Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 7264–7284.
DOI: 10.3934/math.2026299
Received: 20 January 2026
Revised: 07 March 2026
Accepted: 09 March 2026
Published: 19 March 2026

Research article

Metric dimension of line graphs of optimal fault-tolerant token ring networks

Asma Alasmri[1], Nor Muhainiah Mohd Ali[1], Ali Ahmad[2,*] and Muhammad Faisal Nadeem[3]

[1] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, UTM Johor Bahru 81310, Johor, Malaysia

[2] Department of Computer Science, College of Engineering and Computer Science, Jazan University, Jazan, Saudi Arabia

[3] Department of Mathematics, COMSATS University Islamabad, Lahore Campus 54000, Pakistan

* **Correspondence:** Email: ahmadsms@gmail.com; Tel: +966595889726.

Abstract: Interconnection networks are commonly modeled as graphs, where vertices represent processors or devices and edges represent communication links. In such networks, locating or identifying components using a small set of reference points is fundamental for routing, fault diagnosis, monitoring, and navigation. A standard graph-theoretic measure of this capability is the metric dimension (or locating number), defined as the minimum cardinality of a resolving set whose distance vectors uniquely distinguish all vertices. Since determining the metric dimension is NP-hard in general, exact values for structured network families are of both theoretical and practical interest. In this paper, we studied the metric dimension of the line graph of an optimal 2-fault-tolerant token ring network. The underlying network $\mathbb{T}_m^2$ augments a simple ring with additional links to ensure robust connectivity under up to two link or node failures, while the line graph $L(\mathbb{T}_m^2)$ represents the network at the link level. A lemma was established to prove the lower bound of the metric dimension via contradiction, while the upper bound was determined by explicitly constructing resolving sets. The analysis was conducted case by case according to the congruence of the network order modulo 4, which simplified verification of all representation vectors. Our results showed that the fault-tolerant links increase the metric dimension compared with ordinary token rings, highlighting the influence of additional links on network distinguishability and providing insights for the design of robust interconnection networks.

Keywords: metric dimension; line graph; resolving set; token ring network; fault-tolerant network
Mathematics Subject Classification: 05C12, 05C90

1. Introduction and literature review

Communication networks, interconnection topologies for multiprocessors, and a variety of distributed computing architectures can all be described in a natural fashion by graphs, in which vertices model processors (or stations) and edges model communication channels. This approach to interconnection network design and analysis allows graph theory to be applied in a unified mathematical fashion to the problems of connectivity, routing, reliability, and fault tolerance, as described in references [1, 2]. A formal definition for an interconnection network is a connected graph, $\mathbb{G} = (\mathcal{V}, E)$, where the vertex set, $\mathcal{V}$, represents the processing elements, and the edge set, E, represents the communication channels. In traditional interconnection topologies, the ring network, which is a cycle, is a basic and important construction because it is the simplest and most basic unit for the construction of larger topologies. Within ring-based local area networks, the traditional token ring approach controls access to the shared transmission medium using a control packet (token) that passes around the ring, enabling one station at a time to send data packets [3, 4]. Despite token ring technology having long passed out of common usage, its symmetrical cycle and regular structure continue to make it useful as a conceptual model of study for certain graph parameters and resources in interconnection networks [5, 6].

Locating or distinguishing unique objects such as processors, edges, failures, and configurations with a minimal number of resources has been one of the most important concepts in network analysis. The most cited distance-related parameter in this area has been "metric dimension." The metric dimension was first proposed by Slater [7, 8], and then independently by Harary and Melter [9].

The metric dimension has both strong theoretical foundations and wide-ranging applications. Apart from path graphs, it is well known that no connected graph has metric dimension 1 [10]. Cycle graphs satisfy $\dim(C_n) = 2$, capturing the intuition that two appropriately chosen landmarks break the rotational ambiguity in a ring. Beyond classical families, metric dimension has been studied for a broad spectrum of graphs due to its relevance to navigation and localization in robotics, network discovery, pattern recognition, and combinatorial optimization [11, 12]. In artificial intelligence and algorithmic graph theory, resolving sets serve as compact "signatures" that support distance-based classification, map processing, and strategic game analysis. This practical relevance has motivated sustained research on both exact computation and bounding techniques, especially because computing $\dim(\mathbb{G})$ is NP-hard in general and remains challenging even on restricted graph classes (see, e.g., the foundational literature in [13] and the references therein).

