. Thereby, from (3.4) and Lemma 2.1, $\exists\, \epsilon > 0$, $\epsilon \in \Delta$, and subsequences $\{p_{\mathsf{n}_\kappa}\}$ and $\{p_{\mathsf{m}_\kappa}\}$ of $\{p_{\mathsf{n}}\}$ satisfying $\kappa \leq \mathsf{m}_\kappa < \mathsf{n}_\kappa$, $\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa}) > \epsilon$, and $\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa}) \leq \epsilon$. Further, we have

$$\lim_{\mathsf{n}\to\infty} \omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa}) = \epsilon \tag{3.5}$$

and

$$\lim_{\kappa\to\infty} \omega(p_{\mathsf{m}_\kappa+1}, p_{\mathsf{n}_\kappa+1}) = \epsilon. \tag{3.6}$$

Due to local transitivity, we attain $\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa}) \in \Gamma$. Now, by condition (e), we find

$$\tau + \mathrm{F}(\omega(p_{\mathsf{m}_\kappa+1}, p_{\mathsf{n}_\kappa+1})) \leq \mathrm{F}(\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa})) + \min\{\theta(\omega(p_{\mathsf{m}_\kappa}, \mathcal{L}p_{\mathsf{m}_\kappa})), \theta(\omega(p_{\mathsf{n}_\kappa}, \mathcal{L}p_{\mathsf{m}_\kappa})\},$$

or

$$\tau + \mathrm{F}(\omega(p_{\mathsf{m}_\kappa+1}, p_{\mathsf{n}_\kappa+1})) \leq \mathrm{F}(\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa})) + \min\{\theta(\omega_{\mathsf{m}_\kappa}), \theta(\omega(p_{\mathsf{n}_\kappa}, p_{\mathsf{m}_\kappa+1})\}. \tag{3.7}$$

By (3.5) and axiom (θ_2), we find

$$\lim_{\kappa\to\infty} \theta(\omega_{\mathsf{m}_\kappa}) = \lim_{t\to 0^+} \theta(t) = 0. \tag{3.8}$$

Following the lower limit in (3.7) and by (3.8), we get

$$\tau + \liminf_{\kappa\to\infty} \mathrm{F}(\omega(p_{\mathsf{m}_\kappa+1}, p_{\mathsf{n}_\kappa+1})) \le \liminf_{\kappa\to\infty} \mathrm{F}(\omega(p_{\mathsf{m}_\kappa}, p_{\mathsf{n}_\kappa})).$$

Utilizing the continuity of F at ϵ, (3.5), and (3.6), the above inequality reduces to

$$\tau + \mathrm{F}(\epsilon) \le \mathrm{F}(\epsilon),$$

thereby yielding

$$\tau \le 0,$$

which remains a contradiction. Therefore, $\{p_\mathsf{n}\}$ retains Cauchy. As $(\boldsymbol{Q}, \omega)$ is Γ-complete, $\exists\, p^* \in \mathcal{L}$ that verifies $p_\mathsf{n} \xrightarrow{\omega} p^*$.

Step 6. We will employ the condition (d) to ensure that $p^* \in Fix(Q)$. If Q is Γ-continuous, then $p_{\mathsf{n}+1} = \mathcal{L}(p_\mathsf{n}) \xrightarrow{\omega} \mathcal{L}(p^*)$. As a culmination, we acquire $\mathcal{L}(p^*) = p^*$.

Alternately, we assume that Γ remains ω-self-closed. Progressively, we estimate a subsequence of $\{p_{\mathsf{n}_\kappa}\}$ of $\{p_\mathsf{n}\}$ verifying $[p_{\mathsf{n}_\kappa}, p^*] \in \Gamma$, $\forall\, \kappa \in \mathbb{N}_0$. Define $\mathbb{N}_0 = \{\mathsf{n}_\kappa \in \mathbb{N} : \omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*) = 0\}$. If $\mathbb{N}_0$ is denumerable, then we write $\mathbb{N}_0 = \{\mathsf{n}_\iota : \iota \in \mathbb{N}\}$ so that $\{p_{\mathsf{n}_\iota}\}$ as a subsequence of $\{p_{\mathsf{n}_\kappa}\}$ remains also a subsequence of $\{p_\mathsf{n}\}$. Therefore, $p_{\mathsf{n}_\iota+1} \xrightarrow{\omega} \mathcal{L}(p^*)$.

