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Research article

Extended Caputo space-fractional Black-Scholes equation with scale-dependent diffusion

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Abstract: This paper developed an analytical framework for a space-fractional Black-Scholes model formulated with the extended Caputo fractional derivative. Fundamental operational properties of the extended Mellin integral transform, including shift rules, transform formulas for Caputo-type derivatives of orders $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, and a convolution theorem, were established and used to treat scale-invariant fractional differential equations. By applying the extended Mellin transform to the governing Cauchy problem, we derived an explicit integral representation of the solution involving a gamma-function-based time-evolution multiplier. The validity of the representation was rigorously verified, and the classical Black-Scholes model with dividends was recovered as a special case. The model was applied to European put options, with numerical results validating the method and illustrating the impact of fractional dynamics. Calibration to SPY option market data demonstrates that the fractional parameters α and ρ enhance flexibility in fitting observed option prices and capturing market-dependent scaling effects.

Keywords: space-fractional Black-Scholes equation; extended Caputo fractional derivative; extended Mellin integral transform; integral representation; fractional derivative
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1. Introduction

The Black-Scholes equation is a cornerstone of modern option pricing theory [1] and has also served as a benchmark for analytical methods in partial differential equations. Among these, the Mellin integral transform has been shown to provide an effective tool for solving the classical Black-Scholes equation for European options [2, 3]. At the same time, fractional calculus has emerged as a powerful framework for modeling memory, nonlocality, and scale invariance in complex systems. In financial markets, such features appear as long-range dependence, heavy-tailed returns, and fractal-like behavior, motivating the development of fractional extensions of classical pricing models.

Motivated by empirical features such as memory, heavy tails, and anomalous diffusion in asset returns, numerous fractional extensions of the Black-Scholes equation have been proposed in recent years. By incorporating fractional derivatives or fractional stochastic processes, these models introduce nonlocal dynamics that enhance the descriptive power of classical pricing frameworks. Accordingly, growing literature has focused on analytical and approximate solutions of fractional Black-Scholes (FBS) equations [4–6]. For instance, Sawangtong et al. [4] derived analytical solutions for a two-asset model using Liouville-Caputo derivatives and Laplace-transform-based homotopy perturbation methods. Ampun and Sawangtong [6] extended this approach to time-fractional models with solutions expressed in terms of generalized Mittag-Leffler functions, while Thanompolkrang et al. [5] obtained analytic European option prices using Caputo-type Katugampola derivatives. Despite these advances, existing studies largely emphasize time-fractional formulations or classical fractional operators. In particular, space-fractional Black-Scholes equations based on the extended Caputo derivative, as well as Mellin-transform-based analytical solutions for such models, remain largely unexplored.

Recent advances in fractional calculus, particularly the development of operators with non-singular kernels, have further expanded the scope of fractional modeling. Atangana-Baleanu-type fractional derivatives, originally motivated by applications in viscoelasticity, provide realistic representations of hereditary and memory effects and have been successfully employed to generalize classical dynamical models [7]. Their relevance has extended to mathematical finance, where such operators have been explored for capturing memory effects in option pricing dynamics [6, 8]. These developments underscore the importance of selecting fractional operators whose structural properties align with the long-range dependence observed in financial markets.

In this work, we address this gap by introducing a space-fractional Black-Scholes equation formulated with the extended Caputo fractional derivative, in which the classical second-order spatial derivatives are replaced by nonlocal operators that encode scale-dependent diffusion. Specifically, we consider, for $(x,t) \in (0,\infty) \times [0,T)$,

$$\left.\begin{array}{l} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \left(\frac{x^\rho}{\rho}\right)^{1+\alpha} D_x^{\rho,1+\alpha} u + (r-q)\left(\frac{x^\rho}{\rho}\right)^{\alpha} D_x^{\rho,\alpha} u - ru = 0, \\ \text{subject to the terminal condition} \\ u(x,T) = \begin{cases} \left(\frac{K^\rho}{\rho}\right) - \left(\frac{x^\rho}{\rho}\right), & \text{if } x < K, \\ 0, & \text{if } x \geq K, \end{cases} \end{array}\right\} \tag{1.1}$$

where $D_x^{\rho,\alpha}$ denotes the extended Caputo fractional derivative with order $0 < \alpha \leq 1$ and $\rho > 0$, T is the expiration date, and σ, r, q, and K are positive constants.

On the computational side, significant progress has been made through hybrid analytical-numerical techniques for fractional Black-Scholes equations. High-order numerical schemes [9], transform-based perturbation approaches [10,11], and hybrid Laplace finite-difference methods [12] have all contributed to improving accuracy and efficiency. In addition, refined time-stepping strategies, such as variable-step schemes designed to handle solution singularities, have enhanced numerical robustness [13].

It is also important to distinguish these developments from models driven by fractional Brownian motion. While fractional Brownian motion captures long-range dependence through the Hurst exponent, it is non-Markovian and, in general, incompatible with classical arbitrage-free pricing arguments [14, 15]. In contrast, space-fractional Black-Scholes equations arise naturally as generators of Lévy-type processes with jumps and heavy-tailed distributions, leading to nonlocal spatial operators that better reflect market discontinuities [16, 17].

The main objective of this paper is to derive an integral representation of the solution to the model (1.1) using the extended Mellin integral transform [18]. The extended Mellin transform $\mathcal{M}_\rho$, tailored to the structure of the extended Caputo operator, diagonalizes the spatial generator and reduces the pricing equation to a family of scalar ordinary differential equations in transform space. This approach yields analytical insight, preserves standard option boundary conditions, and recovers the classical Black-Scholes solution as a limiting case. Although the resulting market is incomplete due to the underlying Lévy dynamics, the model remains consistent with no-arbitrage pricing under a risk-neutral measure, with remaining degrees of freedom resolved through calibration to market data.

Beyond European put options, the proposed space-fractional Black-Scholes framework is applicable to a broader class of derivative contracts whose payoffs satisfy suitable Mellin integrability conditions, including European calls, digital options, and power-type derivatives. Moreover, the extended Caputo fractional operator can be incorporated into alternative pricing frameworks, such as jump-diffusion or stochastic volatility models, providing a flexible nonlocal structure capable of capturing heavy-tailed return dynamics. In contrast to existing fractional formulations based on Riesz or time-fractional operators, the present approach combines a scale-dependent fractional diffusion mechanism with a geometric Lévy interpretation and a Mellin-transform-based semi-analytical solution, offering both mathematical tractability and enhanced modeling flexibility.

The structure of this article is as follows. Section 2 reviews the preliminary concepts required for the analysis, including fractional derivatives, special functions, and the extended Mellin transform. Section 3 develops the integral representations of solutions to the space-fractional Black-Scholes equations, followed by numerical illustrations in Section 4. Section 5 demonstrates the application of the model to financial data, and Section 6 concludes with a summary and discussion.

2. Preliminaries

This part provides some preliminary information that will be used throughout the study. The definition of fractional derivatives will be the first step.

2.1. Fractional derivatives and special functions

In this subsection we collect the fractional-calculus preliminaries needed for the analysis of model (1.1). We begin by recalling the Caputo fractional derivative and its extended (conformable) Caputo counterpart, highlighting the reductions to the classical Caputo derivative when $\rho = 1$ and to the ordinary derivative when $\rho = \alpha = 1$. We then review the gamma and beta functions together with their basic identities, and record an analytic continuation of the gamma function to negative (non-integer) arguments via a Hankel contour representation, which preserves the fundamental recurrence. These ingredients are used repeatedly in what follows. In particular, the subsequent lemmas provide closed-form evaluations of extended Caputo derivatives on power functions, which will be key in verifying that the proposed integral representation solves model (1.1).

Definition 2.1. *Let α be any positive real constant with $m - 1 < \alpha \leq m$ for some natural number m. The Caputo fractional derivatives of $f : [0, \infty) \to R$ with order α are determined by*

$$ {}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-\xi)^{m-\alpha-1} \frac{d^m}{d\xi^m} f(\xi) d\xi. $$

Definition 2.2. *Let α be any positive real constant with $m - 1 < \alpha \leq m$ for some natural number m. The extended Caputo fractional derivatives of $f : [0, \infty) \to R$ with order α is specified by*

$$ D_{x}^{\rho,\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} \left(\frac{x^{\rho}}{\rho} - \frac{\xi^{\rho}}{\rho} \right)^{m-\alpha-1} \left(T_{\xi}^{\rho} \right)^{m} f(\xi) \frac{d\xi}{\xi^{1-\rho}}, $$

where $\rho > 0$, $T_x^{\rho} f(x) = x^{1-\rho} \frac{d}{dx} f(x)$, called the conformable derivative with order ρ, and $\left(T_x^{\rho}\right)^m f(x) = \underbrace{T_x^{\rho} T_x^{\rho} \ldots T_x^{\rho}}_{m \text{ times}} f(x)$.

Take note of the fact that

(1) if $\rho = 1$ in Definition 2.2, then the extended Caputo fractional derivatives with order α can be simplified to the Caputo fractional derivative with order α in Definition 2.1.
(2) if $\rho = 1$ and $\alpha = 1$ in Definition 2.2, then extended Caputo fractional derivatives become the traditional derivative.

We are going to present the gamma and beta functions, along with their properties.

Definition 2.3. *The gamma function is determined by*

$$ \Gamma(z) = \int_{0}^{\infty} x^{z-1} e^{-x} dx, $$

for any complex number z with $Re(z) > 0$.

Definition 2.4. *The beta function is given by*

$$ B(x, y) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} dt, $$

for any complex number x and y with $Re(x) > 0$ and $Re(y) > 0$.

Lemma 2.1. *The following classical identities for the* Γ *and B functions will be used repeatedly.*

(1) $\Gamma(z+1) = z\Gamma(z)$ *for any complex number z with* $Re(z) > 0$,

(2) $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ *for any complex number x and y with* $Re(x) > 0$ *and* $Re(y) > 0$.

Following that, we will talk about the gamma function in depth. The gamma function of negative real constants can be interpolated as follows.