A substantial body of work is devoted to computing the metric dimension of well-structured graphs and graph operations. Chartrand et al. [13] initiated systematic investigations of resolvability and computed metric dimensions of several fundamental graph classes. For families with high symmetry, circulant graphs have attracted particular attention: Imran et al. [14] analyzed circulant graphs of the form $C_n(1, 2, 3)$ and showed boundedness patterns depending on arithmetic conditions on n. Related investigations include generalizations to Petersen-type structures and multigraph variants [15]. Several works focus on how metric dimension behaves under graph products and composition operations: Yero et al. studied the corona product [16]; Szolonki et al. examined joins of graphs [17]; and other authors developed resolvability results for additional operations and constrained classes [18]. Moreover, metric notions have been explored in digital geometry and discrete models of images, emphasizing the interplay between geometry-like distance structures and combinatorial resolvability [10]. In parallel, evolutionary and game-theoretic contexts have also motivated studying structural identifiability and interaction patterns in networks [17, 19].

Path-related and derived graphs form another active direction, as they provide tractable yet nontrivial instances where exact metric dimension can often be obtained. Ali et al. [20] investigated the metric dimension of middle and power graphs derived from paths and established explicit formulas. Alholi et al. [21] later explored boundedness conditions and exhibited families where the metric dimension remains constant under specific operations. Shahida et al. [22] computed the metric dimension for joins of two arbitrary graphs, while Pan et al. [23] studied splitting graphs of paths and cycles, and showed that their metric dimension can be unbounded. Peng et al. [24] contributed further by determining the metric dimension of Kneser-related graphs, where vertex identification interacts with combinatorial set systems.

In recent years, several refined variants of resolvability parameters and new graph families have been explored. Nadeem et al. [25] investigated the locating number of biswapped interconnection networks, linking network architecture to distance-based identification. Arulperumjothi et al. [26] determined the metric and fault-tolerant metric dimensions of fractal cubic networks, demonstrating how self-similarity shapes the size and structure of resolving sets. Vetrík et al. [27] studied circulant graphs with $2k$ generators and showed that metric dimension can fall below k, challenging earlier intuition about generator counts in Cayley graph resolvability. Hakanen et al. [28] classified unicyclic graphs in which every vertex belongs to all metric and strong metric bases, clarifying structural membership conditions. Assiri et al. [29] investigated fault-tolerant locating sets in biswapped multiprocessor networks, determining the minimum fault-tolerant locating number ℓ to ensure all vertices remain identifiable despite node failures. Nie and Xu [30] derived formulas for doubly metric dimensions of cactus and block graphs, revealing how cycle blocks impose stronger resolving requirements. Mhagama et al. [31] analyzed the edge metric dimension of line graphs of antiviral drug molecular structures, using edge-resolving sets to uniquely identify bonds and study structural properties. Dolžan et al. [32] connected algebraic constructions and metric basis theory by studying semiring total graphs, while Nazeer et al. [33] established constant metric dimension for certain planar pyramid-related families.

In 2025, Hayat et al. determined the fault-tolerant mixed metric dimension and discussed its application in secure and consistence networks [34]. Azhar et al. discussed the fault-tolerant metric dimension of carbon nanostructures in [35]. Recently, the computational and structural aspects of this topic of research have been explored and resolving sets of different families of graphs have been discussed. Knor et al. [36] emphasized that fault-tolerant metric bases can be significantly larger than minimum resolving sets, illustrating the combinatorial cost of robustness. Other studies have focused on graph models and real-world applications like optimization systems, and communication networks, where this topic plays an important role in supporting system consistency [37, 38]. These contributions demonstrate that fault-tolerant metric dimension is an active and rapidly developing research area with strong theoretical and applied significance. In spite of these innovations, existing studies focused on basic graph classes or common network structures. The application of fault-tolerant metric dimension to specific network topologies–particularly line graphs derived from fault-tolerant communication networks such as the token ring network-remains largely unexplored. Therefore, this study aims to bridge this gap by analyzing the metric dimension of line graphs of fault-tolerant token ring networks and provide explicit constructions of resolving sets tailored to this model.

