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Research article

Integrating confidence level-based p, q, r-spherical fuzzy rough Einstein Bonferroni aggregation operators with the CRITIC-CODAS approach for prioritizing waste treatment techniques

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Abstract: Multi-criteria decision-making (MCDM) techniques play a crucial role in solving real-life problems with imprecision and uncertainty. The p, q, r-spherical fuzzy rough set (p, q, r-SFRS) is an important development over FS theory for flexible representation of hesitancy, membership, and non-membership degrees. In this paper, we present a nuanced decision-making approach named confidence levels. Moreover, p, q, r-spherical fuzzy rough Einstein Bonferroni ($C_{p,q,r}$-SFREBOM) aggregation operators, such as the p, q, r-spherical fuzzy rough Einstein Bonferroni weighted geometric operator ($C_{p,q,r}$-SFREBOMWG) and p, q, r-spherical fuzzy rough Einstein Bonferroni weighted average operator ($C_{p,q,r}$-SFREBOMWA), plays a crucial role in providing a strong support for MCDM analysis. The method considers lower and upper approximations of alternatives and synthesizes judgments along criteria based on expert confidence levels. The operational laws, theorems, and properties of p, q, r-SFREBOMWG and q, r-SFREBOMWA are explained, showing the superiority and importance of the suggested work, providing more accurate results than existing approaches. Integrating the newly proposed operator with multi-decision-making techniques like criteria importance through intercriteria correlation (CRITIC) and combinative distance-based assessment (CODAS) is a unique approach. A case study was considered to validate the efficacy and usability of the proposed operators in prioritizing sustainable municipal solid waste treatment techniques with the CRITIC and CODAS techniques. The computed results were compared with approaches to further support the outcomes of the proposed work. The comparison was done with technique for order preference by similarity to ideal solution (TOPSIS), and weighted aggregated sum product assessment (WASPAS) to assess the proposed work's reliability. Additionally, comparative and sensitivity analyses were conducted to demonstrate the robustness and excellence of the $C_{p,q,r}$-SFREBOM approach with that of conventional MCDM methods.

Keywords: p, q, r-spherical fuzzy rough set; Einstein Bonferroni aggregation operators; confidence-based aggregation; fuzzy rough set theory; weighted geometric operator; weighted average operator; CRITIC-CODAS method; decision resilience; spherical fuzzy logic; expert confidence levels
Mathematics Subject Classification: 03E72, 90B50

1. Introduction

Although classical multi-criteria decision-making (MCDM) models have been traditionally used, they tend to be at a loss when dealing with vagueness, uncertainty, and incomplete data. The embedding of fuzzy set (FS) theory within MCDM has provided a strong solution to overcome these challenges. FS makes uncertainty modeling possible by depicting the degrees of membership degree (MD), non-membership degree (NMD), and hesitancy with the constraint that their sum does not exceed one in decision-making. The advent of p, q, r-spherical fuzzy rough set (p, q, r-SFRS) theory has also given more flexibility to FS theory by presenting a more complete method of uncertainty modeling. The p, q, and r parameters allow modeling of differential levels of confidence in the decision process, making possible a richer representation of relationships between alternatives and criteria. Building upon the foundations of FS theory and RS theory, this paper introduces a novel approach that employs confidence levels. These aggregation operators combine the robustness of the Einstein norms with the Bonferroni means and enhance decision-making resilience and robustness. Our attention is centered on two aggregation operators that are of chief importance: p, q, r-spherical fuzzy rough Einstein Bonferroni weighted geometric operator ($C_{p,q,r}$-SFREBOMWG) and p, q, r-spherical fuzzy rough Einstein Bonferroni weighted average operator ($C_{p,q,r}$-SFREBOMWA). Through the use of these operators, one can aggregate expert opinions that recognize different levels of confidence, whereby the decision process becomes more effective and robust.

The real-world application of the proposed method is illustrated in the ranking of municipal waste management treatment methods. Municipal waste management is an important concern for cities globally, as cities expand and struggle to address rising volumes of waste in a sustainable environmental way. Various treatment methods need to be ranked according to a variety of considerations, such as cost, environmental concern, efficiency, and scalability. However, the experts can render varying opinions regarding the performance of such techniques, and these opinions are mostly uncertain. Our suggested method is appropriate in such situations because it enables the optimal selection of alternatives based on the consideration of the lower and upper approximations of the alternatives' judgment factors along with the confidence level of the experts, thereby reflecting the intrinsic uncertainty of expert assessments.

In this research, the criteria importance through intercriteria correlation (CRITIC) method is utilized for determining the criteria weights and hybridizes them with the combinative distance-based assessment (CODAS) method to rank the municipal waste management treatment options. The CRITIC weighting method is an approach used to determine the relative weights of various criteria in a decision-making process. To examine the effectiveness and applicability of the new approach, we carried out a comparison analysis with several well-known MCDM methods, such as technique for order preference by similarity to ideal solution (TOPSIS) and weighted aggregated sum product assessment (WASPAS), which have been frequently applied for ranking alternatives in MCDM problems. Using comparative and sensitivity analysis, it was proved that the $\boldsymbol{C_{p,q,r}}$-SFREBOM methods perform better than conventional approaches do, as they are more robust, accurate in decision-making, and flexible to uncertainty.

1.1. Organization of the workflow

Section 1 deals with an introduction consisting of a literature review, motivation, a research gap, and contributions made in this paper. Section 2 explains the preliminary definitions that are related to this paper, which include q spherical fuzzy number, q-SFRN, p, q, r spherical FS, p, q, r spherical fuzzy rough number. The operational laws, theorems, and properties of p,q,r-SFREBOMWG and q,r-SFREBOMWA are explained clearly in Section 3. Confidence levels based on the p,q,r-spherical fuzzy rough CODAS approach are elucidated in Section 4. Section 5 illustrates the numerical example of municipal solid waste treatment technique selection for the proposed work. The sensitivity analysis, comparative analysis, advantages, and disadvantages are well interpreted in Section 6. Section 7 is the future work, followed by an appendix containing the proof of the theorems.

1.2. Literature survey

FS theory, originally proposed by Zadeh [1], has been a foundation for building mathematical models that handle uncertainty and imprecision. Zadeh's breakthrough concept of making membership be described by a continuum, as opposed to a binary value, paved the way for new applications in modeling systems where exact definitions are difficult to define. Following Zadeh's work, the theory of FSs has developed into many extensions, each aimed at better capturing the richness of uncertainty in real-world situations. Because FS is not capable of representing hesitation or indecision as such, Atanassov [2] introduced intuitionistic FSs (IFS) that combine membership and non-membership degrees with the condition that they add up to 1 or less. However, IFS has the shortcoming of its application when decision makers can obtain high values of membership as well as non-membership degrees, thus deviating from the IFS restriction. To achieve this goal, the author Yager [3] explained the Pythagorean FS (PFS), thereby granting greater flexibility for modeling human perception. This was preceded by the Fermatean FSs, introduced by Senapati and Yager [4], which provided even higher tolerance and decision modeling room. Yager [5] also extended this idea to the q-rung orthopair FS, with larger values of q enabling more freedom in giving values. Moreover, a criteria importance through intercriteria correlation-with measurement of alternatives and ranking according to compromise solution framework based on q-rung fuzzy preferences was studied [6]. Yager [7] suggested some geometric aggregation operators based on intuitionistic FSs. Decision-making approach under Pythagorean fuzzy Yager weighted operators was studied [8]. Ma and Xu [9] established symmetric Pythagorean fuzzy weighted averaging and weighted geometric operators that were designed to accommodate the distinctive form of PFSs. These operators successfully maintain the form of PFSs and are especially useful in handling large-scale decision-making issues that are subject to high uncertainty. Liu and Wang [10] provided important contributions with the introduction of aggregation methods for q-rung orthopair FS in the form of q-rung orthopair FS, q-rung orthopair fuzzy weighted arithmetic and q-rung orthopair fuzzy weighted geometric operators. The Dombi aggregation operator, in terms of complex bipolar FSs with applications in decision-making problems, was analyzed [11]. As an extension of q-rung orthopair FSs, Mandal and Seikh [12] introduced p, q-quasi rung orthopair FSs. In this version, the criterion is relaxed from the q-th power of the membership and nonmembership values to the p-th membership and q-th non-membership. In addition, Torra [13] introduced hesitant

FSs that enable decision-makers to depict possible membership degrees for an element so that hesitation or indecisiveness is captured. This flexibility is particularly useful in regard to group decision-making scenarios, in which the conflicting opinions of experts always lead to numerous plausible assessments. Krishankumar et al. [14] suggested a strong decision-making framework that takes advantage of the probabilistic hesitant fuzzy setting with inherent probability estimation, providing more powerful modeling of human judgment under uncertain situations. Smarandache [15] provided the initial groundwork for neutrosophic sets as an addition of intuitionistic FSs by proposing the idea of independent capture of truth, indeterminacy, and falsity. Ali et al. [16] further extended the theory to neutrosophic cubic sets and proved its effectiveness in pattern recognition problems. Ajay et al. [17] went a step ahead by integrating geometric Bonferroni mean operators with neutrosophic cubic FSs to develop a novel MCDM method that merges decision-makers' opinions more accurately. Furthermore, Gundogdu and Kahraman [18] enhanced the WASPAS method with SFS to provide an additional accurate means for processing vague and inaccurate data in decision-making challenges. Pramanik et al. [19] presented Frank Choquet Bonferroni mean operators in bipolar neutrosophic sets to permit more interdependent and flexible aggregation of criteria and illustrated their application in difficult MCDM situations. In addition, Riaz and Farid [20] proposed spherical fuzzy fairly aggregation operators under a proportional distribution to facilitate equitable consideration of all the criteria and improve group decision-making fairness. Together, these contributions illustrate the increasing sophistication and diversification of neutrosophic and fuzzy MCDM methods with the hope of solving real decision-making problems with high ambiguity and subjectivity. As an expansion to spherical FSs, Kahraman et al. [21] proposed q-spherical FSs, a newer fuzzy logic tool that improves modeling uncertainty and hesitation in multi-attribute decision-making issues. By adding a parameter q to conventional SFS, the authors presented a more versatile and precise tool for intricate decision scenarios illustrated via artificial intelligence and robotics applications. Furthermore, a hyperbolic FS decision framework for construction contracts integrating CRITIC and WASPAS for dispute mitigation and automation in construction is explained, and the researchers in [22] proposed the concept of RSs, which provide the basis for a mathematical method to address vagueness and uncertainty in data analysis. The RS theory presented in this work enables the approximation of imprecise concepts by lower and upper bounds and is thus a valuable tool in data mining and decision support systems. Ganie et al. [23] used the q-rung fuzzy concept to rank barriers in clean energy utilization within the healthcare industry. Moreover, the selection of waste treatment methods for food sources includes an integrated decision model using q-rung fuzzy data, logarithmic percentage change-driven objective weighting), and complex proportional assessment techniques [24]. Monika et al. [25] used T-spherical fuzzy hypersoft sets to solve energy selection issues, treating complicated uncertainty with hypersoft parameterization. Thilagavathy and Mohanaselvi [26] explained a new cubical fuzzy Einstein-Bonferroni mean geometric aggregation operator for tackling intricate multiple criteria group decision-making problems. The authors successfully combined the Einstein t-norm with the Bonferroni mean and cubical fuzzy environments to address the uncertainty and interaction between criteria. The new operators were evaluated via a realistic decision-making scenario, showing their strength, reliability, and flexibility in modeling human-like reasoning with fuzziness. Augmenting human knowledge in this research, Azim et al. [27] created a q-spherical fuzzy rough Einstein geometric aggregation operator for image recognition and interpretation. The technique used approached vagueness and partial truth in image data, and its potential in intelligent image processing and pattern recognition was emphasized. As an expansion of q-spherical FSs, Rahim et al. [28] presented p, q, r-spherical FS (p, q, r-SFS). The link between the membership degree, neuronal membership degree, and non-membership degree is described by the words p, q, and r-SFSs in their suggested framework. When p and q are greater than or equal to 1, and r is the least common multiple of p and q, the relationship is defined by the condition p+r+q≤1. In this paradigm, p and q might be either identical (p=q) or different (p<q or p>q). The flexibility and effectiveness of decision-making are enhanced by the ability to alter the values of a, b, and c due to the nature of p, q, and r-SFSs. Figure 1 presents special cases of p, q, r-SFS. Moreover, the Kaya et. al [29] has studied on combined spherical fuzzy decision-making trial and evaluation laboratory and analytical hierarchy process AHP models for hazard management, emphasizing flexibility toward uncertainty and providing reliable solutions for multicriteria decision-making. Within the context of MCDM methods grounded in CODAS, Abdel-Basset et al. [30] introduced a state-of-the-art multistage MCDM model that integrated the spherical fuzzy CRITIC and CODAS approaches to evaluate sustainable flue gas treatment systems in the iron and steel sector. Decision-making under uncertainty has garnered much attention in various disciplines, including healthcare, transport, and environmental management.

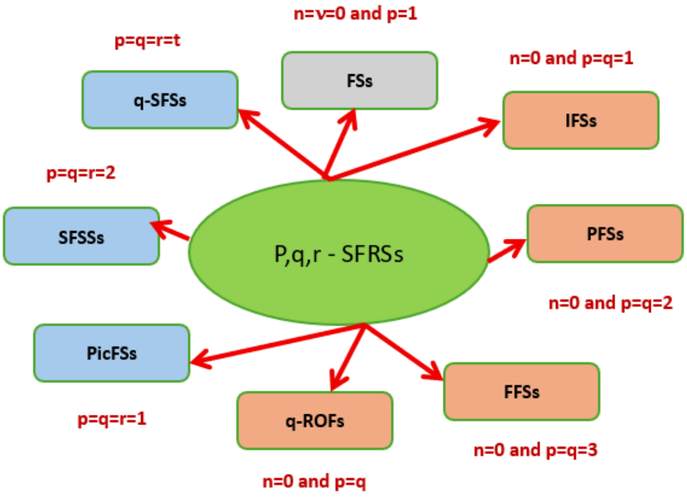


Figure 1. Special cases of p, q, r-SFS.

Akram et al. [31] explored spherical fuzzy graphs for decision-making, presenting an innovative solution that merges graph theory with spherical fuzzy logic to address uncertainty in complex situations. The spherical intelligent fuzzy decision process for improving emergency response systems, notably for COVID-19 diagnosis, confirms the merit of fuzzy logic in real-time decision-making [32]. In the field of renewable energy, Azim et al. [33] presented a q-spherical fuzzy rough CODAS based on a case study to determine renewable energy location selection. The model exhibited excellent discriminatory power in evaluating rival alternatives based on geographic, economic, and environmental criteria. Ayuğdu and Gül [34] developed an entropy-based method for spherical FSs, significantly improving attribute decision-making by handling uncertainty and providing more robust decision support under conditions of imprecision. The establishment of similarity measures in spherical fuzzy settings, as suggested by Shishavan et al. [35], assists in enhancing decision-making processes through enhanced comparison accuracy among alternatives. Ali et al. [36] proceeded with this by developing complex T-spherical fuzzy aggregation operators, providing an enhanced method for addressing multi-attribute decision issues, with far-reaching applicability in numerous fields. Garg et al. [37] worked on T-spherical fuzzy power aggregation operators, providing an efficient tool for decision-making in the case of uncertain and interdependent attributes, as required in multicriteria problems. Liu et al. [38] used linguistic spherical FSs to assess public shared bicycle services in the case of shared mobility systems, offering a real-world example of the use of fuzzy logic in urban transport planning. Guleria and Bajaj [39] presented T-spherical fuzzy soft sets with aggregation operators, further developing decision-making approaches for multifaceted problems in various fields. In the same vein, Sharaf and Khalil [40] utilized spherical fuzzy interactive multi-criteria decision making for green occupational health and safety equipment supplier selection, highlighting the increased importance of sustainability in decision-making.

Finally, confidence-level-based FSs have also been applied in spherical fuzzy MCDM approaches for use in healthcare and quality control systems. Arslan et al. [41] explained how spherical FSs with confidence levels were used in healthcare decision-making to assess treatment possibilities under uncertainty. An integrated fuzzy MCDM approach for manufacturing process selection in Micro, small and medium sized enterprises is explained [42]. Furthermore, p, q, r-spherical fuzzy aggregation operators in decision-making of logo design is narrated [43]. Confidence level-based p, q, r-spherical fuzzy aggregation operators are derived and their application in selection of solar panels has been studied [44]. Multimodal transport route selection is described with regard to an integrated fuzzy hierarchy process approach [45]. Achillas [46] offers a widespread literature review that chronicles the evolution of MCDM applications in waste management, citing the integration of analytical hierarchy process, elimination and choice translating reality, and preference ranking organization method for enrichment evaluations techniques to address complex decision-making problems. In a Brazilian case study, Santos [47] applied the preference ranking organization method for enrichment evaluations approach to evaluate MSW management strategies, with particular emphasis on integrating local socio-economic conditions in decision-making. Jovanovic [48] also applied simple additive weighing and TOPSIS methodologies to identify ideal waste management strategies, proving the utility of the application of more than one MCDM method for comprehensive analysis. Ali et al. [49] used analytical hierarchy process and TOPSIS in Lahore, Pakistan to assess treatment alternatives such as composting and incineration, highlighting the flexibility of MCDM techniques in a wide range of urban settings. Moreover, Soni [50] presented a q-rung orthopair fuzzy based on step-wise weight assessment ratio assessment ratio and Complex proportional assessment model for choosing proper MSW technologies, demonstrating the ability of innovative fuzzy-based MCDM models to handle uncertainty in stakeholder preferences and performance evaluations. CODAS methods for multiple attribute group decision making with interval-valued bipolar uncertain linguistic information and their application is discussed in [51]. Furthermore, geospatial technologies and multi-criteria decision analysis for sustainable and resilient urban planning are greatly explained in [52]. An interesting result about enhancing decision quality in smart manufacturing with T-spherical hesitant fuzzy RSs is presented in [53]. A multi-criteria framework for economic decision support in urban sustainability was made on the basis of comparative insights from European cities [54]. Fuzzy simple weight calculation and ranking alternatives with weights of criterion methods for sustainable waste disposal technology selection are greatly explained in [55].

A comparative study has been conducted with the literature works in Table 1.

Table 1. Comparison of works done in the literature.