If $\mathbb{N}_0$ is finite, then $\exists\, N \in \mathbb{N}$, verifying $\omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*) > 0$, $\forall\, \kappa \ge N$. Employing assumption (e), we get

$$\tau + \mathrm{F}(\omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*)) \le \mathrm{F}(\omega(p_{\mathsf{n}_\kappa}, p^*)) + \min\{\theta(p_{\mathsf{n}_\kappa}, p_{\mathsf{n}_\kappa+1}), \theta(p^*, p_{\mathsf{n}_\kappa+1})\}, \quad \forall\, \kappa \ge N,$$

which yields that

$$\mathrm{F}(\omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*)) \le \mathrm{F}(\omega(p_{\mathsf{n}_\kappa}, p^*)) + \min\{\theta(p_{\mathsf{n}_\kappa}, p_{\mathsf{n}_\kappa+1}), \theta(p^*, p_{\mathsf{n}_\kappa+1})\}, \quad \forall\, \kappa \ge N. \tag{3.9}$$

Following the limit of (3.9) as $\kappa \to \infty$ and using axiom (θ_2) and (3.4), we attain

$$\lim_{\kappa\to\infty} \mathrm{F}(\omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*)) \le \lim_{\kappa\to\infty} \mathrm{F}(\omega(p_{\mathsf{n}_\kappa}, p^*)) + 0,$$

which, using the axiom (F_2) and the fact that $p_{\mathsf{n}_\kappa} \xrightarrow{\omega} p^*$, gives rise to

$$\lim_{\kappa\to\infty} \mathrm{F}(\omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*)) = -\infty.$$

In lieu of axiom (F_2) again, the above equation yields that

$$\lim_{\kappa\to\infty} \omega(p_{\mathsf{n}_\kappa+1}, \mathcal{L}p^*) = 0.$$

Thus, in both cases, we conclude $p_{\mathsf{n}_\kappa+1} \xrightarrow{\omega} \mathcal{L}(p^*)$ so that $\mathcal{L}(p) = p^*$. Thus, p^* serves as a fixed point of $\mathcal{L}$. □

If we substitute $\theta(t) = 0$ in Theorem 3.1, then we find the following finding of Sawangsup et al. [13].

Corollary 3.1. *[13] Let $(\mathbf{Q}, \omega)$ be a MS equipped with a BR Γ and $\mathcal{L}$ a self-map on $\mathbf{Q}$. Also,*

(*a*) $(\mathbf{Q}, \omega)$ *remains* Γ*-complete;*
(*b*) $\exists\ p_0 \in \mathbf{Q}$ *with* $(p_0, \mathcal{L}p_0) \in \Gamma$*;*
(*c*) Γ *remains* $\mathcal{L}$*-closed and locally* $\mathcal{L}$*-transitive;*
(*d*) $\mathcal{L}$ *remains* Γ*-continuous, or* Γ *is* ω*-self-closed;*
(*e*) $\exists\ \tau > 0$, $\mathrm{F} \in \mathfrak{F}$ *and* $\theta \in \Theta$ *that satisfy*

$$(p, q) \in \Gamma \text{ with } \mathcal{L}(p) \neq \mathcal{L}(q) \Rightarrow \tau + \mathrm{F}(\omega(\mathcal{L}p, \mathcal{L}q)) \leq \mathrm{F}(\omega(p, q)).$$

Then, $\mathcal{L}$ *owns a fixed point.*

Under universal BR $\Gamma = \boldsymbol{Q}^2$, Theorem 3.1 leads to the corresponding result in an ordinary MS, which runs given below.

Corollary 3.2. *Let $(\mathbf{Q}, \omega)$ be a complete MS and $\mathcal{L}$ a self-map on $\mathbf{Q}$. If $\exists\ \tau > 0$, and $\mathrm{F} \in \mathfrak{F}$ that verifies*

$$p, q \in \mathbf{Q} \text{ with } \mathcal{L}(p) \neq \mathcal{L}(q) \Rightarrow \tau + \mathrm{F}(\omega(\mathcal{L}p, \mathcal{L}q)) \leq \mathrm{F}(\omega(p, q)) + \min\{\theta(\omega(p, \mathcal{L}p)), \theta(\omega(q, \mathcal{L}p))\},$$

then, $\mathcal{L}$ *owns a fixed point.*

4. Illustrative examples

We provide several scenarios that reveal the accuracy and practicality of our outcomes.

Example 4.1. *Let* $\mathbf{S} = \left\{\frac{1}{n^2} : n \in \mathbb{N}\right\} \cup \{0\}$, $\mathbf{S}' = \{2, 3\}$*, and* $\mathbf{Q} = \mathbf{S} \cup \mathbf{S}'$*. Take the following BR* Γ *on* $\mathbf{Q}$*:*

$$\Gamma := \{(p, q) \in \mathbf{Q}^2 : p = q \ \ or \ \ p, q \in \mathbf{S} \ \ with \ \ q - p \geq 0\}.$$

Clearly, $\mathbf{Q}$ *is a* Γ*-complete MS under standard metric* ω*. Define the following map* $\mathcal{L} : \mathbf{Q} \to \mathbf{Q}$*:*

$$\mathcal{L}(p) = \begin{cases} \dfrac{1}{(n+1)^2}, & \text{if } p = \dfrac{1}{n^2}, \\ p, & \text{if } p \in \{0, 2, 3\}. \end{cases}$$

It is apparent that $\mathcal{L}$ *is* Γ*-continuous, and* Γ *serves as* $\mathcal{L}$*-closed and locally* $\mathcal{L}$*-transitive. Apart from that, we possess* $(p_0, \mathcal{L}p_0) \in \Gamma$ *for* $p_0 = 0$*.*

Consider the following function $\mathrm{F} : (0, \infty) \to \mathbb{R}$*:*

$$\mathrm{F}(t) = \begin{cases} \dfrac{\ln t}{\sqrt{t}}, & \text{if } 0 < t < e^2, \\ t - e^2 + \frac{2}{e}, & \text{if } t \geq e^2. \end{cases}$$

In the interval $(0, e^2)$*, we attain*

$$\mathrm{F}(t) = \frac{\ln t}{\sqrt{t}},$$

so that

$$F'(t) = t^{-3/2}\left(1 - \frac{1}{2}\ln t\right) > 0.$$

Thereby, F *remains strictly increasing in the interval* $(0, e^2)$.

