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Research article

A refinement of Jensen's inequality through abstract convexity

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Abstract: In this study, the Jensen inequality has been refined. To achieve this, we employ the fact that an abstract convex function with certain structural properties can be bounded from below in a neighborhood of its global minimum in terms of the norm of its gradient. Some examples have been provided to show the extent of refinement.

Keywords: Jensen inequality; abstract convexity; refinement; optimization
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1. Introduction

Jensen's inequality is one of the fundamental inequalities in convex analysis, which states that for a convex function, the weighted average of the function values at finitely many points is greater than or equal to the function value at the weighted average of those points. More precisely, for a function defined on a convex subset U of $\mathbb{R}$, $f : U \to \mathbb{R}$, the points $x_1, x_2, ..., x_n \in U$, and the weights $a_1, a_2, ..., a_n \in [0, 1]$ such that $\sum a_i = 1$, and the following inequality holds:

$$f(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \leq a_1f(x_1) + a_2f(x_2) + \cdots + a_nf(x_n).$$

It is a fundamental property of convex functions, which enables it to be related to probability and measure theory. On the other hand, its application to different convex functions leads to new inequalities. Hence, it has a wide range of applications from mathematical analysis to information theory.

Naturally, many extensions, refinements, and generalizations of the Jensen inequality have been established by a wide range of researchers in very different directions. One direction is through abstract convexity. In this paper, a refinement of Jensen's inequality is obtained with the help of abstract convexity.

Abstract convexity of a function with respect to a class of certain functions is based on the principle that an abstract convex function can be written as an envelope of the class of functions it majorizes. A simple example of this principle in classical mathematical analysis is that a differentiable convex function can be written as the supremum of tangent lines that touch its graph. Similarly, abstract concavity of a function is given with respect to the functions it minorizes, and it can be represented as the infimum of these functions. Further examples of abstract convex functions can be seen in [1, 3, 6, 10]. Naturally, the relation between abstract convexity and optimization has been investigated, and the optimality conditions for abstract convex functions have been established. In particular, in [11], the lower bounds for certain abstract concave functions in the neighborhood of its global minimum were obtained, which have been quite useful in refining and generalizing some well-known inequalities to some extent. By means of these results, sharper versions of several inequalities have been derived, including the Hölder inequality in [14], the Brunn–Minkowski inequality in [15], the Bergström inequality in [16] and generalized power mean inequalities with negative exponents in [17]. Note that those inequalities are related to the vectors in nature, whereas Jensen's inequality establishes a functional inequality. To our knowledge, the present work is the first to employ abstract convexity in refining a functional inequality of Jensen type.

We also note that the use of abstract convexity in refining Jensen's inequality in this article requires certain differentiability assumptions on the function. This stems from the fact that functions possessing suitable smoothness properties are abstract concave with respect to a class of quadratic functions [11], and for such functions, necessary and sufficient conditions for the existence of a global minimizer are available. It should be emphasized that the availability of smooth elementary functions with strong optimality properties is exceptional rather than typical in abstract convexity. The results established in [11] are not directly tractable for general abstract concave or convex functions, and in particular, they do not readily extend to nonsmooth settings. Such limitations are intrinsic to the general envelope-based framework of abstract convexity.

At this point, it is worth emphasizing that these limitations are not accidental but stem from the general structure of the theory. Although abstract convexity is formulated for real-valued functions through envelope representations and is theoretically supported by generalized conjugation and subdifferential frameworks (see [9, 10, 13]), its practical applicability often encounters difficulties related to the choice and properties of the elementary functions involved.

In classical convex analysis, subgradients and separation admit canonical computational representations, such as supporting hyperplanes, proximal mappings, and cone-based structures. In contrast, abstract convexity typically replaces these linear objects with families of elementary functions or hull-based separation properties, which may significantly complicate algorithm design and implementation.

For instance, employing min-type representations of increasing positively homogeneous functions in minimization problems leads to subproblems of combinatorial complexity in cutting-plane algorithms due to the intrinsic nonsmoothness of min-type functions [2]. As emphasized in [4], the selection of an appropriate class of elementary functions constitutes a central computational bottleneck: The class must be sufficiently simple to enable efficient computation yet sufficiently rich to provide accurate approximations, and achieving this balance is generally nontrivial.

Historically, the theory of abstract convexity has evolved primarily along duality and structural directions, while geometric intuition and algorithmic development have received comparatively less systematic attention. Even when dual formulations are theoretically well established, the absence of standardized solution procedures may delay practical adoption. These structural and methodological challenges help explain why the expansion of abstract convexity into broader computational applications has progressed more slowly than its classical convex counterpart.