Title	Operators	Application	p,q,r-parameters	MCDM	Confidence level
T spherical fuzzy hybrid aggregation operators [32]	T-spherical fuzzy weighted, ordered weighted, hybrid averaging and geometric operators	Business related	0	×	×
Fermatean fuzzy Einstein aggregation operators [35]	Fermatean fuzzy Einstein aggregation operators	Electric vehicle charging station selection	0	×	×
Complex T-spherical fuzzy frank aggregation operators and their application to decision-making [36]	CT-SF Frank hybrid averaging operator and other related operators	Supplying Nohoor hamlet with water from neighboring villages via subterranean pipeline	0		×
Dombi aggregation operators for p, q, r-SFSs: application in the stability assessment of cryptocurrencies [52]	p, q, r-spherical fuzzy Dombi weighted averaging operator	Cryptocurrencies	3	×	×
Applications of q-spherical fuzzy rough CODAS to the assessment of a problem involving renewable energy site selection [33]	q-spherical fuzzy RSs	Renewable energy site selection	1	✓	×
T-spherical fuzzy power aggregation operators and their applications in multi-attribute decision making [37]	T-spherical fuzzy power aggregationl operators	Software selection problem from a list of packages	0	✓	×
Q-spherical FSs and their usage in multi-attribute decision-making [21]	q-spherical fuzzy rough arithmetic meaning and geometric mean operators	Process of job scheduling	1	✓	×
Frank Choquet Bonferroni mean operators of Bipolar neutrosophic sets and their application to MCDM problems [19]	Frank Choquet Bonferroni mean operators	Plant location selection	0	✓	×
Cubical fuzzy Einstein Bonferroni mean geometric aggregation operators and their applications to multiple criteria group decision making problems [26]	Cubical fuzzy Einstein Bonferroni mean geometric aggregation operators	Identifying the optimal location for constructing a wind power plant	0	✓	×
p, q-spherical fuzzy sets and their aggregation operators with application to third-party logistic provider selection [28]	p, q-spherical FSs	Third-party logistic provider selection	2	×	×
Confidence levels complex q-rung orthopair fuzzy aggregation operators and its application in decision making problem [53]	q-rung orthopair fuzzy aggregation operators	Finding the best and the biggest fire extinguishers	1	✓	✓
Confidence levels-based p, q, r-spherical fuzzy rough Einstein Bonferroni aggregation operators for robust MCDM (proposed work)	p, q, r-spherical fuzzy rough Einstein Bonferroni operator (weighted geometric operator and weighted average operator)	Evaluating municipal solid waste treatment options	3	✓	✓

The research works listed in Table 1 can be highlighted to substantiate the effectiveness of the proposed method:

1) The concept of the p, q, r-spherical fuzzy rough Einstein Bonferroni operator (weighted geometric operator and weighted average operator) has not been researched in the literature p, q, r spherical fuzzy operators with methods in a Pythagorean fuzzy environment were discussed in previous studies. Hence, the concept of spherical fuzzy p, q, r-spherical fuzzy is pondered for this study.
2) Very less work has been made on confidence level are found in the literature. However, our confidence level for q rung orthopair cannot be compared with the proposed p, q, r-spherical fuzzy rough as q rung orthopair generalizes IFS and PFS with a membership and non-membership function, whereas p, q, r-spherical fuzzy rough captures three dimensions of uncertainty: Positive (membership), negative (non-membership), and neutral/hesitation, and then further generalizes with powers. Hence, the proposed work's confidence levels based on p, q, and r-spherical fuzzy rough Einstein Bonferroni aggregation operators is unique and cannot be compared with many studies.

1.3. Research gap

The following are the research gaps that led to this work:

1) A major limitation of the literature is the lack of completely generalized fuzzy rough models, especially p, q, (p, q, r-SFRS)-based models. The augmented model involves three tunable parameters (p, q, r), providing an amenable structure that can fit the broad spectrum of real-world decision contexts better.
2) There is another pressing concern of not incorporating expert confidence levels into aggregation processes. In actual decision-making situations, not all expert opinions are of equal credibility; confidence levels differ based on varying experience, access to information, or subjective belief. However, most aggregation operators take uniform credibility between experts for granted, which can result in skewed or imprecise conclusions. The lack of confidence-sensitive mechanisms in fuzzy rough MCDM models restricts the realism and practicality of such approaches.
3) In addition, aggregation operators, including weighted averages, ordered weighted averaging (OWA), and standard Einstein or Bonferroni means, usually perform independently of expert confidence and intercriteria interactions. In short, the research in this area is not supported by a unified framework of assisting generalized fuzzy rough modeling (through p, q, r-SFRS), expert confidence weighting, and sophisticated aggregation behavior. Filling this void is crucial to the development of MCDM approaches with greater robustness and realism under uncertainty.

1.4. Motivation of the study

The main reason that leads to the motivation of the proposed work is with respect to aggregation operators. The Einstein aggregation operators enable nonlinear, sensitive, and bounded aggregation of fuzzy information without breaking spherical constraints. In contrast, Bonferroni correction remedies the problem of multiple comparisons and dependency among attributes, further strengthening the statistical robustness of decision analysis. This integrated framework enables more accurate boundary areas to be generated, enhanced decision space interpretability, and more effective management of multisource imprecise or conflicting information. The lower approximation in p, q, r-SFRS preserves the nucleus and certain decisions, whereas the upper approximation keeps possible inclusions, both delivering a complete rough-fuzzy picture appropriate for real-world MCDM problems. Einstein Bonferroni means (EBMs), Dombi means, and Heronian means denote three distinct aggregation ideas generally utilized in fuzzy MCDM. Even though the three operators intend to aggregate uncertain or imprecise information, they differ primarily in terms of interaction modeling and parameter flexibility. When criterion interactions are weak or negligible, Dombi means offer a parsimonious and interpretable solution. When interactions exist but are structurally simple, Heronian means provide a balanced compromise between interaction modeling and simplicity. EBMs are beneficial in complex decision environments depicted by correlated criteria, nonlinear trade-offs, and uncertainty, where interaction effects and robustness are vital. EBM are used for intuitionistic FSs, Pythagorean/q-rung orthopair fuzzy environments, and in risk-sensitive MCDM.

Table 2. Comparison table for operators.

	EBMs	Heronian	Dombi means
Interaction modeling	Considers pairwise and nonlinear interactions. Incorporated Einstein t-norms and t-conorms	Considers only pairwise interaction without complexity	Interaction not possible
Parameter flexibility	Two interaction parameters-high flexibility	Fixed interaction and limited flexibility	One strictness parameter very high
Complexity	High	Medium	Low

Table 2 gives the comparison table for various operators. In summary, the new p, q, r-SFRS approach, based on RS approximations, complemented by spherical fuzzy logic and supported by confidence measures and robust aggregation, constitutes a major methodological leap. This approach not only solves the important structural problems of earlier models but also sets the stage for future research into high-dimensional decision systems where uncertainty is represented with accuracy, flexibility, and interpretability. This model is ready to serve sophisticated decision environments from healthcare diagnosis to financial risk assessment, establishing a new standard in knowledge representation for uncertainty.

1.5. Benefits of the study

1) The benefits of the study are that the newly proposed p, q, r-spherical fuzzy RS (p, q, r-SFRS) model presents an interesting expansion of current fuzzy rough models. By adding three independent parameters (p, q, r), this model permits the fine-tuning of membership, non-membership, and hesitation degrees, thus providing greater adaptability and accuracy. This renders the p, q, r-SFRS framework extremely well suited to better capture real-world ambiguity than earlier models such as IFS, PFS, or SFS. Building on this generalization inspires the creation of new aggregation mechanisms tailored to work within this flexible framework.
2) Another fundamental reason is that expert confidence can be brought into the process of aggregation., the incorporation of confidence-based aggregation yields a more realistic aggregation of opinions, supporting more informed and more equitable outcomes. Thus, with our

investigation, we aim to fill this gap by incorporating expert confidence in the p, q, r-SFRS framework.

3) The application of the Einstein t-norm/t-conorm and Bonferroni mean as our introduced aggregation operator is inspired by the virtues of these two individual components - the Einstein operator and the Bonferroni mean operator. The Einstein operator can simulate nonlinear relationships and prevent excessively optimistic or pessimistic results; hence, it is well-suited for sensitive aggregation tasks. In contrast, the Bonferroni mean is good at reflecting intercriteria dependencies, a characteristic lacking in standard arithmetic or geometric means. Through a combination of these methods, our confidence-based p, q, r-spherical fuzzy rough Einstein Bonferroni aggregation operator ($C_{p,q,r}$-SFREBOM) seeks to plan a sophisticated, precise, and resilient decision-making process.

1.6. Contribution of the study

This study proposes a new method using the p, q, r-SFRS theory that integrates decision-maker confidence levels and uncertainty during decision-making. This is made possible by applying new aggregation operators that integrate the merits of FS theory, RS theory, Einstein operations, and Bonferroni means. The research also provides an insightful comparison of CODAS with other MCDM approaches, specifically TOPSIS and WASPAS, providing worthwhile observations on the merits and demerits of these methods in municipal solid waste management settings. The research outcomes emphasize the ability of the presented method to yield a more robust and resilient decision maker tool, particularly in situations where vagueness and imprecision are dominant. In summary, this paper illustrates the importance of incorporating cutting-edge FS and RS theories into MCDM to handle sophisticated decision-making issues. We introduce novel p, q, r-SFREBOM aggregation operators as a powerful system for assessing alternatives in uncertain scenarios, which will prove to be a valuable decision-making tool for decision-makers in any field, particularly in municipal waste management, where decisions have severe environmental, economic, and societal repercussions.

The major contributions of the study are as follows:

1) Two new aggregation operators, $C_{p,q,r}$ -SFREBOMWG and $C_{p,q,r}$ -SFREBOMWA, are proposed in this work to address p, q, r-spherical fuzzy numbers (p, q, r-SFNs) and the confidence levels of experts. The theoretical validity of these operators and their applicability to uncertain settings is confirmed through extensive mathematical examination.
2) Using the suggested operators, a stable MCDM algorithm is constructed to address intricate decision-making issues with the aid of a numerical real-world case study. The study demonstrates the practicality and effectiveness of the model by combining the CRITIC and CODAS methods as a hybrid approach to aggregate expert views and assess alternatives more credibly under uncertainty.
3) The model incorporates Einstein operational laws with Bonferroni mean aggregators to refine the treatment of interrelationships among criteria. This integration supports nuanced aggregation under spherical fuzzy settings, further improving decision accuracy and reliability.

1.7. Practical and theoretical aims

The theoretical aim of the study is to compute an aggregated decision matrix based on the proposed operators $C_{p,q,r}$ -SFREBOMWG and $C_{p,q,r}$ -SFREBOMWA followed by the CRITIC method to find weights. The weights, p, q, r parameters, confidence level, and CRITIC and CODAS methods lead to the effectiveness of the proposed method. The proposed work is based on a case study using the CRITIC-CODAS approach, which proves the practical feasibility of our model in addressing complicated MCDM problems with confidence-aware expert opinions.

2. Preliminaries

In the following section, we will examine several mathematical concepts, initially with a thorough analysis of FS, IFS, PFS, SPS, q-SFS, and RS.

Definition 2.1. For any finite set B, a FS over an element $\mathbf{b} \in B$ can be expressed as:

$$S = \{\boldsymbol{b}, \mu_S(\boldsymbol{b}) | \boldsymbol{b} \in B\}. \quad (1)$$

In this context, $\mu_S(\boldsymbol{b})$ characterize the MD of S, such that $\mu_S(\boldsymbol{b}) \in [0, 1]$, as in [1].

Definition 2.2. For any finite set B, an IFS over an element $b \in B$, as in [2], can be expressed as:

$$I = \{\boldsymbol{b}, (\mu_I(\boldsymbol{b}), v_I(\boldsymbol{b})) | \boldsymbol{b} \in B\}. \quad (2)$$

In this context, $\mu_I(\boldsymbol{b})$ and $v_I(\boldsymbol{b})$ characterize the MD and NMD of I, respectively, such that $\mu_I(\boldsymbol{b}), v_I(\boldsymbol{b}) \in [0, 1]$, and $0 \le \mu_I(\boldsymbol{b}) + v_I(\boldsymbol{b}) \le 1$.

Definition 2.3. Let B be any universal set, and a PFS for an element $\boldsymbol{b} \in B$ as in [3] can be expressed as:

$$Y = \{\boldsymbol{b}, (\mu_Y(\boldsymbol{b}), v_Y(\boldsymbol{b})) | \boldsymbol{b} \in B\}. \quad (3)$$

In this context, $\mu_Y(\boldsymbol{b})$ denote MD, while $v_Y(\boldsymbol{b})$ denote NMD of a component, where $\mu_Y(\boldsymbol{b}), v_Y(\boldsymbol{b}) \in [0,1]$ and $0 \le \mu_Y(\boldsymbol{b}) + v_Y(\boldsymbol{b}) \le 1$.

Definition 2.4. Let B be any universal set, and a picture FS as in [6] is given as

$$P = \{\boldsymbol{b}, (\mu_P(\boldsymbol{b}), \eta_P(\boldsymbol{b}), v_P(\boldsymbol{b})) | \boldsymbol{b} \in B\}, \quad (4)$$

where $\boldsymbol{b} \in B$.

In this context, $\mu_P(\boldsymbol{b})$ denote membership, $\eta_P(\boldsymbol{b})$ denote impartial (neutral) membership, and $v_P(\boldsymbol{b})$ denote non-membership of a component in B with the conditions

$$0 \le \mu_P(\boldsymbol{b}) + \eta_P(\boldsymbol{b}) + v_P(\boldsymbol{b}) \le 1.$$

Definition 2.5. For any universal set B, a spherical FS denoted as H, is represented as:

$$H = \{b, (\mu_H(\boldsymbol{b}), \eta_H(\boldsymbol{b}), v_H(\boldsymbol{b})) | \boldsymbol{b} \in B\}. \quad (5)$$

In this context, $\mu_H(\boldsymbol{b}) \in [0,1]$ represent membership grade, $\eta_H(\boldsymbol{b}) \in [0,1]$ represent neutral membership grade, and $v_H(\boldsymbol{b}) \in [0,1]$ represent non-membership grade of an element $\in B$, such that

$$0 \le \mu_H(\boldsymbol{b}) + \eta_H(\boldsymbol{b}) + v_H(\boldsymbol{b}) \le 1 \text{ and } (\mu_H(\boldsymbol{b}))^2 (\eta_H(\boldsymbol{b}))^2 + (v_H(\boldsymbol{b})^2) \le 1.$$

Definition 2.6. For any finite set B, a q-SFS denoted as Q, is given by:

$$Q = \{\boldsymbol{b}, (\mu_Q(\boldsymbol{b}), \eta_Q(\boldsymbol{b}), v_Q(\boldsymbol{b})) | \boldsymbol{b} \in B\} \quad (6)$$

In this context, $\mu_Q(\boldsymbol{b})$ characterize membership grade, $\eta_Q(\boldsymbol{b})$ characterize impartial membership, and $\nu_Q(\boldsymbol{b})$ represent non-membership, where $(\boldsymbol{b})\in B$ and

$$0\leq\mu_Q(\boldsymbol{b}),\eta_Q(\boldsymbol{b}),\nu_Q(\boldsymbol{b})\leq 1 \text{ and } \left(\mu_Q(\boldsymbol{b})\right)^q+\left(\eta_Q(\boldsymbol{b})\right)^q+\left(\nu_Q(\boldsymbol{b})\right)^q\leq 1.$$

Definition 2.7. The triplet (G1, G2, R) is mentioned as an estimate space when studying an arbitrary binary relation R on G1×G2. As in [22], the $\underline{R(Q)}$ and $\overline{R(Q)}$ are described for sets X⊆G1 and F⊆G2.

$$\begin{pmatrix} \underline{R(Q)} = \left\{\boldsymbol{x}\epsilon G1 : [\boldsymbol{x}]_Q \subseteq \mathrm{G}\right\} \\ \overline{R(Q)} = \left\{\boldsymbol{x}\epsilon G1 : [\boldsymbol{x}]_Q \cap \mathrm{G} \neq \phi\right\} \end{pmatrix}, \tag{7}$$

where $[\boldsymbol{x}]_Q$ denotes the concept of indiscernibility. The set $(\underline{R(Q)},\overline{R(Q)})$ is termed as a RS.

Definition 2.8. A q-spherical fuzzy relation R, is a q-SFS of G1×G2 and is given by

$$R=\left\{\{\langle(\boldsymbol{\varrho},\boldsymbol{x})\text{: }\mu_R(\boldsymbol{\varrho},\boldsymbol{x}),\eta_R(\boldsymbol{\varrho},\boldsymbol{x}),\nu_R(\boldsymbol{\varrho},\boldsymbol{x})\rangle\text{: }\mu_R(\boldsymbol{\varrho},\boldsymbol{x})^p+\eta_R(\boldsymbol{\varrho},\boldsymbol{x})^q+\nu_R(\boldsymbol{\varrho},\boldsymbol{x})^r\leq 1\text{: }\forall\boldsymbol{\varrho}\epsilon\mathrm{G1},\boldsymbol{x}\epsilon\mathrm{G2}\right\}$$

where $\mu_R \longrightarrow [0,1], \eta_R \longrightarrow [0,1]$ and $\nu_R \longrightarrow [0,1]$.