For $e^2 \le t < s$, *we find* $F(t) = t - e^2 + \frac{2}{e} < s - e^2 + \frac{2}{e} = F(s)$. *Henceforth,* F *remains strictly increasing in* $[e^2, \infty)$.

Also, for $0 < t < e^2 \le s$, *we conclude* $F(t) < F(s)$. *It yields that* F *remains strictly increasing.*

Overall, F *confirms axiom* (F_1). *Also,* F *verifies* (F_2). *Thereby, we attain* $F \in \mathfrak{F}$. *It is easy to verify the contraction-condition* (e) *for* $\tau = \ln 2$ *and* $\theta(t) = 2t$. *Because every requirement of Theorem 3.1 is met,* $\mathcal{L}$ *therefore possesses a fixed point. Indeed,* $Fix(\mathcal{L}) = \{0, 2, 3\}$ *here.*

Example 4.2. *Consider the MS* $\mathbf{Q} = [0,1] \cup \mathbb{N}$ *under the metric:*

$$\omega(p,q) = \begin{cases} |p-q|, & \text{when } p, q \in [0,1] \text{ but } p - q \neq 0, \\ p + q, & \text{either } p \text{ or } q \text{ does not lie in } [0,1] \text{ and } p - q \neq 0, \\ 0, & \text{when } p = q. \end{cases}$$

Take the following BR Γ *on* $\mathbf{Q}$*:*

$$\Gamma = \{(p,q) \in \mathbf{Q}^2 : p - q > 0,\ q \neq 2,\ \text{and } p \in \{3,4,5,..\}\}.$$

Clearly, $(\mathbf{Q}, \omega)$ *is* Γ*-complete. Define the following map* $\mathcal{L} : \mathbf{Q} \to \mathbf{Q}$*:*

$$\mathcal{L}(p) = \begin{cases} p - p^3/4, & \text{if } 0 \le p \le 1, \\ p - 1, & \text{if } p \in \{2,3,4,...\}. \end{cases}$$

It is apparent that $\mathcal{L}$ *is* Γ*-continuous, and* Γ *serves as* $\mathcal{L}$*-closed and locally* $\mathcal{L}$*-transitive.*

Consider the following function $F : (0, \infty) \to \mathbb{R}$:

$$F(t) = \begin{cases} t + 1, & \text{if } 0 \le t < 1, \\ t^2, & \text{if } t \ge 1. \end{cases}$$

Then $F \in \mathfrak{F}$. *Set* $\tau = 1/4$.

Let $(p,q) \in \Gamma$ *with* $\mathcal{L}(p) \neq \mathcal{L}(q)$. *Fixing* $p \in \{3,4,....\}$, *there are two options available for selecting* q. *First, we choose* $q \in [0,1]$, *then*

$$\begin{aligned} \omega(\mathcal{L}p, \mathcal{L}q) &= \omega\Big(p - 1, q - \frac{1}{4}q^3\Big) = p - 1 + q - \frac{q^3}{4} \\ &\le p + q - 1. \end{aligned}$$

When $q \in \{3,4,...\}$, *we find*

$$\begin{aligned} \omega(\mathcal{L}p, \mathcal{L}q) &= \omega(p-1, q-1) = p + q - 2 \\ &< p + q - 1. \end{aligned}$$

Consequently, in each scenario, we conclude

$$F(\omega(\mathcal{L}p, \mathcal{L}q)) = (\omega(\mathcal{L}p, \mathcal{L}q))^2 < (p + q - 1)^2$$

$$
\begin{aligned}
&< (p+q-1)(p+q+1) = (p+q)^2 - 1 \\
&< (p+q)^2 - \frac{1}{4} \\
&= \mathrm{F}(\omega(p,q)) - \tau,
\end{aligned}
$$

so that

$$\tau + \mathrm{F}(\omega(\mathcal{L}p, \mathcal{L}q)) < \mathrm{F}(\omega(p,q)) + \min\{\theta(\omega(p, \mathcal{L}p)), \theta(\omega(q, \mathcal{L}p))\},$$

where $\theta \in \Theta$ is an arbitrary. Thus, the contraction-condition (e) holds.

Because every requirement of Theorem 3.1 is met, $\mathcal{L}$ therefore possesses a fixed point, namely: $p = 0$.