In the next section, we present the notion of abstract convexity and some basic results that will be used in obtaining our main result. Then we give the main result in the following section.

2. Preliminaries

We start with the basic notations used in this article. $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ denote n-dimensional Euclidean space and nonnegative and positive hyperoctants, respectively.

X is a Hilbert space with the inner product $\langle .,. \rangle$ and the corresponding norm $\|x\| = \sqrt{\langle x, x \rangle}$; $B(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$ is a closed ball. $B_\infty(x_0, r)$ corresponds to the closed ball in maximum norm ($\|\cdot\|_\infty$).

Let $\Omega \subset X$ and $f, g : \Omega \to \mathbb{R}$. If $g(x) \leq f(x)$ for all $x \in \Omega$, then we write $g \leq f$ on Ω. Assume that H is the class of certain real-valued functions on Ω. The function $f : \Omega \to \mathbb{R}$ is said to be minorized by H if there exists a function $h \in H$ such that $h \leq f$.

Definition 1. Let $f : \Omega \to \mathbb{R}$ and H be the set of functions $h : \Omega \to \mathbb{R}$. The function f is abstract convex with respect to H (or $H - convex$) if there exists a set $U \subset H$ such that

$$f(x) = \sup_{h \in U} h(x)$$

for all $x \in \Omega$.

Note that the original definition in [10] uses extended real line and includes $+\infty$ and $-\infty$. The definition of an abstract concave function differs from that of an abstract convex function; while an abstract convex function is defined as the supremum of elementary functions that minorize it, an abstract concave function is the infimum of those that majorize it.

Let H be the set of all quadratic functions h of the form

$$h(x) = a\,\|x\|^2 + \langle l, x \rangle + c, \quad x \in X, \tag{2.1}$$

where $a > 0$, $l \in X$ and $c \in \mathbb{R}$.

Let $\Omega \subset X$ and let H be the set of quadratic functions given in (2.1). A function $f : \Omega \to \mathbb{R}$ is abstract concave with respect to H if and only if f is majorized by H and f is upper semicontinuous (see [10]).

The proposition below asserts that the function f satisfying certain conditions can be abstract concave with respect to the functions in the form of (2.1) (see [11]).

Proposition 2. [11] Let $\Omega \subset X$ be a convex set and f be a differentiable mapping defined on an open set including Ω. Suppose that the mapping $x \longmapsto \nabla f(x)$ is Lipschitz on Ω :

$$K := \sup_{x,y\in\Omega, x\neq y} \tfrac{\|\nabla f(x)-\nabla f(y)\|}{\|x-y\|} < +\infty.$$

Let $a \geq K$ and, for each $t \in \Omega$,

$$f_t(x) = f(t) + \langle \nabla f(t), x - t\rangle + a\|x - t\|^2, \quad x \in X.$$

Then $f(x) = \min_{t\in\Omega} f_t(x)$, $x \in \Omega$.

In [11], the optimality conditions for the function f that can be expressed as the infimum of quadratic functions over a convex set are studied. One of the important results in that study is about the unconstrained minimization of a function $f : X \to \mathbb{R}$ satisfying $\|\nabla f(x) - \nabla f(y)\| \leq a\,\|x - y\|$ for all $x, y \in X$. Under the mentioned condition on f, the following result is obtained:

If x^* is a global minimum point of f over X, then

$$f(x) - f(x^*) \geq \tfrac{1}{4a}\,\|\nabla f(x)\|^2 \tag{2.2}$$

for all $x \in X$.

We present the following theorem in [11], which enables us to derive the inequality (2.2) on a restricted domain.

Theorem 3. Consider $\mathbb{R}^n$ with norms $\|.\|$ and $\|.\|_\circ$ that are equivalent to $\|.\|$. Let $\Omega \subset \mathbb{R}^n$ be a set with a nonempty interior and $f \in C^1(\Omega)$. Suppose that the mapping $x \longmapsto \nabla f(x)$ is Lipschitz on Ω :

$$K := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \tfrac{\|\nabla f(x)-\nabla f(y)\|}{\|x-y\|} < \infty.$$

Let f have global minimum at $x^* \in \operatorname{int}\Omega$ over Ω. Consider the ball

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leq r\} \subset int\Omega$$

and

$$D := \max\left\{\|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r)\right\}.$$

Let $q > 0$ be a number such that $B_\circ(x^*, r + q) \subset \Omega$ and let $a \geq \max(K, \frac{D}{2q})$. Then

$$\tfrac{1}{4a}\,\|\nabla f(x)\|^2 \leq f(x) - f(x^*),\ \ x \in B_\circ(x^*, r).$$

