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Research article

Liouville type theorems for a class of semilinear biharmonic equations

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Abstract: In this paper, motivated by the techniques developed in X. N. Ma et al., arXiv Preprint, 2025, we proved Liouville type theorems for a class of semi-linear biharmonic equations. The proof was based on a differential identity constructed via the invariant tensor method. We combined this identity with an integral estimate to complete the proof.

Keywords: semilinear biharmonic equations; Liouville type theorems; invariant tensors
Mathematics Subject Classification: 35B45, 35J60

1. Introduction

Liouville type theorems are foundational in partial differential equations, and it is well-known that second-order semi-linear equations in $\mathbb{R}^n$ have been extensively studied. A classic example is the work of Gidas and Spruck [3], who utilized the method of integral estimates to establish Liouville type results for the equation

$$-\Delta u = u^q \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

They showed that Eq (1.1) admits no positive entire solutions when $1 < q < \frac{n+2}{n-2}$. Caffarelli-Gidas-Spruck [1] classified positive solutions for $q = \frac{n+2}{n-2}$ by the moving plane method. Later, Chen-Li [2] gave a more elementary proof with the same approach but requiring fewer restrictions.

Liouville type theorems and classifications of positive solutions have been extensively studied for semi-linear and quasi-linear elliptic equations in $\mathbb{R}^n$. See [7, 11, 12] for results about semi-linear equations, and [6, 13, 14] for quasi-linear equations.

For the fourth-order case, Lin [5] studied the positive solutions of the equation

$$\Delta^2 u = u^q \quad \text{in } \mathbb{R}^n \tag{1.2}$$

for $n \geq 5$ and $1 < q < \frac{n+4}{n-4}$, and classified positive solutions of (1.2) by employing the moving plane method. Recently, Ma-Wu-Wu [10] extended the invariant tensor technique from [9] (for semi-linear equations) to the biharmonic case and provided a new proof of Lin's conclusions on non-compact Riemannian manifolds.

In contrast to the power case treated by Lin [5] via moving planes, our invariant tensor approach applies to general subcritical nonlinearities.

In this paper, we study the Liouville type theorem for the following equation:

$$\Delta^2 u = f(u) \quad \text{in } \mathbb{R}^n. \tag{1.3}$$

For the second-order semi-linear equation, Gidas-Spruck [3] derived some Liouville type results for more general nonlinearity $f(x,u)$ with some extra assumptions on the behavior of f. Serrin and Zou [14] weakened certain technical conditions from [3] (such as $f(u) > 0$ for $u > 0$) and broadened the conclusions to the quasi-linear equation $\Delta_m u + f(u) = 0$ in $\mathbb{R}^n$, with $\Delta_m u = div(|\nabla u|^{m-2}\nabla u)$; see in [14, Theorem II] for details. They introduced the following subcritical condition:

Definition 1.1. *The function f is subcritical if $n > m$ and there exists some $\beta \in (0, \frac{nm}{n-m} - 1)$ such that*

$$f(t) \geq 0, \quad \beta f(t) - t f'(t) \geq 0, \quad for \quad t > 0. \tag{1.4}$$

When $m = 2$, (1.4) implies that $t^{-\beta} f(t)$ is nonincreasing in $(0,\infty)$ for some $\beta \in (0, \frac{n+2}{n-2})$. Especially, a result involving Laplacian case ($m = 2$) is also given. Guo-Zhang [4] provided a new proof for the Laplacian case using the invariant tensor technique.

Throughout the following, we assume $f \in C^1(0,\infty) \cap C[0,\infty)$ satisfies $f(t) > 0$ for all $t > 0$. In place of (1.4), we only impose condition

$$\beta f(t) - t f'(t) \geq 0, \quad \text{for} \quad t > 0. \tag{1.5}$$

This ensures $t^{-\beta} f(t)$ is nonincreasing in $(0,\infty)$ for some $\beta \in (1, \frac{n+4}{n-4})$. For convenience, a function f satisfying (1.5) is said to be subcritical.

Now, we state the main results of this paper.

Theorem 1.2. *Let $n \geq 5$ and u be a nonnegative solution of* (1.3) *in $\mathbb{R}^n$ satisfying $\Delta u \leq 0$. If f is subcritical for some $1 < \beta < \frac{n+4}{n-4}$ and*

$$f(u) \geq c u^{\beta}, \tag{1.6}$$

where c is some positive constant depending on n and β, then $u \equiv 0$.

Theorem 1.3. *Let $n \geq 5$ and u be a nonnegative solution of* (1.3) *in $\mathbb{R}^n$ satisfying $\Delta u \leq 0$. If f is subcritical for some $1 < \beta < \frac{n+4}{n-4}$ and there exists some $\alpha > 1$ such that*

$$f(u) \geq u^{\alpha} \tag{1.7}$$

holds for all sufficiently large u, then $u \equiv 0$.

Indeed, Theorem 1.2 is a special case of Theorem 1.3 below, since condition (1.6) implies (1.7) for any $\alpha \in (1,\beta)$ and sufficiently large u.

Remark 1.4. *The hypothesis $\Delta u \leq 0$ is common in biharmonic problems. In the Euclidean setting, Lin [5] showed that solutions of $\Delta^2 u = e^{4u}$ satisfy $\Delta u < 0$ in $\mathbb{R}^4$; on manifolds, Ma-Wu-Wu [10] proved a Liouville theorem under $\Delta u \leq 0$. We follow their approach.*

The rest of this paper is organized as follows. In Section 2, following the approach of Ma-Wu-Wu [10], we introduce a crucial differential identity. In Section 3, we establish integral estimates to prove Theorem 1.2 using this differential identity. The proof of Theorem 1.3 is presented in Section 4.