A q-spherical fuzzy RS (q-SFRS), which is defined as: Let universal set G1 and G2 be a set of attributes, and let R be a q-SF relation from G1 to G2. Then, the triplet (G1, G2, R) is called q-SF approximation space. Now, for any element $\varrho\in$q-SFRS, the lower and upper approximation space of $\boldsymbol{\varrho}$ w.r.t approximation space (G1, G2, R) are presented and given as:

$$\mathrm{Q}=\underline{Q},\overline{Q}=\left\{\boldsymbol{\varrho},\begin{pmatrix}\left\{\underline{\mu_Q(\boldsymbol{\varrho})},\underline{\eta_Q(\boldsymbol{\varrho})},\underline{\nu_Q(\boldsymbol{\varrho})}\right\} \\ \overline{\mu_Q(\boldsymbol{\varrho})},\overline{\eta_Q(\boldsymbol{\varrho})},\overline{\nu_Q(\boldsymbol{\varrho})}\end{pmatrix}:\boldsymbol{\varrho}\epsilon\mathrm{G1}\right\}, \tag{8}$$

where

$$\underline{\mu_Q(\boldsymbol{\varrho})}=\wedge_{\boldsymbol{x}\epsilon G2}\{\mu_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\mu_Q(\boldsymbol{x})\},\ \underline{\eta_F(\boldsymbol{\varrho})}=\vee_{\boldsymbol{x}\epsilon G2}\{\eta_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\eta_Q(\boldsymbol{x})\}\ ,\underline{\nu_F(\boldsymbol{\varrho})}=\vee_{\boldsymbol{x}\epsilon G2}\{\nu_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\nu_Q(\boldsymbol{x})\},$$

$$\overline{\mu_F(\boldsymbol{\varrho})}=\vee_{\boldsymbol{x}\epsilon G2}\{\mu_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\mu_Q(\boldsymbol{x})\},\ \overline{\eta_F(\boldsymbol{\varrho})}=\wedge_{\boldsymbol{x}\epsilon G2}\{\eta_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\eta_Q(\boldsymbol{x})\},\ \overline{\nu_F(\boldsymbol{\varrho})}=\wedge_{\boldsymbol{x}\epsilon G2}\{\nu_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\nu_Q(\boldsymbol{x})\}.$$

With the condition that

$$0\leq\underline{\mu_R(\boldsymbol{\varrho})^q}+\underline{\eta_R(\boldsymbol{\varrho})^q}+\underline{\nu_R(\boldsymbol{\varrho})^q}\leq 1$$

and

$$0\leq\overline{\mu_R(\boldsymbol{\varrho})^q}+\overline{\eta_R(\boldsymbol{\varrho})^q}+\overline{\nu_R(\boldsymbol{\varrho})^q}\leq 1.$$

The q-spherical fuzzy rough set (q-SFRS) is defined as a pair of p, q, r-spherical fuzzy rough sets (p, q, r-SFSs), where $\underline{Q}$ is distinct from $\overline{Q}$. It denotes the given concept as Q=($\underline{Q},\overline{Q}$), represented as a q-spherical fuzzy rough number (q-SFRN). Q_i denotes the set that includes all q-SFRNs. The q-SFRN Q=($\underline{Q},\overline{Q}$) consists of two components.

Lower approximation space $\underline{Q}$: Represents the lower approximation space of $\boldsymbol{\varrho}$ in the q-SFRS. It captures the lower bound of uncertainty associated with the element $\boldsymbol{\varrho}$.

Upper approximation space $\overline{Q}$: Denotes the upper approximation space of $\boldsymbol{\varrho}$ in the q-SFRS. It captures the upper bound of uncertainty associated with the element $\boldsymbol{\varrho}$.

Definition 2.9. According to [27], let $Q_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}), Q_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2})$ and $Q = (\underline{\mu}, \underline{\eta}, \underline{\nu}, \overline{\mu}, \overline{\eta}, \overline{\nu})$ be any 3 q-SFRNs and $\kappa > 0$, then,

$$Q_1 \oplus Q_2 = \left\langle \begin{pmatrix} \sqrt[q]{\underline{\mu_1}^q + \underline{\mu_2}^q - \underline{\mu_1}^q \underline{\mu_2}^q}, \underline{\eta_1 \eta_2}, \underline{\nu_1 \nu_2} \end{pmatrix} \atop \begin{pmatrix} \sqrt[q]{\overline{\mu_1}^q + \overline{\mu_2}^q - \overline{\mu_1}^q \overline{\mu_2}^q}, \overline{\eta_1 \eta_2}, \overline{\nu_1 \nu_2} \end{pmatrix} \right\rangle, \ Q_1 \otimes Q_2 = \left\langle \begin{pmatrix} \underline{\mu_1 \mu_2}, \underline{\eta_1 \eta_2}, \sqrt[q]{\underline{\nu_1}^q + \underline{\nu_2}^q - \underline{\nu_1}^q \underline{\nu_2}^q} \end{pmatrix} \atop \begin{pmatrix} \overline{\mu_1 \mu_2}, \overline{\eta_1 \eta_2}, \sqrt[q]{\overline{\nu_1}^q + \overline{\nu_2}^q - \overline{\nu_1}^q \overline{\nu_2}^q} \end{pmatrix} \right\rangle$$

$$\kappa Q = \left\langle \begin{pmatrix} \sqrt[q]{1 - \left(1 - \underline{\mu}^q\right)^\kappa}), \underline{\eta}^\kappa, \underline{\nu}^\kappa \end{pmatrix} \atop \begin{pmatrix} \sqrt[q]{1 - (1 - \overline{\mu}^q)^\kappa}), \overline{\eta}^\kappa, \overline{\nu}^\kappa \end{pmatrix} \right\rangle, \ Q^\kappa = \left\langle \begin{pmatrix} \underline{\mu}^\kappa, \sqrt[q]{1 - \left(1 - \underline{\eta}^q\right)^\kappa}, \sqrt[q]{1 - (1 - \underline{\nu}^q)^\kappa} \end{pmatrix} \atop \begin{pmatrix} \overline{\mu^\kappa}, \sqrt[q]{1 - (1 - \overline{\eta}^q)^\kappa}, \sqrt[q]{1 - (1 - \overline{\nu}^q)^\kappa} \end{pmatrix} \right\rangle.$$

Definition 2.10. According to [27], let two q-spherical rough numbers (q-SRNs) be $Q_1 = \left(\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}\right), Q_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2})$. Then,

$$Q_1 \cup Q_2 = \begin{pmatrix} \max\left(\underline{\mu_1}, \underline{\mu_2}\right), \min\left(\underline{\eta_1}, \underline{\eta_2}\right), \min\left(\underline{\nu_1}, \underline{\nu_2}\right) \\ \max(\overline{\mu_1}, \overline{\mu_2}), \min(\overline{\eta_1}, \overline{\eta_2}), \min(\overline{\nu_1}, \overline{\nu_2}) \end{pmatrix},$$

$$Q_1 \cap Q_2 = \begin{pmatrix} \min\left(\underline{\mu_1}, \underline{\mu_2}\right), \max\left(\underline{\eta_1}, \underline{\eta_2}\right), \max\left(\underline{\nu_1}, \underline{\nu_2}\right) \\ \min(\overline{\mu_1}, \overline{\mu_2}), \max(\overline{\eta_1}, \overline{\eta_2}), \max(\overline{\nu_1}, \overline{\nu_2}) \end{pmatrix},$$

$$Q_1^C = \left(\underline{\nu_1}, \underline{\eta_1}, \underline{\mu_1}, \overline{\nu_1}, \overline{\eta_1}, \overline{\mu_1}\right).$$

In this context, Q_1^C represents the complement of Q_1.

Definition 2.11. For any q-SFRNs Q [27], the score function of Q can be expressed as:

$$\text{Sc(Q)} = \frac{2 + \underline{\mu}^q + \overline{\mu}^q - \underline{\eta}^q - \overline{\eta}^q - \underline{\nu}^q - \overline{\nu}^q}{3}. \tag{9}$$

In this context, 0≤Sc(Q)≤1.

Definition 2.12. The accuracy function of q-SFRN [27] is defined as follows:

$$\text{Ac(Q)} = \frac{\underline{\mu}^q + \overline{\mu}^q + \underline{\nu}^q + \overline{\nu}^q}{2}. \tag{10}$$

In this context, -1≤Ac(Q)≤1.

Definition 2.13. Let $Q_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1})$, $Q_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2})$ be any two q-SFRNs [27], then,

Sc(Q_1)<Sc(Q_2), then $Q_1 < Q_2$, Sc(Q_1)>Sc(Q_2), then $Q_1 > Q_2$.
Sc(Q_1)=Sc(Q_2), then $Q_1 < Q_2$, then,
If Ac(Q_1)<Ac(Q_2), then $Q_1 < Q_2$;
If Ac(Q_1)>Ac(Q_2), then $Q_1 > Q_2$;
If Ac(Q_1)=Ac(Q_2), then $Q_1 = Q_2$.

Definition 2.14. Let

$$Q_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}), Q_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2}), \text{ and } Q = (\underline{\mu}, \underline{\eta}, \underline{\nu}, \overline{\mu}, \overline{\eta}, \overline{\nu})$$

be any 3 q-SFRNs and $\kappa > 0$ be any three q-SFRNs, and if $\kappa 1$ and $\kappa 2$ are any positive integers, then the following properties are held as in [27]:

$$Q_1 \oplus Q_2 = Q_2 \oplus Q_1,\ Q_1 \otimes Q_2 = Q_2 \otimes Q_1,\ \kappa(Q_1 \oplus Q_2) = \kappa Q_1 \oplus \kappa Q_2,$$

$$\kappa 1 Q \oplus \kappa 2 Q = (\kappa 1 + \kappa 2)Q,\ (Q_1 \otimes Q_2)^{\kappa} = Q1^{\kappa} \otimes Q2^{\kappa},\ Q^{\kappa 1} \otimes Q^{\kappa 2} = Q^{(\kappa 1 + \kappa 2)}.$$

Definition 2.15. For any finite set B, a p, q, r-SFS F over an element $\boldsymbol{b} \in B$ can be expressed as:

$$F = \{\boldsymbol{b}, (\mu_F(\boldsymbol{b}), \eta_F(\boldsymbol{b}), \nu_F(\boldsymbol{b})) | \boldsymbol{b} \in B\}.$$

In this context, $\mu_F(\boldsymbol{b}) \in [0,1]$ represents membership grade, $\eta_F(\boldsymbol{b}) \in [0,1]$ represents impartial membership grade, and $\nu_F(\boldsymbol{b}) \in [0,1]$ represents non-membership grade of a component $\in B$, satisfying the condition $(\mu_F(\boldsymbol{b}))^p + (\eta_F(\boldsymbol{b}))^r + (\nu_F(\boldsymbol{b}))^q \leq 1$, where p and q are positive integers, and r=max(p,q) [28].

Definition 2.16. [27] The Einstein t-norm and t-conorm are defined as:

$$TN_E(a,b) = \frac{ab}{1+(1-a)(1-b)}. \tag{11}$$

The Einstein t-conorm is defined as:

$$TCN_E(a,b) = \frac{a+b}{1+ab}. \tag{12}$$

Definition 2.17. Let $c_i (i = 1,2,\ldots,n)$ be a set of non-negative real numbers and s, t>0. Then, the Bonferroni mean (BOM) between c_i is mathematically calculated using the following formula [26]:

$$BOM^{s,t}(c_1, c_2, \ldots, c_n) = \left[\frac{1}{n(n-1)}\left(\sum_{i,j=1; i\neq j}^{n} c_i^s c_j^t\right)\right]^{\frac{1}{s+t}}. \tag{13}$$

Let $c_i (i = 1,2,\ldots,n)$ be a set of non-negative real numbers and s,t>0. Then, the geometric Bonferroni mean (GBOM) between c_i is mathematically calculated using the following formula:

$$GBOM^{s,t} = \frac{1}{s+t}\left[\left(\prod_{i,j=1; i\neq j}^{n} (sc_i + tc_j)\right)^{\frac{1}{n(n-1)}}\right]. \tag{14}$$

3. Operational laws and proposed aggregators

Definition 3.1. A p, q, r-spherical fuzzy relation R is a p, q, r-spherical fuzzy subset of G1×G2 and is given by R

$$R = \{\langle (\boldsymbol{\varrho}, \boldsymbol{x}) : \mu_R(\boldsymbol{\varrho}, \boldsymbol{x}), \eta_R(\boldsymbol{\varrho}, \boldsymbol{x}), \nu_R(\boldsymbol{\varrho}, \boldsymbol{x}) \rangle : \eta_R(\boldsymbol{\varrho}, \boldsymbol{x})^p + \eta_R(\boldsymbol{\varrho}, \boldsymbol{x})^r + \nu_R(\boldsymbol{\varrho}, \boldsymbol{x})^q \leq 1 : \forall \boldsymbol{\varrho} \in \mathrm{G1}, \boldsymbol{x} \in \mathrm{G2}\}$$

where $\mu_R \longrightarrow [0,1], \eta_R \longrightarrow [0,1], \nu_R \longrightarrow [0,1]$.

A p, q, r-spherical fuzzy rough set (p, q, r-SFRS), which is defined as: Let universal set G1 and G2 be a set of attributes. Let R be a p, q, r-SF relation from G1 to G2. Then, the triplet (G1, G2, R) is called p, q, r-SF approximation space. Now, for any element $\boldsymbol{\varrho} \in$p, q, r-SFRS, the lower and upper approximation space of $\boldsymbol{\varrho}$ w.r.t approximation space (G1, G2, R) are presented and given as:

$$\mathrm{F}{=}\underline{F},\overline{F}=\left\{\boldsymbol{\varrho},\left(\begin{matrix}\left\{\underline{\mu_F(\boldsymbol{\varrho})},\underline{\eta_F(\boldsymbol{\varrho})},\underline{\nu_F(\boldsymbol{\varrho})}\right\}\\ \overline{\mu_F(\boldsymbol{\varrho})},\overline{\eta_F(\boldsymbol{\varrho})},\overline{\nu_F(\boldsymbol{\varrho})}\end{matrix}\right):\varrho\epsilon\mathrm{G1}\right\},\tag{15}$$

where

$$\underline{\mu_F(\boldsymbol{\varrho})}=\wedge_{x\epsilon G2}\{\mu_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\mu_F(\boldsymbol{x})\},\ \underline{\eta_F(\boldsymbol{\varrho})}=\vee_{x\epsilon G2}\{\eta_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\eta_F(\boldsymbol{x})\},\ \underline{\nu_F(\boldsymbol{\varrho})}=\vee_{x\epsilon G2}\{\nu_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\nu_F(\boldsymbol{x})\},$$

$$\overline{\mu_F(\boldsymbol{\varrho})}=\vee_{x\epsilon G2}\{\mu_R(\boldsymbol{\varrho},\boldsymbol{x})\vee\mu_F(\boldsymbol{x})\},\ \overline{\eta_F(\boldsymbol{\varrho})}=\wedge_{x\epsilon G2}\{\eta_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\eta_F(\boldsymbol{x})\},\ \overline{\nu_F(\boldsymbol{\varrho})}=\wedge_{x\epsilon G2}\{\nu_R(\boldsymbol{\varrho},\boldsymbol{x})\wedge\nu_F(\boldsymbol{x})\},$$

with the condition that

$$0\le\underline{\mu_R(\boldsymbol{\varrho})^p}+\underline{\eta_R(\boldsymbol{\varrho})^r}+\underline{\nu_R(\boldsymbol{\varrho})^q}\le 1$$

and

$$0\le\overline{\mu_R(\boldsymbol{\varrho})^p}+\overline{\eta_R(\boldsymbol{\varrho})^r}+\overline{\nu_R(\boldsymbol{\varrho})^q}\le 1.$$

The p, q, r-SFRS is defined as a pair of p, q, r-spherical fuzzy sets (p, q, r-SFSs), where $\underline{F}$ is distinct from $\overline{F}$. To facilitate comprehension, we denote the given concept as F=($\underline{F},\overline{F}$), which is referred to as a p, q, r-spherical fuzzy rough number. The notation F_i represents the set that encompasses all p, q, r-SFR numbers. The p, q, r-spherical fuzzy rough number (p, q, r-SFRN) F=($\underline{F},\overline{F}$) consists of two components:

Lower approximation space $\underline{F}$: Represents the lower approximation space of $\boldsymbol{\varrho}$ in the p, q, r-SFRS. It captures the lower bound of uncertainty associated with the element $\boldsymbol{\varrho}$.

Upper approximation space $\overline{F}$: Denotes the upper approximation space of $\boldsymbol{\varrho}$ in the p, q, r-SFRS. It captures the upper bound of uncertainty associated with the element $\boldsymbol{\varrho}$.

Definition 3.2. Let

$$F_1=(\underline{\mu_1},\underline{\eta_1},\underline{\nu_1},\overline{\mu_1},\overline{\eta_1},\overline{\nu_1}),F_2=(\underline{\mu_2},\underline{\eta_2},\underline{\nu_2},\overline{\mu_2},\overline{\eta_2},\overline{\nu_2})\ \text{ and }\ F=(\underline{\mu},\underline{\eta},\underline{\nu},\overline{\mu},\overline{\eta},\overline{\nu})$$

be any 3 p, q, r-SFRNs and $\kappa>0$. Then,

$$F_1\oplus F_2{=}\left\langle\begin{matrix}\left(\sqrt[p]{\underline{\mu_1}^p+\underline{\mu_2}^p-\underline{\mu_1}^p\underline{\mu_2}^p},\underline{\eta_1\eta_2},\underline{\nu_1\nu_2}\right)\\ \left(\sqrt[p]{\overline{\mu_1}^p+\overline{\mu_2}^p-\overline{\mu_1}^p\overline{\mu_2}^p},\overline{\eta_1\eta_2},\overline{\nu_1\nu_2}\right)\end{matrix}\right\rangle,\ F_1\otimes F_2{=}\left\langle\begin{matrix}\left(\underline{\mu_1\mu_2},\underline{\eta_1\eta_2},\sqrt[q]{\underline{\nu_1}^q+\underline{\nu_2}^q-\underline{\nu_1}^q\underline{\nu_2}^q}\right)\\ \left(\overline{\mu_1\mu_2},\overline{\eta_1\eta_2},\sqrt[q]{\overline{\nu_1}^q+\overline{\nu_2}^q-\overline{\nu_1}^q\overline{\nu_2}^q}\right)\end{matrix}\right\rangle,$$

$$\kappa F{=}\left\langle\begin{matrix}\left(\sqrt[p]{1-\left(1-\underline{\mu}^p\right)^\kappa}\right),\underline{\eta}^\kappa,\underline{\nu}^\kappa\Big),\\ \left(\sqrt[p]{1-(1-\overline{\mu}^p)^\kappa}\right),\overline{\eta}^\kappa,\overline{\nu}^\kappa\Big)\end{matrix}\right\rangle,\ F^\kappa{=}\left\langle\begin{matrix}\left(\underline{\mu}^\kappa,\sqrt[r]{1-\left(1-\underline{\eta}^r\right)^\kappa},\sqrt[q]{1-\left(1-\underline{\nu}^q\right)^\kappa}\right),\\ \left(\overline{\mu^\kappa},\sqrt[r]{1-\left(1-\overline{\eta}^r\right)^\kappa},\sqrt[q]{1-\left(1-\overline{\nu}^q\right)^\kappa}\right)\end{matrix}\right\rangle.$$

Definition 3.3. Let two p, q, r-SFRNs be

$$F_1=\left(\underline{\mu_1},\underline{\eta_1},\underline{\nu_1},\overline{\mu_1},\overline{\eta_1},\overline{\nu_1}\right),F_2=(\underline{\mu_2},\underline{\eta_2},\underline{\nu_2},\overline{\mu_2},\overline{\eta_2},\overline{\nu_2}).$$

Then,

$$F_1\cup F_2=\left(\begin{matrix}\max\left(\underline{\mu_1},\underline{\mu_2}\right),\min\left(\underline{\eta_1},\underline{\eta_2}\right),\min\left(\underline{\nu_1},\underline{\nu_2}\right),\\ \max(\overline{\mu_1},\overline{\mu_2}),\min(\overline{\eta_1},\overline{\eta_2}),\min(\overline{\nu_1},\overline{\nu_2})\end{matrix}\right),$$

$$F_1 \cap F_2 = \begin{pmatrix} \min\left(\underline{\mu_1}, \underline{\mu_2}\right), \max\left(\underline{\eta_1}, \underline{\eta_2}\right), \max\left(\underline{\nu_1}, \underline{\nu_2}\right), \\ \min(\overline{\mu_1}, \overline{\mu_2}), \max(\overline{\eta_1}, \overline{\eta_2}), \max(\overline{\nu_1}, \overline{\nu_2}) \end{pmatrix},$$

$$F_1^C = \left(\underline{\nu_1}, \underline{\eta_1}, \underline{\mu_1}, \overline{\nu_1}, \overline{\eta_1}, \overline{\mu_1}\right).$$

In this context, F_1^C represents the complement of F_1.

