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Research article

Robust optimal reinsurance and investment with inflation risk: A game-theoretic approach and explicit solutions

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Abstract: This paper investigates robust optimal reinsurance and investment for an insurer facing both inflation risk and model ambiguity. The surplus is described by a diffusion approximation of the Cramér–Lundberg model, and purchasing-power risk is incorporated through a mean-reverting inflation factor. Specifically, the log-inflation index is modeled by an Ornstein–Uhlenbeck process, which yields a two-dimensional real-wealth system and captures state-dependent inflation effects absent under geometric Brownian inflation. Model ambiguity is represented by an adversarial probability distortion with an entropy penalty, allowing the adversary to distort the insurance, financial, and inflation channels in a unified framework. The decision problem is formulated as a zero-sum stochastic differential game. We establish a rigorous duality between the entropy-penalized robust formulation and a risk-sensitive control problem, and derive the associated Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation. Our main theoretical contribution is a tractable characterization of the robust optimal strategies in a finite-horizon setting with non-zero interest rates. By an exponential-quadratic transformation, the nonlinear HJBI equation is reduced to a system of ordinary differential equations of Riccati type, which yields feedback-form optimal reinsurance and investment rules featuring an explicit inflation-hedging component and a novel mean-reversion hedging demand. We further show that the solution collapses to the geometric-Brownian-inflation benchmark as the mean-reversion speed tends to zero. Numerical experiments illustrate that ambiguity aversion amplifies effective risk aversion, strengthens reinsurance demand, and induces a pronounced flight-to-safety effect, with substantially different hedging behavior under mean-reverting versus non-mean-reverting inflation.

Keywords: robust control; stochastic differential game; inflation risk; risk-sensitive control; Hamilton–Jacobi–Bellman–Isaacs equation; reinsurance–investment optimization
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1. Introduction

Reinsurance–investment optimization is a central topic in actuarial science and insurance risk management. In realistic environments, insurers must jointly manage underwriting risk, financial-market risk, and macroeconomic risk. Among macroeconomic factors, inflation is particularly important because long-horizon liabilities, claim severities, and operational costs are naturally stated in real (purchasing-power) terms. A broad literature investigates reinsurance–investment decisions under stochastic economic factors such as interest rates, stochastic volatility, inflation, and exchange rates; see, for example, [1–3] and the references therein.

In parallel, robust control and ambiguity-aware optimization have become influential frameworks for decision-making under model misspecification. Foundational robust and risk-sensitive methodologies can be traced back to [4] and [5]. In the insurance context, robust reinsurance–investment models under ambiguity have been extensively studied, including classical robust formulations and stochastic volatility settings [6–8]. Recent developments include inside information [9], ambiguity aversion and related behavioral or structural extensions [10–12], and game-theoretic interactions among insurers/reinsurers [13–15]. Furthermore, data-driven and machine-learning-based approaches have also emerged; see, for example [16]. Constraint- and correlation-ambiguity settings are studied in, for instance, [17].

Motivation and gap. Despite the extensive literature above, the joint treatment of mean-reverting stochastic inflation and model ambiguity in a tractable robust reinsurance–investment framework remains underdeveloped. Existing works that incorporate inflation typically assume either a non-mean-reverting inflation index (often a geometric Brownian motion) or focus on stochastic inflation without an adversarial probability distortion. Conversely, many robust reinsurance–investment models emphasize ambiguity in claims and/or financial markets while simplifying the inflation mechanism (e.g., deterministic inflation or inflation that does not interact with the ambiguity channel). These simplifications may lead to analytically convenient formulations, but they cannot capture the empirically observed mean-reverting behavior of inflation and may severely bias inflation-hedging demands and reinsurance incentives in real terms.

From a practical standpoint, ignoring the interaction between inflation dynamics and model ambiguity can lead to systematically biased decisions. In particular, strategies optimized under nominal wealth or under a misspecified inflation channel may understate long-horizon liability coverage risk in real terms, resulting in insufficient reinsurance protection and mispricing of reinsurance premia when inflation shocks are persistent. Moreover, misspecified inflation–financial dependence can induce spurious inflation-hedging demands, leading to excessive risky-asset exposure precisely in high-inflation regimes. These considerations motivate a unified framework that simultaneously accounts for (i) inflation risk with mean reversion and (ii) model ambiguity in a mathematically rigorous and tractable way.

Our model and main contributions. To address this gap, we develop a robust reinsurance–investment model in which nominal wealth is explicitly adjusted by a stochastic inflation index, and model ambiguity is represented through an adversarial probability distortion with an entropic penalty. A key modeling feature is that we allow inflation to be mean-reverting: We model the log-inflation factor by an Ornstein–Uhlenbeck process, which leads to a two-dimensional controlled diffusion for real wealth and inflation. The resulting robust optimization problem is formulated as a zero-sum stochastic differential game between the insurer and an adversary who distorts the probability measure.

Our main contributions are summarized as follows:

- **(i)** *Two-dimensional real-wealth dynamics under mean-reverting inflation.* We derive the exact inflation-adjusted real-wealth dynamics when the inflation factor is mean-reverting. This yields state-dependent drift terms and inflation-induced cross effects that are absent under non-mean-reverting benchmarks, and clarifies how inflation persistence interacts with the insurer's overall risk exposure.
- **(ii)** *Entropy-penalized robustness and risk-sensitive representation.* We establish a rigorous duality between the entropy-penalized robust control problem and an equivalent risk-sensitive control problem under the reference measure. This connection provides a transparent economic interpretation of ambiguity aversion as an endogenous amplification of effective risk aversion and links our framework to the risk-sensitive literature.
- **(iii)** *Tractable solution via ODE/Riccati reduction and a novel hedging demand.* We derive the associated Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation in a two-dimensional state space and show that it admits a tractable exponential-quadratic structure. By a suitable transformation, the nonlinear HJBI equation is reduced to a system of ordinary differential equations (of Riccati type), which yields feedback-form optimal reinsurance and investment strategies. In particular, the optimal investment rule features not only an inflation-hedging component but also a mean-reversion hedging demand driven by the inflation state.
- **(iv)** *Benchmark nesting and limiting consistency.* We prove that the mean-reverting inflation model nests the geometric-Brownian inflation benchmark as the mean-reversion speed tends to zero, and the derived strategies collapse to the corresponding benchmark policies. This limiting property provides a clean internal validation of the model and the solution.
- **(v)** *Rigorous verification and numerical implications.* We prove a verification theorem that confirms the optimality of the candidate strategies. The validity of the measure change is justified via a rigorous martingale argument under the derived feedback controls. Numerical experiments demonstrate that ambiguity aversion induces a pronounced flight-to-safety effect: it substantially reduces risky-asset exposure, strengthens reinsurance demand, and leads to materially different inflation-hedging behavior under mean-reverting versus non-mean-reverting inflation.

The remainder of the paper is organized as follows. Section 2 introduces the financial market, the surplus process, and the mean-reverting inflation factor, derives the real-wealth dynamics, and specifies the admissible controls and the entropy-based ambiguity set. Section 3 formulates the robust reinsurance–investment problem as a zero-sum stochastic differential game, establishes the risk-sensitive representation, and derives the associated HJBI equation. Section 4 presents the feedback-form optimal strategies via the ODE/Riccati reduction and proves the verification theorem. Section 5 reports numerical experiments illustrating the effects of ambiguity aversion and inflation mean reversion on real surplus and ruin risk, and Section 6 concludes with a discussion of possible extensions.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, where $T < \infty$ is a fixed time horizon. Let $W(t) = (W_R(t), W_S(t), W_I(t))^\top$ be a three-dimensional

Brownian motion under $\mathbb{P}$. We assume that W_R is independent of (W_S, W_I) and that (W_S, W_I) are correlated with

$$d\langle W_S, W_I\rangle_t = \rho_{SI}\,dt, \qquad \rho_{SI} \in [-1, 1].$$

2.1. The financial market and mean-reverting inflation dynamics

The financial market consists of a risk-free asset $B(t)$ and a risky asset $S(t)$ with dynamics

$$dB(t) = rB(t)\,dt, \qquad \frac{dS(t)}{S(t)} = \mu\,dt + \sigma\,dW_S(t), \tag{2.1}$$

where $r \geq 0$ is the risk-free rate, $\mu > r$ is the expected return, and $\sigma > 0$ is the volatility.

To capture purchasing-power risk with persistence, we model log-inflation by an Ornstein–Uhlenbeck (OU) process. Let $Y(t) := \ln I(t)$, where $I(t)$ is the inflation index. We assume

$$dY(t) = \kappa(\bar{y} - Y(t))\,dt + \sigma_I\,dW_I(t), \qquad Y(0) = \ln I_0, \tag{2.2}$$

where $\kappa > 0$ is the speed of mean reversion, $\bar{y} \in \mathbb{R}$ is the long-run mean, and $\sigma_I > 0$ is the inflation volatility. By Itô's formula, $I(t) = e^{Y(t)}$ satisfies

$$\frac{dI(t)}{I(t)} = \Big(\kappa(\bar{y} - Y(t)) + \frac{1}{2}\sigma_I^2\Big)\,dt + \sigma_I\,dW_I(t), \qquad I(0) = I_0 > 0. \tag{2.3}$$

Remark 2.1 (Benchmark nesting)**.** *The mean-reverting specification* (2.2) *nests the geometric-Brownian-inflation benchmark as a limiting case. Indeed, when $\kappa \to 0$, the drift term in* (2.3) *becomes constant and $I(t)$ reduces to a geometric Brownian motion. This limiting relation will be used later to validate that our optimal strategies collapse to the benchmark policies when mean reversion vanishes.*

2.2. The surplus process and real wealth dynamics

We model the insurer's underwriting risk by a diffusion approximation of the Cramér–Lundberg model in real terms. Let $\lambda > 0$ be the claim arrival intensity, $m > 0$ the mean claim size, and $\sigma_R > 0$ the diffusion volatility. Under the standard actuarial pricing convention with safety loading $\eta_R > 0$, the premium rate (in real terms) is $c = (1 + \eta_R)\lambda m$.