2. preliminaries

In this section, we introduce the necessary notations and then establish a key differential identity via the invariant tensor technique, following the same argument as in [10].

2.1. Invariant tensors

In this subsection, we recall the definition of the invariant tensors introduced in [10]. We use the notation $u_i = \partial_i u = \frac{\partial u}{\partial x_i}$, $u_{ij} = \partial_i \partial_j u$, and summation over repeated indices is understood.

Recall

$$E_{ij} = u_{ij} + b\frac{u_i u_j}{u} - \frac{1}{n}\left(\Delta u + b\frac{|\nabla u|^2}{u}\right)\delta_{ij},$$

where b is a constant to be chosen later (see Proposition 2.1) in order to eliminate certain second-order terms in the differential identity. Here, δ_{ij} denotes the Kronecker delta. A direct computation yields

$$\partial_i E_{ij} = \frac{n-1}{n}\partial_j(\Delta u) + b\frac{\Delta u}{u}u_j + \frac{n-2}{n}b\frac{u_{ij}u_i}{u} - \frac{n-1}{n}b\frac{|\nabla u|^2}{u^2}u_j.$$

Plugging $u_{ij} = E_{ij} - b\frac{u_i u_j}{u} + \frac{1}{n}\left(\Delta u + b\frac{|\nabla u|^2}{u}\right)\delta_{ij}$ into the above formula yields

$$\partial_i E_{ij} = \frac{n-2}{n}b\frac{E_{ij}u_i}{u} + \frac{n-1}{n}F_j,$$

where $F_j = \partial_j(\Delta u) + \frac{n+2}{n}b\frac{\Delta u}{u}u_j - b\left(1 + \frac{n-2}{n}b\right)\frac{|\nabla u|^2}{u^2}u_j$. So, we obtain

$$\begin{aligned}\partial_i F_i =& \Delta^2 u + \frac{n+2}{n}b\left(\frac{\partial_i(\Delta u)}{u}u_i + \frac{(\Delta u)^2}{u} - \frac{\Delta u\,|\nabla u|^2}{u^2}\right)\\ &- b\left(1 + \frac{n-2}{n}b\right)\left(2\frac{u_{ij}u_i u_j}{u^2} + \frac{\Delta u\,|\nabla u|^2}{u^2} - 2\frac{|\nabla u|^4}{u^3}\right).\end{aligned}$$

Similarly, replacing u_{ij} and $\partial_i(\Delta u)$ by E_{ij} and F_i, respectively, we obtain

$$\partial_i F_i = -2b\left(1 + \frac{n-2}{n}b\right)\frac{E_{ij}u_i u_j}{u^2} + \frac{n+2}{n}b\frac{F_i u_i}{u} + G,$$

where

$$G = \Delta^2 u + \frac{n+2}{n}b\frac{(\Delta u)^2}{u} - \frac{2(n+2)}{n}b(1+b)\frac{\Delta u\,|\nabla u|^2}{u^2} + b(3b+2)\left(1 + \frac{n-2}{n}b\right)\frac{|\nabla u|^4}{u^3}.$$

To compute $\partial_i G$, we differentiate equation (1.3), which gives $\partial_i(\Delta^2 u) = f'(u)u_i$, and insert the term $\beta\frac{f(u)}{u}u_i$. Thus,

$$\partial_i G = f'(u)u_i - \beta\frac{f(u)}{u}u_i + \beta\frac{\Delta^2 u}{u}u_i + \frac{n+2}{n}b\left(2\frac{\Delta u\partial_i(\Delta u)}{u} - \frac{(\Delta u)^2}{u^2}u_i\right)$$

$$
\begin{aligned}
&- \frac{2(n+2)}{n} b(1+b)\left(\frac{|\nabla u|^2\, \partial_i(\Delta u)}{u^2} + 2\frac{\Delta u}{u^2} u_{ij}u_j - 2\frac{\Delta u\, |\nabla u|^2}{u^3} u_i\right)\\
&+ b(3b+2)\left(1 + \frac{n-2}{n} b\right)\left(4\frac{|\nabla u|^2}{u^3} u_{ij}u_j - 3\frac{|\nabla u|^4}{u^4} u_i\right).
\end{aligned}
$$

Replacing the terms u_{ij}, $\partial_i(\Delta u)$, and $\Delta^2 u$ by E_{ij}, F_i, and G, respectively, yields

$$
\begin{aligned}
\partial_i G =& 4b\left((3b+2)(1+\frac{n-2}{n}b)\frac{|\nabla u|^2}{u^2} - \frac{n+2}{n}(1+b)\frac{\Delta u}{u}\right)\frac{E_{ij}u_j}{u}\\
&+ \left(f'(u) - \beta\frac{f(u)}{u}\right)u_i + \beta\frac{G}{u}u_i + \frac{2(n+2)}{n}b\left(\frac{\Delta u}{u} - (1+b)\frac{|\nabla u|^2}{u^2}\right)F_i\\
&- \frac{n+2}{n}b\left(\frac{n+4}{n}(1+2b)+\beta\right)\frac{(\Delta u)^2}{u^2}u_i + \frac{2b}{n}\Big(2(n+4)(1+2b)(1+\frac{n-1}{n}b)\\
&+ (n+2)(1+b)\beta\Big)\frac{\Delta u\, |\nabla u|^2}{u^3}u_i - b\left(1+\frac{n-2}{n}b\right)\Big((1+2b)(6+\frac{7n-4}{n}b)\\
&+ (2+3b)\beta\Big)\frac{|\nabla u|^4}{u^4}u_i.
\end{aligned}
$$

By setting $b = -\frac{1}{2}(1+\frac{n\beta}{n+4})$, we eliminate term $\frac{(\Delta u)^2}{u^2}u_i$ in $\partial_i G$, which directly yields relationships between the invariant tensors. Thus, we give the following proposition.