Definition 2.5. *For negative non-integer values* $z \notin \{0, -1, -2, \dots\}$,

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{H}} e^t\, t^{-z}\, dt, \qquad z \notin \{0, -1, -2, \dots\},$$

where $\mathcal{H}$ *encircles the negative real axis once counterclockwise. Collapsing the contour onto the negative axis gives*

$$\int_{\mathcal{H}} e^t t^{-z}\, dt = -2i\sin(\pi z)\,\Gamma(1-z),$$

so that

$$\frac{1}{\Gamma(z)} = \frac{\sin(\pi z)}{\pi}\,\Gamma(1-z).$$

Hence, for negative arguments the gamma function [19] is consistently defined by

$$\Gamma(-z) = -\frac{\pi}{\sin(\pi z)\,\Gamma(1+z)}, \qquad z \notin \{0, 1, 2, \dots\}.$$

This extension preserves the fundamental recurrence relation,

$$\Gamma(z+1) = z\,\Gamma(z).$$

Note: Collapse of $\mathcal{H}$**.** Write points on the branch cut as $t = re^{\pm i\pi} = -r$ with $r > 0$, using the principal branch $\log t = \ln r + i\arg t$ with $-\pi < \arg t < \pi$ so that

$$t^{-z} = e^{-z\log t} = r^{-z}e^{\mp i\pi z} \quad \text{on the upper/lower edge, respectively.}$$

The small circular arc around 0 contributes 0 in the limit (for $\Re(1-z) > 0$, one estimates $|e^t t^{-z}| \lesssim r^{-\Re(z)}$ and the arc length is $O(r)$, giving $O(r^{1-\Re(z)}) \to 0$; the general case follows by analytic continuation).
For the upper edge (above the cut). Parameterize by $t = re^{i\pi}$ with $r : \infty \to 0$. Then $dt = e^{i\pi}dr = -dr$, $e^t = e^{-r}$, and $t^{-z} = r^{-z}e^{-i\pi z}$. Hence

$$\int_{\text{upper}} e^t t^{-z}\, dt = \int_{\infty}^{0} e^{-r}\, r^{-z}e^{-i\pi z}\,(-dr) = e^{-i\pi z}\int_0^{\infty} e^{-r}r^{-z}\, dr.$$

For the lower edge (below the cut). Parameterize by $t = re^{-i\pi}$ with $r : 0 \to \infty$. Then $dt = e^{-i\pi}dr = -dr$, $e^t = e^{-r}$, and $t^{-z} = r^{-z}e^{i\pi z}$. Thus

$$\int_{\text{lower}} e^t t^{-z}\, dt = \int_0^{\infty} e^{-r}\, r^{-z}e^{i\pi z}\,(-dr) = -\,e^{i\pi z}\int_0^{\infty} e^{-r}r^{-z}\, dr.$$

Adding both edges gives

$$\int_{\mathcal{H}} e^t t^{-z}\, dt = (e^{-i\pi z} - e^{i\pi z}) \int_0^{\infty} e^{-r} r^{-z}\, dr = -2i \sin(\pi z) \int_0^{\infty} e^{-r} r^{-z}\, dr.$$

Since $\int_0^{\infty} e^{-r} r^{-z}\, dr = \Gamma(1-z)$ (because $r^{-z} = r^{(1-z)-1}$), we obtain

$$\int_{\mathcal{H}} e^t t^{-z}\, dt = -2i \sin(\pi z)\, \Gamma(1-z).$$

Dividing by $2\pi i$ gives

$$\boxed{\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^t t^{-z}\, dt = \frac{\sin(\pi z)}{\pi}\, \Gamma(1-z)}$$

For example, evaluate $\Gamma\left(-\frac{3}{2}\right)$.

- Using $\Gamma(-z) = \dfrac{\pi}{\sin(-\pi z)\, \Gamma(1+z)}$ and taking $z = \frac{3}{2}$ gives

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\pi}{\sin\left(-\frac{3\pi}{2}\right)\, \Gamma\left(\frac{5}{2}\right)}.$$

  Compute

$$\sin\left(-\tfrac{3\pi}{2}\right) = +1, \qquad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2} \cdot \tfrac{1}{2}\, \Gamma\left(\tfrac{1}{2}\right) = \tfrac{3}{4} \sqrt{\pi}.$$

  Hence

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\pi}{1 \cdot \frac{3}{4} \sqrt{\pi}} = \frac{4}{3} \sqrt{\pi}.$$

- Using the recurrence $\Gamma(z+1) = z\, \Gamma(z) \Rightarrow \Gamma(z) = \dfrac{\Gamma(z+1)}{z}$,

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi},$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4}{3} \sqrt{\pi}.$$

$$\boxed{\Gamma\left(-\tfrac{3}{2}\right) = \frac{4}{3} \sqrt{\pi}}$$

The following lemma deals with the recurrence relation of the gamma function for negative real values.

Lemma 2.2. $\Gamma(-(z+1)) = \dfrac{\Gamma(-z)}{-(z+1)} = -\dfrac{\Gamma(-z)}{z+1}, \quad z > 0.$

Proof. Recurrence of the gamma function. For any $w \notin \{0, -1, -2, \dots\}$,

$$\Gamma(w+1) = w\,\Gamma(w).$$

Applying the recurrence with $w = -(z+1)$ and setting

$$w = -(z+1) \quad \Longrightarrow \quad w + 1 = -z.$$

We then have, by the recurrence,

$$\Gamma(-z) = \Gamma(w+1) = w\,\Gamma(w) = \big(-(z+1)\big)\,\Gamma(-(z+1)).$$

Solving for $\Gamma(-(z+1))$ and rearranging give

$$\Gamma(-(z+1)) = \frac{\Gamma(-z)}{-(z+1)} = -\frac{\Gamma(-z)}{z+1}, \qquad z > 0.$$

□

For example, take $z = \frac{1}{2}$. Then $\Gamma(-(z+1)) = \Gamma\left(-\frac{3}{2}\right)$, as $\Gamma(-z) = \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$.

Applying the formula $\Gamma\left(-\frac{3}{2}\right) = -\frac{\Gamma(-\frac{1}{2})}{\frac{1}{2}+1} = -\frac{\Gamma(-\frac{1}{2})}{\frac{3}{2}} = -\frac{-2\sqrt{\pi}}{\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}$, the formula is confirmed.

The two following lemmas are used in verifying that the integral representation of the solution satisfies the space-Caputo-fractional Black-Scholes (CFBS) equation, which is a part of the main result.

Lemma 2.3. *For any real number $a \neq 0$,*

$$D_x^{\rho,\alpha}\left(\frac{x^\rho}{\rho}\right)^a = \frac{a\Gamma(a)}{\Gamma(1-\alpha+a)}\left(\frac{x^\rho}{\rho}\right)^{a-\alpha}.$$

Proof. Consider:

$$\begin{aligned} D_x^{\rho,\alpha}\left(\frac{x^\rho}{\rho}\right)^a &= \frac{1}{\Gamma(1-\alpha)}\int_0^x \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho}\right)^{-\alpha} \xi^{1-\rho}\frac{d}{d\xi}\left(\frac{\xi^\rho}{\rho}\right)^a \frac{d\xi}{\xi^{1-\rho}} \\ &= \frac{a}{\Gamma(1-\alpha)}\int_0^x \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho}\right)^{-\alpha}\left(\frac{\xi^\rho}{\rho}\right)^{a-1}\frac{d\xi}{\xi^{1-\rho}}. \end{aligned}$$

By letting $y = \dfrac{\xi^\rho}{x^\rho}$, we get that:

$$\begin{aligned} D_x^{\rho,\alpha}\left(\frac{x^\rho}{\rho}\right)^a &= \frac{a}{\Gamma(1-\alpha)}\left(\frac{x^\rho}{\rho}\right)^{a-\alpha}\int_0^1 (1-y)^{-\alpha}\,y^{a-1}dy \\ &= \frac{a}{\Gamma(1-\alpha)}B(1-\alpha, a)\left(\frac{x^\rho}{\rho}\right)^{a-\alpha} \\ &= \frac{a\Gamma(a)}{\Gamma(1-\alpha+a)}\left(\frac{x^\rho}{\rho}\right)^{a-\alpha}. \end{aligned}$$

Consequently, the proof is finished. □

Lemma 2.4. *For any real numbers* β *and* a *with* $1 < \beta \leq 2$, $a \neq 0$, *and* $a\rho \neq 1$,

$$D_x^{\rho,\beta}\left(\frac{x^\rho}{\rho}\right)^a = a(a-1)\frac{\Gamma(a-1)}{\Gamma(1-\beta+a)}\left(\frac{x^\rho}{\rho}\right)^{a-\beta}.$$

Proof. Let us consider

$$\begin{aligned}
& D_x^{\rho,\beta}\left(\frac{x^\rho}{\rho}\right)^a \\
= \;& \frac{1}{\Gamma(2-\beta)}\int_0^x \left(\frac{x^\rho}{\rho}-\frac{\xi^\rho}{\rho}\right)^{1-\beta}\left(\xi^{1-\rho}\frac{d}{d\xi}\right)^2\left(\frac{\xi^\rho}{\rho}\right)^a \frac{d\xi}{\xi^{1-\rho}} \\
= \;& \frac{1}{\Gamma(2-\beta)}\int_0^x \left(\frac{x^\rho}{\rho}-\frac{\xi^\rho}{\rho}\right)^{1-\beta}\left[\xi^{2-2\rho}\frac{d^2}{d\xi^2}\left(\frac{\xi^\rho}{\rho}\right)^a + (1+\rho)\,\xi^{1-2\rho}\frac{d}{d\xi}\left(\frac{\xi^\rho}{\rho}\right)^a\right]\frac{d\xi}{\xi^{1-\rho}} \\
= \;& \frac{a(a-1)}{\Gamma(2-\beta)}\int_0^x \left(\frac{x^\rho}{\rho}-\frac{\xi^\rho}{\rho}\right)^{1-\beta}\left(\frac{\xi^\rho}{\rho}\right)^{a-2}\frac{d\xi}{\xi^{1-\rho}}.
\end{aligned}$$

By letting $y = \frac{\xi^\rho}{x^\rho}$, we have that:

$$\begin{aligned}
D_x^{\rho,\beta}\left(\frac{x^\rho}{\rho}\right)^a &= \frac{a(a-1)}{\Gamma(2-\beta)}\left(\frac{x^\rho}{\rho}\right)^{a-\beta}\int_0^1 (1-y)^{1-\beta}\,y^{a-2}dy \\
&= a(a-1)\frac{\Gamma(a-1)}{\Gamma(1-\beta+a)}\left(\frac{x^\rho}{\rho}\right)^{a-\beta}.
\end{aligned}$$