2.2.1. Inflation indexation of underwriting cash flows

To maintain consistency between proportional reinsurance $u(t) \in [0, 1]$ and real-wealth modeling, we assume that underwriting cash flows are indexed by $I(t)$ in nominal terms. Equivalently, the real underwriting gain under proportional reinsurance has drift $\eta_R\lambda m\,u(t)$ and diffusion $u(t)\sigma_R$. This is standard when claims and premiums are stated in real units and converted into nominal units via the inflation index.

2.2.2. Nominal wealth dynamics

Let $X(t)$ denote the nominal wealth, and let $\pi(t)$ be the nominal amount invested in the risky asset. With proportional reinsurance $u(t) \in [0, 1]$, the nominal wealth evolves as

$$dX(t) = \Big[rX(t) + (\mu - r)\pi(t) + \eta_R\lambda m\,I(t)\,u(t)\Big]dt + u(t)\sigma_R\,I(t)\,dW_R(t) + \pi(t)\sigma\,dW_S(t). \tag{2.4}$$

2.2.3. Real wealth dynamics

Define the real wealth by $\tilde{X}(t) := X(t)/I(t)$ and the real risky investment by $\tilde{\pi}(t) := \pi(t)/I(t)$. Using Itô's formula together with (2.3) and the correlation between (W_S, W_I), we obtain

$$\begin{aligned} d\tilde{X}(t) = \Big[\tilde{X}(t)\Big(r - \mu_I(Y(t)) + \sigma_I^2\Big) + \tilde{\pi}(t)\Big(\mu - r - \sigma\sigma_I\rho_{SI}\Big) + \eta_R\lambda m\, u(t)\Big]dt \\ + u(t)\sigma_R\, dW_R(t) + \tilde{\pi}(t)\sigma\, dW_S(t) - \tilde{X}(t)\sigma_I\, dW_I(t), \end{aligned} \tag{2.5}$$

where we use the shorthand

$$\mu_I(Y) := \kappa(\bar{y} - Y) + \frac{1}{2}\sigma_I^2.$$

In particular,

$$r - \mu_I(Y) + \sigma_I^2 = r - \kappa(\bar{y} - Y) + \frac{1}{2}\sigma_I^2,$$

so the real-wealth drift becomes *state-dependent* through the inflation factor $Y(t)$. Therefore, the controlled state process is two-dimensional:

$$(\tilde{X}(t), Y(t)) \text{ with dynamics given by (2.5) and (2.2).}$$

2.3. Model uncertainty via relative entropy

2.3.1. Admissible controls

We take the control in real terms as $\nu(t) = (u(t), \tilde{\pi}(t))$.

Definition 2.1 (Admissible control set $\mathcal{A}$). *A control* $\nu = \{(u(t), \tilde{\pi}(t))\}_{0\le t\le T}$ *is called* admissible *if*

(1) ν *is* $\mathbb{F}$*-progressively measurable;*
(2) $u(t) \in [0, 1]$ *a.s. for all* $t \in [0, T]$ *and* $\tilde{\pi}(t) \in \mathbb{R}$*;*
(3) The controlled state systems (2.2)–(2.5) *admit a unique strong solution and*

$$\mathbb{E}\Big[\int_0^T (u(t)^2 + \tilde{\pi}(t)^2)\, dt\Big] < \infty.$$

The set of all admissible controls is denoted by $\mathcal{A}$*.*

2.3.2. Admissible distortions and entropy penalty

Let $\theta(t) = (\theta_R(t), \theta_S(t), \theta_I(t))^\top$ be an $\mathbb{R}^3$-valued, progressively measurable process satisfying $\int_0^T \|\theta(t)\|^2 dt < \infty$ a.s. Define the Doléans–Dade exponential

$$\Lambda_T^\theta := \exp\Big(\int_0^T \theta(s)\cdot dW(s) - \frac{1}{2}\int_0^T \|\theta(s)\|^2 ds\Big). \tag{2.6}$$

Definition 2.2 (Admissible distortions Θ). *A distortion* θ *is called* admissible *if* Λ_T^θ *in* (2.6) *is a true martingale, so that the measure* $\mathbb{Q}^\theta$ *defined by*

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \Lambda_T^\theta$$

is well-defined and equivalent to $\mathbb{P}$. A sufficient (but not necessary) condition is Novikov's criterion

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \|\theta(s)\|^2 ds\Big)\Big] < \infty.$$

The set of admissible distortions is denoted by Θ.

Under $\mathbb{Q}^\theta$, the Brownian motion transforms as

$$dW^{\mathbb{Q}^\theta}(t) = dW(t) - \theta(t)\,dt.$$

The robust objective functional with an entropic penalty is

$$J(t,x,y) = \sup_{v\in\mathcal{A}} \inf_{\theta\in\Theta} \mathbb{E}^{\mathbb{Q}^\theta}_{t,x,y}\Big[-e^{-\gamma\tilde{X}(T)} + \int_t^T \frac{1}{2\eta}\|\theta(s)\|^2\,ds\Big], \tag{2.7}$$

where $\eta > 0$ is the ambiguity-aversion parameter and the conditioning means $\tilde{X}(t) = x$ and $Y(t) = y$.

2.4. Theoretical connection: Risk-sensitive control

We now state the risk-sensitive representation of the entropy-penalized robust problem. The key tool is the variational representation of relative entropy (often referred to as the Donsker–Varadhan variational principle); see, for example [18, 19].

Proposition 2.1 (Risk-sensitive equivalence)**.** *The robust control problem* (2.7) *is equivalent to a risk-sensitive control problem under the reference measure $\mathbb{P}$. More precisely, for any* (t,x,y)*,*

$$J(t,x,y) = \sup_{v\in\mathcal{A}}\left\{-\frac{1}{\eta}\log\mathbb{E}^{\mathbb{P}}_{t,x,y}\Big[\exp\big(-\eta\,U(\tilde{X}^v(T))\big)\Big]\right\}, \tag{2.8}$$

where $U(z) = -e^{-\gamma z}$ and $\tilde{X}^v$ denotes the real wealth under v. In particular,

$$J(t,x,y) = \sup_{v\in\mathcal{A}}\left\{-\frac{1}{\eta}\log\mathbb{E}^{\mathbb{P}}_{t,x,y}\Big[\exp(\eta e^{-\gamma\tilde{X}^v(T)})\Big]\right\}. \tag{2.9}$$

Proof. Fix an admissible control $v \in \mathcal{A}$ and define, for this v,

$$\mathcal{V}(v) := \inf_{\theta\in\Theta}\mathbb{E}^{\mathbb{Q}^\theta}_{t,x,y}\Big[U(\tilde{X}^v(T)) + \frac{1}{2\eta}\int_t^T \|\theta(s)\|^2 ds\Big].$$

By Girsanov's theorem and the definition of Λ^θ_T, the entropy term satisfies

$$H(\mathbb{Q}^\theta \mid \mathbb{P}) = \mathbb{E}^{\mathbb{Q}^\theta}_{t,x,y}\Big[\frac{1}{2}\int_t^T \|\theta(s)\|^2 ds\Big],$$

so that

$$\mathcal{V}(v) = \inf_{\mathbb{Q}\sim\mathbb{P}}\left\{\mathbb{E}^{\mathbb{Q}}_{t,x,y}[U(\tilde{X}^v(T))] + \frac{1}{\eta}H(\mathbb{Q}\mid\mathbb{P})\right\}.$$

Using the variational representation of relative entropy, for any measurable Φ with $\mathbb{E}^{\mathbb{P}}_{t,x,y}[e^\Phi] < \infty$,

$$\log\mathbb{E}^{\mathbb{P}}_{t,x,y}[e^\Phi] = \sup_{\mathbb{Q}\sim\mathbb{P}}\Big\{\mathbb{E}^{\mathbb{Q}}_{t,x,y}[\Phi] - H(\mathbb{Q}\mid\mathbb{P})\Big\}.$$

Apply this identity with $\Phi = -\eta\,U(\tilde{X}^v(T))$ to obtain

$$\mathcal{V}(v) = -\frac{1}{\eta}\log\mathbb{E}^{\mathbb{P}}_{t,x,y}\Big[\exp\big(-\eta\,U(\tilde{X}^v(T))\big)\Big].$$

Finally, because $J(t,x,y) = \sup_{v\in\mathcal{A}}\mathcal{V}(v)$, we obtain (2.8). Substituting $U(z) = -e^{-\gamma z}$ yields (2.9). □

3. The HJB–Isaacs equation

Let $V(t, x, y)$ denote the value function, where the state variable is two-dimensional: $x = \tilde{X}(t)$ is the real wealth and $y = Y(t) = \ln I(t)$ is the log-inflation factor. Under an admissible control $\nu(t) = (u(t), \tilde{\pi}(t))$ and an admissible distortion $\theta(t) = (\theta_R(t), \theta_S(t), \theta_I(t))^\top$, the distorted measure $\mathbb{Q}^\theta$ is defined by (2.6), and the Brownian motion transforms as

$$dW^{\mathbb{Q}^\theta}(t) = dW(t) - \theta(t)\,dt, \qquad W = (W_R, W_S, W_I)^\top.$$