3. Main results

For the inequality of the form $f(x) \geq 0$ that is valid for $x \in A \subset \mathbb{R}$, if one can find $u(x) > 0$ for $x \in A$ such that $f(x) \geq u(x)$, then $f(x) \geq u(x)$ is said to be a refinement of the inequality $f(x) \geq 0$. Note that the Jensen inequality can be written in the form of $f(x) \geq 0$ as follows:

$$w_0 := \sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) \geq 0.$$

Note that w_0 is called the Jensen gap in literature. Using Theorem 3, we obtain a refinement of the Jensen inequality in terms of derivatives of given function as follows:

Theorem 4. Let $M \in \mathbb{R}_{++}$ and $f : \mathbb{R}_+ \to \mathbb{R}$. Suppose $x_i \in \mathbb{R}_+$, $a_i \in \mathbb{R}_+$ for $i = 1, 2, ..., n$ such that $\sum_{i=1}^{n} a_i = 1$. If f is convex and f', f'' are bounded such that $|f''| \leq M$, then

$$\sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) \geq \frac{\sum_{i=1}^{n} a_i^2 \left(f'(x_i) - f'\left(\sum_{i=1}^{n} a_i x_i\right)\right)^2}{4M\left(\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i^3 + \left(\sum_{i=1}^{n} a_i^2\right)^2\right)^{\frac{1}{2}}} := w_1. \tag{3.1}$$

Proof. Define $F : \mathbb{R}_+^n \to \mathbb{R}$ as follows:

$$F(\mathbf{x}) = \sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right),$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Note that $F(\mathbf{x})$ is nonnegative and $F(\lambda \mathbf{1}) = 0$ for any vector $\lambda \mathbf{1}$ in the form of $\lambda \mathbf{1} = (\lambda, \lambda, \ldots, \lambda)$, where $\lambda \in \mathbb{R}$. Hence, $\mathbf{x}^* = \lambda \mathbf{1}$ is a global minimum on $\mathbb{R}_+^n$.

The gradient of F follows as

$$\nabla F(\mathbf{x}) = \left[a_1 f'(x_1) - a_1 f'(\sum_{i=1}^{n} a_i x_i), \cdots, a_n f'(x_n) - a_n f'(\sum_{i=1}^{n} a_i x_i)\right],$$

hence

$$\|\nabla F(\mathbf{x})\|^2 = \sum_{i=1}^{n} \left(a_i f'(x_i) - a_i f'(\sum_{i=1}^{n} a_i x_i)\right)^2.$$

Let us define the set

$$V_{\lambda,d} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \|\mathbf{x} - \lambda \mathbf{1}\| \leq d, \text{ where } 0 < d < \lambda\},$$

hence $0 < \lambda - d \leq x_i \leq \lambda + d$ for $i = 1, ..., n$ and the function

$$\rho_i(\mathbf{x}) = a_i f'(x_i) - a_i f'(\sum_{i=1}^{n} a_i x_i).$$

Note that $V_{\lambda,d} \subset \mathbb{R}_{++}^n$ and each ρ_i is the ith component of the gradient of $F(\mathbf{x})$.

Let us estimate $\|\nabla \rho_i(\mathbf{x})\|$ for $\mathbf{x} \in V_{\lambda,d}$. For $\mathbf{x} \in V_{\lambda,d}$, we have

$$\left|\frac{\partial \rho_i}{\partial x_k}(\mathbf{x})\right| = \left|-a_i a_k f''(\sum_{i=1}^{n} a_i x_i)\right| < a_i a_k M \quad (i \neq k),$$
$$\left|\frac{\partial \rho_i}{\partial x_i}(\mathbf{x})\right| = \left|a_i f''(x_i) - a_i^2 f''(\sum_{i=1}^{n} a_i x_i)\right| < (a_i + a_i^2) M.$$

Hence,

$$\|\nabla \rho_i(\mathbf{x})\|^2 \leq \left((a_i + a_i^2)^2 + a_i^2 \sum_{\substack{k=1 \\ k \neq i}}^{n} a_k^2\right) M^2 = \left(a_i^2 + 2a_i^3 + a_i^2 \sum_{i=1}^{n} a_k^2\right) M^2 \tag{3.2}$$

for $i \in \{1, ..., n\}$. Let $\mathbf{x}, \mathbf{z} \in V_{\lambda,d}$. The mean value theorem for ρ_i implies that there are numbers $\zeta_i \in (0, 1)$, $i = 1, ..., n$ such that

$$\|\nabla F(\mathbf{x}) - \nabla F(\mathbf{z})\| = \|\rho_1(\mathbf{x}) - \rho_1(\mathbf{z}), \cdots, \rho_n(\mathbf{x}) - \rho_n(\mathbf{z})\|$$

$$= \left(\sum_{i=1}^{n} \left[\rho_i(\mathbf{x}) - \rho_i(\mathbf{z})\right]^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{n} \left[\langle \nabla\rho_i(\mathbf{x} + \zeta_i(\mathbf{z} - \mathbf{x})), \mathbf{x} - \mathbf{z}\rangle\right]^2\right)^{\frac{1}{2}}.$$