Therefore, we get the proof of this lemma. □

Note that if $\beta = 1+\alpha$, with $0 < \alpha \leq 1$, then, by Lemma 2.4, we obtain that:

$$D_x^{\rho,1+\alpha}\left(\frac{x^\rho}{\rho}\right)^a = a(a-1)\frac{\Gamma(a-1)}{\Gamma(a-\alpha)}\left(\frac{x^\rho}{\rho}\right)^{a-\alpha-1}.$$

2.2. The extended Mellin integral transform and its properties

In this subsection, we recall the Fourier integral transform and use it to motivate an extended Mellin transform tailored to power-law scalings [18]. Beginning with the standard Fourier pair for a real-valued function (g), we apply the logarithmic substitution $e^\xi = x^\rho/\rho$ together with the spectral shift $ik = \varepsilon - s$ to recast the Fourier transform along rays in the complex plane. After a simple renormalization, this leads to the definition of the extended Mellin transform $\mathcal{M}_\rho$ and its inversion, which reduce to the classical Mellin transform when $\rho = 1$. We then establish basic operational properties of $\mathcal{M}_\rho$, including a shift rule, transform formulas for generalized Caputo-type derivatives of orders $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, and a convolution theorem. These results provide the principal tools for treating fractional and scale-invariant differential problems in the x-domain via algebraic manipulation in the s-domain.

Definition 2.6. *Let g be a real-valued function. The Fourier integral transform along with the inverse Fourier integral transform for g are defined by*

$$\widehat{g}(k) = \mathcal{F}\left\{g(\xi)\right\}(k) = \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi) d\xi$$

and

$$g(\xi) = \mathcal{F}^{-1}\left\{\widehat{g}(k)\right\}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}\widehat{g}(k) dk,$$

where k is the complex Fourier integral transform parameter.

Following this, we shall introduce the concept of creating the extended Mellin integral transform. We use a change of variables $e^{\xi} = \frac{x^{\rho}}{\rho}$ and $ik = \varepsilon - s$ where ε is a constant. According to the definitions of the Fourier integral transform as well as the inverse Fourier integral transform, we obtain

$$\widehat{g}(si - \varepsilon i) = \int_{0}^{\infty} \left(\frac{x^{\rho}}{\rho}\right)^{s-\varepsilon-1} g\left(\ln\left(\frac{x^{\rho}}{\rho}\right)\right) dx \tag{2.1}$$

and

$$g\left(\ln\left(\frac{x^{\rho}}{\rho}\right)\right) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \left(\frac{x^{\rho}}{\rho}\right)^{\varepsilon-s} \widehat{g}(si - \varepsilon i) ds. \tag{2.2}$$

By setting $\left(\frac{x^{\rho}}{\rho}\right)^{-\varepsilon} g\left(\ln\left(\frac{x^{\rho}}{\rho}\right)\right) = f(x)$ and $\widehat{g}(si - \varepsilon i) = \widetilde{f}(s)$, we obtain the extended Mellin integral transform and its inverse as demonstrated in the following definition.

Definition 2.7. *The extended Mellin integral transform for $f : [0, \infty) \to R$ is determined by*

$$\widetilde{f}(s) = \mathcal{M}_{\rho}\left\{f(x)\right\}(s) = \int_{0}^{\infty} \left(\frac{x^{\rho}}{\rho}\right)^{s-1} f(x)\frac{dx}{x^{1-\rho}} \text{ for } Re(s) \in \Omega,$$

with $s = \varepsilon + i\omega$ being the complex parameter of the extended Mellin integral transform and Ω being the largest open interval in which the integral exists. Furthermore, the inversion of the extended Mellin integral transform is given by

$$f(x) = \mathcal{M}_{\rho}^{-1}\left\{\widetilde{f}(s)\right\}(x) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \left(\frac{x^{\rho}}{\rho}\right)^{-s} \widetilde{f}(s) ds.$$

It is noted that

(1) if $\rho = 1$, then the extended Mellin integral transform is the Mellin integral transform,
(2) if the extended Mellin integral transform of f exists, then we say that f is the extended Mellin integral transformable function,
(3) the largest open interval (a, b) in which f is the extended Mellin integral transformable is called the fundamental interval.

Using Definition 2.7, we can prove some properties of the extended Mellin integral transforms. We next show some properties of $\mathcal{M}_\rho$. From now on, we assume that ρ, α, and β are any real constants with $\rho > 0$, $0 < \alpha \leq 1$, and $1 < \beta \leq 2$.

Lemma 2.5. *(Shift property) Let $f : [0, \infty) \to R$ be extended Mellin integral transformable. Then,*

$$\mathcal{M}_\rho \left\{ \left(\frac{x^\rho}{\rho} \right)^a f(x) \right\} (s) = \mathcal{M}_\rho \{ f(x) \} (s + a) \text{ for } Re(s + a) \in \Omega,$$

where a is any real constant.

Proof. Through the definition of the extended Mellin integral transform, the proof of this lemma may be achieved immediately. □

Lemma 2.6. *Suppose that $f : [0, \infty) \to R$ is extended Mellin integral transformable and the term*

$$\left(\frac{x^\rho}{\rho} \right)^{s-\alpha} f(x) \text{ converges to } 0 \text{ as } x \text{ approaches } 0 \text{ and } \infty.$$

Then,

$$\mathcal{M}_\rho \{ D_x^{\rho,\alpha} f(x) \} (s) = -\frac{(s-\alpha)\Gamma(\alpha - s)}{\Gamma(1-s)} \mathcal{M}_\rho \{ f(x) \} (s - \alpha) \text{ for } Re(s - \alpha) \in \Omega.$$

Proof. By Definitions 2.2 and 2.7, we get

$$\begin{aligned}
\mathcal{M}_\rho \{ D_x^{\rho,\alpha} f(x) \} (s) &= \int_0^\infty \left(\frac{x^\rho}{\rho} \right)^{s-1} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^x \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right)^{-\alpha} \left(\xi^{1-\rho} \frac{d}{\partial \xi} \right) f(\xi) \frac{d\xi}{\xi^{1-\rho}} \right) \frac{dx}{x^{1-\rho}} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \left(\frac{x^\rho}{\rho} \right)^{s-1} \left(\int_0^x \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right)^{-\alpha} \frac{df(\xi)}{\partial \xi} d\xi \right) \frac{dx}{x^{1-\rho}} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \left(\int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right) \left(\frac{x^\rho}{\rho} \right)^{s-1} \frac{dx}{x^{1-\rho}} \right) \frac{df(\xi)}{\partial \xi} d\xi. \qquad (2.3)
\end{aligned}$$

Let us consider the term $\int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right) \left(\frac{x^\rho}{\rho} \right)^{1-s} \frac{dx}{x^{1-\rho}}$. By using the transformation $\frac{x^\rho}{\rho} = \frac{\xi^\rho}{\rho} y^{-1}$, we obtain

$$\begin{aligned}
\int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right) \left(\frac{x^\rho}{\rho} \right)^{s-1} \frac{dx}{x^{1-\rho}} &= \left(\frac{\xi^\rho}{\rho} \right)^{s-\alpha} \int_0^1 (1-y)^{-\alpha} y^{\alpha-s-1} dy \\
&= \left(\frac{\xi^\rho}{\rho} \right)^{s-\alpha} \frac{\Gamma(1-\alpha)\Gamma(\alpha-s)}{\Gamma(1-s)}. \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4), we now have

$$\mathcal{M}_\rho \{ D_x^{\rho,\alpha} f(x) \} (s) = \frac{\Gamma(\alpha - s)}{\Gamma(1-s)} \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\alpha} \frac{df(\xi)}{\partial \xi} d\xi. \qquad (2.5)$$

Finally, by applying integration by parts to (2.5), we get

$$\begin{aligned}\mathcal{M}_\rho\left\{D_x^{\rho,\alpha} f(x)\right\}(s) &= \frac{\Gamma(\alpha-s)}{\Gamma(1-s)}\left[\left(\frac{\xi^\rho}{\rho}\right)^{s-\alpha} f(\xi)\Big|_{\xi=0}^{\xi=\infty} - (s-\alpha)\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\alpha-1} f(\xi)\frac{d\xi}{\xi^{1-\rho}}\right] \\ &= -\frac{(s-\alpha)\Gamma(\alpha-s)}{\Gamma(1-s)}\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\alpha-1} f(\xi)\frac{d\xi}{\xi^{1-\rho}}.\end{aligned}$$

Therefore, the lemma's proof is complete. □

Lemma 2.7. *Suppose that $f:[0,\infty)\to R$ is extended Mellin integral transformable. Then,*

$$\mathcal{M}_\rho\left\{\left(\frac{x^\rho}{\rho}\right)^\alpha D_x^{\rho,\alpha} f(x)\right\}(s) = -\frac{s\Gamma(-s)}{\Gamma(1-\alpha-s)}\mathcal{M}_\rho\{f(x)\}(s)\, for\ Re(s)\in\Omega.$$

Proof. This lemma's proof is the consequence of Lemma 2.5 and Lemma 2.6, i.e.,

$$\begin{aligned}\mathcal{M}_\rho\left\{\left(\frac{x^\rho}{\rho}\right)^\alpha D_x^{\rho,\alpha} f(x)\right\}(s) &= \mathcal{M}_\rho\left\{D_x^{\rho,\alpha} f(x)\right\}(s+\alpha) \\ &= -\frac{s\Gamma(-s)}{\Gamma(1-\alpha-s)}\mathcal{M}_\rho\{f(x)\}(s).\end{aligned}$$

□

Lemma 2.8. *Let β be any real number with $1<\beta\le 2$. Suppose that $f:[0,\infty)\to R$ is extended Mellin integral transformable,*

$$\left(\frac{x^\rho}{\rho}\right)^{s-\beta+1/\rho}\frac{df}{dx}\ \textit{approaches}\ 0\ \textit{as}\ x\ \textit{goes to}\ 0\ \textit{and}\ \infty,$$

and

$$\left(\frac{x^\rho}{\rho}\right)^{s-\beta} f(x)\ \textit{approaches}\ 0\ \textit{as}\ x\ \textit{goes to}\ 0\ \textit{and}\ \infty.$$