Definition 3.4. For any p, q, r-SFRNs F, the score function of F can be expressed as:

$$\text{Sc(F)}=\frac{2+\underline{\mu}^p+\overline{\mu}^p-\underline{\eta}^r-\overline{\eta}^r-\underline{\nu}^q-\overline{\nu}^q}{3}. \tag{16}$$

In this context, p and q are any positive integers, and r is the maximum of p and q such that $0 \leq \text{Sc}(F) \leq 1$.

Definition 3.5. The accuracy function of p, q, r-SFRN F is defined as follows:

$$\text{Ac(F)}=\frac{\underline{\mu}^p+\overline{\mu}^p+\underline{\nu}^q+\overline{\nu}^q}{2}. \tag{17}$$

In this context, $-1 \leq \text{Ac}(F) \leq 1$.

Definition 3.6. Let

$$F_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}),\ F_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2})$$

be any two p, q, r-SFRNs, then,

1) $\text{Sc}(F_1)<\text{Sc}(F_2)$, then $F_1 < F_2$; Sc $(F_1)>\text{Sc}(F_2)$, then $F_1 > F_2$; $\text{Sc}(F_1)=\text{Sc}(F_2)$, then $F_1 < F_2$.
2) If $\text{Ac}(F_1)<\text{Ac}(F_2)$, then $F_1 < F_2$, If $\text{Ac}(F_1)>\text{Ac}(F_2)$, then $F_1 > F_2$, If $\text{Ac}(F_1)=\text{Ac}(F_2)$, then $F_1 = F_2$.

Definition 3.7. Let

$$F_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}), F_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2}) \text{ and } F = (\underline{\mu}, \underline{\eta}, \underline{\nu}, \overline{\mu}, \overline{\eta}, \overline{\nu})$$

be any 3 p, q, r-SFRNs and $\kappa > 0$ be any three p, q, r-SFRNs, and if $\kappa 1$ and $\kappa 2$ are any positive integers, then the following properties are held.

$$F_1 \oplus F_2 = F_2 \oplus F_1,\ F_1 \otimes F_2 = F_2 \otimes F_1,\ \kappa(F_1 \oplus F_2) = \kappa F_1 \oplus \kappa F,$$

$$\kappa 1 F \oplus \kappa 2 F = (\kappa 1 + \kappa 2) F,\ (F_1 \otimes F_2)^{\kappa} = F1^{\kappa} \otimes F2^{\kappa},\ F^{\kappa 1} \otimes F^{\kappa 2} = F^{(\kappa 1 + \kappa 2)}.$$

Definition 3.8. Let

$$F_1 = (\underline{\mu_1}, \underline{\eta_1}, \underline{\nu_1}, \overline{\mu_1}, \overline{\eta_1}, \overline{\nu_1}),\ F_2 = (\underline{\mu_2}, \underline{\eta_2}, \underline{\nu_2}, \overline{\mu_2}, \overline{\eta_2}, \overline{\nu_2})$$

be any two p, q, r-SFRNs, then the proposed Einstein p, q, r-SFR laws are as follows:

$$\textbf{Addition operation } (\boldsymbol{F_1 \oplus_E F_2}) = \left\langle \begin{matrix} \sqrt[p]{\frac{\underline{\mu_1}^p+\underline{\mu_2}^p}{1+\underline{\mu_1}^p\underline{\mu_2}^p}}, \sqrt[r]{\frac{\underline{\eta_1}^r\underline{\eta_2}^r}{\left(1-\underline{\eta_1}^r\right)\left(1-\underline{\eta_2}^r\right)}}, \sqrt[q]{\frac{\underline{\nu_1}^q\underline{\nu_2}^q}{\left(1-\underline{\nu_1}^q\right)\left(1-\underline{\nu_2}^q\right)}}, \\ \sqrt[p]{\frac{\overline{\mu_1}^p+\overline{\mu_2}^p}{1+\overline{\mu_1}^p\overline{\mu_2}^p}}, \sqrt[r]{\frac{\overline{\eta_1}^r\overline{\eta_2}^r}{(1-\overline{\eta_1}^r)(1-\overline{\eta_2}^r)}}, \sqrt[q]{\frac{\overline{\nu}_1{}^q\overline{\nu}_2^q}{(1-\overline{\nu}_1{}^q)(1-\overline{\nu}_2{}^q)}}, \end{matrix} \right\rangle,$$

$$\textbf{Multiplication operation } (\boldsymbol{F_1 \otimes_E F_2})=\langle \begin{array}{l} \sqrt[p]{\frac{\underline{\mu_1}^p\underline{\mu_2}^p}{\left(1-\underline{\mu_1}^p\right)\left(1-\underline{\mu_2}^p\right)}}, \sqrt[r]{\frac{\underline{\eta_1}^r+\underline{\eta_2}^r}{1+\underline{\eta_1}^q\underline{\eta_2}^q}}, \sqrt[q]{\frac{\underline{\nu_1}^q+\underline{\nu_2}^q}{1+\underline{\nu_1}^q\underline{\nu_2}^q}}, \\ \sqrt[p]{\frac{\overline{\mu_1}^p\overline{\mu_2}^p}{(1-\overline{\mu_1}^p)(1-\overline{\mu_2}^p)}}, \sqrt[r]{\frac{\overline{\eta_1}^r+\overline{\eta_2}^r}{1+\overline{\eta_1}^r\overline{\eta_2}^r}}, \sqrt[q]{\frac{\overline{\nu}_1{}^q+\overline{\nu}_2^q}{1+\overline{\nu}_1{}^q\overline{\nu}_2{}^q}}, \end{array} \rangle,$$

$$\textbf{Scalar multiplication operation}(\boldsymbol{\kappa_E F_1})=\langle \begin{array}{l} \sqrt[p]{\frac{\left(1-\underline{\mu_1}^p\right)^\kappa-\left(1-\underline{\mu_1}^p\right)^\kappa}{\left(1-\underline{\mu_1}^p\right)^\kappa+\left(1-\underline{\mu_1}^p\right)^\kappa}}, \sqrt[r]{\frac{2\left(\underline{\eta_1}^r\right)^\kappa}{\left(2-\underline{\eta_1}^r\right)^\kappa+\left(\underline{\eta_1}^r\right)^\kappa}}, \sqrt[q]{\frac{2\left(\underline{\nu_1}^q\right)^\kappa}{\left(2-\underline{\nu_1}^q\right)^\kappa+\left(\underline{\nu_1}^q\right)^\kappa}}, \\ \sqrt[p]{\frac{(1-\overline{\mu_1}^p)^\kappa-(1-\overline{\mu_1}^p)^\kappa}{(1-\overline{\mu_1}^p)^\kappa+(1-\overline{\mu_1}^p)^\kappa}}, \sqrt[r]{\frac{2(\overline{\eta_1}^r)^\kappa}{(2-\overline{\eta_1}^r)^\kappa+(\overline{\eta_1}^r)^\kappa}}, \sqrt[q]{\frac{2(\overline{\nu}_1{}^q)^\kappa}{(2-\overline{\nu}_1{}^q)^\kappa+(\overline{\nu}_2{}^q)^\kappa}}, \end{array} \rangle,$$

$$\textbf{Power operation } (\boldsymbol{F_1^{\kappa_E}})=\langle \begin{array}{l} \sqrt[p]{\frac{2\left(\underline{\eta_1}^p\right)^\kappa}{\left(2-\underline{\eta_1}^p\right)^\kappa+\left(\underline{\eta_1}^p\right)^\kappa}}, \sqrt[r]{\frac{\left(1-\underline{\eta_1}^r\right)^\kappa-\left(1-\underline{\eta_1}^r\right)^\kappa}{\left(1-\underline{\eta_1}^r\right)^\kappa+\left(1-\underline{\eta_1}^r\right)^\kappa}}, \\ \sqrt[q]{\frac{\left(1-\underline{\nu_1}^q\right)^\kappa-\left(1-\underline{\nu_1}^q\right)^\kappa}{\left(1-\underline{\nu_1}^r\right)^\kappa+\left(1-\underline{\nu_1}^r\right)^\kappa}}, \sqrt[p]{\frac{2(\overline{\mu_1}^p)^\kappa}{(2-\overline{\mu_1}^p)^\kappa+(\overline{\mu_1}^p)^\kappa}}, \\ \sqrt[r]{\frac{(1-\overline{\eta_1}^r)^\kappa-(1-\overline{\eta_1}^r)^\kappa}{(1-\overline{\eta_1}^r)^\kappa+(1-\overline{\eta_1}^r)^\kappa}}, \sqrt[q]{\frac{(1-\overline{\nu}_1{}^q)^\kappa-(1-\overline{\nu}_1{}^q)^\kappa}{(1-\overline{\nu}_1{}^q)^\kappa+(1-\overline{\nu}_1{}^q)^\kappa}}, \end{array} \rangle \text{ for } \kappa>0.$$

Definition 3.9. Let $F = F_1, F_2, \ldots, F_n$ be a group of p, q, r-SFRNs, where $F_i = (\underline{\mu_i}, \underline{\eta_i}, \underline{\nu_i}, \overline{\mu_i}, \overline{\eta_i}, \overline{\nu_i})$, weight vector $w_i = (w_1, w_2, \ldots, w_n)^T$, satisfying $\sum_{i=}^{n} w_i = 1$, and let s,t be any positive integer. A p, q, r-SFREBOMWG operator is a mapping $\mathrm{p,q,r-SFREBOMWG^{s,t}}: \Delta^{\mathrm{n}} \rightarrow \Delta$ and can be expressed as:

$$\mathrm{p,q,r-SFREBOMWG^{s,t}}(F_1, F_2, \ldots\ldots, F_n)=\frac{1}{s+t}\left[\left(\otimes_{E\,i,j;i\neq j}^{n}\left(s\left(F_i^{nw_i}\right)\oplus_E t\left(F_j^{nw_j}\right)\right)^{\left(\frac{1}{n(n-1)}\right)}\right)\right], \quad (18)$$

where s, t>0.

Definition 3.10. Let $F = F_1, F_2, \ldots, F_n$ be a group of p, q, r-SFRNs where $F_i = (\underline{\mu_i}, \underline{\eta_i}, \underline{\nu_i}, \overline{\mu_i}, \overline{\eta_i}, \overline{\nu_i})$ with weight vector $w_i = (w_1, w_2, \ldots, w_n)^T$, satisfying $\sum_{i=}^{n} w_i = 1$, and let s,t be any positive integer. A p, q, r,-SFREBOMWA operator is a mapping $\mathrm{p,q,r-SFREBOMWA}: \Delta^{\mathrm{n}} \rightarrow \Delta$ and can be expressed as:

$$\mathrm{p,q,r-SFREBOMWA}(F_1, F_2, \ldots, F_n)=\left[\frac{1}{n(n-1)}\left(\oplus_{Ei,j=1;i\neq j}^{n}\left((nw_iF_i)^s\otimes_E(nw_jF_j)^t\right)\right)\right]^{\left(\frac{1}{s+t}\right)}, \quad (19)$$

where s, t>0.

The proposed aggregators are confidence-levels-based aggregation operators are an essential means for decision-making processes. These operators integrate the notion of confidence level associated with each input source, thereby extending conventional aggregation approaches and helping in facilitating decision makers to navigate through complex, uncertain scenarios with sensitive accuracy. Here, we consider traditional aggregation operators that have ignored the subtleties of attaching confidence levels, which often lead to skewed outcomes. The inclusion of a confidence level to the input source is a holistic approach that ensures a clear and accurate evaluation of alternatives, and acts as a bridge between membership degrees and decision-makers. The assignment method of the confidence level ξi varies depending on the input source. The decision maker can decide the confidence level depending on the rating they make on a particular case study.

Definition 3.11. Let $F = F_1, F_2, \ldots, F_n$ with corresponding confidence levels ξ_i, such that $0 \le \xi_i \le 1$ and weight vector $w_i = (w_1, w_2, \ldots, w_n)^T$, satisfying $\sum_{i=}^{n} w_i = 1$. Let there be a group of p, q, r-SFRNs with s and t as any positive integer. A confidence level-based p, q, r-spherical fuzzy rough Einstein Bonferroni weighted geometric $C_{p,q,r}$ -SFREBOMWG operator is a mapping $C_{p,q,r}$ -SFREBOMWG: $\Delta^{\mathrm{n}} \to \Delta$ and can be expressed as: $C_{p,q,r}$-SFREBOMWG$^{\mathrm{s,t}}$

$$\left(w_1(\mathrm{F}_1, \xi_1), w_2(\mathrm{F}_2, \xi_2), \ldots, w_n(\mathrm{F}_\mathrm{n}, \xi_n)\right) = \left[\left(\otimes_{E\ i,j;i\neq j}^{n} \left(s\left(F_i^{n\xi_i w_i}\right) \oplus_E t\left(F_j^{n\xi_j w_j}\right)\ \right)^{\left(\frac{1}{n(n-1)}\right)}\right)\right], \tag{20}$$

where s, t>0.

Theorem 3.1. The aggregated value obtained using the $C_{p,q,r}$-SFREBOMWG operator is also a p, q, r-SFRN and can be expressed as follows:

$$C_{p,q,r} - SFREBOMWG^{s,t}(F_1, F_2, \ldots \ldots, F_n) = \left[\otimes_{E\,i,j;i\neq j}^{n} \left(s\left(F_i^{n\xi_i w_i}\right) \oplus_E t\left(F_j^{n\xi_j w_j}\right)\right)\right]^{\left(\frac{1}{n(n-1)}\right)}$$

$$= \left\langle \begin{matrix} \left(\sqrt[p]{\dfrac{\underline{\rho}^{\frac{1}{s+t}} - \underline{\psi}^{\frac{1}{s+t}}}{\underline{\rho}^{\frac{1}{s+t}} + \underline{\psi}^{\frac{1}{s+t}}}}, \sqrt[q]{\dfrac{2\underline{\delta}^{\frac{1}{s+t}}}{\underline{\pi}^{\frac{1}{s+t}} + \underline{\delta}^{\frac{1}{s+t}}}}, \sqrt[r]{\dfrac{2\underline{\zeta}^{\frac{1}{s+t}}}{\underline{\tau}^{\frac{1}{s+t}} + \underline{\zeta}^{\frac{1}{s+t}}}} \right), \\ \left(\sqrt[p]{\dfrac{\overline{\rho}^{\frac{1}{s+t}} - \overline{\psi}^{\frac{1}{s+t}}}{\overline{\rho}^{\frac{1}{s+t}} + \overline{\psi}^{\frac{1}{s+t}}}}, \sqrt[q]{\dfrac{2\overline{\delta}^{\frac{1}{s+t}}}{\overline{\pi}^{\frac{1}{s+t}} + \overline{\delta}^{\frac{1}{s+t}}}}, \sqrt[r]{\dfrac{2\overline{\zeta}^{\frac{1}{s+t}}}{\overline{\tau}^{\frac{1}{s+t}} + \overline{\zeta}^{\frac{1}{s+t}}}} \right) \end{matrix} \right\rangle, \tag{21}$$

where

$$\underline{\rho} = (\underline{2M_{ij}} + 2\underline{N_{ij}} - \underline{L_{ij})}^{\left(\frac{1}{n(n-1)}\right)} + 3\underline{(M_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\rho} = (2\overline{M}_{ij} + 2\overline{N}_{ij} - \overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)} + (\overline{3M}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\psi} = (\underline{2M_{ij}} + 2\underline{N_{ij}} - \underline{L_{ij})}^{\left(\frac{1}{n(n-1)}\right)} - \underline{(M_{ij})}^{\left(\frac{1}{n(n-1)}\right)},,$$

$$\overline{\psi} = 2\overline{M}_{ij} + 2\overline{N}_{ij} - \overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)} + (\overline{3M}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\delta} = (\underline{P_{ij}} + \underline{Q_{ij}} + \underline{O_{ij})}^{\left(\frac{1}{n(n-1)}\right)} - \underline{(P_{ij}} + \underline{Q_{ij}} - \underline{O_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\delta} = (\overline{P}_{ij} + \overline{Q}_{ij} + \overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)} - (\overline{P}_{ij} + \overline{Q}_{ij} - \overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\pi} = (\underline{P_{ij}} + \underline{Q_{ij}} + \underline{O_{ij})}^{\left(\frac{1}{n(n-1)}\right)} + 3\underline{(P_{ij}} + \underline{Q_{ij}} - \underline{O_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\pi} = (\overline{P}_{ij} + \overline{Q}_{ij} + \overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)} + 3(\overline{P}_{ij} + \overline{Q}_{ij} - \overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\zeta} = (\underline{S_{ij}} + \underline{T_{ij}} + \underline{R_{ij})}^{\left(\frac{1}{n(n-1)}\right)} - \underline{(S_{ij}} + \underline{T_{ij}} - \underline{R_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\zeta} = (\overline{S}_{ij} + \overline{T}_{ij} + \overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)} - (\overline{S}_{ij} + \overline{T}_{ij} - \overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\tau} = (\underline{S_{ij}} + \underline{T_{ij}} + \underline{R_{ij}})^{\left(\frac{1}{n(n-1)}\right)} + 3(\underline{S_{ij}} + \underline{T_{ij}} - \underline{R_{ij}})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\tau} = (\overline{S}_{ij} + \overline{T}_{ij} + \overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)} + 3(\overline{S}_{ij} + \overline{T}_{ij} - \overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

and

$$\underline{L}_{ij=} \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} + \left(3\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\overline{L_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} + \left(3\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\underline{M}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} + \left(3\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(2-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\overline{M_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} + \left(3\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j} + 3\left(\bar{\mu}_{F_j}^{p}\right)^{n\xi_j}\right]^{t} + \\ \left[\left(2-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\underline{N}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\overline{N_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(2-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\underline{O}_{ij} = \sum_{i,j=1;i\neq j}^{n} \left\{ \begin{array}{c} 2\left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + 3\left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(1-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{s} \end{array} \right\},$$

$$\overline{O_{ij}} = \sum_{i,j=1;i\neq j}^{n} \left\{ \begin{array}{c} 2\left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + 3\left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_j w_j} + 3\left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_j w_j} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_j w_j}\right]^{s} \end{array} \right\},$$

$$\underline{P}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + 3\left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(1-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\overline{P_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{array}{c} \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + 3\left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(1-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(1-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{array} \right\},$$

$$\underline{Q}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c}\left[\left(1+\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t}\end{array}\right\},$$

$$\overline{Q_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c}\left[\left(1+\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(1-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(1-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t}\end{array}\right\},$$

$$\underline{R_{ij}} = \sum_{i,j=1,i\neq j}^{n}\left\{\begin{array}{c}\left[2\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \\ \left[\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{s}\end{array}\right\},$$

$$\overline{R_{ij}} = \sum_{i,j=1,i\neq j}^{n}\left\{\begin{array}{c}\left[2\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j} - \left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j}\right]^{t} \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j} + 3\left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j} - \left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_j w_j}\right]^{s}\end{array}\right\},$$

$$\underline{S}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c}\left[\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t}\end{array}\right\},$$

$$\overline{S_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c} \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}+3\left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}+3\left(1-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t}+ \\ \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}-\left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}-\left(1-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{array}\right\},$$

$$\underline{T}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c} \left[\left(1+\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}-\left(1-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}-\left(1-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{array}\right\},$$

$$\overline{T_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{array}{c} \left[\left(1+\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}-\left(1-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}-\left(1-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{array}\right\}.$$