5. Existence for a solution to a periodic BVP

In this part, we describe the availability of a solution for a first-order BVP:

$$\begin{cases} \mathsf{v}'(\mathsf{r}) = F(\mathsf{r}, \mathsf{v}(\mathsf{r})), & \mathsf{r} \in [0, \rho], \\ \mathsf{v}(0) = \mathsf{v}(\rho), \end{cases} \tag{5.1}$$

where $\rho > 0$ and the function $F : [0, \rho] \times \mathbb{R} \to \mathbb{R}$ remains continuous.

Definition 5.1. *[31] $v_0 \in C^1[0,\rho]$ is considered a lower solution of* (5.1) *if*

$$\begin{cases} v_0'(r) \le F(r, v_0(r)), & r \in [0, \rho], \\ v_0(0) \le v_0(\rho). \end{cases}$$

Definition 5.2. *[31] $v_0 \in C^1[0,\rho]$ is considered an upper solution of* (5.1) *if*

$$\begin{cases} v_0'(r) \ge F(r, v_0(r)), & r \in [0, \rho], \\ v_0(0) \ge v_0(\rho). \end{cases}$$

We demonstrate the assertions associated with the possibility of solutions to problem (5.1) when a lower solution or an upper solution exists.

Theorem 5.1. *In conjunction with the problem* (5.1)*, if $\exists\, \varepsilon > 1$ such that $\forall\, \alpha, \beta \in \mathbb{R}$ with $\alpha \le \beta$ that*

$$0 \le F(r, \beta) + \epsilon\beta - F(r, \alpha) - \epsilon\alpha \le \beta - \alpha \tag{5.2}$$

and (5.1) *has a lower solution, then problem* (5.1) *admits a solution.*

Proof. Rewrite the problem (5.1) as

$$\begin{cases} \mathsf{v}'(\mathsf{r}) + \varepsilon \mathsf{v}(\mathsf{r}) = F(\mathsf{r}, \mathsf{v}(\mathsf{r})) + \varepsilon \mathsf{v}(\mathsf{r}), & \forall\, \mathsf{r} \in [0, \rho], \\ \mathsf{v}(0) = \mathsf{v}(\rho), \end{cases} \tag{5.3}$$

and (5.3) is equated to the integral equation

$$\mathsf{v}(\mathsf{r}) = \int_0^\rho \mathbb{G}(\mathsf{r}, \upsilon)[F(\upsilon, \mathsf{v}(\upsilon)) + \varepsilon \mathsf{v}(\upsilon)]d\upsilon, \tag{5.4}$$

where

$$\mathbb{G}(\mathsf{r},\upsilon)=\begin{cases}\frac{e^{\varepsilon(\rho+\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}, & 0\le\upsilon<\mathsf{r}\le\rho,\\ \frac{e^{\varepsilon(\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}, & 0\le\mathsf{r}<\upsilon\le\rho,\end{cases}$$

defines the Green function. Define the following map $\mathcal{L}: C[0,\rho]\to C[0,\rho]$:

$$(\mathcal{L}\mathsf{v})(\mathsf{r})=\int_0^\rho \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}(\upsilon))+\varepsilon\mathsf{v}(\upsilon)]d\upsilon,\quad \forall\ \mathsf{r}\in[0,\rho]. \tag{5.5}$$

Thus, $\mathsf{v}\in Fix(\mathcal{L})$ iff $\mathsf{v}\in C^1[0,\rho]$ retains a solution of (5.4), and so, of (5.1).

Consider the following metric ω on $C[0,\rho]$:

$$\omega(\mathsf{v},\mathsf{w})=\sup_{\mathsf{r}\in[0,\rho]}|\mathsf{v}(\mathsf{r})-\mathsf{w}(\mathsf{r})|,\quad \forall\ \mathsf{v},\mathsf{w}\in C[0,\rho]. \tag{5.6}$$

Consider the following BR Γ on $C[0,\rho]$:

$$\Gamma=\{(\mathsf{v},\mathsf{w}):\mathsf{v}(\mathsf{r})\le\mathsf{w}(\mathsf{r}),\ \forall\ \mathsf{r}\in[0,\rho]\}. \tag{5.7}$$

We will now fulfill each of the promises of Theorem 3.1.

(a) The MS $(C[0,\rho],\omega)$ remains Γ-complete.

(b) If $\mathsf{v}_0\in C^1[0,\rho]$ retains the lower solution of (5.1), then we attain

$$\mathsf{v}_0'(\mathsf{r})+\varepsilon\mathsf{v}_0(\mathsf{r})\le F(\mathsf{r},\mathsf{v}_0(\mathsf{r}))+\varepsilon\mathsf{v}_0(\mathsf{r}),\quad \forall\ \mathsf{r}\in[0,\rho].$$

The multiplication of $e^{\varepsilon\mathsf{r}}$ on both sides yields

$$(\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}})'\le[F(\mathsf{r},\mathsf{v}_0(\mathsf{r}))+\varepsilon\mathsf{v}_0(\mathsf{r})]e^{\varepsilon\mathsf{r}},\quad \forall\ \mathsf{r}\in[0,\rho],$$

thereby yielding

$$\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}}\le\mathsf{v}_0(0)+\int_0^{\mathsf{r}}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]e^{\varepsilon\upsilon}d\upsilon,\quad \forall\ \mathsf{r}\in[0,\rho]. \tag{5.8}$$