Applying the Cauchy–Schwarz inequality and using (3.2), one has

$$\begin{aligned}
\left(\sum_{i=1}^{n} \left[\langle \nabla\rho_i(\mathbf{x} + \zeta_i(\mathbf{z} - \mathbf{x})), \mathbf{x} - \mathbf{z}\rangle\right]^2\right)^{\frac{1}{2}} &\leq (\sum_{i=1}^{n} \|\nabla\rho_i(\mathbf{x} + \zeta_i(\mathbf{z} - \mathbf{x}))\|^2)^{\frac{1}{2}} \|\mathbf{x} - \mathbf{z}\| \\
&\leq \left(\sum_{i=1}^{n} \left(a_i^2 + 2a_i^3 + a_i^2 \sum_{i=1}^{n} a_i^2\right)\right)^{\frac{1}{2}} M \|\mathbf{x} - \mathbf{z}\| \\
&\leq \left(\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i^3 + \left(\sum_{i=1}^{n} a_i^2\right)^2\right)^{\frac{1}{2}} M \|\mathbf{x} - \mathbf{z}\|.
\end{aligned}$$

Letting

$$C_1(\lambda, d) = \left(\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i^3 + \left(\sum_{i=1}^{n} a_i^2\right)^2\right)^{\frac{1}{2}} M$$

yields

$$\|\nabla F(\mathbf{x}) - \nabla F(\mathbf{z})\| \leq C_1(\lambda, d) \|\mathbf{x} - \mathbf{z}\| \text{ for } \mathbf{x}, \mathbf{z} \in V_{\lambda,d}.$$

This inequality concludes that the mapping $x \to \nabla f(x)$ is Lipschitz continuous on $V_{\lambda,d}$ with its Lipschitz constant (say K) satisfying $K \leq C_1(\lambda, d)$.

Let us choose positive numbers q and r such that $r \in (0, d)$ and $q = d - r$ so that $B_\infty(\lambda\mathbf{1}, r + q) \subset V_{\lambda,d}$. By boundedness of $f'(x)$, we know there is a positive number T that $|f'(x)| \leq T$.

Let us estimate $D := \max\{\|\nabla F(\mathbf{x})\|_\infty : \mathbf{x} \in B_\infty(\lambda\mathbf{1}, r)\}$:

$$\begin{aligned}
D &= \max_{\mathbf{x}\in B_\infty(\lambda,r)} \left\{\max_{k\in\{1,\ldots,n\}} |a_k f'(x_k) - a_k f'(\textstyle\sum a_i x_i)|\right\} \\
&\leq \max_{k\in\{1,\ldots,n\}} \left\{\max_{\mathbf{x}\in B_\infty(\lambda,r)} 2a_k T\right\} \\
&\leq 2T \max_{k\in\{1,\cdots,n\}} \{a_k\}.
\end{aligned}$$

Thus,

$$\frac{D}{2q} \leq \frac{1}{(d-r)} 2T \max_{k\in\{1,n\}} \{a_k\}.$$

Let $C_2(\lambda, r, d) := \frac{1}{(d-r)} 2T \max_{k\in\{1,\ldots,n\}} \{a_k\}$. Now we get an upper bound for $\max(K, \frac{D}{2q})$ in Theorem 3:

$$\max(K, \tfrac{D}{2q}) \leq \max\{C_1(\lambda, d), C_2(\lambda, d, r)\}.$$

Considering that $C_1(\lambda, d)$ is constant with respect to d and $C_2(\lambda, d, r)$ is decreasing on (r, λ) with respect to d and $\lim_{d\to r^+} C_2(\lambda, d, r) = +\infty$, we can conclude that the minimum value of the function $\max\{C_1(\lambda,d), C_2(\lambda,d,r)\}$ attained with respect to d on the interval (r,λ) equals to $C_1(\lambda,d)$. Thus, we get

$$\min_{r<d<\lambda} \max\{C_1(\lambda,d), C_2(\lambda,d,r)\} = \left(\sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} a_k^3 + \left(\sum_{k=1}^{n} a_k^2\right)^2\right)^{\frac{1}{2}} M.$$