A closely related and highly studied transformation is the *line graph*. Given a simple graph $\mathbb{G}$, its line graph $L(\mathbb{G})$ has the vertex set $E(\mathbb{G})$, where two vertices in $L(\mathbb{G})$ are adjacent whenever the corresponding edges in $\mathbb{G}$ share an endpoint. Line graphs are important in network science because they convert link-based questions into vertex-based questions, enabling the modeling of link conflicts, routing transitions, and resource contention. They also appear naturally in chemistry (where edges encode bonds) and in communication settings, where links rather than processors are the primary objects of interest. The metric dimension of line graphs has therefore attracted attention as a link-identification analogue of classical resolvability. Feng et al. [39] investigated $\dim(L(\mathbb{G}))$ and established bounds such as

$$\lceil \log_2 \Delta(\mathbb{G}) \rceil \leq \dim(L(\mathbb{G})) \leq |V(\mathbb{G})| - 2,$$

where $\Delta(\mathbb{G})$ is the maximum degree. Eroh et al. [40] computed the metric dimension and zero-forcing numbers of wheel and bouquet line graphs, exhibiting strong connections between these parameters. Farooq et al. [41] studied line graphs arising from molecular families and showed that chemical structures often yield constant or efficiently computable resolving sets.

Alongside exact results, algorithmic and learning-based approaches have recently emerged for estimating metric dimension on large instances. Wu et al. [42] proposed a hybrid method combining graph learning with greedy strategies to approximate $\dim(\mathbb{G})$ effectively on random graph models. Khan et al. [43] proved constant metric dimension for certain bicyclic families regardless of size, while Pranjali et al. [44] studied unit graphs over commutative rings and classified cases with small metric dimension values. Mhagama et al. [45] refined the edge metric dimension edim of splitting graphs, providing smaller edge-resolving sets for paths and cycles, enhancing prior results and achieving $\mathrm{edim}(G) = \dim(G)$. Vidya et al. [46] investigated infinite planar graphs and showed that their metric bases form independent sets and remain constant across each family.

Several studies have investigated the metric dimension of ordinary token ring networks, which correspond to cycle graphs C_m and contain only adjacent connections between vertices. It is well known that for such networks, the metric dimension of the underlying cycle is $\dim(C_m) = 2$, and since the line graph of a cycle is also a cycle, we have $\dim(L(C_m)) = 2$ [7,9]. These results provide a baseline understanding of resolving sets in simple ring networks.

Motivated by the token passing mechanism in ring networks, a natural way to increase modeling fidelity is to study not only the underlying communication graph, but also the *state space* formed by all possible placements of a fixed number of indistinguishable tokens on the vertices of that graph. This leads to the notion of a *token graph*. In this model, each vertex of the token graph represents a configuration of tokens (that is, a choice of vertices occupied by the tokens), and two configurations are adjacent precisely when one token can be moved along a single edge of the underlying graph while all other tokens remain fixed. Token graphs arise naturally in reconfiguration problems, routing of indistinguishable agents, and distributed systems where global states are determined by the locations of multiple identical resources. When the underlying graph is a cycle (the ring network), the associated token graph provides a mathematically precise framework for describing multiple tokens circulating around the ring and interacting through adjacency constraints. This perspective connects classical ring topology with modern state-space methods for analyzing network dynamics.

Although the metric dimension has been extensively studied for many graph families and for line graphs, the *line graph of a token graph* has received comparatively little attention from the viewpoint of resolvability. The token graph encodes configurations as vertices and single-token moves as edges, while its line graph transforms these moves into vertices, declaring two moves adjacent whenever they share a common configuration endpoint. Consequently, studying the metric dimension in this setting corresponds to identifying a minimum collection of landmark moves whose distance information uniquely distinguishes every possible move in the configuration space. This viewpoint is natural in communication and reconfiguration contexts, since in many protocols, the primary objects of interest are the *events* (moves, link activations, transitions) rather than static configurations. In particular, when the underlying network is a ring, the induced symmetry provides a rich yet structured family in which one may expect sharp bounds and exact results.

Despite substantial progress in understanding the metric dimension of graphs and their line graphs, several challenges remain open for token-based constructions. First, passing from an underlying graph to its token graph can substantially alter fundamental distance properties, such as shortest-path structure and diameter, and can also change the nature of symmetry in the resulting model. These changes make resolvability questions significantly more delicate than in the base graph. Second, applying the line graph operation introduces an additional layer in which vertices represent moves in the configuration space, so resolving sets must distinguish transitions rather than states. Third, much of the existing work on metric dimension in line graphs and configuration graphs does not address robustness requirements such as fault tolerance, even though real monitoring devices may fail and identification must remain reliable under such failures. Therefore, a systematic investigation of the metric dimension, and subsequently, fault-tolerant variants, for line graphs of token graphs over ring-based networks remains largely unexplored and strongly motivates the present study.

In this paper, we initiate a detailed study of distance-based resolvability for the line graphs associated with token graphs, with particular emphasis on token ring networks and closely related structured base graphs. We develop techniques for constructing resolving sets that exploit the layered structure inherited from token configurations and move adjacency, and we obtain exact values. These results extend existing literature on metric dimension in interconnection networks.