With regard to $\mathsf{v}_0(0)\le\mathsf{v}_0(\rho)$, we find

$$\mathsf{v}_0(0)e^{\varepsilon\rho}\le\mathsf{v}_0(\rho)e^{\varepsilon\rho}\le\mathsf{v}_0(0)+\int_0^\rho[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]e^{\varepsilon\upsilon}d\upsilon,$$

so that

$$\mathsf{v}_0(0)\le\int_0^\rho\frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon. \tag{5.9}$$

By (5.8) and (5.9), we find

$$\begin{aligned}\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}} &\le \int_0^\rho\frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon+\int_0^{\mathsf{r}}e^{\varepsilon\upsilon}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon\\ &= \int_0^{\mathsf{r}}\frac{e^{\varepsilon(\rho+\upsilon)}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon+\int_{\mathsf{r}}^\rho\frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon,\end{aligned}$$

so that

$$\begin{aligned} \mathsf{v}_0(\mathsf{r}) &\leq \int_0^{\mathsf{r}} \frac{e^{\varepsilon(\rho+\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon + \int_{\mathsf{r}}^{\rho} \frac{e^{\varepsilon(\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon \\ &= \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon \\ &= (\mathcal{L}\mathsf{v}_0)(\mathsf{r}),\ \forall\ \mathsf{r}\in[0,\rho], \end{aligned}$$

which yields that $(\mathsf{v}_0, \mathcal{L}\mathsf{v}_0) \in \Gamma$.

(c) Take $\mathsf{v}, \mathsf{w} \in C[0,\rho]$ verifying $(\mathsf{v}, \mathsf{w}) \in \Gamma$. Then, for any $\forall\ \mathsf{r} \in [0,\rho]$, we conclude $\mathsf{v}(\mathsf{r}) \leq \mathsf{w}(\mathsf{r})$, thereby yielding $\varepsilon\mathsf{v}(\mathsf{r}) \leq \varepsilon\mathsf{w}(\mathsf{r})$. Also, from (5.2), we attain $F(\mathsf{r},\mathsf{v}(\mathsf{r})) \leq F(\mathsf{r},\mathsf{w}(\mathsf{r}))$. On adding, foregoing inequalities, we find

$$F(\mathsf{r},\mathsf{v}(\mathsf{r}))+\varepsilon\mathsf{v}(\mathsf{r}) \leq F(\mathsf{r},\mathsf{w}(\mathsf{r}))+\varepsilon\mathsf{w}(\mathsf{r}), \quad \forall\ \mathsf{r}\in[0,\rho]. \tag{5.10}$$

Using $\mathbb{G}(\mathsf{r},\upsilon) > 0$, $\forall\ (\mathsf{r},\upsilon) \in [0,\rho]^2$; and by (5.5) and (5.10), we conclude

$$\begin{aligned} (\mathcal{L}\mathsf{v})(\mathsf{r}) &= \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}(\upsilon))+\varepsilon\mathsf{v}(\upsilon)]d\upsilon \\ &\leq \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{w}(\upsilon))+\varepsilon\mathsf{w}(\upsilon)]d\upsilon \\ &= (\mathcal{L}\mathsf{w})(\mathsf{r}), \quad \forall\ \mathsf{r}\in[0,\rho], \end{aligned}$$

which, through (5.7), leads to $(\mathcal{L}\mathsf{v}, \mathcal{L}\mathsf{w}) \in \Gamma$ and so Γ is $\mathcal{L}$-closed.

(d) If $\{\mathsf{v}_n\} \subset C[0,\rho]$ is an Γ-preserving sequence and converging to $\mathsf{v} \in C[0,\rho]$, then for any $\mathsf{r} \in [0,\rho]$, the sequence $\{\mathsf{v}_n(\mathsf{r})\} \subset \mathbb{R}$ will remain monotonically increasing as it converges to $\mathsf{v}(\mathsf{r})$. Thus, with $\forall\ n \in \mathbb{N}_0$ and $\forall\ \mathsf{r} \in [0,\rho]$, we conclude $\mathsf{v}_n(\mathsf{r}) \leq \mathsf{v}(\mathsf{r})$. Hence from (5.7), we find that $(\mathsf{v}_n, \mathsf{v}) \in \Gamma,\ \forall\ n \in \mathbb{N}$. This means that Γ is ω-self-closed.