Proposition 2.1. *Invariant tensors are taken as*

$$
\begin{aligned}
E_{ij} =& u_{ij} - \frac{1}{2}\left(1+\frac{n\beta}{n+4}\right)\frac{u_iu_j}{u} - \frac{1}{n}\left(\Delta u - \frac{1}{2}\left(1+\frac{n\beta}{n+4}\right)\frac{|\nabla u|^2}{u}\right)\delta_{ij},\\
F_j =& \partial_j(\Delta u) - \frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)\frac{\Delta u}{u}u_j + \frac{1}{4}\left(1+\frac{n\beta}{n+4}\right)\left(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\right)\frac{|\nabla u|^2}{u^2}u_j,\\
G =& \Delta^2 u - \frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)\frac{(\Delta u)^2}{u} + \frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)\left(1-\frac{n\beta}{n+4}\right)\frac{\Delta u\, |\nabla u|^2}{u^2}\\
&- \frac{1}{8}\left(1+\frac{n\beta}{n+4}\right)\left(1-\frac{3n\beta}{n+4}\right)\left(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\right)\frac{|\nabla u|^4}{u^3}.
\end{aligned}
$$

Relationships between them via differentiation are as follows:

$$
\begin{aligned}
\partial_i\left(\frac{E_{ij}u_j}{u}\right) =& u^{-1}\left|E_{ij}\right|^2 - \left(\frac{n-1}{n} - \frac{\beta}{n+4}\right)\frac{E_{ij}u_iu_j}{u^2} + \frac{n-1}{n}\frac{F_iu_i}{u},\\
\partial_i F_i =& \frac{1}{2}\left(1+\frac{n\beta}{n+4}\right)\left(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\right)\frac{E_{ij}u_iu_j}{u^2} - \frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)\frac{F_iu_i}{u} + G,\\
\partial_i G =& \left(1+\frac{n\beta}{n+4}\right)\Bigg[-\frac{1}{2}\left(1-\frac{3n\beta}{n+4}\right)\left(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\right)\frac{|\nabla u|^2}{u^2}\\
&+ \frac{n+2}{n}\left(1-\frac{n\beta}{n+4}\right)\frac{\Delta u}{u}\Bigg]\frac{E_{ij}u_j}{u} + \Bigg[\frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)\Bigg(\left(1-\frac{n\beta}{n+4}\right)\frac{|\nabla u|^2}{u^2}
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{\Delta u}{u}\Big)\Big]F_i+\Big(f'(u)-\beta\frac{f(u)}{u}\Big)u_i+\beta\frac{G}{u}u_i\\
&-\frac{\beta}{2(n+4)}\Big(1+\frac{n\beta}{n+4}\Big)\Big(1-\frac{n-4}{n+4}\beta\Big)\Big(n+2-\frac{n(n-2)}{n+4}\beta\Big)\frac{|\nabla u|^4}{u^4}u_i\\
&+\frac{\beta}{2}\Big(1+\frac{n\beta}{n+4}\Big)\Big(1-\frac{n-4}{n+4}\beta\Big)\frac{\Delta u\,|\nabla u|^2}{u^3}u_i.
\end{aligned}
$$

The crucial differential identity is established by Proposition 2.1.

2.2. A crucial identity

Our crucial identity is given as follows.

Proposition 2.2. *For* $c_1=-\frac{n^2(3n-10)}{4(n-1)(n+4)^2}\beta^2+\frac{2(n+2)}{(n-1)(n+4)}\beta+\frac{3(n+2)}{4(n-1)}$ *and*

$$c_2=\frac{n^2+2n+4}{(n-1)(n+4)}\beta+\frac{n+2}{n-1},$$

the following identity holds:

$$
\begin{aligned}
&u^{\frac{2\beta}{n+4}}\partial_i\Big\{u^{-\frac{2\beta}{n+4}}\Big[\Big(c_1\frac{|\nabla u|^2}{u}-c_2\Delta u\Big)\frac{E_{ij}u_j}{u}+\Big(\frac{n+2}{n+4}\beta\frac{|\nabla u|^2}{u}+\Delta u\Big)F_i\\
&-Gu_i+\frac{n\beta}{2(n+4)}\Big(1+\frac{n\beta}{n+4}\Big)\Big(1-\frac{n-4}{n+4}\beta\Big)\frac{|\nabla u|^4}{u^3}u_i\Big]\Big\}\\
=&\Big(c_1\frac{|\nabla u|^2}{u^2}-c_2\frac{\Delta u}{u}\Big)\left|E_{ij}\right|^2+2c_1\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}+|F_i|^2-\Big(f'(u)-\beta\frac{f(u)}{u}\Big)|\nabla u|^2\\
&+2A_{12}\frac{E_{ij}u_j}{u}F_i+2A_1\frac{|\nabla u|^2}{u^3}E_{ij}u_iu_j+2A_{23}\frac{|\nabla u|^2}{u}\frac{F_iu_i}{u}+A_{33}\frac{|\nabla u|^6}{u^4},
\end{aligned}
\tag{2.1}
$$

where the coefficients are

$$
\begin{aligned}
&A_{12}=\frac{n^2-8}{2(n-1)(n+4)}\beta-\frac{n+2}{2(n-1)},\quad A_{13}=\frac{\beta}{n+4}\Big(\frac{n^2\beta}{n+4}+n+1\Big)\Big(1-\frac{n-4}{n+4}\beta\Big),\\
&A_{23}=-\frac{\beta}{4}\Big(1-\frac{n-4}{n+4}\beta\Big),\quad A_{33}=\frac{n(n-2)}{2(n+4)^2}\beta^2\Big(1+\frac{n\beta}{n+4}\Big)\Big(1-\frac{n-4}{n+4}\beta\Big).
\end{aligned}
$$