Properties for idempotency, monotonicity, and boundedness based on Theorem 3.1 are given below:

Property 3.1. (Idempotency) Let $\tilde{F}_i = \langle\left(\underline{\mu_{F_i}}, \underline{\eta_{F_i}}, \underline{\nu_{F_i}}\right), \left(\overline{\mu_{F_i}}, \overline{\eta_{F_i}}, \overline{\nu_{F_i}}\right), \xi_i\rangle, (i = 1,2,\dots,n)$ be a set of n $C_{p,q,r}$ -SFRNs, and if $\tilde{F}_i = \tilde{F} = \langle\left(\underline{\mu_F}, \underline{\eta_F}, \underline{\nu_F}\right), \left(\overline{\mu_F}, \overline{\eta_F}, \overline{\nu_F}\right), \xi\rangle$ for all i, then $C_{p,q,r} -$ $\text{SFREBOMWG}^{\text{s,t}}\left((w_1, \tilde{F}_1), (w_2, \tilde{F}_2), \dots, (w_n, \tilde{F}_n)\right) = \tilde{F}$.

Property 3.2. (Monotonicity) For any two sets of $C_{p,q,r}$-SRFNs

$$\tilde{F}_i = \langle\left(\underline{\mu_{F_i}}, \underline{\eta_{F_i}}, \underline{\nu_{F_i}}\right), \left(\overline{\mu_{F_i}}, \overline{\eta_{F_i}}, \overline{\nu_{F_i}}\right), \xi_i\rangle,\ \widetilde{F'}_{i} = \langle\left(\underline{\eta'_{F_i}}, \underline{\eta'_{F_i}}, \underline{\nu'_{F_i}}\right), \left(\overline{\eta'_{F_i}}, \overline{\eta'_{F_i}}, \overline{\nu'_{F_i}}\right), \xi'_i\rangle$$

such that $\tilde{F}_i \leq \widetilde{F'}_i$ for all i=1,2,…, n, we have

$$C_{p,q,r} - \text{SFREBOMWG}^{\text{s,t}}\left((w_1, \tilde{F}_1), (w_2, \tilde{F}_2), \dots, (w_n, \tilde{F}_n)\right) \leq C_{p,q,r} - \text{SFREBOMWG}^{\text{s,t}}\left((w_1, \widetilde{F'}_1), (w_2, \widetilde{F'}_2), \dots, (w_n, \widetilde{F'}_n)\right).$$

Property 3.3. (Boundedness) For a collection of $C_{p,q,r}$ -SFRNs $\tilde{F}_i = \langle\left(\underline{\mu_{F_i}}, \underline{\eta_{F_i}}, \underline{\nu_{F_i}}\right), \left(\overline{\mu_{F_i}}, \overline{\eta_{F_i}}, \overline{\nu_{F_i}}\right), \xi_i\rangle$, $(i = 1,2,\dots.,n)$, if $\widetilde{F^l} = \min_i \tilde{F}_i$ and $\widetilde{F^u} = \max_i \tilde{F}_i$.

Then, $(w, \widetilde{F^l}) \leq C_{p,q,r} - \text{SFREBOMWG}^{\text{s,t}}\left((w_1, \tilde{F}_1), (w_2, \tilde{F}_2), \dots, (w_n, \tilde{F}_n)\right) \leq (w, \widetilde{F^u})$.

Definition 3.12. Let $F = F_1, F_2, \dots, F_n$ with corresponding confidence levels ξ_i, such that $0 \leq \xi_i \leq 1$ and weight vector $w_i = (w_1, w_2, \dots\dots, w_n)^T$, satisfying $\sum_{i=}^{n} w_i = 1$. Thus, there is a group of p, q, r-SFRNs. A confidence level-based $C_{p,q,r}$-SFREBOMWA operator is a mapping $C_{p,q,r}$-SFREBOMWA:$\Delta^{\text{n}} \rightarrow \Delta$ and can be expressed as: $C_{p,q,r}$-$\text{SFREBOMWA}^{\text{s,t}}$

$$\left(w_1(\mathrm{F}_1, \xi_1), w_2(\mathrm{F}_2, \xi_2), \dots, w_n(\mathrm{F}_\mathrm{n}, \xi_n)\right) = \left[\frac{1}{n(n-1)}\left(\oplus_{Ei,j=1;i\neq j}^{n}\left((n\xi_i w_i F_i)^s \otimes_E (n\xi_j w_j F_j)^t\right)\right)\right]^{\left(\frac{1}{s+t}\right)}, \quad (22)$$

where s, t>0.

Theorem 3.2. The aggregated value obtained using the$C_{p,q,r}$--SFREBOMWA operator is also a p, q, r-SFRN and can be expressed as follows:

$$C_{p,q,r}-SFREBOMWA^{s,t}(F_1,F_2,\dots,F_n)=\left[\frac{1}{n(n-1)}\left(\oplus_{Ei,j=1;i\neq j}^{n}\left((n\xi_i w_i F_i)^s\otimes_E(n\xi_j w_j F_j)^t\right)\right)\right]^{\left(\frac{1}{s+t}\right)}$$

$$=\left\langle\begin{matrix}\left(\sqrt[p]{\frac{2\underline{\rho}^{\frac{1}{s+t}}}{\underline{\rho}^{\frac{1}{s+t}}+\underline{\psi}^{\frac{1}{s+t}}}},\sqrt[r]{\frac{\underline{\delta}^{\frac{1}{s+t}}-\underline{\pi}^{\frac{1}{s+t}}}{\underline{\delta}^{\frac{1}{s+t}}+\underline{\pi}^{\frac{1}{s+t}}}},\sqrt[q]{\frac{\underline{\zeta}^{\frac{1}{s+t}}-\underline{\tau}^{\frac{1}{s+t}}}{\underline{\zeta}^{\frac{1}{s+t}}+\underline{\tau}^{\frac{1}{s+t}}}}\right),\\ \left(\sqrt[p]{\frac{2\overline{\rho}^{\frac{1}{s+t}}}{\overline{\rho}^{\frac{1}{s+t}}+\overline{\psi}^{\frac{1}{s+t}}}},\sqrt[r]{\frac{\overline{\delta}^{\frac{1}{s+t}}-\overline{\pi}^{\frac{1}{s+t}}}{\overline{\delta}^{\frac{1}{s+t}}+\overline{\pi}^{\frac{1}{s+t}}}},\sqrt[q]{\frac{\overline{\zeta}^{\frac{1}{s+t}}-\overline{\tau}^{\frac{1}{s+t}}}{\overline{\zeta}^{\frac{1}{s+t}}+\overline{\tau}^{\frac{1}{s+t}}}}\right),\end{matrix}\right\rangle, \tag{23}$$

where

$$\underline{\rho}=(\underline{M_{ij}}+\underline{N_{ij}}+\underline{L_{ij}})^{\left(\frac{1}{n(n-1)}\right)}-\underline{M_{ij}}+\underline{N_{ij}}-\underline{L_{ij}})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\rho}=(\overline{M}_{ij}+\overline{N}_{ij}+\overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)}-(\overline{M}_{ij}+\overline{N}_{ij}-\overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\psi}=(\underline{M_{ij}}+\underline{N_{ij}}+\underline{L_{ij}})^{\left(\frac{1}{n(n-1)}\right)}+3(\underline{M_{ij}}+\underline{N_{ij}}-\underline{L_{ij}})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\psi}=(\overline{M}_{ij}+\overline{N}_{ij}+\overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)}+3(\overline{M}_{ij}+\overline{N}_{ij}-\overline{L}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\delta}=(2\underline{P_{ij}}+2\underline{Q_{ij}}-\underline{O_{ij}})^{\left(\frac{1}{n(n-1)}\right)}+3\underline{(O_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\delta}=(2\overline{P}_{ij}+2\overline{Q}_{ij}-\overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)}+(\overline{3O}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\pi}=(2\underline{P_{ij}}+2\underline{Q_{ij}}-\underline{O_{ij}})^{\left(\frac{1}{n(n-1)}\right)}-\underline{(O_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\pi}=2\overline{P}_{ij}+2\overline{Q}_{ij}-\overline{O}_{ij})^{\left(\frac{1}{n(n-1)}\right)}+(\overline{3O}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\zeta}=(2\underline{S_{ij}}+2\underline{T_{ij}}-\underline{R_{ij}})^{\left(\frac{1}{n(n-1)}\right)}+3\underline{(R_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\zeta}=(2\overline{S}_{ij}+2\overline{T}_{ij}-\overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)}+(\overline{3R}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

$$\underline{\tau}=(2\underline{S_{ij}}+2\underline{T_{ij}}-\underline{R_{ij}})^{\left(\frac{1}{n(n-1)}\right)}-\underline{(R_{ij})}^{\left(\frac{1}{n(n-1)}\right)},$$

$$\overline{\tau}=2\overline{S}_{ij}+2\overline{T}_{ij}-\overline{R}_{ij})^{\left(\frac{1}{n(n-1)}\right)}+(\overline{3R}_{ij})^{\left(\frac{1}{n(n-1)}\right)},$$

and

$$\underline{L}_{ij} = \textstyle\sum_{i,j=1;i\neq j}^{n} \left\{ \begin{matrix} 2\left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} + 3\left(1-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(1-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{s} \end{matrix} \right\},$$

$$\overline{L}_{ij} = \textstyle\sum_{i,j=1;i\neq j}^{n} \left\{ \begin{matrix} 2\left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} + 3\left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_j w_j} + 3\left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_j w_j}\right]^{s} + \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{t} \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_j w_j} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_j w_j}\right]^{s} \end{matrix} \right\},$$

$$\underline{M}_{ij} = \textstyle\prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} + 3\left(1-\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\eta_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(1-\eta_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\eta_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\overline{M_{ij}} = \textstyle\prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} + 3\left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} + 3\left(1-\mu_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(1-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\underline{N}_{ij} = \textstyle\prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(1+\underline{\eta}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\eta_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(1-\underline{\eta}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\overline{N_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(1+\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i} - \left(1-\bar{\mu}_{F_i}^{p}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(1+\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j} - \left(1-\bar{\mu}_{F_j}^{p}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\underline{O}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(2-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + \left(3\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\overline{O_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(2-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + \left(3\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\underline{P}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(2-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + \left(3\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(2-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\overline{P_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(2-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} + \left(3\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} + 3\left(\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(2-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\underline{Q}_{ij} = \prod_{i,j=1;i\neq j}^{n} \left\{\begin{matrix} \left[\left(2-\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\underline{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\underline{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\overline{Q_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2\left\{\begin{matrix} \left[\left(2-\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{r}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j} - \left(\bar{\eta}_{F_j}^{r}\right)^{n\xi_j w_j}\right]^{t} \end{matrix}\right\},$$

$$\underline{R}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\underline{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\nu_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\underline{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\nu_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\overline{R_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} - \\ \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{q}\right)^{n\xi_i} w_i\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\underline{S}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + \left(3\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\overline{S_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} + 3\left(\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} + 3\left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} + \\ \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\underline{T}_{ij} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\},$$

$$\overline{T_{ij}} = \prod_{i,j=1;i\neq j}^{n} 2 \left\{ \begin{matrix} \left[\left(2-\bar{\nu}_{F_i}^{q}\right)^{n\xi_i w_i} - \left(\bar{\eta}_{F_i}^{q}\right)^{n\xi_i w_i}\right]^{s} \\ \left[\left(2-\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j} - \left(\bar{\nu}_{F_j}^{q}\right)^{n\xi_j w_j}\right]^{t} \end{matrix} \right\}.$$

Properties for idempotency, monotonicity, and boundedness based on Theorem 3.2 are given below:

Property 3.4. (Idempotency) Let $\tilde{F}_i = \langle\left(\underline{\mu_{F_i}}, \underline{\eta_{F_i}}, \underline{\nu_{F_i}}\right), \left(\overline{\mu_{F_i}}, \overline{\eta_{F_i}}, \overline{\nu_{F_i}}\right), \xi_i\rangle, (i = 1,2, \dots, n)$ be a set of n $C_{p,q,r}$ -SFRNs, and if $\tilde{F}_i = \tilde{F} = \langle\left(\underline{\mu_F}, \underline{\eta_F}, \underline{\nu_F}\right), \left(\overline{\mu_F}, \overline{\eta_F}, \overline{\nu_F}\right), \xi\rangle$ for all i, then $C_{p,q,r} -$

$-\text{SFREBOMWA}^{\text{s,t}}\big((w_1,\tilde{F}_1),(w_2,\tilde{F}_2),\dots,(w_n,\tilde{F}_n)\big)=\tilde{F}$.

Property 3.5. (Monotonicity) For any two sets of $C_{p,q,r}$-SRFNs

$$\tilde{F}_i=\langle\big(\underline{\mu_{F_i}},\underline{\eta_{F_i}},\underline{\nu_{F_i}}\big),\big(\overline{\mu_{F_i}},\overline{\eta_{F_i}},\overline{\nu_{F_i}}\big),\xi_i\rangle,\ \widetilde{F'}_{i}=\langle\big(\underline{\mu'_{F_i}},\underline{\eta'_{F_i}},\underline{\nu'_{F_i}}\big),\big(\overline{\mu'_{F_i}},\overline{\eta'_{F_i}},\overline{\nu'_{F_i}}\big),\xi'_i\rangle$$

such that $\tilde{F}_i\le\widetilde{F'}_i$ for all i=1,2,…,n, we have

$$C_{p,q,r}-\text{SFREBOMWA}^{\text{s,t}}\big((w_1,\tilde{F}_1),(w_2,\tilde{F}_2),\dots,(w_n,\tilde{F}_n)\big)\le C_{p,q,r}-\text{SFREBOMWA}^{\text{s,t}}\big((w_1,\widetilde{F'}_1),(w_2,\widetilde{F'}_2),\dots,(w_n,\widetilde{F'}_n)\big).$$

Property 3.6. (Boundedness) For a collection of $C_{p,q,r}$ -SFRNs $\tilde{F}_i=\langle\big(\underline{\mu_{F_i}},\underline{\eta_{F_i}},\underline{\nu_{F_i}}\big),\big(\overline{\mu_{F_i}},\overline{\eta_{F_i}},\overline{\nu_{F_i}}\big),\xi_i\rangle$, $(i=1,2,\dots.,n)$, if $\widetilde{F^l}=\min_i\tilde{F}_i$ and $\widetilde{F^u}=\max_i\tilde{F}_i$. Then,

$$(w,\widetilde{F^l})\le C_{p,q,r}-\text{SFREBOMWA}^{\text{s,t}}\big((w_1,\tilde{F}_1),(w_2,\tilde{F}_2),\dots,(w_n,\tilde{F}_n)\big)\le(w,\widetilde{F^u}).$$

4. Confidence level-based p, q, r-spherical fuzzy rough codas approach

One of the most commonly used methods for solving MCDM problems is the application of a decision matrix. This method is very useful in situations dominated by a p, q, r-spherical fuzzy rough environment, where uncertainty and imprecision are of primary concern. In such a setting, the decision matrix represents the evaluation values of all feasible alternatives with respect to given criteria. The collection of potential alternatives is represented as A={A_1,A_2,…,A_l} (l≥2), representing a discrete set of viable choices. Each alternative is evaluated according to a list of predetermined criteria under the auspices of the q-SFRS. The decision-influencing criteria are designated C={C_1,C_2,…,C_m}, covering a broad spectrum of considerations essential to the evaluation process. To ensure an organized analysis, a weight vector is defined, measuring the relative significance of each decision maker. These weights W={w_1,w_2,...,w_n} need to meet the requirements so that they are in the closed interval [0,1], and their sum equals 1 to provide validity and consistency in assessment. Additionally, a confidence level measure is also being collected from the decision-makers to validate how certain they are about the accuracy of the information provided. These confidence levels are in the closed interval [0,1]. Through the utilization of this system, decision-makers can efficiently rank and prioritize alternatives, even within the intricacies of a q-SFRS, thereby opening doors to well-informed and well-balanced choices. The p, q, r-SFRNs with linguistic terms were fixed by decision makers.