(e) Take $\mathsf{v}, \mathsf{w} \in C[0,\rho]$ with $(\mathsf{v}, \mathsf{w}) \in \Gamma$. Employing (5.2), (5.5), and (5.6), we attain

$$\begin{aligned} \omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}) &= \sup_{\mathsf{r}\in[0,\rho]} |(\mathcal{L}\mathsf{v})(\mathsf{r})-(\mathcal{L}\mathsf{w})(\mathsf{r})| = \sup_{\mathsf{r}\in[0,\rho]} ((\mathcal{L}\mathsf{w})(\mathsf{r})-(\mathcal{L}\mathsf{v})(\mathsf{r})) \\ &\leq \sup_{\mathsf{r}\in[0,\rho]} \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{w}(\upsilon))+\varepsilon\mathsf{w}(\upsilon)-F(\upsilon,\mathsf{v}(\upsilon))-\varepsilon\mathsf{v}(\upsilon)]d\upsilon \\ &\leq \sup_{\mathsf{r}\in I} \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)\,(\mathsf{w}(\upsilon)-\mathsf{v}(\upsilon))\,d\upsilon. \end{aligned} \tag{5.11}$$

Because, $0 \leq \mathsf{w}(\upsilon) - \mathsf{v}(\upsilon) \leq \omega(\mathsf{v}, \mathsf{w}))$, (5.11) reduces to

$$\begin{aligned} \omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}) &\leq \omega(\mathsf{v},\mathsf{w}) \sup_{\mathsf{r}\in[0,\rho]} \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)\omega\upsilon \\ &= \omega(\mathsf{v},\mathsf{w}) \sup_{\mathsf{r}\in[0,\rho]} \frac{1}{e^{\varepsilon\rho}-1}\Big(\frac{1}{\varepsilon}e^{\varepsilon(\rho+\upsilon-\mathsf{r})}\Big]_0^{\mathsf{r}} + \frac{1}{\varepsilon}e^{\varepsilon(\upsilon-\mathsf{r})}\Big]_{\mathsf{r}}^{\rho}\Big) \\ &= \omega(\mathsf{v},\mathsf{w})\frac{1}{\varepsilon(e^{\varepsilon\rho}-1)}(e^{\varepsilon\rho}-1), \end{aligned}$$

so that

$$\varepsilon\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}) \leq \omega(\mathsf{v},\mathsf{w}).$$

Taking the logarithm on both sides, we obtain

$$\ln\varepsilon + \ln\omega(\mathcal{L}\mathsf{v}, \mathcal{L}\mathsf{w}) \le \ln\omega(\mathsf{v}, \mathsf{w}),$$

thereby implying

$$\tau + \mathrm{F}(\omega(\mathcal{L}\mathsf{v}, \mathcal{L}\mathsf{w})) \le \mathrm{F}(\omega(\mathsf{v}, \mathsf{w})) + \min\{\theta(\omega(\mathsf{v}, \mathcal{L}\mathsf{v})), \theta(\omega(\mathsf{w}, \mathcal{L}\mathsf{v}))\},$$

$$\forall\, \mathsf{v}, \mathsf{w} \in C[0, \rho] \text{ with } (\mathsf{v}, \mathsf{w}) \in \Gamma,$$

and $\mathcal{L}(\mathsf{v}) \neq \mathcal{L}(\mathsf{w})$, where $\tau = \ln\varepsilon > 0$, $\mathrm{F}(t) = \ln t$, and $\theta \in \Theta$ is arbitrary.

Thereby, owing to Theorem 3.1, $\mathcal{L}$ admits a fixed point, which forms a solution to problem (5.1). □

Theorem 5.2. *In conjunction with the problem* (5.1)*, if* $\exists\, \varepsilon > 1$ *such that* $\forall\, \alpha, \beta \in \mathbb{R}$ *with* $\alpha \le \beta$*, we have*

$$0 \le F(\mathsf{r}, \beta) + \epsilon\beta - F(\mathsf{r}, \alpha) - \epsilon\alpha \le \beta - \alpha \tag{5.12}$$

and (5.1) *has an upper solution, then problem* (5.1) *admits a solution.*

Proof. Take $C[0, \rho]$ with a metric ω and a map $\mathcal{L} : C[0, \rho] \to C[0, \rho]$ to Theorem 5.1. Consider a BR **S** on $C[0, \rho]$ as below:

$$\Gamma' = \{(\mathsf{v}, \mathsf{w}) : \mathsf{v}(\mathsf{r}) \ge \mathsf{w}(\mathsf{r}),\ \forall\, \mathsf{r} \in [0, \rho]\}. \tag{5.13}$$

We will now fulfill each of the promises of Theorem 3.1.

(a) The MS $(C[0, \rho], \omega)$ remains Γ'-complete.

(b) If $\mathsf{v}_0 \in C^1[0, \rho]$ forms an upper solution of (5.1), then we conclude

$$\mathsf{v}_0'(\mathsf{r}) + \varepsilon\mathsf{v}_0(\mathsf{r}) \ge F(\mathsf{r}, \mathsf{v}_0(\mathsf{r})) + \varepsilon\mathsf{v}_0(\mathsf{r}), \quad \forall\, \mathsf{r} \in [0, \rho].$$

The multiplication of $e^{\varepsilon\mathsf{r}}$ on both sides yields

$$(\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}})' \ge [F(\mathsf{r}, \mathsf{v}_0(\mathsf{r})) + \varepsilon\mathsf{v}_0(\mathsf{r})]e^{\varepsilon\mathsf{r}}, \quad \forall\, \mathsf{r} \in [0, \rho]$$

thereby yielding

$$\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}} \ge \mathsf{v}_0(0) + \int_0^{\mathsf{r}} [F(\upsilon, \mathsf{v}_0(\upsilon)) + \varepsilon\mathsf{v}_0(\upsilon)]e^{\varepsilon\upsilon}d\upsilon, \quad \forall\, \mathsf{r} \in [0, \rho]. \tag{5.14}$$