This yields $\max(K, \frac{D}{2q}) \le C_1(\lambda,d)$, which implies that

$$\frac{1}{4M}\left(\sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} a_k^3 + \left(\sum_{k=1}^{n} a_k^2\right)^2\right)^{-\frac{1}{2}} \|\nabla f(\mathbf{x})\|^2 \le F(\mathbf{x})$$

is valid for all $\mathbf{x} \in \mathbb{R}_+^n$ such that $\|\mathbf{x}-\lambda\|_\infty \le r$. Due to the fact that we can select Ω to be as large as necessary with the extending d on $\mathbb{R}_+$ and the upper bound $C_1(\lambda,d)$ for $\max(K, \frac{D}{2q})$ is independent of the choice of d and r, the inequality is true for all vectors in $\mathbb{R}_+^n$. □

Corollary 5. Let $M \in \mathbb{R}_{++}$ and $f : [a,b] \subset \mathbb{R}_+ \to \mathbb{R}$. Suppose $x_i \in [a,b]$, $a_i \in \mathbb{R}_+$ for $i = 1,2,...,n$ such that $\sum_{i=1}^{n} a_i = 1$. If f is convex and f'' are bounded on $[a,b]$ such that $|f''| \le M$, then the inequality (3.1) remains true.

The corollary above sets (3.1) for all $x_i \in [a,b]$. If the numbers x_i are certain, one can obtain more precise results by setting $M = \max_{x\in[\min\{x_i\},\max\{x_i\}]} |f''(x)|$ in (3.1).

4. Numerical experiments and comparisons

Some numerical examples are presented, and numerical comparisons with certain refinements of the Jensen inequality are carried out to see the efficiency of the obtained refinement.

Example 6. Let $x_i = \dfrac{i}{15}$ and $a_i = \dfrac{2i-1}{25}$ for $i = 1,...,5$. Assume $f(x) = x^3$. Then $M = 6\max\{x_i\} = 2$, $\sum_{i=1}^{n} a_i f(x_i) \approx 0.020539$, $f\left(\sum_{i=1}^{n} a_i x_i\right) \approx 0.016258$, and $u(x) \approx 0.000561$, where $u(\mathbf{x})$ denotes the right-hand side term in (3.1).

Also applying the theorem for negative entropy function $f(x) = x\ln x$ that is convex on $\mathbb{R}_{++}$, we can get the following result.

Example 7. Suppose $x_i \in \mathbb{R}_{++}$, $a_i \in \mathbb{R}_+$ for $i = 1,2,...,n$ such that $\sum_{i=1}^{n} a_i = 1$:

$$\ln\left(\frac{\prod x_i^{a_i x_i}}{\left(\sum_{i=1}^{n} a_i x_i\right)^{\left(\sum_{i=1}^{n} a_i x_i\right)}}\right) > \frac{\min_{i\in\{1,...,n\}}\{x_i\}}{\left(\sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} a_k^3 + \left(\sum_{k=1}^{n} a_k^2\right)^2\right)^{\frac{1}{2}}} \sum_{i=1}^{n}\left(a_i \ln\left(\frac{x_i}{\sum a_i x_i}\right)\right)^2.$$

Another example can be given via log gamma function $f(x) = \log\Gamma(x)$ that is convex on $\mathbb{R}_{++}$. Using Theorem 4, one can obtain an inequality involving psi (digamma) function.

Example 8. For $x \in (0,1)$, the following inequality holds:

$$-4\sqrt{5}\log(\sin(\pi x))\left(\frac{1}{\min\{x^2,(1-x)^2\}}+\frac{\pi^2}{6}\right) \geq (\psi(x)+\gamma+2\ln 2)^2+(\psi(1-x)+\gamma+2\ln 2)^2,$$

where γ is Euler–Masheroni constant, i.e., $\gamma \approx 0.577216$, and $\Psi(x)$ is digamma function, i.e.,

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Proof. In Theorem 4, suppose $a_1 = a_2 = \frac{1}{2}$ and $x_1 = x$, $x_2 = 1-x$ such that $x \in (0,1)$ and $f(x) = \log\Gamma(x)$ that is convex on $(0,1)$. It is known that its derivative is digamma function (also known as psi function), i.e., $f'(x) = \Psi(x)$. Also, $f''(x) = \sum\limits_{k=0}^{\infty}\frac{1}{(x+k)^2}$, thereby $f''(x) < \frac{1}{(\min\{x,1-x\})^2} + \sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{(\min\{x,1-x\})^2} + \frac{1}{6}\pi^2$ for $x \in (0,1)$. Using the fact via [12] that $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ and $\Psi\left(\frac{1}{2}\right) = -\gamma - 2\ln(2)$, we get the result. □