Remark 2.3. *The constants c_1 and c_2 are chosen following [10] so that the quadratic form in* (2.1) *is positive definite for* $1<\beta<\frac{n+4}{n-4}$. *This ensures the coercivity needed in Lemma 3.1.*

Proof. By Proposition 2.1, we derive the following differential identities:

$$
\begin{aligned}
u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}-1}|\nabla u|^2E_i)_i=&\ \frac{|\nabla u|^2}{u^2}\left|E_{ij}\right|^2+2\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}\\
&+\Big(\frac{n-2}{n+4}\beta-1\Big)\frac{|\nabla u|^2}{u^3}E_{ij}u_iu_j+\frac{2}{n}\Delta u\frac{E_{ij}u_iu_j}{u^2}+\frac{n-1}{n}\frac{|\nabla u|^2}{u}\frac{F_iu_i}{u},
\end{aligned}
$$

$$u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}}\Delta u E_i)_i = \frac{\Delta u}{u}\left|E_{ij}\right|^2 + \frac{E_{ij}u_j}{u}F_i + \Big(\frac{n\beta}{2(n+4)} - \frac{n-4}{2n}\Big)\Delta u\frac{E_{ij}u_iu_j}{u^2}$$
$$- \frac{1}{4}\Big(1 + \frac{n\beta}{n+4}\Big)\Big(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\Big)\frac{|\nabla u|^2}{u}\frac{E_{ij}u_iu_j}{u^2} + \frac{n-1}{n}\Delta u\frac{F_iu_i}{u},$$

$$u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}-1}|\nabla u|^2F_i)i = 2\frac{E_{ij}u_j}{u}F^i + \frac{1}{2}\Big(1 + \frac{n\beta}{n+4}\Big)\Big(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\Big)\frac{|\nabla u|^2}{u}\frac{E_{ij}u_iu_j}{u^2}$$
$$+ \Big(\frac{n-8}{2(n+4)}\beta - \frac{n+4}{2n}\Big)\frac{|\nabla u|^2}{u}\frac{F_iu_i}{u} + \frac{2}{n}\Delta u\frac{F_iu_i}{u} + \frac{|\nabla u|^2}{u}G,$$

$$u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}}\Delta u F_i)_i = |F_i|^2 - \frac{2\beta}{n+4}\Delta u\frac{F_iu_i}{u} + \Delta u G$$
$$+ \frac{1}{4}\Big(1 + \frac{n\beta}{n+4}\Big)\Big(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\Big)\Big(2\Delta u\frac{E_{ij}u_iu_j}{u^2} - \frac{|\nabla u|^2}{u}\frac{F_iu_i}{u}\Big),$$

$$u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}}Gu_i)_i$$
$$= \Big(1 + \frac{n\beta}{n+4}\Big)\Big(-\frac{1}{2}\Big(1 - \frac{3n\beta}{n+4}\Big)\Big(\frac{n+2}{n} - \frac{n-2}{n+4}\beta\Big)\frac{|\nabla u|^2}{u} + \frac{n+2}{n}\Big(1 - \frac{n\beta}{n+4}\Big)\Delta u\Big)\frac{E_{ij}u_iu_j}{u^2}$$
$$+ \frac{n+2}{2n}\Big(1 + \frac{n\beta}{n+4}\Big)\Big(\Big(1 - \frac{n\beta}{n+4}\Big)\frac{|\nabla u|^2}{u} - 2\Delta u\Big)\frac{F_iu_i}{u} + \frac{n+2}{n+4}\beta\frac{|\nabla u|^2}{u}G + \Delta u G + \left(f'(u) - \beta\frac{f(u)}{u}\right)|\nabla u|^2$$
$$+ \frac{\beta}{2}\Big(1 + \frac{n\beta}{n+4}\Big)\Big(1 - \frac{n-4}{n+4}\beta\Big)\Big[\Big(\frac{n(n-2)}{(n+4)^2}\beta - \frac{n+2}{n+4}\Big)\frac{|\nabla u|^2}{u} + \Delta u\Big]\frac{|\nabla u|^4}{u^3}.$$

$$u^{\frac{2\beta}{n+4}}(u^{-\frac{2\beta}{n+4}-3}|\nabla u|^4u_i)_i = 4\frac{|\nabla u|^2}{u}\frac{E_{ij}u_iu_j}{u^2} + \Big(\frac{2(n-2)}{n+4}\beta - \frac{n+2}{n}\Big)\frac{|\nabla u|^6}{u^4} + \frac{n+4}{n}\frac{\Delta u|\nabla u|^4}{u^3}.$$

We finish the proof by linearly combining the above six identities. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by adapting the methods employed by Ma-Wu-Wu [10], specifically by utilizing the approach of integral estimates.

First, we prepare the following lemma for proving Theorem 1.2.