With regard to $\mathsf{v}_0(0) \ge \mathsf{v}_0(\rho)$, we find

$$\mathsf{v}_0(0)e^{\varepsilon\rho} \ge \mathsf{v}_0(\rho)e^{\varepsilon\rho} \ge \mathsf{v}_0(0) + \int_0^{\rho} [F(\upsilon, \mathsf{v}_0(\upsilon)) + \varepsilon\mathsf{v}_0(\upsilon)]e^{\varepsilon\upsilon}d\upsilon$$

so that

$$\mathsf{v}_0(0) \ge \int_0^{\rho} \frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho} - 1}[F(\upsilon, \mathsf{v}_0(\upsilon)) + \varepsilon\mathsf{v}_0(\upsilon)]d\upsilon. \tag{5.15}$$

By (5.14) and (5.15), we find

$$\mathsf{v}_0(\mathsf{r})e^{\varepsilon\mathsf{r}} \ \ge\ \int_0^{\rho} \frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho} - 1}[F(\upsilon, \mathsf{v}_0(\upsilon)) + \varepsilon\mathsf{v}_0(\upsilon)]d\upsilon + \int_0^{\mathsf{r}} e^{\varepsilon\upsilon}[F(\upsilon, \mathsf{v}_0(\upsilon)) + \varepsilon\mathsf{v}_0(\upsilon)]d\upsilon$$

$$= \int_0^{\mathsf{r}} \frac{e^{\varepsilon(\rho+\upsilon)}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon + \int_{\mathsf{r}}^{\rho} \frac{e^{\varepsilon\upsilon}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon,$$

so that

$$\begin{aligned}
\mathsf{v}_0(\mathsf{r}) &\geq \int_0^{\mathsf{r}} \frac{e^{\varepsilon(\rho+\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon + \int_{\mathsf{r}}^{\rho} \frac{e^{\varepsilon(\upsilon-\mathsf{r})}}{e^{\varepsilon\rho}-1}[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon \\
&= \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}_0(\upsilon))+\varepsilon\mathsf{v}_0(\upsilon)]d\upsilon \\
&= (\mathcal{L}\mathsf{v}_0)(\mathsf{r}),\ \forall\ \mathsf{r}\in[0,\rho],
\end{aligned}$$

which yields that $(\mathsf{v}_0,\mathcal{L}\mathsf{v}_0)\in\Gamma'$.

(*c*) Take $\mathsf{v},\mathsf{w}\in C[0,\rho]$ verifying $(\mathsf{v},\mathsf{w})\in\Gamma'$. Then, for any $\forall\ \mathsf{r}\in[0,\rho]$, we conclude $\mathsf{v}(\mathsf{r})\geq\mathsf{w}(\mathsf{r})$, thereby yielding $\varepsilon\mathsf{v}(\mathsf{r})\geq\varepsilon\mathsf{w}(\mathsf{r})$. Also, from (5.12), we attain $F(\mathsf{r},\mathsf{v}(\mathsf{r}))\geq F(\mathsf{r},\mathsf{w}(\mathsf{r}))$. On adding, foregoing inequalities, we find

$$F(\mathsf{r},\mathsf{v}(\mathsf{r}))+\varepsilon\mathsf{v}(\mathsf{r})\geq F(\mathsf{r},\mathsf{w}(\mathsf{r}))+\varepsilon\mathsf{w}(\mathsf{r}),\quad \forall\ \mathsf{r}\in[0,\rho]. \tag{5.16}$$

Employing $\mathbb{G}(\mathsf{r},\upsilon)>0$, $\forall\ (\mathsf{r},\upsilon)\in[0,\rho]^2$ and Eqs (5.5) and (5.16), we obtain

$$\begin{aligned}
(\mathcal{L}\mathsf{v})(\mathsf{r}) &= \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}(\upsilon))+\varepsilon\mathsf{v}(\upsilon)]d\upsilon \\
&\geq \int_0^{\rho} \mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{w}(\upsilon))+\varepsilon\mathsf{w}(\upsilon)]d\upsilon \\
&= (\mathcal{L}\mathsf{w})(\mathsf{r}),\quad \forall\ \mathsf{r}\in[0,\rho],
\end{aligned}$$

which, through (5.13), leads to $(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w})\in\Gamma'$, and so Γ' is $\mathcal{L}$-closed.

(*d*) If $\{\mathsf{v}_n\}\subset C[0,\rho]$ is an Γ'-preserving sequence and converges to $\mathsf{v}\in C[0,\rho]$, then for any $\mathsf{r}\in[0,\rho]$, the sequence $\{\mathsf{v}_n(\mathsf{r})\}\subset\mathbb{R}$ will remain monotonically decreasing as it converges to $\mathsf{v}(\mathsf{r})$. Thus, $\forall\ n\in\mathbb{N}_0$, and $\forall\ \mathsf{r}\in[0,\rho]$, so we conclude $\mathsf{v}_n(\mathsf{r})\geq\mathsf{v}(\mathsf{r})$. Hence from (5.7), we find that $(\mathsf{v}_n,\mathsf{v})\in\Gamma'$, $\forall\ n\in\mathbb{N}$. This means that Γ' is ω-self-closed.