Step 1. Construct $\boldsymbol{D_n}=\big[(\boldsymbol{A_{ij}})\big]^{n}_{l\times m}$.

The decision-maker starts with filling in the assessment matrix using the linguistic words given in Table 1. A_{ij} denotes the performance value of the ith alternative on the jth criterion, i=1,2,…,l and j=1,2,…,m. $\boldsymbol{D_n}=\big[(\boldsymbol{A_{ij}})\big]^{n}_{l\times m}$. The decision matrix is created based on each expert, where n=1,2,…,t.

Step 2. Construct the normalized decision matrices by $U^{(s)},s=1,2,3,$...,t

$$U^{(s)}=\begin{Bmatrix}\big[A^{(\boldsymbol{n})}_{\boldsymbol{ij}}\big]\ for\ benefit\ criteria\\ \big[\big(A^{(\boldsymbol{n})}_{\boldsymbol{ij}}\big)\big]^{C}\ for\ cost\ criteria\end{Bmatrix},$$

where $\big[\big(A^{(\boldsymbol{n})}_{\boldsymbol{ij}}\big)\big]^{C}$ is a complement of $\big[A^{(\boldsymbol{n})}_{\boldsymbol{ij}}\big]$.

Step 3. Utilize the proposed operators Eq (20) or (22), $C_{p,q,r}-\text{SFREBOMWG}^{\text{s,t}}$ (20) or $C_{p,q,r}-$

SFREBOMAG^{s,t} (22) to aggregate the decision matrices $U^{(s)} = \left[A_{\boldsymbol{ij}}^{(\boldsymbol{n})}\right]$ into $U = \left[A_{\boldsymbol{ij}}\right]_{l\times m}$.

Step 4. The weighted decision matrix is defuzzied using a score function specifically intended for this function. The equation used is as in (16)

$$\text{Sc }(C_j(A_{iw}))=\frac{2+\underline{\mu}^p+\overline{\mu^p}-\underline{\eta}^r-\overline{\eta^r}-\underline{\nu}^q-\overline{\nu^q}}{3}.$$

This step converts fuzzy data into crisp values for better interpretability.

Step 5. CRITIC method [26]

The procedure involved in the CRITIC approach is outlined in the following steps:

Step 5.1. Equation (24) calculates the correlation coefficient for the criteria.

$$r_{jk} = \frac{(\sum_{i=1}^{l} Sc_{ij}-\overline{Sc_j})(\sum_{i=1}^{l} Sc_{ij}-\overline{Sc_t})}{\sqrt{\sum_{i=1}^{l}(Sc_{ij}-\overline{Sc_j})^2(Sc_{ij}-\overline{Sc_t})}}, \tag{24}$$

where r_{jk} is the correlation coefficient between the vectors r_j and r_k, where j, k=1,2,…,m.

Step 5.2. Compute an approximation of the standard deviations of the criterion from Eq (25)

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{l} Sc_{ij}-\overline{Sc_j}}{l}}, \tag{25}$$

where $\overline{Sc_{ij}} = \sum_{i=1}^{l} Sc_{ij}\,/l.$

Step 5.3. Analyze each criterion's information using (26)

$$c_j = \sigma_j \sum_{i=1}^{l}(1 - r_{jt}). \tag{26}$$

With increasing c_j, a particular criterion possesses more information than others. Therefore, the weight assigned to that criterion increases relative to other factors.

Step 5.4. The objective weight for each criterion is calculated using Eq (27):

$$we_j = \frac{c_j}{\sum_{j=1}^{m} c_j}. \tag{27}$$

Step 6. The CODAS method [33]

The next step procedure for applying the CODAS method is outlined below:

Step 6.1. Get the negative ideal solution against each criterion using Eq (28).

$$A^{-} = \{C_j, \min_i < \text{Sc}\left(C_j(A_{iw})\right) > \text{j} = 1,2,\dots,\text{m}\}. \tag{28}$$

Step 6.2. Find the normalized Hamming distance from the aggregated p, q, r-SFRNs for alternatives using Eq (29) as in [33,51]

$$D_{HAM}(A_i, A^{-}) = \sqrt[\min(p,q,r)]{\frac{1}{6n}\sum_{i=1}^{l} we_i \begin{pmatrix} \left|\underline{\eta}_{A_i}^{p} - \underline{\eta}_{A^-}^{p}\right| + \left|\underline{\eta}_{A_i}^{r} - \underline{\eta}_{A^-}^{r}\right| + \\ \left|\underline{\nu}_{A_i}^{q} - \underline{\nu}_{A^-}^{q}\right| + \left|\overline{\mu_{A_i}^{p}} - \overline{\mu_{A^-}^{p}}\right| + \\ \left|\overline{\eta_{A_i}^{r}} - \overline{\eta_{A^-}^{r}}\right| + \left|\overline{\nu_{A_i}^{q}} - \overline{\nu_{A^-}^{q}}\right| \end{pmatrix}}. \tag{29}$$

Step 6.3. Find the normalized Euclidean distance from the aggregated p, q, r-SFRNs for alternatives using (30) as in [33]

$$D_{ECL}(A_i, A^-) = \sqrt[\min(p,q,r)]{\frac{1}{6n}\sum_{i=1}^{l} we_i \begin{pmatrix} \left(\underline{\eta^p_{A_i}} - \underline{\eta^p_{A^-}}\right)^2 + \left(\underline{\eta^r_{A_i}} - \underline{\eta^r_{A^-}}\right)^2 \\ + \left(\underline{\nu^q_{A_i}} - \underline{\nu^q_{A^-}}\right)^2 \left(\overline{\mu^p_{A_i}} - \overline{\mu^p_{A^-}}\right)^2 + \\ \left(\overline{\eta^r_{A_i}} - \overline{\eta^r_{A^-}}\right)^2 + \left(\overline{\nu^q_{A_i}} - \overline{\nu^q_{A^-}}\right)^2 \end{pmatrix}}. \tag{30}$$

Step 6.4. Evaluate the relative assessment matrix for the alternatives using Eq (31).

$$r_{ik} = (D_{ECL_i} - D_{ECL_k}) + \omega(D_{ECL_i} - D_{ECL_k})(D_{HAM_i} - D_{HAM_k}), \tag{31}$$

where $k \in \{1,2,\ldots,m\}$ and ω is a threshold function that is defined in Eq (11). The decision maker can define the threshold value. In this study, we follow ω=0.02,

$$\omega(x) = \begin{Bmatrix} 1 \; if \; |x| \geq \Omega \\ 0 \; if \; |x| < \Omega \end{Bmatrix}.$$

Step 6.5. Obtain the score for each using Eq (32)

$$AS_i = \sum_{k=1}^{l} rik. \tag{32}$$

Rank the alternatives and obtain the optimal solution.

The algorithm is described via a flowchart presented in Figure 2, which provides a visual representation of its step-by-step reasoning and decision-making process.

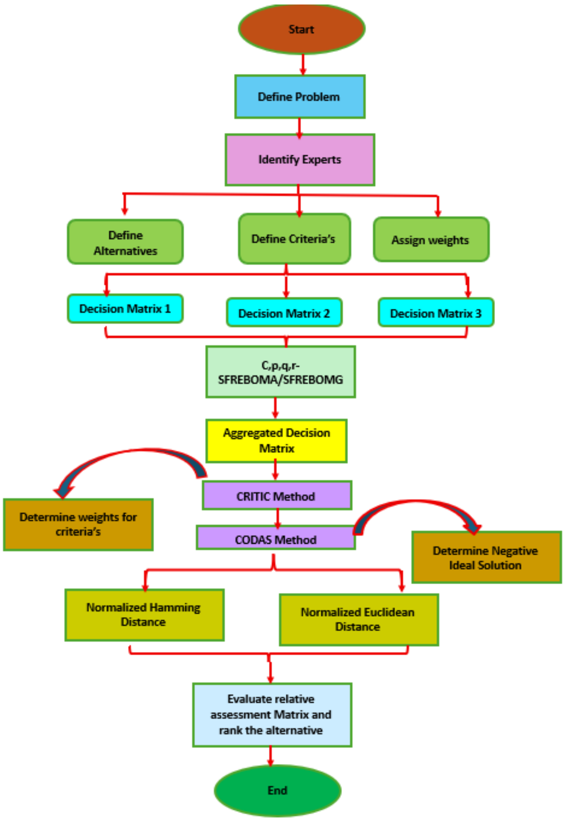


Figure 2. Flowchart depicting the overall MCDM process.

5. Numerical example

MCDM techniques have been identified as essential in evaluating municipal solid waste (MSW) treatment options, offering a structured approach to balancing environmental, economic, and social considerations.

Together, these studies verify the tractability and value of MCDM methods in informing sustainable MSW treatment decisions in diverse geographical and socio-economic contexts.

MSW management is a complex decision-making problem with uncertainty, vagueness, and incomplete information. Factors such as waste generation rates, working efficiency, treatment performance, environmental impact, and social acceptance are frequently vague or partially known. Hence, such uncertainties are incapable of practical MSW planning. Thus, to overcome these issues, RS theory, specifically rough boundary approximations, provides a suitable mathematical framework by enabling lower and upper approximation sets. The lower approximation represents waste management strategies (e.g., recycling, composting, incineration, and landfilling) that certainly satisfy predefined sustainability or efficiency criteria, whereas the upper approximation includes strategies that possibly satisfy these criteria. The resulting boundary region portrays the essential ambiguity occurring from overlapping attributes, inconsistent data, or conflicting expert assessments. Rough boundary approximations boost the assessment of MSW management techniques by reducing loss of information and avoiding overconfident rankings in case of MCDM frameworks. They facilitate the classification of alternatives into acceptable, conditionally acceptable, and unacceptable categories, which aligns well with policy-making and regulatory processes. Moreover, rough boundaries can be combined with fuzzy or probabilistic models to further accommodate linguistic assessments and uncertainty in stakeholder judgments.

The complexity of the system in municipal waste management is promptly progressing, such that the sophisticated decision–making approaches are needed to process high levels of uncertainty. The traditional methodologies are suitable but are incapable of managing with the uncertainty and imprecision generated by real-world phenomena. Hence, we aim to develop a sharp decision-making model that can successfully accept uncertainty within the environment of municipal waste management. Despite various treatment techniques, the states need to face continual challenges, incorporating inefficient segregation, inadequate recycling infrastructure, and the accumulation of unprocessed waste at landfills. Therefore, good prioritization of strategies is important for optimizing resource utilization and minimizing environmental impact in the states of the country. There are many profitable and economical strategies for handling the waste management treatment techniques, but it is necessary to choose the optimum method. Confidence level-based p, q, r- spherical fuzzy rough Einstein Bonferroni aggregation operators with the CRITIC-CODAS approach is a novel application in waste management systems that can handle inaccurate and vague information by considering the spherical representation, which is more effective than traditional FS theories in handling higher levels of uncertainty. The spherical representation is SFS that are constructed based on the hesitancy of a decision maker, satisfying the condition that the squared sum of membership, non-membership, and hesitancy degrees is at most equal to 1. Hence, a study is conducted in a district of south Kerala, comprising experts in the field of waste treatment techniques, including higher officials in municipal authorities with more than 15 years of experience in waste treatment management techniques.

Here, the p, q, r-spherical fuzzy rough approach is utilized by taking $p=3$, $q=2$, and $r=\max(p, q)=3$ to assess and rank different waste treatment methods. The method targets finding the best solutions by examining key criteria designed for urban waste management. In this regard, analyses are performed on the waste treatment techniques listed in Table 3.

Table 3. Waste treatment techniques for ranking.

Ai	Method	Description
A1	Upcycling/upgradation	Reuse of waste products with improved quality.
A2	Recycling	Reprocessing waste products.
A3	Gasification	Conversion of carbonaceous materials into a gaseous product.
A4	Incineration	Waste products are burnt.
A5	Dumping/landfill	Disposal of wastes in a particular area.
A6	Composting	Conversion of organic waste into nutrient-rich soil.
A7	Pyrolysis/thermal decomposition	Converting waste plastics into energy.
A8	Biotechnological process	Conversion of organic waste into solid and liquid fuels.
A9	Hydrolysis/fermentation	Anaerobic decomposition of organic waste to produce liquid fuel.

Based on in-depth critical analysis of studies, important criteria are selected to meet community expectations and standards, which are outlined in Table 3. The assessment process is conducted by a group of decision-makers based on linguistic terms. Three decision makers with respective weights (0.25, 0.36, and 0.39) are involved in the evaluation and mark their choices along with their confidence level. The decision maker decides the confidence level depending on the rating and the normalize aggregated decision matrix using $C_{p,q,r}$-SFREBOMG is computed. A decision tree for municipal waste treatment technique selection is given in Figure 3.

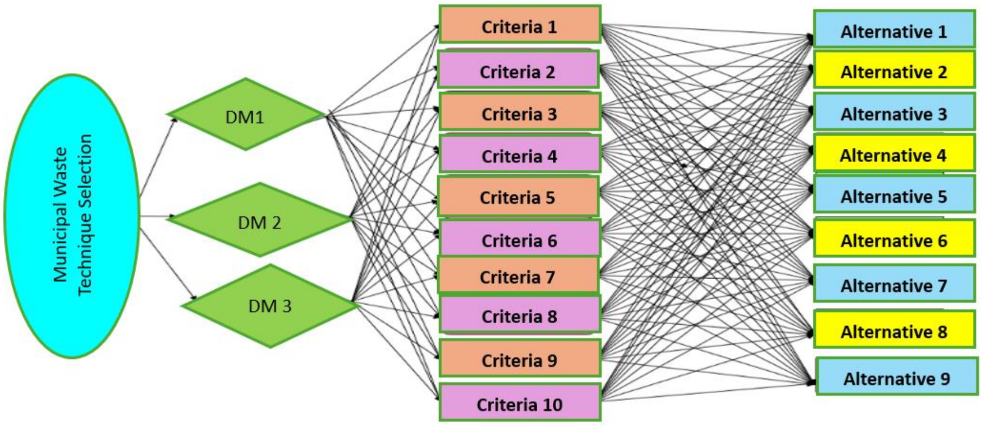


Figure 3. Decision tree for municipal waste treatment technique selection.

The gained assessments are then synthesized based on a structured p, q, r-spherical fuzzy rough model, which involves weighted criteria to facilitate balanced and impartial rankings. This approach perfectly aligns with the waste management environment, enabling a thorough and effective process of decision-making. The outcomes provide actionable information for the choice of green municipal waste treatment methods, keeping with environmental and urban planning objectives. The selected criteria and their description is given in Table 4.

Table 4. Selected criteria and their descriptions.

Criteria	Parameter	Description
C1	Investment requirement	The amount required for establishing and operating the waste management system.
C2	Eco-impact	Environmental effects of the waste management process with regard to air, water, and land.
C3	Method aptness	Degree of suitability of the chosen waste management process.
C4	Processing efficiency	The ratio of waste processed to the input resources used.
C5	Implementation viability	The feasibility of adopting and executing the waste management system.
C6	Handling capacity	The input and capacity of waste that can be effectively managed.
C7	Public approval	The level of acceptance and support from the community and stakeholders.
C8	Sustainability potential	The long-term environmental, economic, and social viability of the system.
C9	Regulatory compliance	Adherence to legal and policy frameworks governing waste management.
C10	Operational complexity	The difficulty level in managing and maintaining the system.

The following criteria are explained in detail:

1) **Investment requirement**

This criterion deals with the financial resources necessary for the growth, implementation, and maintenance of financial feasibility.

2) **Eco-impact**

Eco-impact is an important criterion in this application to examine the environmental consequences, involving resource consumption, emissions, and waste generation to promote ecological sustainability.

3) **Method aptness**

Method aptness is the criterion that evaluates a technique's alignment with the physical, chemical, and biological properties of waste, as well as with the operational, environmental, and regulatory context.

4) **Processing efficiency**

Processing efficiency implies the competence of the process to maximize output by minimizing input resources, time, and energy. High efficiency indicates optimal performance with decreased costs or defects and operational inefficiencies.

5) **Implementation viability**

The implementation of viability consists of technical complexity, infrastructure needs, readiness of skilled personnel, implementation of timelines for a job, and allied risks during the solid waste treatment technique process.

6) **Handling capacity**

Handling capacity is considered a key performance indicator in the selection of waste treatment technologies. It is the ability of a waste treatment technique to handle a specified quantity of waste related to waste composition, moisture content, contaminant concentration, and the degree of waste segregation without compromising operational efficiency.

7) **Public approval**

Public approval deals with the level of acceptance and support from stakeholders and the broader community.

8) **Sustainability potential**

Sustainability potential is a decisive criterion for assessing the contribution of treatment methods to resource conservation, pollution reduction, and sustainable development goals.

9) **Regulatory compliance**

Regulatory compliance evaluates the extent to which the waste treatment techniques adhere to relevant legal, safety, environmental, and quality standards.

10) **Operational complexity** is considered to be an important parameter to analyze how difficult a system is to operate, control, and maintain. It needs continuous monitoring and is often sensitive to input variations. Few factors affecting operational complexity are technology sophistication,scale of operation, infrastructure availability

The $\boldsymbol{C_{p,q,r}}$-SFREBOMWG operator is used to aggregate the decision matrices from the three experts, considering their relative importance. To find the criterion weights, the CRITIC method is used, where intitially, the correlation between the criteria is obtained using Eq (24), and the standard deviations is also obtained using Eq (25). The normalized aggregated decision matrix using $\boldsymbol{C_{p,q,r}}$-SFREBOMG, the score values of the $\boldsymbol{C_{p,q,r}}$-SFRNs of the aggregated decision matrix, correlation coefficient matrix and standard deviation of the criteria is given in Tables 5–8.