3.1. Controlled state dynamics under $\mathbb{Q}^\theta$

Recall the OU inflation dynamics (2.2) and the real-wealth dynamics (2.5). Under $\mathbb{Q}^\theta$, we have

$$\begin{cases} dY(t) = \Big(\kappa(\bar{y} - Y(t)) - \sigma_I\,\theta_I(t)\Big)dt + \sigma_I\,dW_I^{\mathbb{Q}^\theta}(t), \\ d\tilde{X}(t) = \Big[\tilde{X}(t)(r - \mu_I(Y(t)) + \sigma_I^2) + \tilde{\pi}(t)(\mu - r - \sigma\sigma_I\rho_{SI}) + \eta_R\lambda m\,u(t) \\ \qquad - u(t)\sigma_R\,\theta_R(t) - \tilde{\pi}(t)\sigma\,\theta_S(t) + \tilde{X}(t)\sigma_I\,\theta_I(t)\Big]dt \\ \qquad + u(t)\sigma_R\,dW_R^{\mathbb{Q}^\theta}(t) + \tilde{\pi}(t)\sigma\,dW_S^{\mathbb{Q}^\theta}(t) - \tilde{X}(t)\sigma_I\,dW_I^{\mathbb{Q}^\theta}(t), \end{cases} \tag{3.1}$$

where $\mu_I(Y) = \kappa(\bar{y} - Y) + \frac{1}{2}\sigma_I^2$.

3.2. Generator and the HJBI equation

Let $V \in C^{1,2,2}([0, T]\times\mathbb{R}\times\mathbb{R})$. For fixed (ν, θ), the infinitesimal generator of $(\tilde{X}, Y)$ under $\mathbb{Q}^\theta$ is given by

$$\mathcal{L}^{\nu,\theta}V = V_t + b_x^{\nu,\theta}(t, x, y)\,V_x + b_y^\theta(t, y)\,V_y + \frac{1}{2}a_{xx}^\nu(t, x)\,V_{xx} + \frac{1}{2}a_{yy}\,V_{yy} + a_{xy}^\nu(t, x)\,V_{xy}, \tag{3.2}$$

where

$$\begin{aligned} &b_y^\theta(t, y) = \kappa(\bar{y} - y) - \sigma_I\theta_I, \\ &b_x^{\nu,\theta}(t, x, y) = x(r - \mu_I(y) + \sigma_I^2) + \tilde{\pi}(\mu - r - \sigma\sigma_I\rho_{SI}) + \eta_R\lambda m\,u - u\sigma_R\theta_R - \tilde{\pi}\sigma\theta_S + x\sigma_I\theta_I, \\ &a_{xx}^\nu(t, x) = u^2\sigma_R^2 + \tilde{\pi}^2\sigma^2 + x^2\sigma_I^2 - 2x\tilde{\pi}\sigma\sigma_I\rho_{SI}, \\ &a_{yy} = \sigma_I^2, \\ &a_{xy}^\nu(t, x) = \tilde{\pi}\,\sigma\sigma_I\rho_{SI} - x\sigma_I^2. \end{aligned}$$

The cross term a_{xy}^ν arises because the same Brownian motion W_I drives both Y and $\tilde{X}$, and W_S is correlated with W_I.

The robust optimization problem (2.7) leads to the following HJBI equation with terminal condition $V(T, x, y) = -e^{-\gamma x}$:

$$\begin{cases} 0 = \sup\limits_{\nu=(u,\tilde{\pi})}\inf\limits_{\theta\in\mathbb{R}^3}\Big\{\mathcal{L}^{\nu,\theta}V(t, x, y) + \frac{1}{2\eta}\|\theta\|^2\Big\}, \\ V(T, x, y) = -e^{-\gamma x}. \end{cases} \tag{3.3}$$

3.3. Worst-case distortion and the reduced HJBI equation

To compute the inner minimization in (3.3), isolate the θ-dependent terms. Using (3.2) and (3.1), we can rewrite

$$\mathcal{L}^{\nu,\theta}V = \mathcal{L}^{\nu,0}V - \theta_R\,(u\sigma_R V_x) - \theta_S\,(\tilde{\pi}\sigma V_x) + \theta_I\,(x\sigma_I V_x - \sigma_I V_y),$$

so the inner objective becomes, pointwise in (t, x, y),

$$\Psi(\theta) = -\theta_R\, A_R - \theta_S\, A_S + \theta_I\, A_I + \frac{1}{2\eta}\|\theta\|^2,$$

where

$$A_R := u\sigma_R V_x, \qquad A_S := \tilde{\pi}\sigma V_x, \qquad A_I := x\sigma_I V_x - \sigma_I V_y.$$

Because Ψ is strictly convex in θ, the unique minimizer satisfies

$$\nabla_\theta \Psi(\theta) = 0 \quad \Longleftrightarrow \quad \theta^* = \eta\,(A_R, A_S, -A_I)^\top,$$

that is

$$\theta_R^* = \eta\, u\sigma_R V_x, \qquad \theta_S^* = \eta\,\tilde{\pi}\sigma V_x, \qquad \theta_I^* = -\eta\,\sigma_I(xV_x - V_y). \tag{3.4}$$

Substituting θ^* back yields the minimum value

$$\inf_\theta \Psi(\theta) = -\frac{\eta}{2}\Big(A_R^2 + A_S^2 + A_I^2\Big) = -\frac{\eta}{2}\Big[(u\sigma_R V_x)^2 + (\tilde{\pi}\sigma V_x)^2 + \sigma_I^2(xV_x - V_y)^2\Big].$$

Therefore, (3.3) reduces to the following nonlinear HJBI equation:

$$\begin{cases} 0 = V_t + \sup\limits_{u\in[0,1],\,\tilde{\pi}\in\mathbb{R}} \Big\{ b_x^{\nu,0}(t,x,y)\, V_x + \kappa(\bar{y} - y)V_y + \dfrac{1}{2}a_{xx}^{\nu}(t,x)\, V_{xx} + \dfrac{1}{2}\sigma_I^2 V_{yy} + a_{xy}^{\nu}(t,x)\, V_{xy} \\ \qquad - \dfrac{\eta}{2}\Big[(u\sigma_R V_x)^2 + (\tilde{\pi}\sigma V_x)^2 + \sigma_I^2(xV_x - V_y)^2\Big]\Big\}, \\ V(T,x,y) = -e^{-\gamma x}, \end{cases} \tag{3.5}$$

where the drift $b_x^{\nu,0}$ and diffusion coefficients $a_{xx}^{\nu}, a_{xy}^{\nu}$ are given above, with $\theta = 0$, that is

$$b_x^{\nu,0}(t,x,y) = x(r - \mu_I(y) + \sigma_I^2) + \tilde{\pi}(\mu - r - \sigma\sigma_I\rho_{SI}) + \eta_R\lambda m\, u.$$

Equation (3.5) shows that ambiguity enters through a quadratic penalty on the distorted drift loadings, including a state-coupled term $\sigma_I^2(xV_x - V_y)^2$ that is induced by mean-reverting inflation and is absent in one-dimensional benchmark models. This structural coupling is the source of the additional mean-reversion hedging component in the optimal investment strategy derived in Section 4.

4. Solution and theoretical analysis

Throughout this section, the state variable is the real wealth $x = \tilde{X}(t)$ governed by (2.5), and the controls are the retention ratio $u(t) \in [0, 1]$ and the *real* amount $\tilde{\pi}(t)$ invested in the risky asset. (Recall that the nominal risky position satisfies $\pi(t) = I(t)\tilde{\pi}(t)$.)

We solve the reduced HJBI equation (3.5) with the terminal condition

$$V(T, x) = -e^{-\gamma x}. \tag{4.1}$$

4.1. Closed-form robust optimal strategies

4.1.1. Effective risk aversion under entropy-penalized ambiguity

A convenient way to summarize the impact of ambiguity is through the effective risk aversion coefficient

$$\Gamma := \gamma(1 + \eta\gamma), \tag{4.2}$$

which is increasing in η and reduces to γ when $\eta = 0$. This is exactly the channel behind the "flight-to-safety" comparative statics documented in Section 5.

4.1.2. Exponential-affine Ansatz

Motivated by the CARA terminal utility (4.1) and the linear-quadratic structure of the drift/variance in (3.5), we postulate

$$V(t, x) = -\exp\{-\alpha(t)x + A(t)\}, \qquad A(T) = 0, \tag{4.3}$$

where $\alpha(\cdot)$ and $A(\cdot)$ are deterministic C^1 functions. Direct differentiation yields

$$V_t = (-\alpha'(t)x + A'(t))V, \qquad V_x = -\alpha(t)V, \qquad V_{xx} = \alpha(t)^2 V.$$

4.1.3. Reduction to a concave static maximization

Substituting (4.3) into (3.5) and dividing by $-V(t, x) > 0$ reduces the HJBI to a pointwise maximization over $(u, \tilde{\pi})$. The resulting Hamiltonian is strictly concave in $(u, \tilde{\pi})$ (quadratic with negative definite Hessian under $\sigma_R > 0$ and $\sigma > 0$); hence, the maximizer is characterized by first-order conditions.