2. Preliminaries and basic definitions

Throughout this paper, all graphs are finite, simple, and connected. For a graph $\mathbb{G}$, we write $V(\mathbb{G})$ and $E(\mathbb{G})$ for its vertex set and edge set, respectively. The distance between vertices $x, y \in V(\mathbb{G})$ is denoted by $d_{\mathbb{G}}(x, y)$ (or simply $d(x, y)$ when the graph is clear from context), and equals the length of a shortest x-y path in $\mathbb{G}$.

2.1. Basic graph notions

Definition 2.1. *A **graph** $\mathbb{G}$ is an ordered pair $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$, where $V(\mathbb{G})$ is a nonempty set whose elements are called vertices and $E(\mathbb{G}) \subseteq \{\{u, v\} \mid u, v \in V(\mathbb{G}), u \neq v\}$ is a set of unordered pairs of distinct vertices called edges.*

Definition 2.2. *Let $\mathbb{G}$ be a simple connected graph and let $x, y \in V(\mathbb{G})$ be two vertices. The* distance *between x and y, denoted by $d_{\mathbb{G}}(x, y)$, is the length of a shortest path connecting x and y in $\mathbb{G}$.*

Definition 2.3. *Let $\mathbb{G}$ be a graph. The **neighborhood** of a vertex $x \in V(\mathbb{G})$ is*

$$N_{\mathbb{G}}(x) = \{\, y \in V(\mathbb{G}) \mid \{x, y\} \in E(\mathbb{G}) \,\}.$$

*More generally, for an integer $k \geq 1$, the k**-neighborhood** of x is*

$$N_k^{\mathbb{G}}(x) = \{\, y \in V(\mathbb{G}) \mid d_{\mathbb{G}}(x, y) \leq k \,\}.$$

Clearly, $N_1^{\mathbb{G}}(x) = N_{\mathbb{G}}(x)$.

Definition 2.4. *In applied settings, a **network** is a graph $\mathbb{N} = (V(\mathbb{N}), E(\mathbb{N}), \phi)$, where ϕ is a mapping that assigns additional information (such as weights, directions, or capacities) to elements of $V(\mathbb{N})$ or $E(\mathbb{N})$.*

2.2. Line graphs

Definition 2.5. *Let $\mathbb{G}$ be a graph. The **line graph** of $\mathbb{G}$, denoted by $L(\mathbb{G})$, is the graph defined by*

$$V(L(\mathbb{G})) = E(\mathbb{G}), \qquad E(L(\mathbb{G})) = \{\, e_i e_j \mid e_i, e_j \in E(\mathbb{G}),\ e_i \cap e_j \neq \varnothing \,\}.$$

Equivalently, each vertex of $L(\mathbb{G})$ represents an edge of $\mathbb{G}$, and two vertices of $L(\mathbb{G})$ are adjacent if and only if the corresponding edges of $\mathbb{G}$ are incident.

2.3. Resolving sets and metric dimension

Definition 2.6. *Let $\mathbb{G}$ be a connected graph. A set $\mathbb{S} \subseteq V(\mathbb{G})$ is a **resolving set** for $\mathbb{G}$ if for every pair of distinct vertices $x, y \in V(\mathbb{G})$, there exists $s \in \mathbb{S}$ such that*

$$d_{\mathbb{G}}(x, s) \neq d_{\mathbb{G}}(y, s).$$

Definition 2.7. *The **metric dimension** of a connected graph $\mathbb{G}$, denoted by $\dim(\mathbb{G})$, is the minimum cardinality of a resolving set of $\mathbb{G}$, that is,*

$$\dim(\mathbb{G}) = \min\{\, |\mathbb{S}| \ : \ \mathbb{S} \subseteq V(\mathbb{G}) \text{ and } \mathbb{S} \text{ resolves } \mathbb{G} \,\}.$$

*Any resolving set of size $\dim(\mathbb{G})$ is called a **metric basis** of $\mathbb{G}$.*

2.4. Optimal $\mathbb{K}$-fault-tolerant networks for token rings

We also recall a classical augmentation of the cycle used as an optimal $\mathbb{K}$-fault-tolerant topology for ring-like networks.

Definition 2.8. *(Circulant graph) Let $n \geq 3$ and let $D = \{d_1, \ldots, d_t\} \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. The **circulant graph** $\mathrm{Circ}(n; D)$ is the graph with vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edges*

$$\{v_i, v_j\} \in E \quad \Longleftrightarrow \quad (i - j) \bmod n \in D \text{ or } (j - i) \bmod n \in D.$$

Equivalently, v_i is adjacent to $v_{i \pm d}$ for each $d \in D$, with indices computed modulo n.