Then,

$$\mathcal{M}_\rho\left\{D_x^{\rho,\beta} f(x)\right\}(s) = \frac{(s-\beta)(s-\beta+1)\Gamma(\beta-s-1)}{\Gamma(1-s)}\mathcal{M}_\rho\{f(x)\}(s-\beta)\ \ for\ Re(s-\beta)\in\Omega.$$

Proof. Let us consider

$$\begin{aligned}&\mathcal{M}_\rho\left\{D_x^{\rho,\beta} f(x)\right\}(s) \\ &= \int_0^\infty \left(\frac{x^\rho}{\rho}\right)^{s-1}\left(\frac{1}{\Gamma(2-\beta)}\int_0^x \left(\frac{x^\rho}{\rho}-\frac{\xi^\rho}{\rho}\right)^{1-\beta}\left(\xi^{1-\rho}\frac{d}{\partial\xi}\right)^2 f(\xi)\frac{d\xi}{\xi^{1-\rho}}\right)\frac{dx}{x^{1-\rho}} \\ &= \frac{1}{\Gamma(2-\beta)}\int_0^\infty \left(\frac{x^\rho}{\rho}\right)^{s-1}\left(\int_0^x \left(\frac{x^\rho}{\rho}-\frac{\xi^\rho}{\rho}\right)^{1-\beta}\left[\xi^{2-2\rho}\frac{d^2 f}{d\xi^2}+(1-\rho)\xi^{1-2\rho}\frac{df}{d\xi}\right]\frac{d\xi}{\xi^{1-\rho}}\right)\frac{dx}{x^{1-\rho}}\end{aligned}$$

$$= \frac{1}{\Gamma(2-\beta)} \int_0^\infty \left(\int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right)^{1-\beta} \left(\frac{x^\rho}{\rho} \right)^{s-1} \frac{dx}{x^{1-\rho}} \right) \left[\xi^{2-2\rho} \frac{d^2 f}{d\xi^2} + (1-\rho)\xi^{1-2\rho} \frac{df}{d\xi} \right] \frac{d\xi}{\xi^{1-\rho}}. \tag{2.6}$$

Consider the term $\int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right)^{1-\beta} \left(\frac{x^\rho}{\rho} \right)^{s-1} \frac{dx}{x^{1-\rho}}$. By setting $\frac{x^\rho}{\rho} = \frac{\xi^\rho}{\rho} y^{-1}$, we obtain

$$\begin{aligned} \int_\xi^\infty \left(\frac{x^\rho}{\rho} - \frac{\xi^\rho}{\rho} \right)^{1-\beta} \left(\frac{x^\rho}{\rho} \right)^{s-1} \frac{dx}{x^{1-\rho}} &= \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \int_0^1 (1-y)^{1-\beta} y^{-s+\beta-2} dy \\ &= \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \frac{\Gamma(\beta-s-1)\Gamma(2-\beta)}{\Gamma(1-s)}. \end{aligned} \tag{2.7}$$

We substitute (2.7) into (2.6) and then we have

$$\begin{aligned} & \mathcal{M}_\rho \left\{ D_x^{\rho,\beta} f(x) \right\}(s) \\ = \; & \frac{\Gamma(\beta-s-1)}{\Gamma(1-s)} \left[\int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \xi^{2-2\rho} \frac{d^2 f}{d\xi^2} \frac{d\xi}{\xi^{1-\rho}} + (1-\rho) \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \xi^{1-2\rho} \frac{df}{d\xi} \frac{d\xi}{\xi^{1-\rho}} \right] \\ = \; & \frac{\Gamma(\beta-s-1)}{\Gamma(1-s)} \left[I_1 + I_2 \right], \end{aligned} \tag{2.8}$$

where $I_1 = \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \xi^{2-2\rho} \frac{d^2 f}{d\xi^2} \frac{d\xi}{\xi^{1-\rho}}$ and $I_2 = (1-\rho) \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \xi^{1-2\rho} \frac{df}{d\xi} \frac{d\xi}{\xi^{1-\rho}}$.

Consider the term I_1. We have

$$\begin{aligned} I_1 &= \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+1} \xi^{2-2\rho} \frac{d^2 f}{d\xi^2} \frac{d\xi}{\xi^{1-\rho}} \\ &= \rho^{\frac{1-\rho}{\rho}} \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+\frac{1}{\rho}} \frac{d^2 f}{d\xi^2} d\xi. \end{aligned}$$

With the help of integration by parts, we obtain

$$\begin{aligned} I_1 &= \rho^{\frac{1-\rho}{\rho}} \left[\left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+\frac{1}{\rho}} \frac{df}{d\xi} \Bigg|_{\xi=0}^{\xi=\infty} - \left(s - \beta + \frac{1}{\rho} \right) \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+\frac{1}{\rho}-1} \frac{df}{d\xi} \frac{d\xi}{\xi^{1-\rho}} \right] \\ &= -\rho^{\frac{1-\rho}{\rho}} \left(s - \beta + \frac{1}{\rho} \right) \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta+\frac{1}{\rho}-1} \frac{df}{d\xi} \frac{d\xi}{\xi^{1-\rho}} \\ &= -\left(s - \beta + \frac{1}{\rho} \right) \int_0^\infty \left(\frac{\xi^\rho}{\rho} \right)^{s-\beta} \frac{df}{d\xi} d\xi. \end{aligned}$$

We use integration by parts again, and then we have

$$\begin{aligned} I_1 &= -\left(s-\beta+\frac{1}{\rho}\right)\left[\left(\frac{\xi^\rho}{\rho}\right)^{s-\beta} f(\xi)\Big|_{\xi=0}^{\xi=\infty} - (s-\beta)\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\beta-1} f(\xi)\frac{d\xi}{\xi^{1-\rho}}\right] \\ &= \left(s-\beta+\frac{1}{\rho}\right)(s-\beta)\,\mathcal{M}_\rho\left\{f(x)\right\}(s-\beta). \end{aligned} \tag{2.9}$$

We next consider the term I_2 :

$$\begin{aligned} I_2 &= (1-\rho)\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\beta+1} \xi^{1-2\rho}\frac{df}{d\xi}\frac{d\xi}{\xi^{1-\rho}} \\ &= \left(\frac{1}{\rho}-1\right)\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\beta} \frac{df}{d\xi}d\xi \\ &= \left(\frac{1}{\rho}-1\right)\left[\left(\frac{\xi^\rho}{\rho}\right)^{s-\beta} f(\xi)\Big|_{\xi=0}^{\xi=\infty} - (s-\beta)\int_0^\infty \left(\frac{\xi^\rho}{\rho}\right)^{s-\beta-1} f(\xi)\frac{d\xi}{\xi^{1-\rho}}\right] \\ &= \left(1-\frac{1}{\rho}\right)(s-\beta)\,\mathcal{M}_\rho\left\{f(x)\right\}(s-\beta). \end{aligned} \tag{2.10}$$

By substituting (2.9) and (2.10) into (2.8), we obtain the desired outcome:

$$\mathcal{M}_\rho\left\{D_x^{\rho,\beta} f(x)\right\}(s) = \frac{(s-\beta)(s-\beta+1)\,\Gamma(\beta-s-1)}{\Gamma(1-s)}\mathcal{M}_\rho\left\{f(x)\right\}(s-\beta).$$

□

Lemma 2.9. *Let β be any real number with $1<\beta\le 2$. If $f:[0,\infty)\to R$ is extended Mellin integral transformable, then*

$$\mathcal{M}_\rho\left\{\left(\frac{x^\rho}{\rho}\right)^\beta D_x^{\rho,\beta} f(x)\right\}(s) = \frac{s(s+1)\Gamma(-s-1)}{\Gamma(1-s-\beta)}\mathcal{M}_\rho\left\{f(x)\right\}(s)\,for\,Re(s)\in\Omega.$$

Proof. This lemma's proof derives immediately from Lemma 2.5 and Lemma 2.8. □

Definition 2.8. *The convolution for the extended Mellin integral transform of functions f and g is determined by*

$$(f *_\rho g)(x) = \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1} f\left(\left(\frac{\rho x^\rho}{y^\rho}\right)^{\frac{1}{\rho}}\right)g(y)\frac{dy}{y^{1-\rho}}.$$

Lemma 2.10. *(The convolution theorem for the extended Mellin integral transform) Suppose that $f:[0,\infty)\to R$ and $g:[0,\infty)\to R$ are extended Mellin integral transformable. Then,*

$$\mathcal{M}_\rho\left\{(f *_\rho g)(x)\right\}(s) = \mathcal{M}_\rho\left\{f(x)\right\}(s)\mathcal{M}_\rho\left\{g(x)\right\}(s)\,for\,Re(s)\in\Omega.$$

Proof. Consider:

$$\begin{aligned}\mathcal{M}_\rho\left\{(f *_\rho g)(x)\right\}(s) &= \int_0^\infty \left(\frac{x^\rho}{\rho}\right)^{s-1}\left(\int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1} f\left(\left(\frac{\rho x^\rho}{y^\rho}\right)^{\frac{1}{\rho}}\right) g(y)\frac{dy}{y^{1-\rho}}\right)\frac{dx}{x^{1-\rho}}\\ &= \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\left(\int_0^\infty \left(\frac{x^\rho}{\rho}\right)^{s-1} f\left(\left(\frac{\rho x^\rho}{y^\rho}\right)^{\frac{1}{\rho}}\right)\frac{dx}{x^{1-\rho}}\right) g(y)\frac{dy}{y^{1-\rho}}.\end{aligned}$$

Changing variables in the inside integral by setting $\frac{u^\rho}{\rho} = \frac{x^\rho}{y^\rho}$ yields

$$\begin{aligned}\mathcal{M}_\rho\left\{(f *_\rho g)(x)\right\}(s) &= \int_0^\infty\left(\int_0^\infty \left(\frac{u^\rho}{\rho}\frac{y^\rho}{\rho}\right)^{s-1} f(u)\frac{du}{u^{1-\rho}}\right) g(y)\frac{dy}{y^{1-\rho}}\\ &= \left(\int_0^\infty \left(\frac{u^\rho}{\rho}\right)^{s-1} f(u)\frac{du}{u^{1-\rho}}\right)\left(\int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{s-1} g(y)\frac{dy}{y^{1-\rho}}\right).\end{aligned}$$

The proof of Lemma 2.10 is therefore finished. □

3. The space-fractional Black-Scholes model and its integral representations of solutions

In this section, we establish an explicit integral representation for the solution of the model (1.1) by employing the extended Mellin integral transform. The approach begins with transforming the governing Cauchy problem, which leads to the identification of the time-evolution multiplier $e^{p_\alpha(s)(T-t)}$, where $p_\alpha(s)$ is expressed in terms of gamma-function ratios. The solution is then recovered via Mellin inversion, yielding a generalized Mellin convolution between the option payoff ω and the inverse transform of the multiplier, as stated in Theorem 3.1. Subsequently, we verify that this representation indeed satisfies the original model (1.1) dynamics (Theorem 3.2), and highlight two important special cases: the fractional model in the original asset variable ($\rho = 1$, Corollary 5.1) and the classical Black-Scholes model with dividends ($\alpha = \rho = 1$, Corollary 5.2).