4.1.4. First-order conditions

The optimal retention and investment satisfy

$$u^*(t) = \left[\frac{\eta_R \lambda m}{\Gamma\,\sigma_R^2}\, e^{-k(T-t)}\right]_{[0,1]}, \tag{4.4}$$

$$\tilde{\pi}^*(t) = \frac{\mu - r - \sigma\sigma_I\rho_{SI}}{\Gamma\,\sigma^2}\, e^{-k(T-t)} + \frac{\sigma_I\rho_{SI}}{\sigma}\, x, \tag{4.5}$$

where $[\cdot]_{[0,1]}$ denotes projection onto $[0, 1]$ (which enforces the admissible retention constraint), and k is determined below. The second term in (4.5) is the inflation-hedging demand induced by the correlation between S and I.

4.1.5. Determination of the time-separation rate k

Matching the x-coefficient after substituting (4.4)–(4.5) into the reduced HJBI yields

$$k = r - \mu_I + \sigma_I^2 - \frac{1}{2}\Gamma\,\sigma_I^2\,(1 - \rho_{SI}^2). \tag{4.6}$$

This quantity governs the finite-horizon time-separation in the nonlinear HJBI and vanishes in several practically relevant parameter regimes, explaining why the optimal controls can be nearly time-homogeneous in numerical experiments.

4.1.6. Closed-form expression for $A(t)$.

With k given by (4.6), the remaining (state-independent) terms yield a linear ODE for $A(\cdot)$. Solving it under $A(T) = 0$ gives

$$A(t) = \begin{cases} \dfrac{1 - e^{-2k(T-t)}}{4k}\left(\dfrac{(\eta_R \lambda m)^2}{\Gamma \sigma_R^2} + \dfrac{(\mu - r - \sigma\sigma_I \rho_{SI})^2}{\Gamma \sigma^2}\right), & k \neq 0, \\ \dfrac{T-t}{2}\left(\dfrac{(\eta_R \lambda m)^2}{\Gamma \sigma_R^2} + \dfrac{(\mu - r - \sigma\sigma_I \rho_{SI})^2}{\Gamma \sigma^2}\right), & k = 0. \end{cases} \tag{4.7}$$

The explicit form (4.7) is useful for implementation and for verifying the HJBI directly.

We summarize the analytical solution.

Theorem 4.1 (Closed-form robust optimal strategies)**.** *Let* $\Gamma = \gamma(1 + \eta\gamma)$ *and* k *be given by* (4.6)*. Define* V *by* (4.3)–(4.7)*. Then* $V \in C^{1,2}([0, T] \times \mathbb{R})$ *satisfies the reduced HJBI* (3.5) *with terminal condition* (4.1)*. Moreover, the optimal controls are* (4.4) *and* (4.5)*. Equivalently, the optimal* nominal *risky position is*

$$\pi^*(t) = I(t)\tilde{\pi}^*(t).$$

Remark 4.1 (On mean-reverting inflation and the GBM benchmark)**.** *Although the baseline analytical model uses geometric Brownian motion (GBM) inflation for tractability, our numerical robustness checks under mean-reverting inflation (OU-type log-inflation) confirm that the hedging component is state-dependent and vanishes in the limit* $\kappa \to 0$*, whereas the myopic component remains stable. The "OU* $\to$ *GBM as* $\kappa \to 0$*" degradation check reported in Section 5 provides an explicit consistency validation of the benchmark.*

4.2. Verification theorem

We now verify that the candidate solution in Theorem 4.1 indeed achieves the value of the robust control problem.

Lemma 4.1 (Regularity and growth)**.** *The function* V *defined by* (4.3)–(4.7) *belongs to* $C^{1,2}([0, T] \times \mathbb{R})$ *and satisfies the exponential growth bound*

$$|V(t, x)| + |V_x(t, x)| + |V_{xx}(t, x)| \leq C\, e^{C|x|}, \qquad (t, x) \in [0, T] \times \mathbb{R},$$

for some constant $C > 0$*.*

Lemma 4.2 (Admissibility of the candidate optimizer)**.** *Let* $\nu^* = (u^*, \tilde{\pi}^*)$ *be defined by* (4.4) *and* (4.5)*. Then,* $\nu^* \in \mathcal{A}$*. In particular:*

(1) $u^*(t) \in [0, 1]$ *for all* t *by construction, and* $\tilde{\pi}^*(t)$ *has at most linear growth in* x*;*

(2) The stochastic differential equation (2.5) *under* ν^* *admits a unique strong solution, and*

$$\mathbb{E}\left[\int_0^T \left(|u^*(t)|^2 + |\tilde{\pi}^*(t)|^2\right) dt\right] < \infty;$$

(3) The associated worst-case Girsanov kernel θ^* *from the inner minimization (Section 3) is progressively measurable and satisfies Novikov's condition under the standing parameter assumptions used in Section 5.*

Proof. Items (1) and (2) follow from the projection $[\cdot]_{[0,1]}$ and the affine form of $\tilde{\pi}^*$, together with standard moment estimates for linear-growth diffusions. Item (3) follows by combining the explicit feedback form of θ^* with the exponential-moment bound in Lemma 4.1 and the finite-horizon moment bounds of $\tilde{X}^{\nu^*}$. □

Theorem 4.2 (Verification)**.** *Let V be defined by* (4.3)–(4.7)*. Then, V coincides with the value function of the robust control problem* (2.7)*, and the control* $\nu^* = (u^*, \tilde{\pi}^*)$ *in* (4.4) *and* (4.5) *is optimal.*

Proof. Fix (t, x) and any admissible control $\nu \in \mathcal{A}$. Let $\tilde{X}^\nu$ be the corresponding real-wealth process and define the localization $\tau_n := \inf\{s \geq t : |\tilde{X}^\nu(s)| \geq n\} \wedge T$. Applying Itô's formula to $V(s, \tilde{X}^\nu(s))$ on $[t, \tau_n]$ and using the reduced HJBI inequality (i.e., V is a supersolution under any admissible control) yields that the stopped process is a supermartingale. Taking expectations, sending $n \to \infty$, and using Lemma 4.1 together with admissibility to justify dominated convergence gives

$$V(t, x) \geq J(t, x; \nu).$$

For $\nu = \nu^*$, the first-order conditions attain the supremum in the HJBI, so the supermartingale becomes a martingale and the following equality holds:

$$V(t, x) = J(t, x; \nu^*).$$

Therefore, V is the value function, and ν^* is optimal. □

5. Numerical analysis and discussion

This section validates the qualitative implications of the closed-form strategies in Theorem 4.1 and quantifies how ambiguity aversion and inflation/market parameters shape the distribution of real surplus and downside risk. In particular, we document a robust "flight-to-safety" pattern: Increasing ambiguity aversion reduces risky-asset exposure and increases reinsurance demand, leading to a sizable contraction of tail risk.

5.1. Baseline parameters and reported risk metrics

Unless otherwise specified, we use the following baseline parameters:

$$\lambda m = 1.2, \quad c = 1.6, \quad \sigma_R = 0.8, \quad \mu = 0.08, \quad r = 0.03, \quad \sigma = 0.25,$$

$$\mu_I = 0.02, \quad \sigma_I = 0.03, \quad \rho_{SI} = 0.4, \quad \gamma = 2, \quad T = 1.$$

We set the initial real wealth $\tilde{X}_0 = 1$ and the initial inflation index $I_0 = 1$. The time grid is $t_n = n\Delta t$ with $\Delta t = T/N$. Unless stated otherwise, we use $N = 250$ time steps and $M = 10{,}000$ Monte Carlo paths, and we fix the random seed for reproducibility.

We report the following terminal and pathwise risk metrics:

$$\mathbb{E}[\tilde{X}_T], \qquad \text{Std}[\tilde{X}_T], \qquad \text{VaR}_\alpha(\tilde{X}_T), \qquad \text{CVaR}_\alpha(\tilde{X}_T),$$

with $\alpha = 5\%$, where CVaR_α is the conditional mean in the lower α-tail. For strategy interpretation, we also report the time-averaged controls

$$\bar{u} := \frac{1}{T}\,\mathbb{E}\left[\int_0^T u^*(t)\,dt\right], \qquad \overline{\mathrm{Exp}} := \frac{1}{T}\,\mathbb{E}\left[\int_0^T \frac{\tilde{\pi}^*(t)}{\tilde{X}(t)}\,\mathbf{1}_{\{\tilde{X}(t)>\varepsilon\}}\,dt\right],$$

where $\varepsilon > 0$ is a small numerical safeguard (we take $\varepsilon = 10^{-6}$). In addition, for actuarial interpretation we compute the finite-horizon ruin probability

$$\psi(T) := \mathbb{P}\Big(\min_{0\le t\le T} \tilde{X}_t < 0\Big),$$

estimated by the fraction of simulated paths that cross below zero at any time.

5.2. Simulation scheme and implementation details

We simulate the inflation index I using an exact discretization of the GBM model, and the real-wealth stochastic differential equation (2.5) by Euler–Maruyama under the feedback controls (4.4) and (4.5). Correlated Brownian increments for (W_S, W_I) are generated from i.i.d. standard normals (Z_S, Z_I) via

$$\Delta W_S^{(n)} = \sqrt{\Delta t}\,Z_S^{(n)}, \qquad \Delta W_I^{(n)} = \sqrt{\Delta t}\big(\rho_{SI} Z_S^{(n)} + \sqrt{1-\rho_{SI}^2}\,Z_I^{(n)}\big),$$

while $\Delta W_R^{(n)} = \sqrt{\Delta t}\,Z_R^{(n)}$ is independent.

For numerical stability and admissibility, we enforce the retention constraint by projection:

$$u^*(t) = [\ u_{\mathrm{uncon}}(t)\ ]_{[0,1]} := \min\{1, \max\{0, u_{\mathrm{uncon}}(t)\}\}.$$

Algorithm 1 Monte Carlo simulation of robust real wealth under stochastic inflation.