Lemma 3.1. *Under assumptions of Proposition 2.2, when $1 < \beta < \frac{n+4}{n-4}$ and $n \geq 5$, there exist constants δ_1, $\delta_2 > 0$, depending on n and β, such that*

$$\partial_i\left\{u^{-\frac{2\beta}{n+4}}\left[\left(c_1\frac{|\nabla u|^2}{u} - c_2\Delta u\right)\frac{E_{ij}u_j}{u} + \frac{n+2}{n+4}\beta\frac{|\nabla u|^2}{u}F_i + \left(1 - \frac{n+4}{n+2}\frac{\delta_2}{\beta}\right)(\Delta u F_i - Gu_i)\right.\right.$$
$$\left.\left.+\frac{n\beta}{2(n+4)}\left(1 + \frac{n\beta}{n+4}\right)\left(1 - \frac{n-4}{n+4}\beta\right)\frac{|\nabla u|^4}{u^3}u_i + \delta_1\frac{\Delta u\,|\nabla u|^2}{u^2}u_i\right]\right\} \tag{3.1}$$
$$\geq \delta_2 u^{-\frac{2\beta}{n+4}}\left(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2} + |F_i|^2 + \frac{|\nabla u|^6}{u^4} + \frac{(\Delta u)^2|\nabla u|^2}{u^2} + \frac{|\nabla u|^2}{u}\Delta^2 u - \left(f'(u) - \beta\frac{f(u)}{u}\right)|\nabla u|^2\right).$$

Proof. We work in the flat Euclidean space $\mathbb{R}^n$ (i.e., $R_{ij} = 0$). From [10, Proposition 2.2] we have the differential identity, and together with the estimates

$$-\Delta u \geq \frac{2}{n-4}\frac{|\nabla u|^2}{u^2}, \qquad |\nabla u|^2|E_{ij}|^2 \geq \frac{4}{3}\frac{|E_{ij}|^2|\nabla u|^2}{u^2},$$

(the latter follows from a similar argument as in [8, Lemma 2.2]), we substitute them into the identity. After collecting all terms, the left-hand side can be written as a divergence while the remaining part becomes a quadratic form in the vector $(E_i, F_i, \frac{|\nabla u|^2}{u^2}u_i)$. More precisely, we obtain

$$\begin{aligned} & u^{\frac{2\beta}{n+4}}\partial_i\Bigg\{u^{-\frac{2\beta}{n+4}}\Bigg[\left(\left(c_1\frac{|\nabla u|^2}{u} - c_2\Delta u\right)\frac{E_{ij}u_j}{u} + \left(\frac{n+2}{n+4}\beta\frac{|\nabla u|^2}{u} + \Delta u\right)F_i\right. \\ & \qquad -Gu_i + \frac{n\beta}{2(n+4)}\Big(1+\frac{n\beta}{n+4}\Big)\Big(1-\frac{n-4}{n+4}\beta\Big)\frac{|\nabla u|^4}{u^3}u_i\Bigg]\Bigg\} \\ & \geq (E_i, F_i, \tfrac{|\nabla u|^2}{u^2}u_i)\, A\, (E^i, F^i, \tfrac{|\nabla u|^2}{u^2}u^i)^T - \left(f'(u) - \beta\frac{f(u)}{u}\right)|\nabla u|^2, \end{aligned} \tag{3.2}$$

where $A = (A_{kl})_{3\times 3}$ is a symmetric matrix whose entries are polynomials in n and β. The explicit expressions of A and the proof of its positive definiteness for $1 < \beta < \frac{n+4}{n-4}$ and $n \geq 5$ are given in [10, Lemma 3.3]; in the flat Euclidean case, the same argument applies without modification because all curvature terms vanish.

Note that by condition (1.5), we have $f'(u) - \beta\frac{f(u)}{u} \leq 0$; hence, the term $-(f'(u) - \beta\frac{f(u)}{u})|\nabla u|^2$ in (3.2) is nonnegative.

Consequently, with δ_1, δ_2 chosen as in Lemma 3.1 (their dependence on β is also detailed in [10]), the desired inequality (3.1) follows immediately. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.2. First, we choose a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$, and

$$\begin{cases} \eta = 1, & \text{in } B_R, \\ 0 \leq \eta \leq 1, & \text{in } B_{2R}, \\ \eta \equiv 0, & \text{in } \mathbb{R}^n \setminus B_{2R}, \\ |\partial\eta| \leq \dfrac{C(n)}{R}, & \text{in } \mathbb{R}^n. \end{cases}$$

Then, let γ be a sufficiently large constant (in particular, $\gamma > 6$) to be determined later. Multiplying (3.1) by η^γ and integrating yields

$$\begin{aligned} & \int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Bigg(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2} + |F_i|^2 + \frac{|\nabla u|^6}{u^4} + \frac{(\Delta u)^2|\nabla u|^2}{u^2} + \frac{|\nabla u|^2}{u}\Delta^2 u \\ & - \left(f'(u) - \beta\frac{f(u)}{u}\right)|\nabla u|^2\Bigg)\eta^\gamma \\ \leq & C\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Bigg[\left(\frac{|\nabla u|^2}{u} - \Delta u\right)\left(\frac{\left|E_{ij}\right|\left|u_j\right|}{u} + |F_i|\right) + \Bigg(|G| + \frac{|\nabla u|^4}{u^3} \\ & + \frac{\Delta u\,|\nabla u|^2}{u^2}\Bigg)|u_i|\Bigg]\eta^{\gamma-1}|\eta_i|. \end{aligned} \tag{3.3}$$

By Young's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\frac{|\nabla u|^2}{u}-\Delta u\right)\left(\frac{\left|E_{ij}\right|\left|u_j\right|}{u}+|F_i|\right)\eta^{\gamma-1}\left|\eta_i\right|\\ \le&\varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}+|F_i|^2\right)\eta^{\gamma}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\frac{|\nabla u|^4}{u^2}+(\Delta u)^2\right)\eta^{\gamma-2}\\ \le&\varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}+|F_i|^2+\frac{|\nabla u|^6}{u^4}\right)\eta^{\gamma}\\ &+CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(|G|+\frac{|\nabla u|^4}{u^3}+\frac{\Delta u\,|\nabla u|^2}{u^2}\right)|u_i|\,\eta^{\gamma-1}\left|\eta_i\right|\\ \le&C\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\Delta^2 u+\frac{(\Delta u)^2}{u}+\frac{|\Delta u|\,|\nabla u|^2}{u^2}+\frac{|\nabla u|^4}{u^3}\right)|u_i|\,\eta^{\gamma-1}\left|\eta_i\right|,\end{aligned}\tag{3.5}$$

where we used $|G|\le C\left(\Delta^2 u+\frac{(\Delta u)^2}{u}+\frac{|\Delta u||\nabla u|^2}{u^2}+\frac{|\nabla u|^4}{u^3}\right)$.