(*e*) Take $\mathsf{v},\mathsf{w}\in C[0,\rho]$ with $(\mathsf{v},\mathsf{w})\in\Gamma'$. By (5.2), (5.5) and (5.6), we have

$$\begin{aligned}
\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}) &= \sup_{\mathsf{r}\in[0,\rho]}|(\mathcal{L}\mathsf{v})(\mathsf{r})-(\mathcal{L}\mathsf{w})(\mathsf{r})| = \sup_{\mathsf{r}\in[0,\rho]}((\mathcal{L}\mathsf{v})(\mathsf{r})-(\mathcal{L}\mathsf{w})(\mathsf{r})) \\
&\leq \sup_{\mathsf{r}\in[0,\rho]}\int_0^{\rho}\mathbb{G}(\mathsf{r},\upsilon)[F(\upsilon,\mathsf{v}(\upsilon))+\varepsilon\mathsf{v}(\upsilon)-F(\upsilon,\mathsf{w}(\upsilon))-\varepsilon\mathsf{w}(\upsilon)]d\upsilon \\
&\leq \sup_{\mathsf{r}\in I}\int_0^{\rho}\mathbb{G}(\mathsf{r},\upsilon)\,(\mathsf{v}(\upsilon)-\mathsf{w}(\upsilon))\,d\upsilon.
\end{aligned}$$

Because, $0\leq\mathsf{v}(\upsilon)-\mathsf{w}(\upsilon)\leq\omega(\mathsf{v},\mathsf{w}))$, (5.17) reduces to

$$\begin{aligned}
\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}) &\leq \omega(\mathsf{v},\mathsf{w})\sup_{\mathsf{r}\in[0,\rho]}\int_0^{\rho}\mathbb{G}(\mathsf{r},\upsilon)\omega\upsilon \\
&= \omega(\mathsf{v},\mathsf{w})\sup_{\mathsf{r}\in[0,\rho]}\frac{1}{e^{\varepsilon\rho}-1}\Big(\frac{1}{\varepsilon}e^{\varepsilon(\rho+\upsilon-\mathsf{r})}\Big]_0^{\mathsf{r}}+\frac{1}{\varepsilon}e^{\varepsilon(\upsilon-\mathsf{r})}\Big]_{\mathsf{r}}^{\rho}\Big) \\
&= \omega(\mathsf{v},\mathsf{w})\frac{1}{\varepsilon(e^{\varepsilon\rho}-1)}(e^{\varepsilon\rho}-1),
\end{aligned}$$

so that

$$\varepsilon\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w})\leq\omega(\mathsf{v},\mathsf{w}).$$

Taking the logarithm on both sides, we obtain

$$\ln\varepsilon+\ln\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w})\leq\ln\omega(\mathsf{v},\mathsf{w}),$$

thereby implying

$$\tau+\mathrm{F}(\omega(\mathcal{L}\mathsf{v},\mathcal{L}\mathsf{w}))\leq\mathrm{F}(\omega(\mathsf{v},\mathsf{w}))+\min\{\theta(\omega(\mathsf{v},\mathcal{L}\mathsf{v})),\theta(\omega(\mathsf{w},\mathcal{L}\mathsf{v}))\},$$

$$\forall\ \mathsf{v},\mathsf{w}\in\mathcal{C}[0,\rho]\text{ with }(\mathsf{v},\mathsf{w})\in\Gamma'$$

and

$$\mathcal{L}(\mathsf{v})\neq\mathcal{L}(\mathsf{w}),$$

where $\tau=\ln\varepsilon>0$, $\mathrm{F}(t)=\ln t$, and $\theta\in\Theta$ is arbitrary.

Thereby, owing to Theorem 3.1, $\mathcal{L}$ admits a fixed point, which retains a solution to problem (5.1). □

6. Conclusions

We investigated specific fixed-point findings over relational nonlinear almost F-contractions. We incorporated two instances and an application to first-order periodic BVP to emphasize the relevance of the rationale and the scope of our outcomes. An adequate contraction condition that merely pertains to the pairings of comparative elements—not all elements—was one of the achievements of this effort.

Regarding the impact of the relation-theoretic fixed point model, we suggest the following possible further investigation programs:

(1) Enhancing the properties of the test functions F and θ;
(2) Sharpening our findings in different metrical frameworks, viz, dislocated MS, quasi-MS, partial MS, fuzzy MS, and so forth;
(3) Expanding our findings by examining the common fixed-point theorems;
(4) Integrating our ideas to specific first-order differential equations (cf. [32]), matrix equations (cf. [33]), integral equations (cf. [34]), fractional differential equations (cf. [35]), and elastic beam equations (cf. [36]).

Author contributions

Faizan Ahmad Khan: Writing–original draft, formal analysis, project administration; Ahmed Alamer: Writing–original draft, methodology, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors announce that they are free from conflicting interests.

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