Require: Parameters $\lambda, m, c, \eta_R, \sigma_R, \mu, r, \sigma, \mu_I, \sigma_I, \rho_{SI}, \gamma, \eta$, initial conditions $(I_0, \tilde{X}_0)$, horizon T, steps N, paths M.

Set $\Delta t = T/N$ and $\alpha = 5\%$.
Compute $\Gamma = \gamma(1 + \eta\gamma)$ and k from (4.6).
for $j = 1$ to M **do**
  Initialize $I_0^{(j)} = I_0$, $\tilde{X}_0^{(j)} = \tilde{X}_0$, and set $\tilde{X}_{\min}^{(j)} = \tilde{X}_0$.
  for $n = 0$ to $N - 1$ **do**
    $t_n = n\Delta t$.
    Compute the feedback controls:

$$u_{\text{uncon}}(t_n) = \frac{\eta_R \lambda m}{\Gamma \sigma_R^2} e^{-k(T-t_n)}, \qquad u^*(t_n) = \left[u_{\text{uncon}}(t_n)\right]_{[0,1]},$$

$$\tilde{\pi}^*(t_n) = \frac{\mu - r - \sigma\sigma_I\rho_{SI}}{\Gamma\sigma^2} e^{-k(T-t_n)} + \frac{\sigma_I \rho_{SI}}{\sigma} \tilde{X}_n^{(j)}.$$

    Generate i.i.d. normals $Z_S^{(n)}, Z_I^{(n)}, Z_R^{(n)} \sim \mathcal{N}(0,1)$ and set $(\Delta W_S^{(n)}, \Delta W_I^{(n)}, \Delta W_R^{(n)})$ as above.
    Update the inflation index using the exact GBM step:

$$I_{n+1}^{(j)} = I_n^{(j)} \exp\left((\mu_I - \tfrac{1}{2}\sigma_I^2)\Delta t + \sigma_I \Delta W_I^{(n)}\right).$$

    Update real wealth by Euler–Maruyama for (2.5):

$$\begin{aligned}\tilde{X}_{n+1}^{(j)} = \tilde{X}_n^{(j)} &+ \left[\tilde{X}_n^{(j)}(r - \mu_I + \sigma_I^2) + \tilde{\pi}^*(t_n)(\mu - r - \sigma\sigma_I\rho_{SI}) + \eta_R \lambda m\, u^*(t_n)\right]\Delta t \\ &+ u^*(t_n)\sigma_R \Delta W_R^{(n)} + \tilde{\pi}^*(t_n)\sigma \Delta W_S^{(n)} - \tilde{X}_n^{(j)}\sigma_I \Delta W_I^{(n)}.\end{aligned}$$

    Update $\tilde{X}_{\min}^{(j)} = \min\{\tilde{X}_{\min}^{(j)}, \tilde{X}_{n+1}^{(j)}\}$.
  end for
end for
Compute $\mathbb{E}[\tilde{X}_T]$, $\text{Std}[\tilde{X}_T]$, VaR_α and CVaR_α from the empirical distribution of $\{\tilde{X}_N^{(j)}\}_{j=1}^M$.
Estimate ruin probability by $\widehat{\psi}(T) = \frac{1}{M}\sum_{j=1}^M \mathbf{1}_{\{\tilde{X}_{\min}^{(j)} < 0\}}$.
return Risk metrics and simulated paths.

5.3. Numerical results

We organize the numerical evidence around four key questions:

(i) Does robustness reduce dispersion and downside risk?
(ii) How does ambiguity aversion η reshape the optimal controls and tail risk?
(iii) How do inflation parameters (σ_I, ρ_{SI}) affect exposure and risk?
(iv) Is the GBM inflation benchmark consistent with a mean-reverting inflation extension as $\kappa \to 0$?

5.3.1. Neutral vs. robust strategies: Dispersion and downside risk

Figure 1 compares representative sample paths and moment dynamics under ambiguity-neutral ($\eta = 0$) and robust ($\eta = 0.5$) controls. The robust strategy lowers the mean but substantially compresses dispersion and, importantly, improves left-tail outcomes.

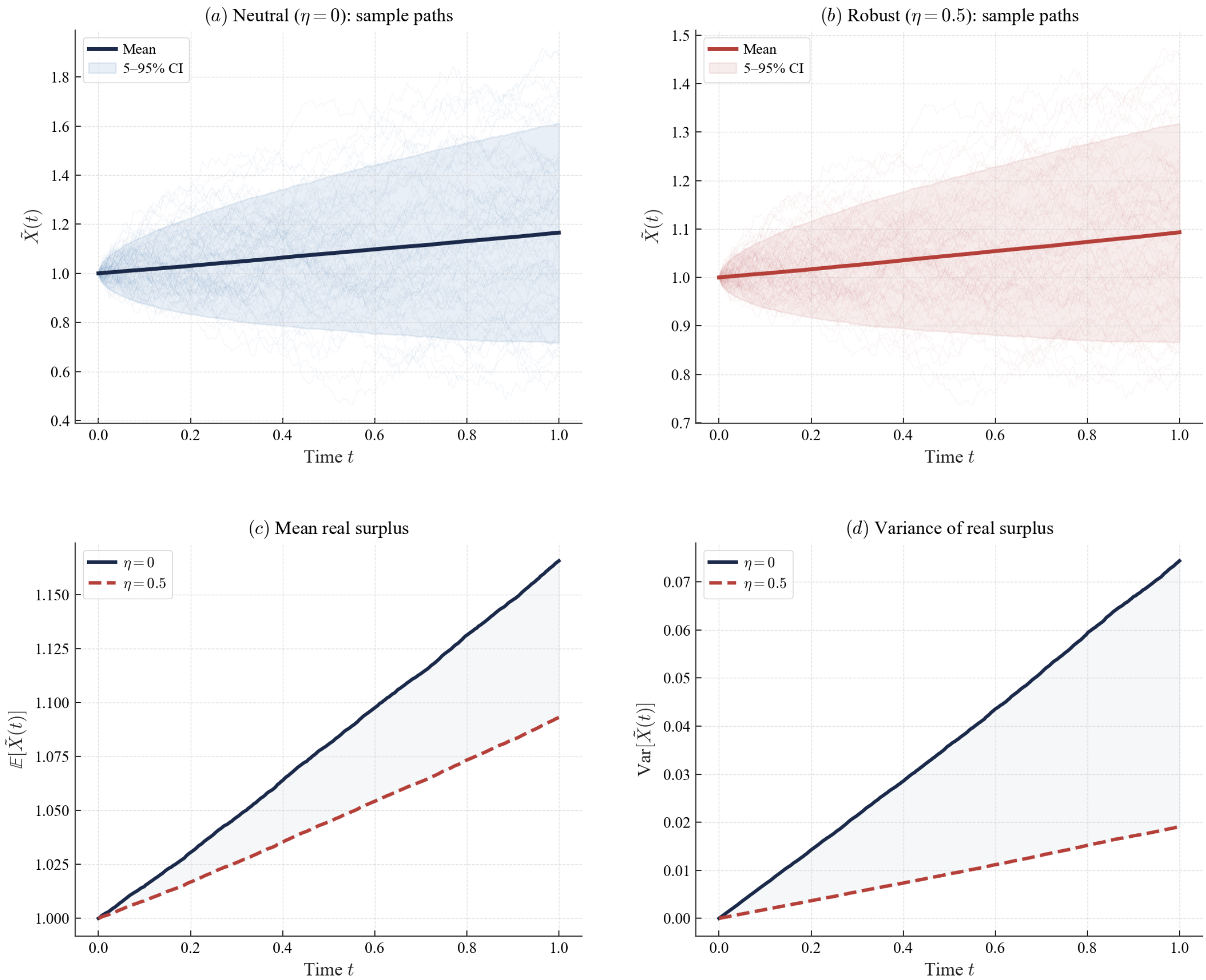

Figure 1. Neutral vs robust policies: Sample paths and moment dynamics of real wealth. The robust strategy reduces risky exposure and increases reinsurance usage, resulting in materially lower dispersion and improved downside risk.

5.3.2. Effect of ambiguity aversion: Flight-to-safety

We next vary η while keeping all other parameters at baseline. Figure 2 and Table 1 show that increasing η monotonically decreases risky exposure and the retention ratio, consistent with the effective risk aversion amplification $\Gamma = \gamma(1 + \eta\gamma)$ in Theorem 4.1. The reduction in downside risk is pronounced, as reflected by the increase in $\text{VaR}_{5\%}$ and $\text{CVaR}_{5\%}$.