Theorem 3.1. *The integral representation for the model* (1.1) *is*

$$u(x,t) = \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1} \omega(y)\mathcal{M}_\rho^{-1}\left\{e^{p_\alpha(s)(T-t)}\right\}\left(\left(\frac{\rho x^\rho}{y^\rho}\right)^{\frac{1}{\rho}}\right)\frac{dy}{y^{1-\rho}}, \tag{3.1}$$

where $p_\alpha(s)$ is defined in a gamma-ratio form as

$$p_\alpha(s) = \frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}s^2 + \left(\frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)} - (r-q)\frac{\Gamma(-s)}{\Gamma(1-\alpha-s)}\right)s - r.$$

Proof. We will obtain the desired result with the help of the extended Mellin integral transform. Let $\mathcal{M}_\rho\{u(x,t)\}(s) = \widetilde{u}(s,t)$ and $\mathcal{M}_\rho\{u(x,T)\}(s) = \mathcal{M}_\rho\{\omega(x)\}(s) = \widetilde{\omega}(s)$. By taking the extended Mellin integral transform $\mathcal{M}_\rho$ with respect to the independent variable x on both sides of the model (1.1), we have

$$\mathcal{M}_\rho\left\{\frac{\partial u}{\partial t}\right\}(s)+\frac{1}{2}\sigma^2\mathcal{M}_\rho\left\{\left(\frac{x^\rho}{\rho}\right)^{1+\alpha}D_x^{\rho,1+\alpha}u\right\}(s)+(r-q)\mathcal{M}_\rho\left\{\left(\frac{x^\rho}{\rho}\right)^{\alpha}D_x^{\rho,\alpha}u\right\}(s)-r\mathcal{M}_\rho\{u\}(s)=0.$$

By Lemma 2.7 and Lemma 2.9, we obtain

$$\frac{d}{dt}\widetilde{u}(s,t)+\frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}(s^2+s)\widetilde{u}(s,t)-(r-q)\frac{s\Gamma(-s)}{\Gamma(1-\alpha-s)}\widetilde{u}(s,t)-r\widetilde{u}(s,t)=0.$$

The Mellin transform of model (1.1) becomes

$$\left.\begin{aligned}&\frac{d}{dt}\widetilde{u}(s,t)+p_\alpha(s)\widetilde{u}(s,t)=0 \text{ for } t\in[0,T),\\ &\widetilde{u}(s,T)=\widetilde{\omega}(s),\end{aligned}\right\}\tag{3.2}$$

where

$$p_\alpha(s)=\frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}s^2+\left(\frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}-(r-q)\frac{\Gamma(-s)}{\Gamma(1-\alpha-s)}\right)s-r.$$

The solution of (3.2) is determined by

$$\widetilde{u}(s,t)=\widetilde{\omega}(s)e^{p_\alpha(s)(T-t)} \text{ for } t\in[0,T],$$

where $\widetilde{\omega}(s)$ is defined by taking the Mellin integral transform to the terminal conditions for put option prices. By using the inversion of the extended Mellin integral transform and Lemma 2.10, we have

$$\begin{aligned}u(x,t)&=\mathcal{M}_\rho^{-1}\left\{\widetilde{\omega}(s)e^{p_\alpha(s)(T-t)}\right\}(x)\\&=\left(\omega *_\rho \mathcal{M}_\rho^{-1}\left\{e^{p_\alpha(s)(T-t)}\right\}\right)(x).\end{aligned}$$

□

In the next theorem, the solution u in Eq (3.1) is then shown to satisfy model (1.1).

Theorem 3.2. *The integral representation of u in Eq* (3.1) *satisfies model* (1.1).

Proof. By (3.1) and the inverse of the extended Mellin integral transform in Definition 2.7, we have

$$\begin{aligned}u(x,t)&=\int_0^\infty\left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\mathcal{M}_\rho^{-1}\left\{e^{p_\alpha(s)(T-t)}\right\}\left(\left(\frac{\rho x^\rho}{y^\rho}\right)^{\frac{1}{\rho}}\right)\frac{dy}{y^{1-\rho}}\\&=\int_0^\infty\left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s}e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}.\end{aligned}$$

In the following, we will find the first derivative of u with respect to the independent variable t. Consider

$$\frac{\partial}{\partial t}u(x,t)=\frac{\partial}{\partial t}\int_0^\infty\left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s}e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}$$

$$= -\frac{1}{2\pi i}\int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} p_\alpha(s)\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}. \quad (3.3)$$

Next, we find the fractional-order derivative of u with order $0 < \alpha \leq 1$ with respect to x. Consider

$$\begin{aligned} D_x^{\rho,\alpha}u(x,t) &= \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(\frac{y^\rho}{\rho}\right)^{s} D_x^{\rho,\alpha}\left(\frac{x^\rho}{\rho}\right)^{-s} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}} \\ &= \frac{1}{2\pi i}\int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(\frac{y^\rho}{\rho}\right)^{s}\frac{-s\Gamma(-s)}{\Gamma(1-s-\alpha)}\left(\frac{x^\rho}{\rho}\right)^{-s-\alpha} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}. \end{aligned}$$

Then, we have

$$\left(\frac{x^\rho}{\rho}\right)^{\alpha} D_x^{\rho,\alpha}u(x,t) = \frac{1}{2\pi i}\int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)}\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}. \quad (3.4)$$

In the following, we find the fractional-order derivative of u with order $1+\alpha$ with respect to x. Consider

$$D_x^{\rho,1+\alpha}u(x,t) = \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left(\frac{y^\rho}{\rho}\right)^{s} D_x^{\rho,1+\alpha}\left(\frac{x^\rho}{\rho}\right)^{-s} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}.$$

To evaluate $D_x^{\rho,1+\alpha}\left(\frac{x^\rho}{\rho}\right)^{-s}$ we use Lemma 2.4 with $a = -s$ and $\beta = 1+\alpha$:

$$D_x^{\rho,1+\alpha}\left(\frac{x^\rho}{\rho}\right)^{a} = a(a-1)\,\frac{\Gamma(a-1)}{\Gamma(1-(1+\alpha)+a)}\left(\frac{x^\rho}{\rho}\right)^{a-(1+\alpha)}.$$

Substituting $a = -s$ gives

$$D_x^{\rho,1+\alpha}\left(\frac{x^\rho}{\rho}\right)^{-s} = (-s)(-s-1)\,\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}\left(\frac{x^\rho}{\rho}\right)^{-s-1-\alpha} = (s^2+s)\,\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}\left(\frac{x^\rho}{\rho}\right)^{-s-1-\alpha}.$$

Inserting this back into the inner s-integral yields

$$\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}(s^2+s)\,\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s-1-\alpha} e^{p_\alpha(s)(T-t)}\,ds.$$

Finally, multiplying both sides by $\left(\frac{x^\rho}{\rho}\right)^{1+\alpha}$, we have

$$\left(\frac{x^\rho}{\rho}\right)^{1+\alpha} D_x^{\rho,1+\alpha}u(x,t) = \int_0^\infty \left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}(s^2+s)\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s} e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}. \quad (3.5)$$

By using (3.3)–(3.5) and the definition of $p_\alpha(s)$, we have

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2\left(\frac{x^\rho}{\rho}\right)^{1+\alpha}D_x^{\rho,1+\alpha}u+(r-q)\left(\frac{x^\rho}{\rho}\right)^{\alpha}D_x^{\rho,\alpha}u-ru\\
=\;&\frac{1}{2\pi i}\int_0^\infty\left(\frac{y^\rho}{\rho}\right)^{-1}\omega(y)\left(\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\left[-p_\alpha(s)+\frac{\sigma^2}{2}(s^2+s)\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}+(r-q)\frac{-s\Gamma(-s)}{\Gamma(1-s-\alpha)}-r\right]\right.\\
&\times\left.\left(\frac{y^\rho}{\rho}\right)^{s}\left(\frac{x^\rho}{\rho}\right)^{-s}e^{p_\alpha(s)(T-t)}ds\right)\frac{dy}{y^{1-\rho}}\\
=\;&0.
\end{aligned}
$$

Therefore, according to the preceding reasoning, this integral representation of u is the analytical solution for model (1.1). □

4. Stochastic representation and the infinitesimal generator

4.1. Lévy-driven asset dynamics

In the classical Black-Scholes framework, the asset price process S_t follows the Itô diffusion

$$dS_t=\mu S_t dt+\sigma S_t dW_t, \tag{4.1}$$

where W_t is a standard Brownian motion. The infinitesimal generator of this diffusion is the second-order differential operator

$$\mathcal{A}_{BS}f(x)=\mu x f'(x)+\frac{1}{2}\sigma^2x^2f''(x). \tag{4.2}$$

In the proposed space-fractional extension, the second-order derivative is replaced by an extended Caputo fractional operator. Such nonlocal operators naturally arise as infinitesimal generators of Lévy processes.

Let $(L_t)_{t\geq 0}$ be a pure-jump Lévy process with a characteristic exponent given by the Lévy-Khintchine formula

$$\mathbb{E}\left[e^{i\xi L_t}\right]=\exp\{t\psi(\xi)\}, \tag{4.3}$$

where

$$\psi(\xi)=ib\xi-\frac{1}{2}\sigma^2\xi^2+\int_{\mathbb{R}\setminus\{0\}}\left(e^{i\xi y}-1-i\xi y\mathbf{1}_{|y|<1}\right)\nu(dy), \tag{4.4}$$

and ν denotes the Lévy measure.