By Young's inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(|G|+\frac{|\nabla u|^4}{u^3}+\frac{\Delta u\,|\nabla u|^2}{u^2}\right)|u_i|\,\eta^{\gamma-1}\left|\eta_i\right|\\ \le&\varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(\frac{|\nabla u|^6}{u^4}+\frac{(\Delta u)^2|\nabla u|^2}{u^2}+\frac{|\nabla u|^2}{u}\Delta^2 u\right)\eta^{\gamma}\\ &+CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}\Delta^2 u\,\eta^{\gamma-2}.\end{aligned}\tag{3.6}$$

Substituting (3.6) and (3.4) into (3.3) yields

$$\begin{aligned}&\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Big(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}+|F_i|^2+\frac{|\nabla u|^6}{u^4}+\frac{(\Delta u)^2|\nabla u|^2}{u^2}+\frac{|\nabla u|^2}{u}\Delta^2 u\\ &-\left(f'(u)-\beta\frac{f(u)}{u}\right)|\nabla u|^2\Big)\eta^{\gamma}\\ \le&CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}\Delta^2 u\,\eta^{\gamma-2}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2}.\end{aligned}\tag{3.7}$$

To complete the proof of Theorem 1.2, we now prove the following two lemmas.

Lemma 3.2. *For any $\gamma>6$, there exists a constant $C>0$ depending on n,β,γ such that*

$$\begin{aligned}R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}\Delta^2 u\,\eta^{\gamma-2}\le&\varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Big(\frac{\left|E_{ij}\right|^2|\nabla u|^2}{u^2}+|F_i|^2+\frac{|\nabla u|^6}{u^4}\Big)\eta^{\gamma}\\ &+CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}+CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2}.\end{aligned}$$

Proof. From Proposition 2.1, the definition of G, and Cauchy's inequality, we have

$$\begin{aligned} u^{\frac{2\beta}{n+4}}\partial_i\left(u^{-\frac{2\beta}{n+4}+1}F_i\right) =& uG + \frac{1}{2}\left(1+\frac{n\beta}{n+4}\right)\left(\frac{n+2}{n}-\frac{n-2}{n+4}\beta\right)\frac{E_{ij}u_iu_j}{u} \\ &- \frac{n+2}{2n}\left(1+\frac{n\beta}{n+4}\right)F_iu_i \\ \geq& u\Delta^2u - C\Big(\Big(\frac{|E_{ij}|\,|u_j|}{u}+|F_i|\Big)|u_i| + \frac{|\nabla u|^4}{u^2}+(\Delta u)^2\Big). \end{aligned}$$

By testing $u^{-\frac{2\beta}{n+4}}\eta^{\gamma-2}$ on it, and applying Young's inequality, we obtain

$$\begin{aligned} R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}\Delta^2u\eta^{\gamma-2} \leq& CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Big(\Big(\frac{|E_{ij}|\,|u_j|}{u}+|F_i|\Big)|u_i| + \frac{|\nabla u|^4}{u^2} \\ &+ (\Delta u)^2\Big)\eta^{\gamma-2} + CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}|F_i|\,\eta^{\gamma-3}\,|\eta_i|\,. \end{aligned}$$

Thus, we complete the proof by Young's inequality. □

Lemma 3.3. *For any $\gamma > 6$, there exists a constant $C > 0$ depending on n, β, γ such that*

$$R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2} \leq \varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(|F_i|^2+\frac{|\nabla u|^6}{u^4}\right)\eta^{\gamma} + CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}.$$

Proof. From Proposition 2.1 and Cauchy's inequality, we have

$$\begin{aligned} u^{\frac{2\beta}{n+4}}\partial_i\left(u^{-\frac{2\beta}{n+4}}\Delta u u_i\right) =& (\Delta u)^2 + F_iu_i + \frac{1}{2}\left(\frac{n-2}{n+4}\beta+\frac{n+2}{n}\right)\frac{\Delta u\,|\nabla u|^2}{u} \\ &+ \frac{1}{4}\left(1+\frac{n\beta}{n+4}\right)\left(\frac{n-2}{n+4}\beta-\frac{n+2}{n}\right)\frac{|\nabla u|^4}{u^2} \\ \geq& \frac{1}{2}(\Delta u)^2 - C\left(|F_i|\,|u_i|+\frac{|\nabla u|^4}{u^2}\right). \end{aligned}$$

By testing $u^{-\frac{2\beta}{n+4}}\eta^{\gamma-2}$ on it, and applying Young's inequality, we obtain

$$\begin{aligned} &R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2} \\ \leq& CR^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(|F_i|\,|u_i|+\frac{|\nabla u|^4}{u^2}\right) + CR^{-3}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\Delta u\,|\nabla u|\,\eta^{\gamma-3} \\ \leq& \frac{1}{2}R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}(\Delta u)^2\eta^{\gamma-2} + \varepsilon\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}}\left(|F_i|^2+\frac{|\nabla u|^6}{u^4}\right)\eta^{\gamma} + CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}. \end{aligned}$$