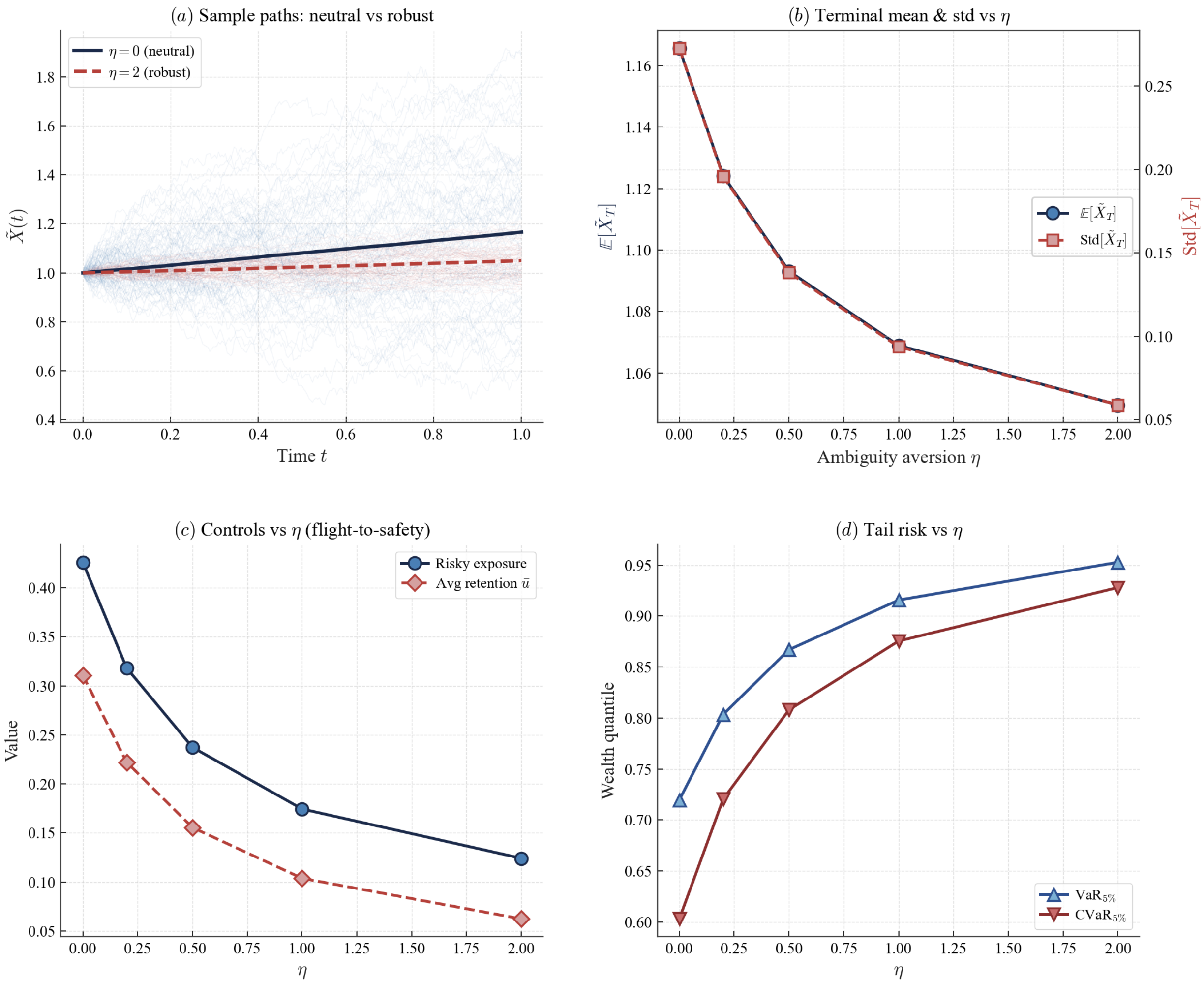

Figure 2. Sensitivity to ambiguity aversion η: Terminal moments, control intensities (flight-to-safety), and tail risk.

Table 1. Effect of ambiguity aversion η (baseline parameters).

η	$\mathbb{E}[\tilde{X}_T]$	Std	$\mathrm{VaR}_{5\%}$	$\mathrm{CVaR}_{5\%}$	$\overline{\mathrm{Exp}}$	$\bar{u}$
0.0	1.1657	0.2727	0.7196	0.6030	0.4261	0.3109
0.2	1.1242	0.1957	0.8035	0.7204	0.3182	0.2221
0.5	1.0931	0.1382	0.8670	0.8080	0.2373	0.1555
1.0	1.0689	0.0937	0.9157	0.8755	0.1744	0.1037
2.0	1.0495	0.0589	0.9528	0.9281	0.1241	0.0623

5.3.3. Inflation volatility and correlation: Exposure and risk transmission

We vary the inflation volatility σ_I and the stock–inflation correlation ρ_{SI}. Figure 3 and Tables 2–3 show: (i) larger σ_I increases surplus variance and worsens tail metrics, reflecting stronger inflation-induced diffusion in (2.5); (ii) larger ρ_{SI} increases the optimal risky-asset exposure due to the hedging term in (4.5), and impacts downside risk through cross-variation effects.

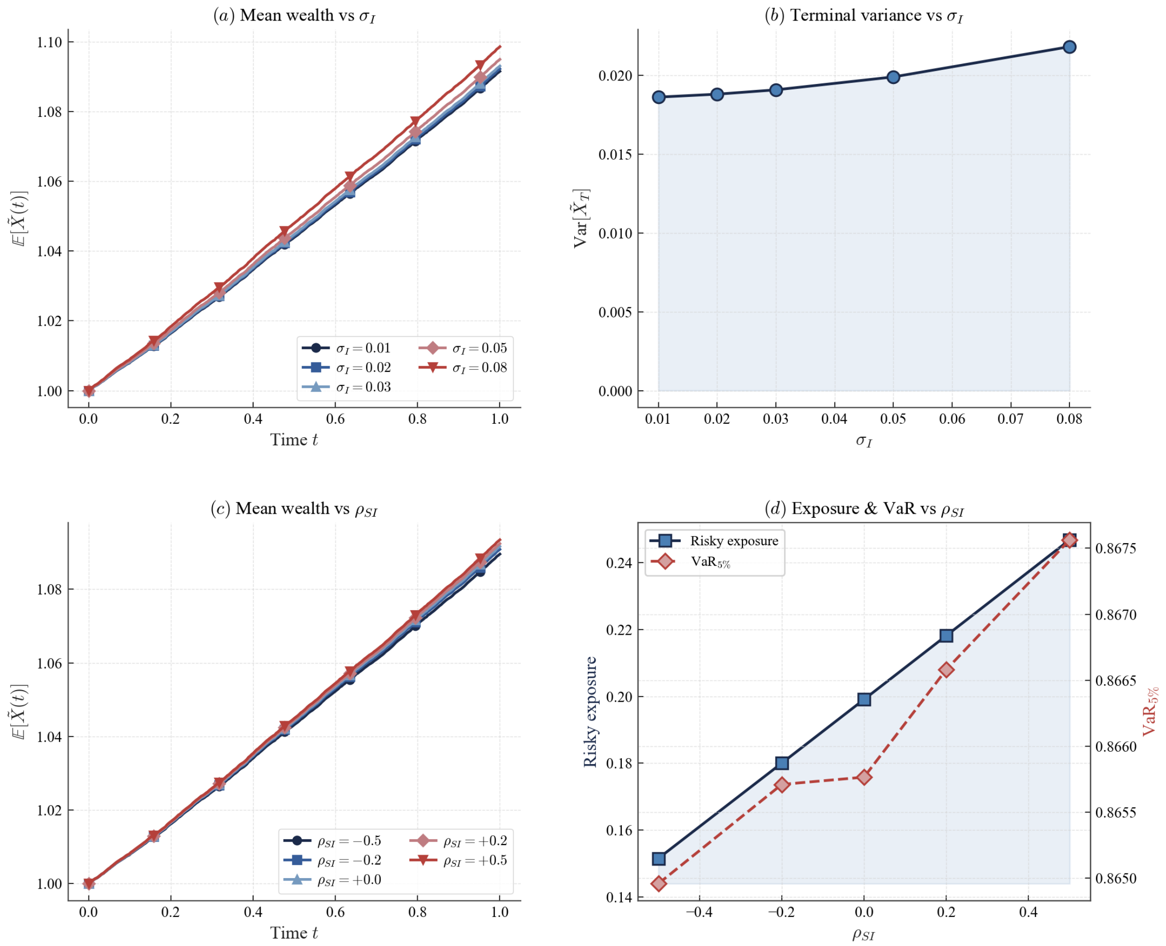

Figure 3. Sensitivity to inflation volatility σ_I and stock–inflation correlation ρ_{SI}.

Table 2. Effect of inflation volatility σ_I (baseline parameters).

σ_I	$\mathbb{E}[\tilde{X}_T]$	Std	Var	$\mathrm{VaR}_{5\%}$	$\mathrm{CVaR}_{5\%}$
0.01	1.0916	0.1365	0.0186	0.8685	0.8095
0.02	1.0923	0.1372	0.0188	0.8677	0.8090
0.03	1.0931	0.1382	0.0191	0.8670	0.8080
0.05	1.0950	0.1411	0.0199	0.8646	0.8049
0.08	1.0987	0.1477	0.0218	0.8581	0.7973

Table 3. Effect of stock–inflation correlation ρ_{SI} (baseline parameters).

ρ_{SI}	$\mathbb{E}[\tilde{X}_T]$	Std	$\overline{\mathrm{Exp}}$	$\mathrm{VaR}_{5\%}$
−0.5	1.0896	0.1363	0.1515	0.8650
−0.2	1.0909	0.1372	0.1800	0.8657
0.0	1.0917	0.1376	0.1991	0.8658
0.2	1.0924	0.1379	0.2182	0.8666
0.5	1.0935	0.1382	0.2469	0.8676

5.3.4. High-inflation regime stress test

To stress-test the robust strategy under a high-inflation regime, we increase the initial inflation state in the mean-reverting log-inflation specification used for robustness checks (Appendix/robustness experiment). Figure 4 shows that elevated initial inflation leads to higher dispersion of real wealth, and the robust strategy mitigates this dispersion by reducing risky exposure and increasing reinsurance usage.