For an α-stable Lévy process ($0<\alpha<2$), the generator is the fractional Laplacian

$$\mathcal{L}f(x)=-(-\Delta)^{\alpha/2}f(x). \tag{4.5}$$

We consider the geometric Lévy model

$$dS_t=\mu S_t dt+\sigma S_{t-}dL_t, \tag{4.6}$$

where L_t is an α-stable Lévy process. Under appropriate integrability conditions, the infinitesimal generator of S_t acting on smooth test functions f is given by

$$\mathcal{A}f(x) = \mu x f'(x) + \int_{\mathbb{R}\setminus\{0\}} \left[f(x + \sigma x y) - f(x) - \sigma x y f'(x) \mathbf{1}_{|y|<1} \right] \nu(dy). \tag{4.7}$$

This nonlocal operator reduces to the classical second-order operator when the Lévy measure degenerates to the Gaussian case ($\alpha = 2$).

4.2. Backward Kolmogorov equation

Let

$$V(x,t) = \mathbb{E}\left[e^{-r(T-t)} \Phi(S_T) \mid S_t = x \right]. \tag{4.8}$$

Then V satisfies the backward Kolmogorov equation

$$\frac{\partial V}{\partial t} + \mathcal{A}V - rV = 0, \quad V(x,T) = \Phi(x). \tag{4.9}$$

For α-stable dynamics, the jump operator is equivalent to a fractional spatial derivative. In particular, the generator can be written in pseudo-differential form as

$$\mathcal{A}f(x) = \mu x f'(x) + \sigma^{\alpha} x^{\alpha} \mathcal{D}_x^{\alpha} f(x), \tag{4.10}$$

where $\mathcal{D}_x^{\alpha}$ denotes the extended Caputo fractional derivative introduced in Section 3.

Consequently, the proposed space-fractional Black-Scholes equation is rigorously interpreted as the backward Kolmogorov equation of a geometric Lévy process.

4.3. Forward equation and fractional Fokker-Planck dynamics

Let $p(x,t)$ denote the probability density function of S_t. Then p satisfies the forward Kolmogorov (Fokker-Planck) equation

$$\frac{\partial p}{\partial t} = \mathcal{A}^* p, \tag{4.11}$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$.

In the α-stable case, this yields the fractional Fokker-Planck equation

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu x p) + \sigma^{\alpha} \mathcal{D}_x^{\alpha}(x^{\alpha} p). \tag{4.12}$$

This establishes the precise stochastic-PDE duality:

- Geometric Lévy SDE,
- Fractional Fokker-Planck equation (forward),
- Space-fractional Black-Scholes equation (backward).

4.4. *Connection with stylized facts*

The fractional parameter α directly controls tail behavior. For $\alpha < 2$, the transition density exhibits power-law decay

$$p(x) \sim |x|^{-(1+\alpha)}, \tag{4.13}$$

consistent with empirically observed heavy-tailed returns. When $\alpha \to 2$, the Lévy measure converges to the Gaussian case and the fractional operator reduces to the classical Laplacian. Hence, the standard Black-Scholes equation is recovered as a limiting case.

Moreover, α-stable processes exhibit anomalous scaling

$$\mathbb{E}|S_t - S_0|^q \sim t^{q/\alpha}, \tag{4.14}$$

which deviates from classical Brownian scaling and aligns with empirical findings reported in econophysics literature.

Finally, deviations from log-normality induced by the nonlocal generator produce non-flat implied volatility surfaces, allowing the model to capture volatility smiles and skewness observed in option markets.

5. Mellin transform solution

The following two corollaries are derived from Theorems 3.1 and 3.2.

Corollary 5.1. *The integral representation for the solution of model* (1.1) *with fractional order* $0 < \alpha \leq 1$ *and* $\rho = 1$ *is*

$$u(x,t) = \int_0^\infty y^{-1}\omega(y)\mathcal{M}_\rho^{-1}\left\{e^{p_\alpha(s)(T-t)}\right\}\left(\frac{x}{y}\right)dy,$$

where

$$p_\alpha(s) = \frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)}s^2 + \left(\frac{1}{2}\sigma^2\frac{\Gamma(-s-1)}{\Gamma(-s-\alpha)} - (r-q)\frac{\Gamma(-s)}{\Gamma(1-\alpha-s)}\right)s - r. \tag{5.1}$$

Proof. The proof of the corollary is obtained from Theorem 3.1 by setting $\rho = 1$. □

The Black-Scholes problem solution is obtained as a specific instance of the integral form of the solution that is provided in Theorem 3.1, as the following conclusion shows.

Corollary 5.2. *The integral representation of the solution for the sCfBC model with* $\alpha = \rho = 1$ *(the classical Black-Scholes model with dividends),*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\sigma^2 x^2\dfrac{\partial^2 u}{\partial x^2} + (r-q)x\dfrac{\partial u}{\partial x} - ru = 0 \ \text{for} \ (x,t) \in (0,\infty)\times[0,T), \\ u(x,T) = \omega(x) \ \text{for} \ x \in (0,\infty), \end{cases}$$

is that

$$u(x,t) = \frac{x^{\left(\frac{1}{2}-\frac{r-q}{\sigma^2}\right)}e^{-\frac{1}{8}(T-t)\left(\sigma+\frac{2(r-q)}{\sigma}\right)^2}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty y^{\left(\frac{r-q}{\sigma^2}-\frac{3}{2}\right)}\omega(y)e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{x}{y}\right)}{\sigma(T-t)}\right)^2}dy.$$

Proof. Set $\rho = \alpha = 1$ in Theorem 3.1 and and then we have

$$u(x,t) = \int_0^\infty y^{-1}\,\omega(y)\,\mathcal{M}^{-1}\Big\{e^{\,p_{1,q}(s)\,(T-t)}\Big\}\Big(\frac{x}{y}\Big)\,dy, \tag{5.2}$$

where

$$p_{1,q}(s) = \tfrac{1}{2}\sigma^2(s^2 - s) - (r-q)\,s - r = \tfrac{1}{2}\sigma^2\Big(s^2 + (1-\psi_q)s - \psi_q\Big), \qquad \psi_q := \frac{2(r-q)}{\sigma^2}.$$

Let us consider the term $\mathcal{M}^{-1}\Big\{e^{\,p_{1,q}(s)\,(T-t)}\Big\}(x)$. By the shift property of the Mellin transform, we have

$$\begin{aligned}
\mathcal{M}^{-1}\Big\{e^{\,p_{1,q}(s)\,(T-t)}\Big\}(x) &= \mathcal{M}^{-1}\Big\{\exp\Big(\tfrac{1}{2}\sigma^2[s^2 + (1-\psi_q)s - \psi_q](T-t)\Big)\Big\}(x)\\
&= \exp\Big(-\mu\Big(\tfrac{\psi_q+1}{2}\Big)^2\Big)\,\mathcal{M}^{-1}\Big\{\exp\big(\mu(s+\eta_q)^2\big)\Big\}(x)\\
&= \exp\Big(-\mu\Big(\tfrac{\psi_q+1}{2}\Big)^2\Big)\,x^{\eta_q}\,\mathcal{M}^{-1}\Big\{e^{\mu s^2}\Big\}(x),
\end{aligned} \tag{5.3}$$

where $\mu = \frac{1}{2}\sigma^2(T-t)$ and $\eta_q := \frac{1-\psi_q}{2}$.

Let

$$f(x) := \mathcal{M}^{-1}\Big\{e^{\mu s^2}\Big\}(x), \qquad \widetilde{f}(s) := e^{\mu s^2}.$$

Using the relation between the Fourier and Mellin transforms (via $x = e^{\xi}$ and $ik = -s$), one obtains (see, e.g., [20])

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}e^{-\mu k^2}\,dk = \frac{1}{2\sqrt{\pi\mu}}\,\exp\Big(-\frac{\xi^2}{4\mu}\Big),$$

which yields

$$\mathcal{M}^{-1}\Big\{e^{\mu s^2}\Big\}(x) = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\,\exp\left[-\frac{1}{2}\left(\frac{\ln x}{\sigma(T-t)}\right)^2\right]. \tag{5.4}$$

Combining (5.2)–(5.4), we obtain the integral representation

$$\begin{aligned}
u(x,t) &= \frac{x^{\eta_q}\,\exp\Big(-\mu(\frac{\psi_q+1}{2})^2\Big)}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty y^{-(1+\eta_q)}\,\omega(y)\,\exp\left[-\frac{1}{2}\left(\frac{\ln\big(\frac{x}{y}\big)}{\sigma(T-t)}\right)^2\right]dy\\
&= \frac{x^{\left(\frac{1}{2}-\frac{r-q}{\sigma^2}\right)}\exp\Big(-\frac{1}{8}(T-t)(\sigma + \frac{2(r-q)}{\sigma})^2\Big)}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty y^{\left(\frac{r-q}{\sigma^2}-\frac{3}{2}\right)}\,\omega(y)\,\exp\left[-\frac{1}{2}\left(\frac{\ln\big(\frac{x}{y}\big)}{\sigma(T-t)}\right)^2\right]dy.
\end{aligned}$$

□

5.1. Comparison with classical numerical methods

Fractional partial differential equations are commonly solved using finite difference, finite element, or Petrov-Galerkin methods. While these approaches are powerful, they often require dense discretization matrices and careful treatment of nonlocal terms, which may increase computational complexity.

In contrast, the extended Mellin transform method provides a semi-analytical representation of the solution, reducing the problem to algebraic manipulation in transform space followed by inversion. This approach offers several advantages including no spatial mesh discretization, no stability constraints associated with time-stepping schemes, reduced computational cost for one-dimensional problems, and exact handling of nonlocal operators in transform space.

Therefore, the proposed methodology complements classical numerical schemes and provides an efficient alternative for fractional option pricing problems.