Thus, the proof is completed. □

Combining Lemmas 3.2 and 3.3, and inequalities (1.5), (3.8), we arrive at

$$R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1}\Delta^2u\eta^{\gamma-2} + \int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}-4}\,|\nabla u|^6\,\eta^{\gamma}$$

$$\leq CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}. \tag{3.8}$$

By Eqs (1.3) and (1.6), we can get

$$\begin{aligned} &R^{-2}\int_{\mathbb{R}^n} u^{\frac{n+2}{n+4}\beta+1}\eta^{\gamma-2}+\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}-4}\,|\nabla u|^6\,\eta^{\gamma}\\ \leq& CR^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}. \end{aligned} \tag{3.9}$$

Since $1<\beta<\frac{n+4}{n-4}$, we choose $\gamma=\max\left\{7,\frac{1}{\beta-1}\left(\frac{6n+16}{n+4}\beta-2\right)\right\}$, and use Young's inequality to obtain

$$R^{-6}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+2}\eta^{\gamma-6}\leq \varepsilon R^{-2}\int_{\mathbb{R}^n} u^{\frac{n+2}{n+4}\beta+1}\eta^{\gamma-2}+CR^{n-\frac{1}{\beta-1}\left(\frac{6n+16}{n+4}\beta+2\right)}.$$

We now verify that the exponent on the righthand side is negative for all admissible β. A direct computation yields

$$n-\frac{1}{\beta-1}\left(\frac{6n+16}{n+4}\beta+2\right)=\frac{(n^2-2n-16)\beta-(n+2)(n+4)}{(n+4)(\beta-1)}.$$

Since $(n+4)(\beta-1)>0$ for $\beta>1$, the sign is determined by the numerator. We consider two cases.

Case 1. $n\geq 6$. Then, $n^2-2n-16\geq 8>0$. Using the upper bound $\beta<\dfrac{n+4}{n-4}$, we obtain

$$(n^2-2n-16)\beta-(n+2)(n+4)<(n^2-2n-16)\frac{n+4}{n-4}-(n+2)(n+4)=-\frac{8(n+4)}{n-4},$$

and, consequently,

$$n-\frac{1}{\beta-1}\left(\frac{6n+16}{n+4}\beta+2\right)<\frac{-8(n+4)}{(n+4)(\beta-1)(n-4)}=-\frac{8}{(n-4)(\beta-1)}<0.$$

Case 2. $n=5$. Substituting $n=5$ gives

$$\frac{(n^2-2n-16)\beta-(n+2)(n+4)}{(n+4)(\beta-1)}=\frac{-\beta-63}{9(\beta-1)}<0.$$

Thus, in both cases, the exponent is strictly negative. Letting $R\to\infty$ in (3.9) and using that the righthand side tends to zero, we obtain

$$\lim_{R\to\infty}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}-4}|\nabla u|^6\eta^{\gamma}=0.$$

Since $\eta\equiv 1$ on B_R, it follows that $\int_{B_R} u^{-\frac{2\beta}{n+4}-4}|\nabla u|^6\to 0$; by monotone convergence,

$$\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}-4}|\nabla u|^6=0.$$

The integrand is nonnegative, hence, $|\nabla u|=0$ almost everywhere and u is constant. Write $u\equiv c\geq 0$. If $c>0$, inserting into (1.3) yields $0=\Delta^2 c=f(c)$, contradicting $f(t)>0$ for all $t>0$. Therefore, $c=0$ and $u\equiv 0$. This completes the proof. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which employs a method similar to that of Theorem 1.2.

Proof of Theorem 1.3. Since f is subcritical and condition (1.5) holds, it follows that

$$\left(u^{-\beta} f(u)\right)' \leq 0, \tag{4.1}$$

where the derivative is with respect to u. Meanwhile, (1.7) implies the existence of some u_0, chosen sufficiently large with $u_0 > 1$, for which $f(u) \geq u^\alpha$ holds whenever $u \geq u_0$. Therefore, $f(u_0) > 0$. Integrating (4.1) from u_0 to u yields

$$u^\alpha \leq f(u) \leq \frac{f(u_0)}{u_0^\beta} u^\beta, \quad \text{for} \quad u \geq u_0. \tag{4.2}$$

From (4.2), we observe that $\alpha \leq \beta$. Integrating (4.1) from u to u_0 gives

$$f(u) \geq \frac{f(u_0)}{u_0^\beta} u^\beta, \quad \text{for} \quad u < u_0. \tag{4.3}$$

We now decompose $\mathbb{R}^n$ into the regions $\{u < u_0\}$ and $\{u \geq u_0\}$.

Our goal is to establish an integral estimate analogous to (3.8). Applying Hölder's inequality with the exponent pair $\left(\frac{(n+2)\beta+n+4}{2(n+4-\beta)}, \frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}\right)$, we obtain

$$\begin{aligned} R^{-6} \int_{\{u<u_0\}} u^{-\frac{2\beta}{n+4}+2} \eta^{\gamma-6} \leq & \left(R^{-2} \int_{\{u<u_0\}} u^{\frac{n+2}{n+4}\beta+1} \eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+2)\beta+n+4}} \\ & \times \left(R^{-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}} \int_{\{u<u_0\}} \eta^{\gamma-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}}\right)^{\frac{(n+4)(\beta-1)}{(n+2)\beta+n+4}} \\ \leq & C(n,\beta) \left(R^{-2} \int_{\{u<u_0\}} f(u) u^{-\frac{2\beta}{n+4}+1} \eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+2)\beta+n+4}} \\ & \times \left(R^{-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}} \int_{\{u<u_0\}} \eta^{\gamma-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}}\right)^{\frac{(n+4)(\beta-1)}{(n+2)\beta+n+4}}. \end{aligned} \tag{4.4}$$