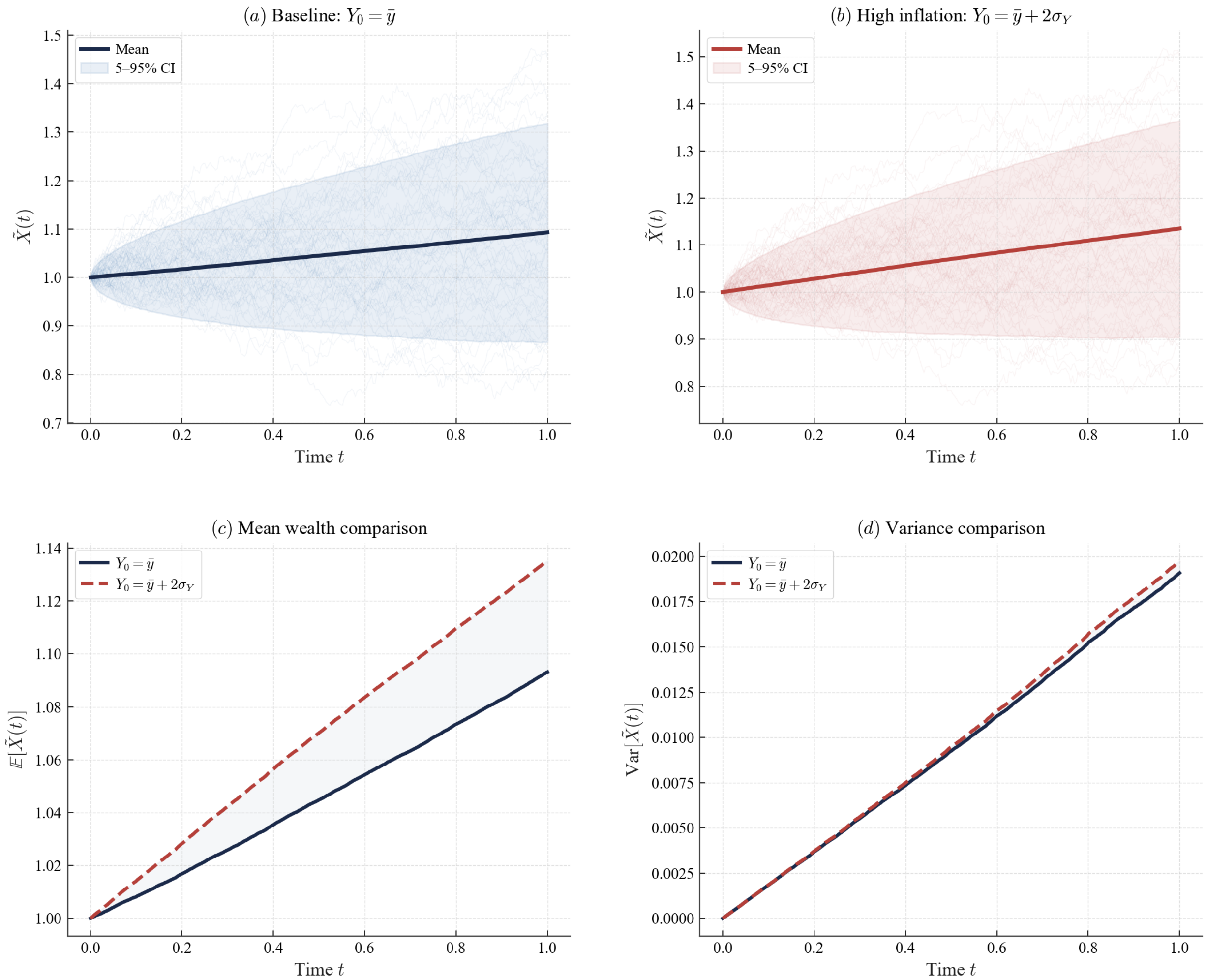

Figure 4. High-inflation regime stress test: Comparison of real-wealth dispersion under baseline vs elevated initial inflation state.

5.3.5. Robustness to mean-reverting inflation and the GBM benchmark

A common concern is that GBM inflation lacks mean reversion. To address this, we implement a robustness experiment in which log-inflation follows an OU-type mean-reverting dynamics with speed κ. Figure 5 and Table 4 report sensitivity with respect to κ. We also provide an explicit degradation check: As $\kappa \to 0$, the mean-reverting model approaches the GBM benchmark, and the optimal strategies (and wealth distribution) converge accordingly; see Figure 6. This numerical consistency check supports the GBM specification as a tractable benchmark in the main analysis.

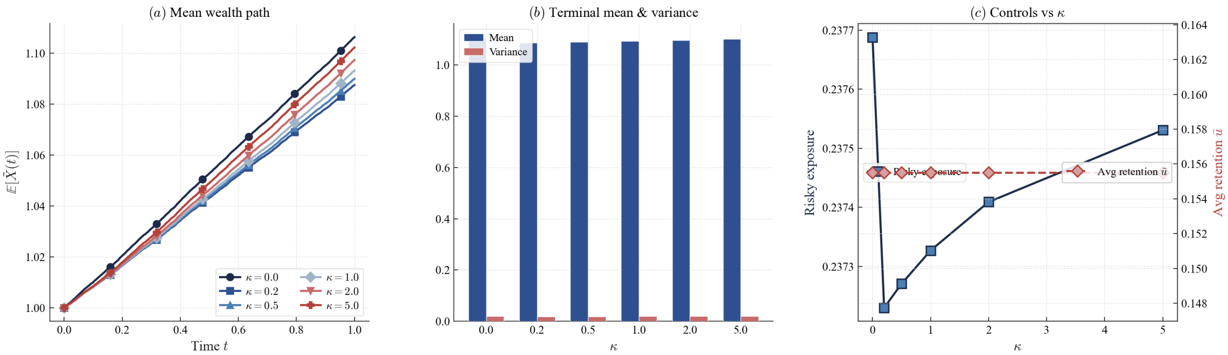

Figure 5. Mean-reversion speed scan under OU-type log-inflation: Terminal statistics and control sensitivity versus κ.

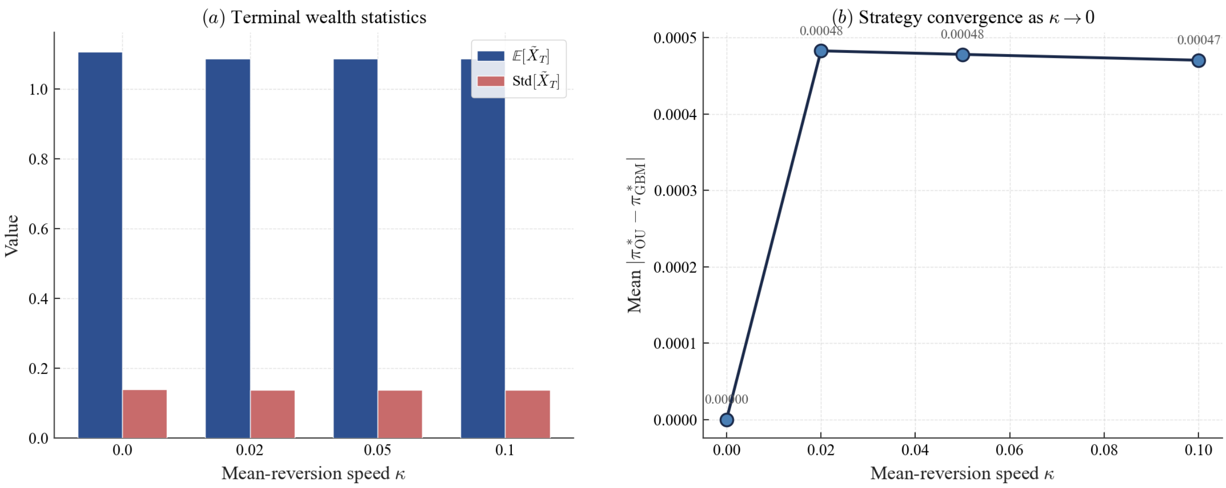

Figure 6. Degradation check: The mean-reverting inflation specification converges to the GBM benchmark as $\kappa \to 0$ (strategy convergence, wealth convergence, and vanishing hedge component).

Table 4. Effect of mean-reversion speed κ (OU-type log-inflation robustness experiment).

κ	$\mathbb{E}[\tilde{X}_T]$	Std	Var	$\overline{\mathrm{Exp}}$	$\bar{u}$
0.0	1.1063	0.1388	0.0193	0.2377	0.1555
0.2	1.0875	0.1375	0.0189	0.2372	0.1555
0.5	1.0899	0.1377	0.0190	0.2373	0.1555
1.0	1.0931	0.1382	0.0191	0.2373	0.1555
2.0	1.0973	0.1390	0.0193	0.2374	0.1555
5.0	1.1022	0.1401	0.0196	0.2375	0.1555

Overall, the numerical evidence supports the theoretical mechanism: Ambiguity aversion increases effective risk aversion and produces a strong flight-to-safety effect, especially under elevated inflation uncertainty, substantially reducing risky-asset exposure and strengthening reinsurance demand.

6. Conclusions

This paper studies a finite-horizon reinsurance–investment problem for an insurer facing stochastic inflation and model ambiguity. By working with real wealth and explicitly incorporating the inflation index into the wealth dynamics, we obtain an inflation-adjusted controlled diffusion in which both the drift and the instantaneous variance are affected by inflation and stock–inflation dependence. Model uncertainty is represented through an adversarial probability distortion penalized by relative entropy, which yields a robust zero-sum stochastic differential game. Using the entropy-penalized formulation, we establish a precise variational duality between ambiguity-averse robust control and risk-sensitive control, and derive the corresponding HJBI equation.