6. Numerical examples

In this section, we apply model (1.1) to a European put option. The extended Mellin integral transform of a function f is defined as

$$\mathcal{M}_\rho\{f(x)\} = \widetilde{f}(s) = \int_0^\infty \left(\frac{x^\rho}{\rho}\right)^{s-1} f(x)\frac{dx}{x^{1-\rho}}.$$

For the terminal condition $u(x,T)$, we have

$$\widetilde{u}(s,T) = \int_0^\infty u(x,T)\left(\frac{x^\rho}{\rho}\right)^{s-1}\frac{dx}{x^{1-\rho}} = \int_0^K \left(\frac{K^\rho}{\rho}-\frac{x^\rho}{\rho}\right)\left(\frac{x^\rho}{\rho}\right)^{s-1}\frac{dx}{x^{1-\rho}}. \tag{6.1}$$

We break the integral of the extended Mellin integral transform into two parts, i.e.,

$$\widetilde{u}(s,T) = \frac{K^\rho}{\rho}\int_0^K \left(\frac{x^\rho}{\rho}\right)^{s-1}\frac{dx}{x^{1-\rho}} - \int_0^K \left(\frac{x^\rho}{\rho}\right)^{s}\frac{dx}{x^{1-\rho}}.$$

Since

$$\int_0^K \left(\frac{x^\rho}{\rho}\right)^{s-1}\frac{dx}{x^{1-\rho}} = \frac{(K^\rho/\rho)^s}{s},$$

we substitute this result back into the expression for $\widetilde{u}(s,T)$ in (6.1). Then

$$\begin{aligned}\widetilde{u}(s,T) &= \left(\frac{K^\rho}{\rho}\right)\cdot\frac{(K^\rho/\rho)^s}{s} - \frac{(K^\rho/\rho)^{s+1}}{s+1}\\ &= \left(\frac{K^\rho}{\rho}\right)^{s+1}\left(\frac{(s+1)-s}{s(s+1)}\right) = \frac{(K^\rho/\rho)^{s+1}}{s(s+1)}.\end{aligned}$$

This compact form $\frac{(K^\rho/\rho)^{s+1}}{s(s+1)}$ corresponds to the Mellin transform of the put payoff, emphasizing the scale-invariant structure of the problem.

The general solution is of the form

$$\widetilde{u}(s,t) = \widetilde{u}(s,T)e^{p_\alpha(s)(T-t)} = \frac{(K^\rho/\rho)^{s+1}}{s(s+1)}e^{p_\alpha(s)(T-t)}.$$

The put option price is

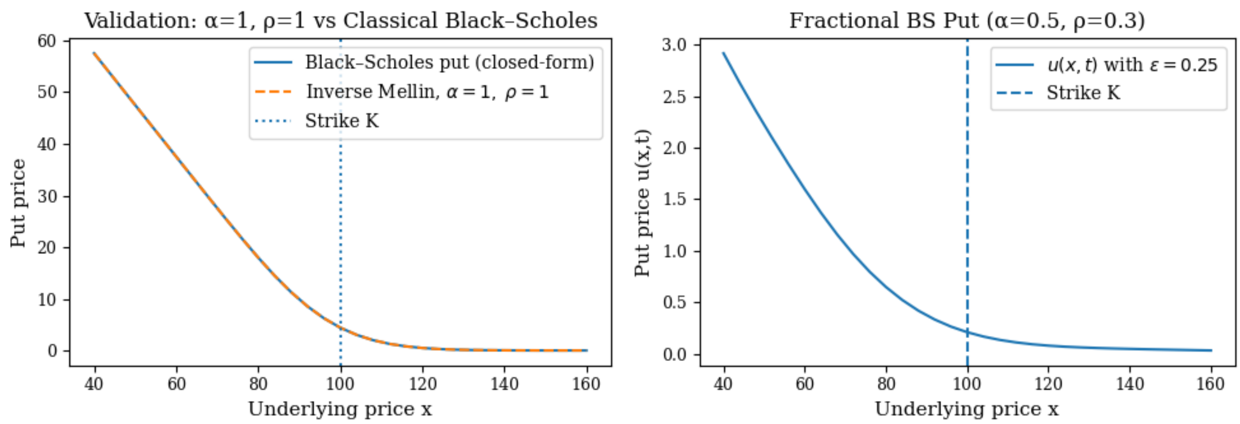
$$u(x,t) = \mathcal{M}_\rho^{-1}\{\widetilde{u}(s,t)\} = \frac{1}{2\pi i}\int_{\mathcal{R}(s)=\varepsilon}\left(\frac{x^\rho}{\rho}\right)^{-s}\frac{(K^\rho/\rho)^{s+1}}{s(s+1)}e^{p_\alpha(s)(T-t)}\,ds.$$

with $p_\alpha(s)$ as defined in Theorem 2.1 (gamma-ratio form). The exponential factor $e^{p_\alpha(s)(T-t)}$ reflects the fractional diffusion and distinguishes this model from the classical Black-Scholes case.
The procedure involves two main steps as follows.

(1) Discretizing the integral along a contour in the complex plane.
(2) Using numerical integration methods (e.g., Gaussian quadrature or the trapezoidal rule) to compute the solution.

We use following common market parameters, including strike price $K = 100$, risk-free rate $r = 0.05$, volatility $\sigma = 0.20$, maturity $T = 1.0$, and valuation time $t = 0.5$. The inverse extended Mellin transform was evaluated along the contour $\Re(s) = \varepsilon = 0.25$, with numerical quadrature performed using truncation width $W = 30$, $N = 1200$ nodes, and precision set to 50 decimal places. Figure 1(a) demonstrates validation of the method for $\alpha = 1, \rho = 1$. The computed put prices via the inverse extended Mellin transform overlap almost perfectly with the classical Black-Scholes closed-form solution across the range of underlying asset prices. The maximum absolute error is of order 10^{-6}, confirming that the numerical inversion scheme is accurate and stable under the chosen parameters.

Figure 1(b) illustrates fractional Black-Scholes put prices for $\alpha = 0.5, \rho = 0.3$. Compared with the standard case, the option values are systematically lower, especially near the money. This reflects the effect of fractional dynamics, where memory and anomalous diffusion reduce the option premium. The prices decline smoothly as the underlying price x increases, tending toward zero far above the strike.



(a) (b)

Figure 1. (a) Validation of the inverse extended Mellin transform method for the case $\alpha = 1, \rho = 1$; (b) fractional Black-Scholes put price with parameters $\alpha = 0.5, \rho = 0.3$.

Application to finance. In order to empirically assess the proposed model (1.1), we conducted a calibration exercise using publicly available option market data. Option quotes were retrieved from Yahoo Finance via the `yfinance` Python interface, focusing on European-style exchange-traded options written on the SPDR S&P 500 ETF (SPY), which is commonly employed as a liquid proxy for the S&P 500 index. For a given expiration date, we extracted the put option chain and constructed mid-prices as the average of bid and ask quotes, with the last traded price used as a fallback in cases of missing or zero bid-ask spreads. To ensure reliability, quotes were filtered to exclude extremely illiquid strikes and to restrict attention to options with moneyness in the range of approximately 75% to 130% of the underlying spot price.

Figure 2 illustrates the sensitivity of the fractional Black-Scholes put option value to changes in the conformable parameter ρ, with the fractional order fixed at $\alpha = 0.5$. The horizontal axis represents the underlying asset price x, while the vertical axis shows the corresponding put option value $u(x,t)$. As expected, the put price decreases as the underlying price rises above the strike $K = 100$, eventually approaching zero in the far out-of-the-money region. The comparison across curves reveals that larger values of ρ lead to significantly higher put prices for the same underlying level, particularly when the option is in-the-money. For example, at low values of x, the case $\rho = 0.9$ produces a much steeper price function than the cases $\rho = 0.3$ or $\rho = 0.5$. This demonstrates that the conformable parameter ρ acts as a scaling factor that amplifies the sensitivity of the option value to movements in the underlying price, thereby providing additional flexibility in capturing market pricing patterns beyond the classical Black-Scholes framework.

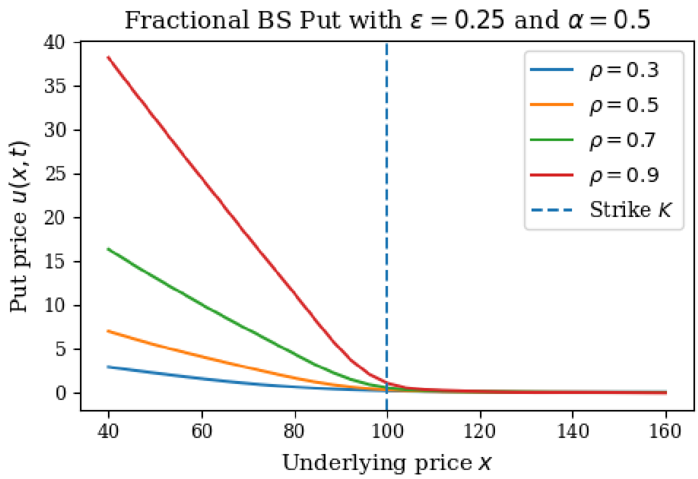


Figure 2. Put option prices under the fractional Black-Scholes model with fixed fractional order $\alpha = 0.5$ and varying conformable parameters $\rho \in \{0.3, 0.5, 0.7, 0.9\}$.

6.1. Sensitivity and robustness analysis

To assess the robustness of the model, we examine the sensitivity of option prices to the fractional parameters (α, ρ).

Sensitivity to α. For $1 < \alpha \leq 2$, decreasing α increases tail thickness and produces higher option premiums for out-of-the-money contracts. This behavior reflects the increased probability of large price movements. The model smoothly converges to the classical Black-Scholes price as $\alpha \to 2$.

Sensitivity to ρ. The parameter ρ controls the degree of scale dependence in the diffusion term. Numerical experiments indicate that varying ρ modifies the slope of the implied volatility curve, providing additional flexibility in capturing skew effects.

Stability across market conditions. The model remains numerically stable across different volatility regimes. In high-volatility environments, fractional dynamics yield larger deviations from the classical model, while in low-volatility conditions, the results approach the standard Black-Scholes case.

These observations confirm that the model is structurally robust and well-behaved for admissible parameter ranges.