Table 5. Normalized aggregated decision matrix using $\boldsymbol{C_{p,q,r}}$-SFREBOMG.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
A1	(0.5312, 0.0269, 0.0523), (0.6701, 0.0484, 0.0059)	(0.8139, 0.0423, 0.0175), (0.5421, 0.0387, 0.0253)	(0.9124, 0.0218, 0.0083), (0.4087, 0.0293, 0.0249)	(0.635, 0.0785, 0.015), (0.585, 0.0218, 0.0415)	(0.6892, 0.0742, 0.0192), (0.6436, 0.0418, 0.028)	(0.8888, 0.0288, 0.0089), (0.4934, 0.0321, 0.0245)	(0.7262, 0.0512, 0169), (0.5141, 0.0376, 0.0326)	(0.9224, 0.0182, 0.0064), (0.4493, 0.0386, 0.0155)	(0.7509, 0.0331, 0.0372), (0.7185, 0.0655, 0.0205)	(0.8736, 0.0311, 0.0237), (0.6624, 0.0779, 0.0355)
A2	(0.7442, 0.0534, 0.0281), (0.7735, 0.04, 0.0191)	(0.7432, 0.0531, 0.0256), (0.683, 0.0572, 0.0236)	(0.7169, 0.0677, 0.0175), (0.6696, 0.0466, 0.0279)	(0.8375, 0.031, 0.0114), (0.4496, 0.0461, 0.021)	(0.7414, 0.047, 0.034), (0.708, 0.0571, 0.0216)	(0.7414, 0.047, 0.034), (0.708, 0.0571, 0.0216)	(0.8914, 0.0285, 0.0088), (0.4962, 0.0319, 0.0244)	(0.8109, 0.0223, 0.0107), (0.283, 0.0698, 0.0078)	(0.7735, 0.0335, 0.0159), (0.3639, 0.0541, 0.0217)	(0.8628, 0.032, 0.0252), (0.6399, 0.081, 0.0377)
A3	(0.5442, 0.0213, 0.0399), (0.4678, 0.0672, 0.0038)	(0.8523, 0.0303, 0.0125), (0.4797, 0.0354, 0.0245)	(0.585, 0.0871, 0.0223), (0.6272, 0.0283, 0.0418)	(0.694, 0.0722, 0.0266), (0.6876, 0.0786, 0.0155)	(0.8296, 0.0302, 0.0113), (0.4531, 0.05, 0.0198)	(0.6911, 0.0664, 0.0211), (0.6609, 0.0509, 0.028)	(0.7115, 0.0498, 0.0446), (0.8076, 0.0785, 0.0169)	(0.7836, 0.034, 0.0158), (0.3699, 0504, 0.0224)	(0.7455, 0.0335, 0.0376), (0.716, 0.0658, 0.021)	(0.5442, 0.0213, 0.0399), (0.4678, 0.0672, 0.0038)
A4	(0.6614, 0.0437, 0.0317), (0.6679, 0.0533, 0.0115)	(0.7472, 0.0643, 0.0171), (0.5216, 0.0349, 0.0416)	(0.7558, 0.0526, 0.0203), (0.5778, 0.0552, 0.0238)	(0.6325, 0.0937, 0.0316), (0.7597, 0.0719, 0.0285)	(0.7499, 0.0604, 0.0261), (0.6785, 0.0446, 0.0267)	(0.6115, 0.0813, 0.0158), (0.5234, 0.0233, 0.0457)	(0.7159, 0.0653, 0.0257), (0.7239, 0.0653, 0.0208)	(0.8658, 0.0197, 0.0086), (0.3731, 0.0564, 0.0113)	(0.7435, 0.0332, 0.0387), (0.711, 0.0674, 0.0206)	(0.601, 0.025, 0.0518), (0.5963, 0.0644, 0.0148)
A5	(0.529, 0.0314, 0.045), (0.669, 0.0526, 0.0071)	(0.745, 0.0476, 0.0193), (0.5098, 0.0648, 0.0182)	(0.6766, 0.068, 0.0211), (0.5487, 0.0508, 0.0266)	(0.66, 0.0786, 0.0384), (0.7524, 0.0628, 0.0233)	(0.8338, 0.0272, 0.0119), (0.4092, 0.0476, 0.0192)	(0.7877, 0.0512, 0.0172), (0.6297, 0.0522, 0.0238)	(0.73, 0.0456, 0.0338), (0.6677, 0.0665, 0.0175)	(0.8039, 0.0227, 0.011), (0.2787, 0.0709, 0.008)	(0.7818, 0.0328, 0.0154), (0.393, 0.0536, 0.0208)	(0.5244, 0.0223, 0.0421), (0.4488, 0.0696, 0.004)

Continued on next page

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
A6	(0.7333, 0.064, 0.0254), (0.6996, 0.0561, 0.0236)	(0.8025, 0.0308, 0.022), (0.5523, 0.0502, 0.024)	(0.8057, 0.0485, 0.0164), (0.5932, 0.0403, 0.0285)	(0.7459, 0.05, 0.024), (0.5648, 0.0446, 0.0272)	(0.6983, 0.0786, 0.0155), (0.5787, 0.023, 0.0433)	(0.7479, 0.042, 0.0308), (0.6879, 0.0674, 0.0158)	(0.6663, 0.0561, 0.0292), (0.6033, 0.052, 0.0319)	(0.8133, 0.0223, 0.0106), (0.2832, 0.0694, 0.0077)	(0.7058, 0.0353, 0.0433), (0.6836, 0.0721, 0.0234)	(0.4662, 0.023, 0.0462), (0.3807, 0.0731, 0.0042)
A7	(0.7291, 0.0699, 0.0253), (0.6795, 0.0612, 0.0273)	(0.7102, 0.0877, 0.0184), (0.5933, 0.0263, 0.0503)	(0.7498, 0.0599, 0.0166), (0.535, 0.0359, 0.0404)	(0.649, 0.0849, 0.0339), (0.7924, 0.0827, 0.0182)	(0.787, 0.0325, 0.0151), (0.3946, 0.0532, 0.0204)	(0.6608, 0.0833, 0.0329), (0.7996, 0.081, 0.0177)	(0.811, 0.0507, 0.015), (0.5974, 0.0514, 0.0236)	(0.8062, 0.0227, 0.0109), (0.2818, 0.0705, 0.0079)	(0.7407, 0.0294, 0.036), (0.6946, 0.0786, 0.0144)	(0.8559, 0.0326, 0.0262), (0.6237, 0.0832, 0.0394)
A8	(0.7275, 0.0776, 0.0219), (0.7932, 0.0396, 0.0201)	(0.7059, 0.0664, 0.0265), (0.7152, 0.0665, 0.0216)	(0.7378, 0.0624, 0.0247), (0.6515, 0.0458, 0.0365)	(0.7617, 0.0495, 0.0213), (0.5905, 0.0433, 0.0281)	(0.7072, 0.0668, 0.0265), (0.7147, 0.0668, 0.0213)	(0.7125, 0.068, 0.0177), (0.6576, 0.0474, 0.0291)	(0.8468, 0.044, 0.0122), (0.5814, 0.0451, 0.0234)	(0.7215, 0.0347, 0.0409), (0.6971, 0694, 0.0225)	(0.8087, 0.0223, 0.0108), (0.2731, 0.0702, 0.0078)	(0.5277, 0.0218, 0.028), (0.4407, 0.0693, 0.004)
A9	(0.5696, 0.0365, 0.0443), (0.7245, 0.0407, 0.0089)	(0.6671, 0.0659, 0.0225), (0.5869, 0.0547, 0.0252)	(0.5994, 0.09, 0.0284), (0.7138, 0.0657, 0.0286)	(0.6842, 0.0724, 0.039), (0.7448, 0.0495, 0.0251)	(0.795, 0.0272, 0.0128), (0.3367, 0.0646, 0.013)	(0.5673, 0.0917, 0.0235), (0.6542, 0.0417, 0.0378)	(0.794, 0.0311, 0.0234), (0.5853, 0.0523, 0.0226)	(0.6523, 0.0844, 0.0336), (0.7948, 0.0822, 0.018)	(0.7974, 0.023, 0.0112), (0.2556, 0.0722, 0.0081)	(0.6298, 0.0379, 0.0343), (0.7352, 0.0436, 0.0073)

Table 6. Score values of the aggregated decision matrix.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
A1	0.8160	0.9020	0.9424	0.8472	0.8711	0.9405	0.8492	0.9584	0.9262	0.9116
A2	0.9373	0.9262	0.8969	0.8925	0.8791	0.9251	0.9432	0.8518	0.8367	0.8987
A3	0.9656	0.9096	0.8511	0.8678	0.8878	0.8804	0.9459	0.8436	0.9219	0.7539
A4	0.8524	0.8715	0.8671	0.8924	0.9249	0.8175	0.8994	0.9002	0.9180	0.7841
A5	0.8151	0.8454	0.8488	0.9081	0.8825	0.9056	0.8997	0.8469	0.8459	0.7442
A6	0.9103	0.8937	0.9167	0.8702	0.8805	0.9169	0.8519	0.8534	0.8843	0.7180
A7	0.8997	0.8968	0.8715	0.8947	0.8494	0.9045	0.9071	0.8486	0.9083	0.8899
A8	0.8978	0.8895	0.9173	0.8882	0.8898	0.8895	0.9303	0.8991	0.8496	0.7438
A9	0.8543	0.8562	0.8569	0.9341	0.8467	0.8391	0.8983	0.8976	0.8410	0.8819

Table 7. Correlation coefficient matrix of criteria.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
C1	1	0.7	-0.08	-0.2	0.03	0.08	0.61	-0.59	0.3	0.17
C2	0.7	1	0.48	-0.6	-0	0.51	0.26	-0.58	0.28	0.23
C3	-0.1	0.48	1	-0.6	0	-0.6	-0.45	0.16	0.11	0.16
C4	-0.2	-0.63	-0.64	1	-0.3	-0.6	-0.31	-0.27	-0.69	-0.1
C5	0.03	-0.01	0	-0.3	1	-0.1	0.13	0	0.22	0.1
C6	0.08	0.51	-0.59	-0.6	-0.1	1	-0.34	0.13	0.06	-0.4
C7	0.61	0.26	-0.45	-0.3	0.13	-0.3	1	-0.18	0.25	0.07
C8	-0.6	-0.58	0.16	-0.3	0	0.13	-0.18	1	-0.03	0.2
C9	0.3	0.28	0.11	-0.7	0.22	0.06	0.25	-0.03	1	0.53
C10	0.17	0.23	0.16	-0.1	0.1	-0.4	0.07	0.2	0.53	1

Table 8. Standard deviations of the criteria.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
σ_j	0.0523	0.0258	0.0340	0.0250	0.0232	0.0401	0.0349	0.0389	0.0381	0.0795

The criteria index is standardized to find the criteria weights shown in Tables 9 and 10 using Eq (27).

Table 9. Index of criteria.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
c_j	0.4487	0.1861	0.2816	0.2957	0.2254	0.3548	0.3226	0.3591	0.3455	0.6887

Table 10. Weights of criteria.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
we_j	0.1279	0.0530	0.0803	0.0843	0.0643	0.1011	0.0919	0.1024	0.0985	0.1963

Next, the negative ideal solution for each criterion using Eq (28) is computed and given in the Table 11 to determine the optimal waste treatment alternative.

Table 11. Negative ideal solution (NIS) values of each criterion.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
NIS	(0.529, 0.0776, 0.0523), (0.4678, 0.0672, 0.0273)	(0.6671, 0.0877, 0.0265), (0.4797, 0.0665, 0.0503)	(0.585, 0.09, 0.0284), 0.4087, 0.0657, 0.0418)	(0.6325, 0.0937, 0.039), (0.4496, 0.0827, 0.0415)	(0.6892, 0.0786, 0.034), (0.3367, 0.0668, 0.0433)	(0.5673, 0.0917, 0.034), (0.4934, 0.081, 0.0457)	(0.6663, 0.0653, 0.0446), (0.4962, 0.0785, 0.0326)	(0.6523, 0.0844, 0.0409), (0.2787, 0.0822, 0.0225)	(0.7058, 0.0353, 0.0433), (0.2556, 0.0786, 0.0234)	(0.4662, 0.0379, 0.0518), (0.3807, 0.0832, 0.0394)

Normalized hamming and Euclidean distances for each alternative are obtained using Eqs (29) and (30) and are given in Table 12.

Table 12. Normalized hamming and Euclidean distances of each alternative from the NIS values.

	NHAM distance	NECL distance
A1	0.0546	0.0849
A2	0.0503	0.0873
A3	0.0305	0.0635
A4	0.0341	0.0714
A5	0.0272	0.0615
A6	0.0314	0.0656
A7	0.0473	0.0840
A8	0.0363	0.0738
A9	0.0385	0.0730

The relative assessment matrix is given in Table 13 using Eq (31), which inculcates the normalized hamming and Euclidean distances of the alternatives. Figure 4 gives the graphical representation of normalized hamming and Euclidean distances. The appraisal scores for each alternative from the relative assessment matrix are computed and given in Table 14 using Eq (32).

Table 13. Relative assessment matrix of the alternatives.

	A1	A2	A3	A4	A5	A6	A7	A8	A9
A1	0.0000	0.0044	0.0242	0.0205	0.0275	0.0233	0.0074	0.0183	0.0161
A2	-0.0044	0.0000	0.0198	0.0162	0.0231	0.0189	0.0030	0.0140	0.0118
A3	-0.0241	-0.0198	0.0000	-0.0036	0.0033	-0.0009	-0.0168	-0.0058	-0.0080
A4	-0.0205	-0.0161	0.0036	0.0000	0.0069	0.0028	-0.0132	-0.0022	-0.0044
A5	-0.0274	-0.0231	-0.0033	-0.0069	0.0000	-0.0042	-0.0201	-0.0091	-0.0113
A6	-0.0233	-0.0189	0.0009	-0.0028	0.0042	0.0000	-0.0159	-0.0049	-0.0071
A7	-0.0074	-0.0030	0.0168	0.0132	0.0201	0.0159	0.0000	0.0110	0.0088
A8	-0.0183	-0.0140	0.0058	0.0022	0.0091	0.0049	-0.0110	0.0000	-0.0022
A9	-0.0161	-0.0118	0.0080	0.0044	0.0113	0.0071	-0.0088	0.0022	0.0000

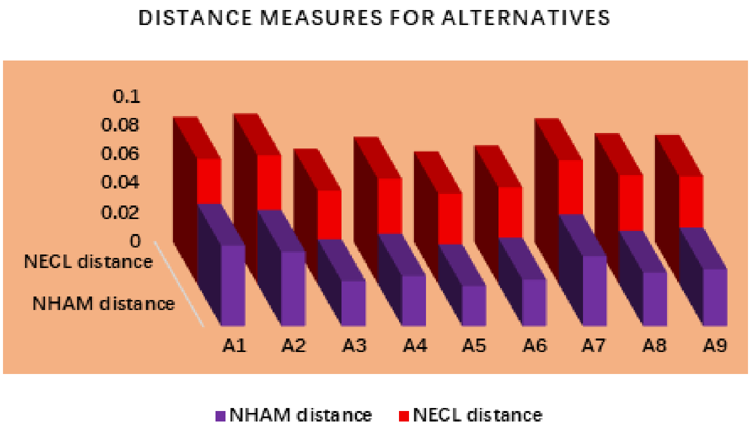


Figure 4. Graphical representation of normalized hamming and Euclidean distances.

Table 14. Appraisal score and ranking of the alternatives.

Alternatives	Appraisal score	Rank
A1	0.1417	1
A2	0.1023	2
A3	-0.0758	8
A4	-0.0431	6
A5	-0.1055	9
A6	-0.0679	7
A7	0.0755	3
A8	-0.0234	5
A9	-0.0036	4

From the appraisal scores, the optimal waste treatment technique from the nine alternatives is A1, which is upcycling/upgradation, and the worst technique is dumpfill/landfill. The rankings of the respective alternatives are visually represented in Figure 5.

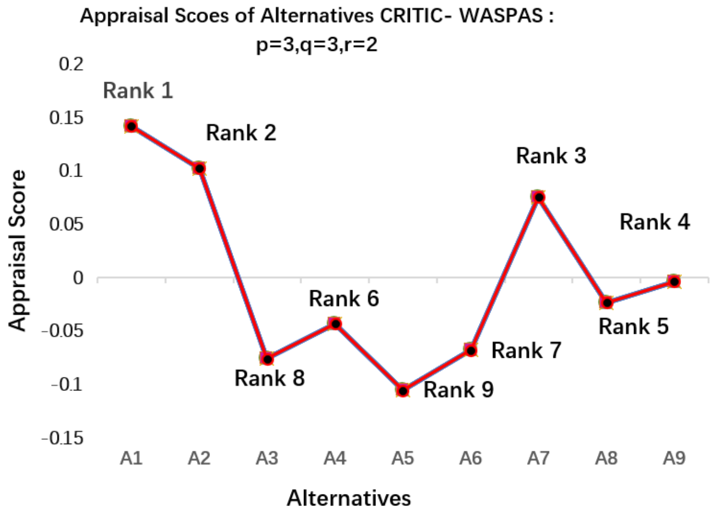


Figure 5. Appraisal scores of alternatives.

6. Managerial implications

In this study, we propose a powerful MCDM model based on p, q, r-SFR Einstein Bonferroni aggregation operators with important managerial implications in diverse fields. Incorporating confidence levels in the spherical FSs makes the decision-maker capable of handling different levels of certainty, doubt, and vagueness regularly faced when making strategic choices. In the field of environmental management, and especially in selecting waste treatment techniques, such as those illustrated in this case study, environmental managers and planners can compare alternatives not only based on quantitative factors such as costs and efficiency but also on qualitative factors such as sustainability, public acceptance, and adherence to regulations. The MCDM model has wide implications across sectors of industry apart from waste reduction. Hospital administration can apply the model to select medical technology or treatment procedures in the health sector by weighing specialist advice against evidence-based practice, particularly in risky and uncertain cases. In manufacturing, the proposed method can help in technology adoption, machinery purchases, or quality improvement initiatives where multiple criteria have to be assessed under data uncertainty. Here, we introduce a new methodological innovation that is useful to apply practically toward strategic planning and operation decision-making within industries.

6.1. Sensitivity analysis

Sensitivity analysis is a necessary component of MCDM processes and provides a means of analyzing how parameter variations will affect the outputs of the model. This facilitates ensuring the firmness and quality of the decision-making system. For the model of p, q, r-spherical fuzzy rough MCDM, the parameters p, q, and r determine the ability of the model to deal with fuzziness and uncertainty in an appropriate manner.

Different confidence values for various combinations of criteria and alternatives are given by each expert. Based on different confidence values, the aggregated decision matrix is created followed by the correlation matrix, standard deviation of the criteria, index of the criteria and weights of the criteria. It can be noted that weights of the criteria are obtained from the final decision matrix. Hence, it may not be possible to vary the weights of the criteria. Additionally, the confidence value is fixed by the expert for the proposed case study and cannot vary. Due to these reasons, we performed the sensitivity analysis by varying the parametric values.