We now apply Hölder's inequality analogously, with the exponent pair $\left(\frac{(n+4)(\alpha+1)-2\beta}{2(n+4-\beta)}, \frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}\right)$. This gives the following estimate:

$$\begin{aligned} R^{-6} \int_{\{u\geq u_0\}} u^{-\frac{2\beta}{n+4}+2} \eta^{\gamma-6} \leq & \left(R^{-2} \int_{\{u\geq u_0\}} u^{\alpha-\frac{2\beta}{n+4}+1} \eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+4)(\alpha+1)-2\beta}} \\ & \times \left(R^{-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}} \int_{\{u\geq u_0\}} \eta^{\gamma-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}}\right)^{\frac{(n+4)(\alpha-1)}{(n+4)(\alpha+1)-2\beta}} \\ \leq & \left(R^{-2} \int_{\{u\geq u_0\}} f(u) u^{-\frac{2\beta}{n+4}+1} \eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+4)(\alpha+1)-2\beta}} \\ & \times \left(R^{-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}} \int_{\{u\geq u_0\}} \eta^{\gamma-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}}\right)^{\frac{(n+4)(\alpha-1)}{(n+4)(\alpha+1)-2\beta}}. \end{aligned} \tag{4.5}$$

By Combining (3.8), (4.4), (4.5), we obtain

$$
\begin{aligned}
&R^{-2}\int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}+1} f(u)\eta^{\gamma-2} + \int_{\mathbb{R}^n} u^{-\frac{2\beta}{n+4}-4}|\nabla u|^6\eta^{\gamma}\\
\le& C(n,\alpha,\beta,\gamma)\left(R^{-2}\int_{\{u<u_0\}} f(u)u^{-\frac{2\beta}{n+4}+1}\eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+2)\beta+n+4}}\\
&\times\left(R^{-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}}\int_{\{u<u_0\}}\eta^{\gamma-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}}\right)^{\frac{(n+4)(\beta-1)}{(n+2)\beta+n+4}}\\
&+C(n,\alpha,\beta,\gamma)\left(R^{-2}\int_{\{u\ge u_0\}} f(u)u^{-\frac{2\beta}{n+4}+1}\eta^{\gamma-2}\right)^{\frac{2(n+4-\beta)}{(n+4)(\alpha+1)-2\beta}}\\
&\times\left(R^{-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}}\int_{\{u\ge u_0\}}\eta^{\gamma-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}}\right)^{\frac{(n+4)(\alpha-1)}{(n+4)(\alpha+1)-2\beta}}.
\end{aligned}
\tag{4.6}
$$

Taking

$$\gamma=\max\left\{3+4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)},\,3+4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}\right\}$$

in (4.6) and applying Young's inequality, it follows from the definition of η that

$$
\begin{aligned}
&R^{-2}\int_{B_R} u^{-\frac{2\beta}{n+4}+1} f(u)\eta^{\gamma-2} + \int_{B_R} u^{-\frac{2\beta}{n+4}-4}|\nabla u|^6\eta^{\gamma}\\
\le& C(n,\alpha,\beta,\gamma)\left(R^{n-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}}+R^{n-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}}\right).
\end{aligned}
\tag{4.7}
$$

Note that the first exponent satisfies

$$n-2-4\frac{(n+2)\beta+n+4}{(n+4)(\beta-1)}\le n-2-4\frac{(n+2)\times\frac{n+4}{n-4}+n+4}{(n+4)\times\frac{8}{n-4}}=-1<0.$$

For the second exponent, a direct computation yields

$$n-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)}=\frac{(n^2-2n-24)\alpha-(n^2+6n+8)+8\beta}{(n+4)(\alpha-1)}.$$

We now verify its negativity by considering two cases.
Case 1. $n\ge 6$. Since $n^2-2n-24\ge 0$ for $n\ge 6$, using $\alpha<\beta$ we obtain

$$\frac{(n^2-2n-24)\alpha-(n^2+6n+8)+8\beta}{(n+4)(\alpha-1)}\le\frac{(n^2-2n-16)\beta-(n^2+6n+8)}{(n+4)(\alpha-1)}.$$

Moreover, from $\beta<\dfrac{n+4}{n-4}$, it follows that

$$(n^2-2n-16)\beta-(n^2+6n+8)<(n^2-2n-16)\frac{n+4}{n-4}-(n^2+6n+8)=-\frac{8(n+4)}{n-4}<0,$$

and the denominator $(n+4)(\alpha-1) > 0$. Hence, the expression is negative.
Case 2. $n = 5$. Substituting $n = 5$ gives

$$\frac{(n^2-2n-24)\alpha-(n^2+6n+8)+8\beta}{(n+4)(\alpha-1)} = \frac{-9\alpha-63+8\beta}{9(\alpha-1)}.$$

Using $1 < \alpha < \beta < \frac{n+4}{n-4} = 9$, we have $-9\alpha - 63 + 8\beta < -9\alpha - 63 + 72 < -9\alpha + 9 < 0$, so the expression is negative as well.

Thus, in all cases,

$$n-2-4\frac{(n+4)(\alpha+1)-2\beta}{(n+4)(\alpha-1)} < 0.$$

By letting $R \to \infty$ in (4.7), we obtain $u \equiv 0$. This completes the proof of Theorem 1.3. □

5. Conclusions

Using the invariant tensor method to overcome the lack of a maximum principle, we prove Liouville type theorems for positive solutions of the biharmonic equation $\Delta^2 u = f(u)$ in $\mathbb{R}^n$ ($n \geq 5$), showing that $u \equiv 0$ under subcritical conditions on f.

Use of Generative-AI tools declaration

The author declares she has not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author state no conflict of interest.

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