6.2. Calibration to market data and limitations

6.2.1. Calibration of (α, ρ)

The model was then calibrated by minimizing the sum of squared errors between observed market put prices and model-implied values computed via the inverse extended Mellin transform. In this procedure, the volatility parameter σ was fixed at 20%, and the risk-free rate r was taken to be 4.5% per annum (consistent with prevailing short-term U.S. Treasury yields). A continuous dividend yield q was estimated from Yahoo Finance trailing yield data for the ETF, and the effective discount rate $r_{\text{eff}} = r - q$ was applied.

The fractional parameters $\alpha \in (0, 1]$ and $\rho > 0$ were treated as free variables, initialized at $(0.7, 0.4)$, and optimized using a derivative-free Nelder–Mead routine. Numerical inversion of the Mellin integral employed a contour with real part $\varepsilon = 0.25$, truncation width $W = 28$, and $N = 900$ trapezoidal nodes, with computations performed at 50-digit precision in `mpmath`. The resulting estimates of (α, ρ) were obtained for the selected maturity, thereby linking the fractional model directly to observable market data.

Given observed option prices $\{P_j^{\text{mkt}}\}_{j=1}^N$ for strikes K_j, maturities T_j, and underlying levels x_j, we estimate (α, ρ) by minimizing the squared error across strikes, namely,

$$\min_{\alpha,\rho} \sum_{K \in \mathcal{K}} \left(u^{\text{model}}(S_t, t; K, \alpha, \rho, \sigma) - P^{\text{mkt}}(K) \right)^2,$$

with bounds $\alpha \in (0, 1]$ and $\rho > 0$.

6.2.2. Model calibration results

In this subsection, calibration of model (1.1) was performed on the SPY put option expiring on October 1, 2025, with fixed volatility $\sigma = 20\%$, risk-free rate $r = 4.50\%$, and dividend yield $q = 0.86\%$.

Figure 3 illustrates the outcome of calibrating model (1.1) to option market data. The dataset consists of SPY put options expiring on October 1, 2025, with a very short time to maturity ($\tau \approx 0.003$ years). The blue markers represent market mid-prices obtained from Yahoo Finance, while the orange curve shows the model fit using the inverse extended Mellin transform. The optimal fractional parameters were estimated as $\hat{\alpha} = 0.598$ and $\hat{\rho} = 0.960$, suggesting a deviation from the classical Black–Scholes case ($\alpha = 1, \rho = 1$). The fitted curve captures the overall shape of the option price profile across strikes, though discrepancies are visible near deep out-of-the-money and in-the-money regions. These differences are partly attributable to the extremely short maturity, where market quotes may be affected by microstructure noise and bid–ask frictions. Nevertheless, the calibration demonstrates that the fractional model can be successfully linked to real option prices, and that the parameters α and ρ provide additional flexibility in adapting the pricing kernel to empirical data.

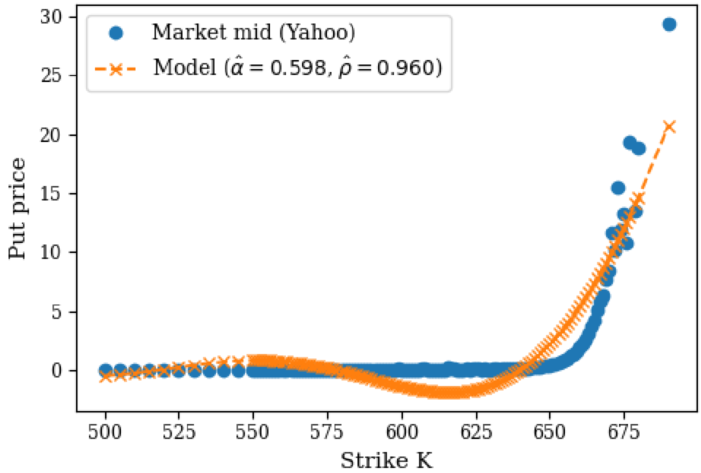


Figure 3. Calibration of model (1.1) to SPY put option data.

Figure 4 illustrates the behavior of the fractional Black-Scholes put price as a function of the underlying asset value when parameters are fixed to those obtained from the calibration exercise. As expected, the put price decreases monotonically with an increasing underlying price, approaching zero when the option is far out-of-the-money and rising steeply as the underlying price falls below the strike. The inclusion of fractional parameters α and ρ modifies the decay profile relative to the classical Black-Scholes case, reflecting the impact of memory and anomalous diffusion effects embedded in the model.

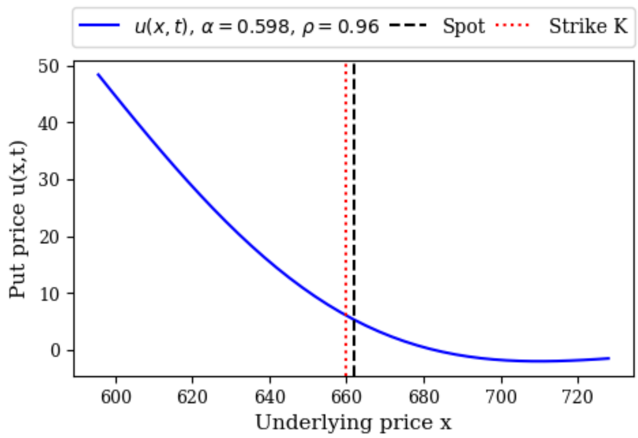


Figure 4. Put option values under model (1.1), computed via the inverse extended Mellin transform.

Compared with the previous result (Figure 3), which displayed the fit of the model against quoted option prices across strikes, this figure highlights the dynamic mapping from the underlying price to the option value at fixed maturity. Whereas Figure 3 focused on cross-sectional calibration against market data, Figure 4 provides a model-driven sensitivity analysis that shows how the calibrated parameters shape the option pricing curve. Together, the two figures confirm both the empirical relevance and the internal consistency of model (1.1) framework.

6.2.3. Dataset limitations

The calibration was performed using SPY option data covering a range of strikes and maturities. While the dataset includes both in-the-money and out-of-the-money contracts, it is limited to a single asset class and a specific time window. Potential limitations include:

- Restricted maturity spectrum,
- Market microstructure noise,
- Liquidity variations across strikes.

Although the calibration results demonstrate improved flexibility compared to the classical Black–Scholes model, further validation across multiple asset classes (e.g., index options, equity options, commodities) would strengthen the empirical generalization of the framework. This constitutes an important direction for future research.

6.3. Extension to other derivatives and pricing frameworks

The proposed space-fractional Black-Scholes framework is not restricted to European put options. Since the Mellin transform approach applies to payoff functions satisfying appropriate integrability conditions, the methodology is not restricted to a specific contract type. In particular, it can be extended to European call options, digital and power options, as well as basket options under suitable separability assumptions. Moreover, structured products with polynomial-type payoff structures can also be accommodated within the same analytical framework. Furthermore, the fractional operator can be incorporated into alternative pricing frameworks. For example, replacing the diffusion term in Merton's jump-diffusion model or local volatility models with the extended Caputo operator yields a nonlocal pricing equation that captures heavy-tailed dynamics. Similarly, fractional extensions of stochastic volatility models (e.g., Heston-type dynamics with nonlocal generators) may provide additional flexibility in modeling skewness and kurtosis.

Compared with existing fractional approaches (e.g., Riesz fractional derivatives or time-fractional models), the present framework combines a scale-dependent extended Caputo operator, a geometric Lévy interpretation, and a Mellin-transform-based semi-analytical solution. This integrated structure distinguishes the model from previously proposed fractional Black–Scholes formulations.

7. Conclusions

The Black-Scholes equation is widely recognized as one of the most fundamental models in option pricing theory. In this work, we introduced a space-fractional extension of the Black-Scholes equation by incorporating the extended Caputo fractional derivative into the diffusion term. The resulting nonlocal pricing equation was solved using the extended Mellin integral transform, for which we established the shift property and convolution theorem, together with transform-domain identities for $(x^\rho/\rho)^\alpha D_x^{\rho,\alpha}$ and $(x^\rho/\rho)^{1+\alpha} D_x^{\rho,1+\alpha}$. These results led to the integral representation in Theorem 3.1 and its verification in Theorem 3.2, while the classical Black-Scholes formulation was recovered as a limiting case. Hence, the proposed framework provides a mathematically consistent generalization that preserves the standard model when the fractional order approaches two.

From a financial perspective, numerical experiments for European put options illustrate the impact of fractional dynamics on option prices. In particular, fractional models yield systematically different valuations near the money compared with the classical case, reflecting the influence of nonlocal diffusion and heavy-tailed behavior. The fractional parameter α governs tail thickness and deviations from Gaussian dynamics, whereas the spatial parameter ρ acts as a scale-dependent mechanism that modifies sensitivity to the underlying asset and influences the slope of the implied volatility structure. Sensitivity analysis confirms that the model behaves smoothly across admissible parameter ranges and converges continuously to the classical Black–Scholes case as $\alpha \to 2$.

Calibration to SPY option market data demonstrates that the additional parameters α and ρ provide enhanced flexibility in fitting observed prices. However, the empirical study is restricted to a single asset class and a specific time window. Although the dataset includes a range of strikes and maturities, further validation across multiple asset classes, broader maturity spectra, and different volatility regimes would strengthen the empirical generalization of the model. Such extensions constitute a natural direction for future research.

From a methodological standpoint, the extended Mellin transform approach offers a semi-analytical alternative to classical finite difference and finite element methods commonly used for fractional partial differential equations. By operating directly in transform space, the method avoids spatial mesh discretization and handles nonlocal operators in a structured manner, providing computational efficiency for one-dimensional pricing problems.

Finally, the proposed framework is not limited to European put options. The transform-based methodology can be extended to other payoff structures satisfying appropriate integrability conditions and may be incorporated into alternative pricing models, including jump-diffusion or stochastic volatility settings with fractional generators. Overall, the space-fractional Black-Scholes formulation developed in this paper establishes a bridge between fractional calculus, geometric Lévy dynamics, and transform-based analytical techniques, offering a flexible and mathematically tractable extension of the classical option pricing paradigm.

Author contributions

P. Sawangtong and D. Wiwatanapataphee: Conceptualization, investigation; W. Sawangtong and D. Wiwatanapataphee: Software, visualization; W. Sawangtong, D. Wiwatanapataphee, and P. Sawangtong: Formal analysis, writing—original draft; D. Wiwatanapataphee and P. Sawangtong: Writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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