Here, in the model, p and q are variable parameters, and r is the maximum of p and q and is adjusted dynamically depending on their values. The initial parameter setting of p=3, q=2, and r=3 serves as the default for sensitivity analysis.

Sensitivity analysis considers variation in one parameter whereas another parameter is kept as constant. Upon setting q as a constant and changing p, minor shifts are noted among rankings, signaling moderate sensitivity. When setting p as a constant and q as varied, the rankings do not change, emphasizing the insensitivity of the model. Ranking of the alternatives for different combinations of parameters is given in Table 15. Hence, it can be concluded that p influences the rankings more strongly than q, even though the differences between them are small. The key point here is that optimal and worst solutions do not vary across all possible scenarios, thus emphasizing the robustness of the decision-making process of the model.

Table 15. Ranking of alternatives for combinations of parameters.

p	q	r	Ranking
3	2	3	1>2>7>9>8>4>6>3>5
3	3	3	1>2>7>9>8>4>6>3>5
3	4	4	1>2>7>9>8>4>6>3>5
3	5	5	1>2>7>9>8>4>6>3>5
4	2	4	1>2>7>9>8>6>4>3>5
5	2	5	1>2>7>8>6>9>4>3>5

The regular identification of optimal and worst solutions over different parameter configurations is an indicator of the reliability and strength of the p, q, r-spherical fuzzy rough MCDM framework. The definition of r is the max of p, and this introduces a component of adaptability, as well as further enhancing the model's reliability.

The findings of this sensitivity analysis reveal relevant information about how the p, q, r-spherical fuzzy rough MCDM model performs under various scenarios. This illustrates the ability of the framework to effectively identify the optimal and poorest solutions while highlighting the importance of p and q as parameters affecting the output of the model. These results enhance the credibility of the approach and confirm its value in dealing with complex decision-making scenarios with factors in the real world. By gaining in depth knowledge about the parameters, researchers and practitioners can optimize the framework more efficiently for a particular application, achieving its usability in various contexts. This analysis makes a valuable contribution to the validation and fine-tuning of the decision-making method, opening the door to its application in solving real-world problems. Figure 6 gives bump chart depicting ranking of alternatives for parameter combinations.

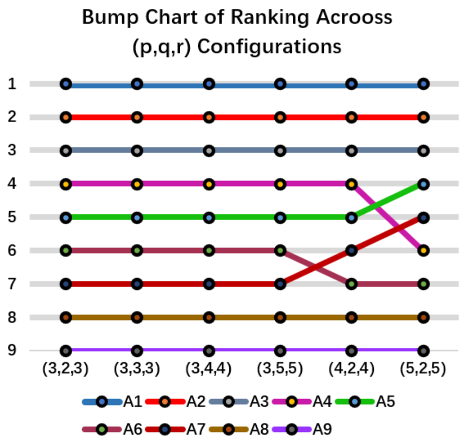


Figure 6. Bump chart depicting ranking of alternatives for parameter combinations.

6.2. Comparative analysis

A comparison was made to assess the performance of the developed aggregation model compared to other MCDM approaches. The findings illustrate that the confidence levels based p, q, r-spherical fuzzy rough Einstein Bonferroni operators yield more stable and strong decision results, particularly in situations involving high uncertainty and vague information. Ranking of alternatives for MCDM techniques for different combinations of p,q, and r is computed in Table 16.

Table 16. Ranking of alternatives for MCDM techniques for combinations of parameters.

Technique	p	q	r	Alternative ranking	Technique	p	q	r	Alternative ranking
CRITIC-CODAS	3	2	3	1>2>7>9>8>4>6>3>5	CRITIC-WASPAS	3	3	3	2>1>7>9>8>4>6>3>5
	3	3	3	1>2>7>9>8>4>6>3>5		3	4	4	2>1>7>9>8>4>6>3>5
	3	4	4	1>2>7>9>8>4>6>3>5		3	5	5	2>1>7>9>8>4>6>3>5
	3	5	5	1>2>7>9>8>4>6>3>5		4	2	4	2>7>1>8>9>4>6>5>3
	4	2	4	1>2>7>9>8>6>4>3>5		5	2	5	2>1>7>8>9>4>6>5>3
	5	2	5	1>2>7>8>6>9>4>3>5					
	3	2	3	1>2>7>9>8>4>6>3>5					
CRITIC-TOPSIS	3	3	3	1>2>7>9>4>8>6>3>5					
	3	4	4	1>2>7>9>4>8>6>3>5					
	3	5	5	1>2>7>9>4>8>6>3>5					
	4	2	4	1>2>7>8>4>6>9>5>3					
	5	2	5	1>2>7>8>6>4>9>5>3					
	3	2	3	2>1>7>8>9>4>6>3>5					

The comparative ranking table indicates the strength and resilience of the CRITIC-CODAS method over CRITIC-TOPSIS and CRITIC-WASPAS for different values of parameters p, q, r. One can see that CRITIC-CODAS is consistent in having the same rank pattern for all combinations of parameters, with the alternatives A1, A2, A7, and A9 as the top-ranking alternatives, and A5 as the worst. This illustrates CODAS's strength in managing fuzzy parameter changes without significantly changing the ranking order, a key requirement for sound decision-making in uncertain settings. In contrast, CRITIC-TOPSIS exhibits significant variability, particularly among the mid- and lower-ranked alternatives, which reflects its sensitivity to parameter changes. CRITIC-WASPAS does slightly worse than TOPSIS in being unstable. Additionally, CODAS is flexible with parameter settings without compromising ranking reliability, which makes it especially suitable for applications where decision-makers prefer to adjust the contributions of membership, non-membership, and hesitation. Thus, from the results observed, CRITIC-CODAS stands out as the most consistent and reliable method among the three, particularly in situations requiring high sensitivity to uncertain and imprecise data. Comparison of parameters p,q,r and the confidence level for the existing and proposed fuzzy approaches is given in Table 17.

Table 17. Comparison of concepts for the existing and the proposed fuzzy approaches.

Approaches	MD	MD	NMD	p	q	r	Approx.	Conf. Level
PFS	Yes	Yes	Yes	No	No	No	No	No
SFS	Yes	Yes	Yes	Yes	No	No	No	No
q–SFS	Yes	Yes	Yes	Yes	Yes	No	No	No
r–SFRS	Yes	Yes	Yes	Yes	No	Yes	No	No
p, q, r–SFS	Yes	Yes	Yes	Yes	Yes	Yes	No	No
$C_{p,x,r}$–SFS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
$C_{p,x,r}$–SFRS	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

1) In contrast to conventional models such as PFS and SFS that consider membership and non-membership alone, p, q, r-SFRS captures all three aspects-membership (MD), non-membership (NMD), and neutral membership (Neu-MD). This triple-valued modelling enables a much richer and more detailed representation of human judgments and uncertain data.

2) The p, q, and r parameters enable control of the relative importance of each fuzzy component. For MCDM applications and aggregation operators, the parameters $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ describe decision attitudes, where higher $\boldsymbol{p}$ values signify optimism toward benefits, higher $\boldsymbol{q}$ values represent risk aversion, and higher $\boldsymbol{r}$ values denote caution occurring from informational uncertainty.

3) In contrast to earlier fuzzy models, the p, q, r-SFRS model incorporates RS theory, making it possible to utilize lower and upper approximations in addressing fuzzy or partially known information. This imparts a robust aspect to the decision-making model, which can better address vagueness and granularity within data.

4) The development of $\boldsymbol{C_{p,q,r}}$-SFREBOMWG and $\boldsymbol{C_{p,q,r}}$-SFREBOMWA operators places confidence weights on each expert, reflecting their reliability and knowledge depth. This provides credibility and reliability to group decision-making via proper weighting of expert opinions.

5) Einstein and Bonferroni mean-based aggregation implies a nonlinear, interaction-sensitive approach that extends additive averaging. The operations more appropriately represent the synergy or weakening effects between criteria, resulting in more realistic and context-sensitive ratings.

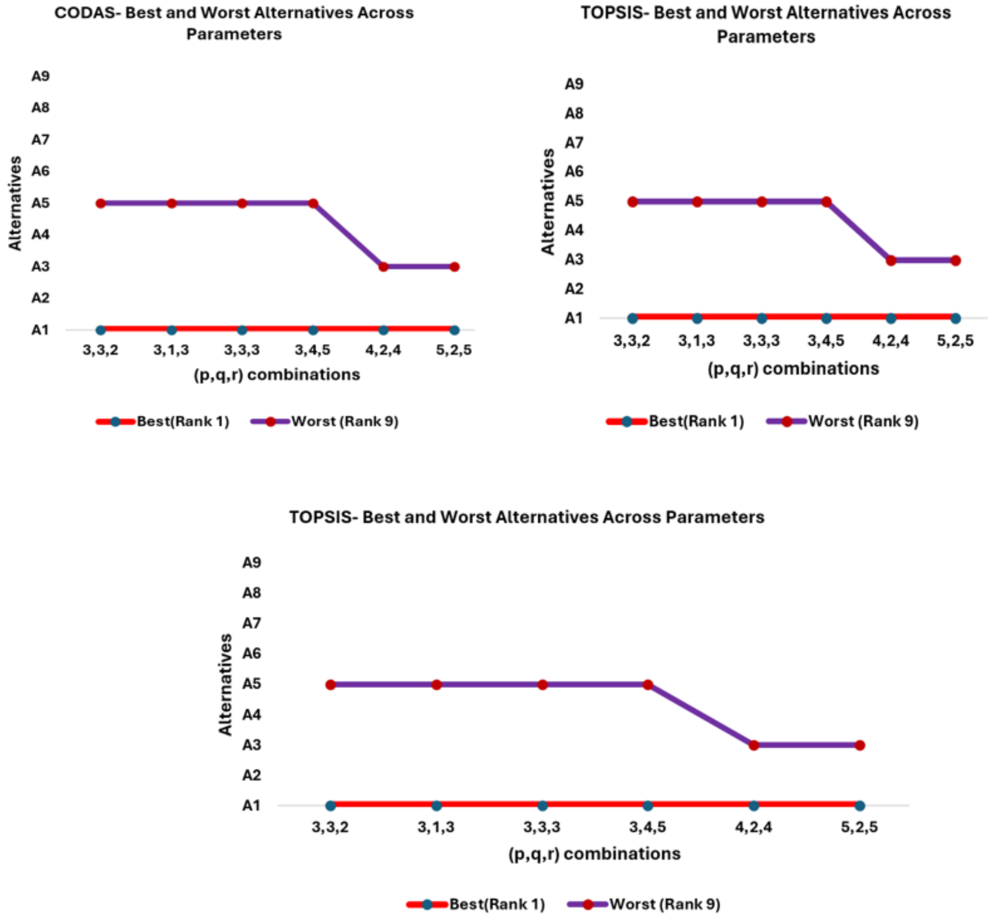
6) The p, q, r-SFRS framework generalizes older fuzzy systems like q-SFRS and SFRS such that it introduces more capabilities, including parameter tunability, rough approximation, and dealing with confidence, making it more robust and extensive than any FS framework.

7) Coupled with the CODAS approach, the framework offers higher discrimination among alternatives even when scores are near one another, something that most conventional TOPSIS and

WASPAS tend not to attain. The use of CODAS based on Euclidean and negative distances serves to prevent rank reversals and yields more stable results.

8) Balanced weighting through CRITIC-CODAS is the integration of CRITIC (objective weighting) and CODAS (subjective dominance-based ranking), which ensures a well-balanced evaluation process.

This provides a balance between information-driven data and expert intuition, leading to equitable, stable, and understandable decision outcomes. Figure 7 explains the best and worst alternatives across parameters for techniques.


CODAS- Best and Worst Alternatives Across Parameters

TOPSIS- Best and Worst Alternatives Across Parameters

TOPSIS- Best and Worst Alternatives Across Parameters

Figure 7. Best and worst alternatives across parameters for techniques.

6.3. Advantages

The suggested methodology presents a thorough and adaptable decision-making model with several benefits that dramatically enhance its performance in real, uncertain environments.

1) The designs of two high-level aggregation operators, $\boldsymbol{C}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$-SFREBOMWG and $\boldsymbol{C}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}$-SFREBOMWA, are purposely conceived for dealing with p, q, r-SFNs and the confidence levels of experts. These operators enhance the expressiveness and flexibility of the pattern and are, hence, appropriate for realistic-life decision-making circumstances accompanied by vagueness and subjectivity.

2) Owing to its nonlinear nature, the Einstein mean can be used to define interactive relationships between criteria, particularly where one criterion's influence is conditional upon the others. The Bonferroni mean operation, however, enables the model to incorporate pairwise relationships between inputs without losing intercriteria correlations and producing more context-sensitive aggregation. Combining these means leads to a richer and more flexible mathematical form that enhances the fidelity and balance of expert information aggregation.

3) The proposed work is integrated into a robust MCDM algorithm that combines the best of the CRITIC (for weighing the objective criteria based on contrast intensity and conflict) and CODAS (for alternative evaluation in a stable manner via Euclidean and Hamming distances) techniques. The results of the hybrid model enhance the overall objectivity and consistency of the work.

4) The model also has another major strength of incorporating parameter-based tailings. By adjusting the p, q, and r values (max (p, q)) through value modification, users can dynamically control the contributions of membership, non-membership, and hesitation degrees.

5) Finally, the combination of p, q, r-spherical fuzzy RS theory with confidence-aware aggregation fills an important gap in decision-making models.

6.4. *Limitations*

Acknowledging these limitations is key to understanding the scope and avenues for future expansion.

1) First, the use of the proposed method may be limited to some fields or situations in which decisions are made. Although the model is flexible, its performance could depend on the type of problem and the degree to which the decision-makers are aware of FS theories.

2) As in the case of most analytical techniques, the method is developed under some assumptions and simplifications with the intention of economizing the analysis. When unavoidable, these assumptions may not always hold in real situations and may detract from the realism and generalizability of the findings.

3) While the model provides variable parameters p, q, and r for enabling tailoring membership, non-membership, and hesitation values, it is not simple to set them to optimal or best values. This would necessitate additional investigations on parameters for decision-making in line with the problem type.

4) The demonstration and success of the suggested framework rely mostly on a case study with few alternatives and criteria. Although this is considered good validation, it might not reflect the model behavior for large or complex decision-making problems.

5) The computational complexity of the method increases when more advanced operations, such as the Einstein and Bonferroni means, are incorporated and the spherical fuzzy RSs are utilized. These factors make the model more expressive, but they may involve greater computational requirements, especially in the case of large applications or real-time environments for decision-making.

7. Conclusions

In this investigation, a strong and adaptable decision-making model is proposed by introducing two new aggregation operators, $\boldsymbol{C_{p,q,r}}$-SFREBOMWG and $\boldsymbol{C_{p,q,r}}$SFREBOMWA, which are meant to handle p, q, r-SFNs with the introduction of decision experts' confidence levels. These operators are developed in addition to the Einstein and Bonferroni mean operations, which have a profoundly positive effect on the process of aggregation by capturing nonlinear interdependencies and keeping pairwise interdependencies of expert assessments intact. A significant contribution of this work is the incorporation of spherical fuzzy RS theory, which enables the proposed framework to handle not only fuzzy uncertainty but also rough approximations, successfully tackling vagueness and boundary-based ambiguity in expert opinions. This two-layer modeling enables the framework to model imprecise data more comprehensively, thereby enhancing the decision-making process in highly uncertain settings. The suggested operators are implemented within a hybrid MCDM procedure via the CRITIC technique for criterion weighting and CODAS for alternative ranking. Notably, CODAS features a greater discriminatory capacity than usual alternatives such as TOPSIS and WASPAS since CODAS considers Euclidean and Hamming distances from the negative-ideal solution in ranking the alternatives. The flexibility of the model is also illustrated by adding adjustable parameters p, q, and r, thereby permitting decision-makers to regulate the relative weights of membership, non-membership, and hesitation degrees according to their risk attitudes or the nature of the problem. A sensitivity analysis validates the robustness of the model against parameter variation, and a comparative analysis reveals its superiority over current methods. Moreover, the appraisal score and ranking of the alternative is computed. The results reveal, from the appraisal scores, that the optimal waste treatment technique from the nine alternatives is A1, which is upcycling/upgradation, and the worst technique is dumpfill/landfill. In sensitivity analysis, setting q as a constant and changing p leads to minor shifts among rankings, signaling moderate sensitivity on the part of the model. When setting p as a constant and q as varied, the rankings do not change, emphasizing the insensitivity of the model. The above remarks imply that p influences the rankings more strongly than q, while the differences between them are small. Additionally, it can be seen that CRITIC-CODAS is consistent in having the same rank pattern for all combinations of parameters, with the alternatives A1, A2, A7, and A9 as the top-ranking alternatives and A5 as the worst. In contrast, CRITIC-TOPSIS exhibits significant variability, particularly among the mid- and lower-ranked alternatives, which reflects its sensitivity to parameter changes. CRITIC-WASPAS does slightly worse than TOPSIS in being unstable, especially in the case of rank 1. Moreover, the weights obtained by the CRITIC method from the decision matrix, obtained using the $\boldsymbol{C_{p,q,r}}$-SFREBOMWG or $\boldsymbol{C_{p,q,r}}$-SFREBOMWA operator framework, shows the ability to identify the optimal solution, thereby increasing the credibility in complex decision-making scenarios. Thus, from the results, CRITIC-CODAS stands out as the most consistent and reliable method among the three, particularly in situations requiring high sensitivity to uncertain and imprecise data. In conclusion, the proposed method effectively integrates confidence-aware aggregation, rough approximation processing, and dominance-based ranking to provide a robust and flexible solution for contemporary decision-making problems. In the future, the proposed operators can be extended to a more complex p, q, r spherical rough fuzzy environment. Distance measures and entropy measures can also be used for comparisons other than NHAM and NECL. Additionally, the proposed method can be integrated with a complex decision-making situation that involves numerous alternatives and diverse criteria. The framework can be further applied to design evaluations, site selections, policy planning, and decision-making in soft-set contexts areas that require high uncertainty management and expert collaborations.

Author contributions

O. S. Deepa: conceptualization, supervision, investigation visualization, writing–review & editing the manuscript; Nandana Vasudevan: writing, investigation, data collection and validation of the manuscript. All authors collaborated on writing, reviewing, and editing the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors affirm that there are no conflicts of interest.

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