A main analytical contribution is that the HJBI equation admits an explicit, closed-form solution under non-zero interest rates in a finite-horizon setting. By exploiting a separable exponential ansatz, we obtain closed-form optimal controls for both proportional reinsurance and risky investment. The resulting strategies admit transparent economic interpretation. First, ambiguity aversion amplifies risk aversion through an endogenous effective coefficient $\gamma_{\text{eff}} = \gamma(1 + \eta\gamma)$, which uniformly scales down speculative risk-taking. Second, inflation affects the optimal portfolio through an interaction between the inflation diffusion and the stock diffusion, generating an explicit hedging component that depends on ρ_{SI} and σ_I. Third, the reinsurance demand responds nonlinearly to underwriting volatility, consistent with the quadratic variance structure of the robust penalty term.

The numerical experiments corroborate the theoretical comparative statics and address robustness concerns about the inflation specification. In particular, increasing ambiguity aversion produces a pronounced flight-to-safety pattern: risky-asset exposure declines sharply, reinsurance demand strengthens, and terminal tail risk (VaR/CVaR) improves substantially, albeit at the cost of a lower mean terminal surplus. Sensitivity analyses with respect to σ_I and ρ_{SI} quantify how inflation uncertainty and dependence transmit into real-wealth dispersion. Moreover, an additional mean-reverting inflation robustness check demonstrates that, as the mean-reversion speed tends to zero, the OU-type specification converges to the GBM benchmark and the corresponding strategies and wealth distribution converge as well, supporting the GBM inflation model as a tractable benchmark for explicit analysis.

Several extensions are natural. One direction is to incorporate jump or heavy-tailed claim dynamics (e.g., jump–diffusion or spectrally negative Lévy models) while preserving robust tractability via appropriate approximation or asymptotic regimes. A second direction is to embed a term-structure or regime-switching inflation model, allowing the inflation risk premium and inflation uncertainty to vary over time, and to study whether partial explicitness or semi-closed-form solutions can be retained. A third direction is to consider multi-agent interactions (insurer–reinsurer or multiple insurers) under model ambiguity and inflation risk, leading to robust Nash equilibria in incomplete markets. These avenues would further strengthen the applicability of the entropy-based robust framework to realistic insurance and macro-financial environments.

Author contributions

QiongLin Li: Supervision, writing – review & editing, project administration; XuJiang Tang: Conceptualization, methodology, formal analysis, investigation, software, validation, writing – original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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A. A modular verification framework and technical details

This appendix complements Section 4 by (i) collecting auxiliary regularity and integrability facts used in the verification argument, and (ii) highlighting a structural feature that is central to the analytical tractability and, in our view, constitutes a key theoretical contribution: Under stochastic inflation and entropy-penalized ambiguity, the closed-loop dynamics induced by the explicit optimal feedback controls remain linear, which allows a fully rigorous verification theorem with explicit Novikov verification for the worst-case measure distortion.

A.1. Why the verification step is nontrivial under stochastic inflation

Compared with robust reinsurance–investment models without inflation, the real-wealth dynamics in (2.5) contain an additional multiplicative diffusion term $-\sigma_I \tilde{X}_t\, dW_I(t)$ and a correlation-induced cross term between W_S and W_I. As a consequence, the reduced HJBI equation in Section 3 features a nonlinear ambiguity penalty of the form $-\frac{\eta}{2}\|\sigma_{\text{eff}}\|^2 V_x^2$, where $\sigma_{\text{eff}} = (\tilde{u}\sigma_R, \tilde{\pi}\sigma, -\tilde{X}\sigma_I)^\top$. This coupling creates two technical issues that do not appear (or are much simpler) in many existing settings: (i) the worst-case Girsanov kernel depends on V_x and on the state-dependent diffusion loading through $\sigma_{\rm eff}$, so one must ensure that the candidate θ^* indeed generates an equivalent measure (Novikov); (ii) the multiplicative inflation diffusion may induce exponential growth of moments if the closed-loop coefficients are not carefully controlled.

Our explicit exponential-affine ansatz and the resulting affine feedback controls resolve both issues: V_x/V becomes a deterministic function of time, and the closed-loop state process is a linear diffusion with finite-horizon moment bounds. These two facts permit a transparent, verifiable Novikov check for the worst-case distortion and complete the verification theorem rigorously.

A.2. Regularity and growth bounds

Lemma A.1 (Regularity and exponential growth bounds)**.** *Let*

$$V(t,x) = -\exp\{-\alpha(t)x + A(t)\}, \qquad (t,x) \in [0,T] \times \mathbb{R},$$

where $\alpha(\cdot)$ and $A(\cdot)$ are deterministic C^1 functions on $[0,T]$. Then $V \in C^{1,2}([0,T] \times \mathbb{R})$. Moreover, there exist constants $C_1, C_2 > 0$ such that for all (t,x),

$$|V(t,x)| + |V_x(t,x)| + |V_{xx}(t,x)| \le C_1\, e^{C_2|x|}.$$

Proof. The exponent is affine in x, and α, A are continuous on $[0,T]$. Hence V is $C^{1,2}$ and the derivatives satisfy $V_x = -\alpha(t)V$, $V_{xx} = \alpha(t)^2 V$. Since $\alpha(\cdot)$ is bounded on $[0,T]$, the asserted exponential growth bound follows with $C_2 = \sup_{t\in[0,T]} |\alpha(t)|$ and an appropriate C_1. □

A.3. Closed-loop linearity and Novikov verification for the worst-case kernel

Recall from Section 3 that the worst-case distortion is given pointwise by

$$\theta^*(t) = -\eta\, \sigma_{\rm eff}(t,\tilde{X}_t)\, V_x(t,\tilde{X}_t), \qquad \sigma_{\rm eff} = (\tilde{u}\sigma_R, \tilde{\pi}\sigma, -\tilde{X}\sigma_I)^\top. \tag{A.1}$$

A key simplifying feature of the exponential-affine form is that

$$\frac{V_x(t,x)}{V(t,x)} = -\alpha(t),$$

which is deterministic. This allows us to express the worst-case kernel in a way that makes integrability transparent.

Lemma A.2 (Linear closed-loop dynamics)**.** *Under the feedback controls in Theorem 4.1, $\tilde{u}^*(t)$ is deterministic and $\tilde{\pi}^*(t)$ is affine in $\tilde{X}_t$. Consequently, the closed-loop real wealth process $\tilde{X}_t$ solves a linear stochastic differential equation of the form*

$$d\tilde{X}_t = (a_0(t) + a_1(t)\tilde{X}_t)\,dt + b_R(t)\,dW_R(t) + (b_0(t) + b_1(t)\tilde{X}_t)\,dW_S(t) + c_1(t)\tilde{X}_t\,dW_I(t),$$

with bounded deterministic coefficients on $[0,T]$. In particular, $\sup_{t\in[0,T]} \mathbb{E}[|\tilde{X}_t|^p] < \infty$ for any $p \ge 1$.

Proof. The explicit formulas imply taht $\tilde{u}^*(t)$ is deterministic and $\tilde{\pi}^*(t) = \pi_0(t) + \pi_1\tilde{X}_t$ with deterministic $\pi_0(t)$ and constant $\pi_1 = \sigma_I\rho_{SI}/\sigma$. Substituting into (2.5) yields a linear stochastic differential equation with bounded coefficients. Standard moment estimates for linear diffusions on a finite horizon give the stated bounds. □

Lemma A.3 (Novikov condition for the worst-case measure)**.** *Let* θ^* *be defined by* (A.1) *under the optimal feedback control. Then* θ^* *is admissible in the sense of the admissible distortion set* Θ *(Section 2):*

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \|\theta^*(s)\|^2\, ds\Big)\Big] < \infty,$$

so the exponential martingale is a true martingale and the worst-case measure is well-defined.

Proof. Using $V_x = -\alpha(t)V$ and writing σ_{eff} under the feedback control, we can rewrite θ^* as

$$\theta^*(t) = \eta\, \alpha(t)\, V(t, \tilde{X}_t)\, \sigma_{\text{eff}}(t, \tilde{X}_t).$$

Because $\alpha(\cdot)$ is bounded on $[0, T]$ and $\sigma_{\text{eff}}(t, \tilde{X}_t)$ is at most affine in $\tilde{X}_t$, Lemma A.2 yields finite moments of any order for $\tilde{X}_t$. Moreover, $|V(t, \tilde{X}_t)| = \exp\{-\alpha(t)\tilde{X}_t + A(t)\}$ has exponential growth controlled by Lemma A.1 and the finite-horizon moment bounds for linear diffusions. Combining these estimates implies that $\int_0^T \|\theta^*(s)\|^2 ds$ admits exponential moments, which is sufficient for Novikov's condition. □

A.4. Implication for novelty and transferability

The above lemmas show that the verification theorem in Section 4 rests on a structural mechanism: exponential-affine value function ⇒ deterministic V_x/V ⇒ explicit control of worst-case kernel θ^*; and affine feedback controls ⇒ closed-loop linear diffusion ⇒ finite-horizon moment bounds and an explicit Novikov check. This mechanism remains valid for a broad class of robust insurance–finance problems in which (i) the ambiguity set is entropy-penalized, (ii) the state is real wealth, and (iii) the candidate optimal controls are affine feedback laws. In this sense, the verification argument is not merely a routine step but a reusable theoretical template that supports explicit solvability under stochastic inflation and ambiguity.