There are numerous studies dedicated to the refinement or enhancement of Jensen's inequality; some focus on the integral version, some on the matrix version, and others on random variables, thus making the comparison of them with our results impossible. Furthermore, the counterparts in the literature regarding the refinements or improvements of the Jensen inequality presented in this work may have utilized additional variables or auxiliary parameters. For the purpose of a direct comparison, we will examine two specific studies related to the discrete version of Jensen's inequality as expressed in (3.1). It is worth noting that while many improvements to Jensen's inequality have been proposed over the years, numerical comparisons remain relatively scarce in the literature. Because of its nature, one refined version may outperform another for a specific convex function and a given set of x_i points, yet the results may reverse when different points are selected. Therefore, it is generally not possible to establish a definitive theoretical superiority numerically. In this context, we will perform a numerical comparison using two studies that we consider to be both relevant and up-to-date. The first refinement is the following theorem given in [7], which uses only one additional parameter η_i:

Theorem 9. Suppose that $f : I \to R$ is a convex function on the real interval I. Let $x_i \in I$, a_i, $\eta_i \in (0,\infty)$ for all $i \in \{1,2,\ldots,n\}$, $\sum_{i=1}^{n}, a_i = 1$. Then,

$$\sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) \geq w_2, \tag{4.1}$$

where

$$w_2 = \sum_{i=1}^{n}\eta_i a_i f\left(\frac{\sum_{i=1}^{n} a_i\eta_i x_i}{\sum_{i=1}^{n} a_i\eta_i}\right) + \sum_{i=1}^{n} a_i(1-\eta_i) f\left(\frac{\sum_{i=1}^{n} a_i(1-\eta_i)x_i}{\sum_{i=1}^{n} a_i(1-\eta_i)}\right) - f\left(\sum_{i=1}^{n} a_i x_i\right).$$

The second refinement is given as in [8, corollary 1] as follows:

Theorem 10. Let $f : I \to R$ be a convex function and let $\sum_{i=1}^{n} a_i = 1$ and $a_i \geq 0$, $x_i \in I$, $i, k \in \{1, 2, ...n\}$. Then,

$$\sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) \geq w_3, \tag{4.2}$$

where

$$w_3 = n \min_{1\leq i\leq n}\{a_i\}\mathcal{I}_n(f, \mathbf{x}) + n \min_{1\leq i\leq n}\{a_i^*\}\mathcal{I}_n(f, \bar{\mathbf{x}}_k)$$

and

$$\mathcal{I}_n(f, \mathbf{x}) = \sum_{i=1}^{n} \frac{1}{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right),$$

$$\begin{aligned}
\mathbf{x} &= (x_1, x_2, \ldots, x_n),\\
\bar{\mathbf{x}}_k &= (x_1, .., x_{k-1}, \frac{1}{n}\sum_{i=1}^{n} x_i, x_{k+1}, .., x_n),\\
a_i^* &= \begin{cases} a_i - a_k \text{ if } i \neq k \\ n a_k \quad \text{ if } i = k \end{cases} \text{ such that } \min_{1\ \leq i\ \leq n}\{a_i\} = a_k.
\end{aligned}$$

Since all inequalities (3.1), (4.1) and (4.2) are presented in a way that the Jensen gap (w_0) is written on the left side of the inequalities, a larger value on the right-hand side indicates a sharper inequality. Furthermore, the sharpening ratio will be calculated by dividing the right-hand side by the Jensen gap. In calculations, both functions $f(x) = x^2$ and $f(x) = x^4$ are taken into account. M is calculated by $M = \max f''(x_i)$. In all calculations, for the sake of simplicity of the comparison, $\eta_i = \frac{i}{2n}$ are fixed since they affect only the value of w_2.

In Tables 1 and 2, x_i, a_i values are chosen in such a way that x_i go to zero by decreasing and a_i have quadratic increase. It is clear that the suggested refinement in [8] has by far the best results, which provide the tightest lower bounds under the selected parameters in both cases, so our proposed refinement amount (w_1) shows superior performance compared to the work of [7] in both scenarios. Also, switching $f(x) = x^2$ to $f(x) = x^4$ makes a shift that, while sharpening ratios of w_1 and w_2 decrease, almost catches Jensen's gap (w_0) for w_2, especially for larger n values.

Table 1. $f(x) = x^2$, $a_i = \frac{6i^2}{n(2n+1)(n+1)}$, $x_i = \frac{1}{i^2}$, $\eta_i = \frac{i}{2n}$, and $M = 2$.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	0.018347	0.000622	0.000407	0.014669	3.39	2.22	79.95
10	0.003351	0.000046	0.000039	0.002626	1.37	1.16	78.36
15	0.001128	0.000009	0.000009	0.000888	0.8	0.8	78.72

Table 2. $f(x) = x^4$, $a_i = \frac{6i^2}{n(2n+1)(n+1)}$, $x_i = \frac{1}{i^2}$, $\eta_i = \frac{i}{2n}$, and $M = 12\max\{x_i^2\}$.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	0.018428	0.000134	0.000019	0.018069	0.73	0.10	98.05
10	0.002642	0.000004	0.000000	0.002627	0.15	0.00	99.43
15	0.000820	0.000001	0.000000	0.000818	0.12	0.00	99.76

In Tables 3 and 4, the x_i values are selected to converge to zero with an oscillatory behavior, and the a_i weights have cubic increase. For both functions, while w_3 exhibits the highest performance at $n = 5$, our proposed result demonstrates a superior approximation compared to the others as n increases. In the case of choosing $f(x) = x^4$, all three results show significantly lower performance levels as n grows larger. Furthermore, it is observed that w_2 consistently exhibits the poorest performance across all scenarios.

Table 3. $f(x) = x^2$, $a_i = \frac{4i^3}{n^2(n+1)^2}$, $x_i = \left|\frac{\sin(i)}{n}\right|$, $\eta_i = \frac{i}{2n}$, and $M = 2$.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	0.002684	0.000288	0.000041	0.000458	10.73	1.53	17.06
10	0.000509	0.000062	0.000001	0.000020	12.18	0.20	3.93
15	0.000235	0.000032	0.000000	0.000003	13.62	0.00	1.28

Table 4. $f(x) = x^4$, $a_i = \frac{4i^3}{n^2(n+1)^2}$, $x_i = \left|\frac{\sin(i)}{n}\right|$, $\eta_i = \frac{i}{2n}$, and $M = 12\max\{x_i^2\}$H.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	0.000285	0.000027	0.000006	0.000041	9.47	2.11	14.39
10	0.000013	0.000001	0.000000	0.000000	7.69	< 1	< 1
15	0.000003	0.000000	0.000000	0.000000	< 1	< 1	< 1

In Tables 5 and 6, where x_i increases quadratically, the a_i weights have cubic increase. Despite this rapid growth, our proposed refinement w_1 maintains a remarkably stable ratio to w_0, consistently outperforming w_2 and w_3 for larger values of n. While w_3 provides a competitive approximation at $n = 5$, its relative impact diminishes sharply as the system scales. Conversely, w_2 consistently represents the least effective refinement across both quadratic and quartic cases of the function.

Table 5. $f(x) = x^2$, $a_i = \frac{4i^3}{n^2(n+1)^2}$, $x_i = i^2$, $\eta_i = \frac{i}{2n}$, and $M = 2$.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	43.555556	6.026994	1.278591	7.306667	13.84	2.94	16.78
10	648.000000	86.221892	18.706978	24.216818	13.31	2.89	3.74
15	3146.888889	379.992542	89.529103	38.655844	12.08	2.84	1.23

Table 6. $f(x) = x^4$, $a_i = \frac{4i^3}{n^2(n+1)^2}$, $x_i = i^2$, $\eta_i = \frac{i}{2n}$, and $M = 12\max\{(x_i)^2\}$.

n	w_0	w_1	w_2	w_3	$\frac{w_1}{w_0}$ (%)	$\frac{w_2}{w_0}$ (%)	$\frac{w_3}{w_0}$ (%)
5	86854.62	11743.00	2998.70	9172.64	13.52	3.45	10.56
10	18664344.00	2405741.23	606308.93	367672.10	12.89	3.25	1.97
15	437734692.02	49865050.16	13895193.85	2682610.47	11.39	3.17	0.61

Overall, under the restrictions of the selection of functions and fixed η values, the numerical experiments indicate that for smaller values of n, the refinement proposed by [8] generally yields relatively better results. However, for larger values of n, the proposed result in Theorem 4 mostly demonstrates superior performance. Conversely, the refinement presented by [7] is observed to show the relatively lowest performance in all scenarios.

5. Conclusions

In this paper, we obtained a refinement of Jensen's inequality via abstract convexity. The refinement arises from a quadratic lower estimate for abstract concave functions whose gradients satisfy Lipschitz conditions. This mechanism strengthens the classical Jensen inequality by an explicit nonnegative term depending on derivatives of the function. The refinement follows from a known result combining abstract convexity and quantitative optimality conditions, which we apply in the present setting, and it relies essentially on smoothness assumptions, in particular Lipschitz continuity of the gradient.

A natural direction for further research is to extend these results to nonsmooth functions. In the absence of differentiability, subdifferential mappings replace gradients, but establishing comparable quadratic lower bounds becomes substantially more delicate. Developing refinement inequalities under weaker regularity assumptions therefore remains an open problem. At this point, it is worth noting that the work by Dolgopolik [5] provides a unified abstract convex framework for generalized derivatives and establishes optimality conditions for nonsmooth functions. While primarily structural in nature, this framework offers important insight into possible extensions of quantitative results.

Another possible direction for further investigation is the refinement of other functional inequalities beyond Jensen's inequality, especially in infinite-dimensional settings or under alternative abstract convex structures. Obtaining sharper constants and exploring connections with strong convexity and uniform convexity also appear promising directions.

We hope that this work stimulates further development in the quantitative aspects of abstract convexity and contributes to narrowing the gap between its structural foundations and its computational applications.

Author contributions

Ramazan Tinaztepe: Conceptualization, writing-review & editing, validation; Gultekin Tinaztepe: Formal analysis, writing-original draft, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. G. Adilov, I. Yeşilce, B^{-1}-convex functions, *J. Convex Anal.*, **24** (2017), 505–517.

2. A. Bagirov, A. M. Rubinov, Global minimization of increasing positively homogeneous function over the unit simplex, *Ann. Oper. Res.*, **98** (2000), 171–187. https://doi.org/10.1023/A:1019204407420
3. W. Briec, C. D. Horvath, B-convexity, *Optimization*, **53** (2004), 103–127. https://doi.org/10.1080/02331930410001695283
4. R. D. Millán, N. Sukhorukova, J. Ugon, Applications and issues in abstract convexity, *Commun. Optimiz. Theory*, **2023** (2023), 1–16. https://doi.org/10.23952/cot.2023.1
5. M. V. Dolgopolik, Abstract convex approximations of nonsmooth functions, *Optimization*, **64** (2015), 1439–1469. https://doi.org/10.1080/02331934.2013.869811
6. S. Kemali, I. Yeşilce, G. Adilov, B-convexity, B^{-1}-convexity, and their comparison. *Numer. Func. Anal. Opt.*, **36** (2015), 133–146. https://doi.org/10.1353/ner.2015.0121
7. M. A. Khan, D. Pečarić, J. Pečarić, A new refinement of the Jensen inequality with applications in information theory, *B. Malays. Math. Sci. So.*, **44** (2021), 267–278. https://doi.org/10.1007/s40840-020-00944-5
8. M. Krnić, N. Minculete, F. C. M. Symeonidis, On further refinements of the Jensen inequality and applications, *Math. Method. Appl. Sci.*, **47** (2024), 7488–7501. https://doi.org/10.1002/mma.9984
9. D. E. Pallaschke, S. Rolewicz, *Foundations of mathematical optimization: Convex analysis without linearity*, Springer Science & Business Media, **388** (2013).
10. A. M. Rubinov, *Abstract convexity and global optimization*, Springer Science & Business Media, **44** (2013).
11. A. M. Rubinov, Z. Y. Wu, Optimality conditions in global optimization and their applications, *Math. Program.*, **120** (2009), 101–123. https://doi.org/10.1007/s10107-007-0142-4
12. I. S. Ryzhik, I. M. Gradshteyn, A. Jeffrey, D. Zwillinger, *Table of integrals, series, and products*, 7 Eds., Academic Press. 2007.
13. I. Singer, *Abstract convex analysis*, New York: Wiley-Interscience, 1997.
14. G. Tınaztepe, The sharpening Hölder inequality via abstract convexity, *Turk. J. Math.*, **40** (2016), 438–444. https://doi.org/10.3906/mat-1505-58
15. G. Tınaztepe, S. Kemali, S. Sezer, Z. Eken, The sharper form of a Brunn-Minkowski type inequality for boxes, *Hacet. J. Math. Stat.*, **50** (2020), 377–386. https://doi.org/10.15672/hujms.657839
16. G. Tınaztepe, İ. Yeşilce, G. Adilov, A refinement of the Bergström inequality, *Turk. J. Math.*, **45** (2021), 1619–1625. https://doi.org/10.3906/mat-2001-54
17. R. Tınaztepe, G. Tınaztepe, Z. Eken, S. Sezer, S. Kemali, İ. Y. Işık, et al., The sharper version for generalized power mean inequalities with negative exponent, *J. Math. Inequal.*, **17** (2023), 99–111. http://doi.org/10.7153/jmi-2023-17-07

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