Definition 2.9. *(Optimal $\mathbb{K}$-fault-tolerant token ring augmentation [47]) Let m and $\mathbb{K}$ be integers with $m > 2\mathbb{K}$ and $V = \{v_0, v_1, \ldots, v_{m-1}\}$ labeled cyclically in clockwise order. The graph $\mathbb{T}_m^{\mathbb{K}}$ is constructed on V so that it contains the* ***elementary cycle***

$$\langle v_0, v_1, \ldots, v_{m-1}, v_0 \rangle,$$

and additional links are added to provide $\mathbb{K}$-fault tolerance as follows:
***(i)** $\mathbb{K}$ **even.** In this case, $\mathbb{T}_m^{\mathbb{K}}$ is the circulant graph with connection set $D = \{1, 2, \ldots, \frac{\mathbb{K}}{2} + 1\}$, i.e.,*

$$\{v_i, v_{i\pm j}\} \in E(\mathbb{T}_m^{\mathbb{K}}) \quad \text{for all } i \text{ and } 1 \le j \le \frac{\mathbb{K}}{2} + 1.$$

***(ii)** $\mathbb{K}$ **odd and** m **even.** Then $\mathbb{T}_m^{\mathbb{K}}$ is the circulant graph with distance set $D = \{1, 2, \ldots, \frac{\mathbb{K}+1}{2}, \frac{m}{2}\}$.*
***(iii)** $\mathbb{K}$ **odd and** m **odd.** Then $\mathbb{T}_m^{\mathbb{K}}$ is obtained by joining each v_i to v_{i+j} for $1 \le j \le \frac{\mathbb{K}+1}{2}$ (indices modulo m), and adding the additional long chords prescribed in [47] (including the special edge $\{v_0, v_{\frac{m-1}{2}}\}$ when required) to achieve the optimal $\mathbb{K}$-fault-tolerant design.*

Visual construction of the 2-fault-tolerant token ring. To illustrate Definition 2.9, we consider the case $\mathbb{K} = 2$ and $m = 9$. Since $\mathbb{K}$ is even, case (i) of the definition applies and the graph $\mathbb{T}_9^2$ is the circulant graph with connection set $D = \{1, 2\}$.

First, the elementary cycle $\langle v_0, v_1, v_2, \ldots, v_8, v_0 \rangle$ is constructed, forming a simple token ring. Next, additional links are introduced so that each vertex v_i is joined not only to its immediate neighbors $v_{i\pm1}$ but also to the vertices $v_{i\pm2}$ (indices taken modulo 9). These additional chords increase the connectivity of the ring and ensure tolerance to two link failures.

Figure 1 illustrates the resulting 2-fault-tolerant token ring augmentation $\mathbb{T}_9^2$, where the circular edges represent the elementary cycle and the inner chords correspond to the additional connections of distance two. Figure 2 shows the corresponding line graph $L(\mathbb{T}_9^2)$, whose vertices represent the edges of $\mathbb{T}_9^2$ and where two vertices are adjacent whenever the corresponding edges in $\mathbb{T}_9^2$ share a common endpoint.

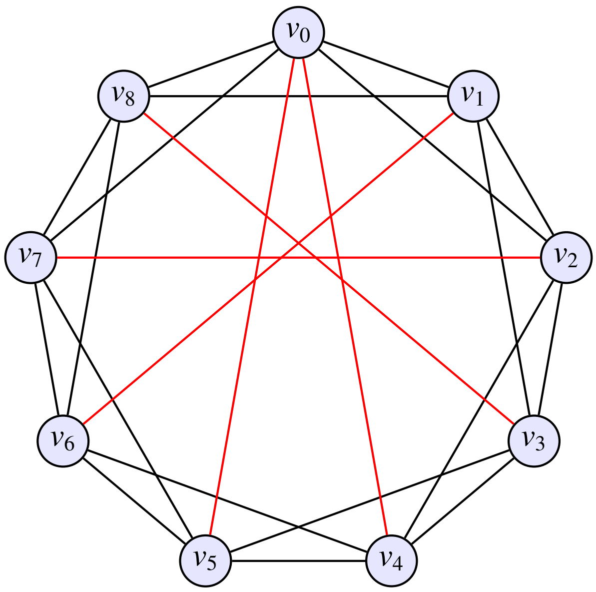

Figure 1. $\mathbb{T}_9^2$.

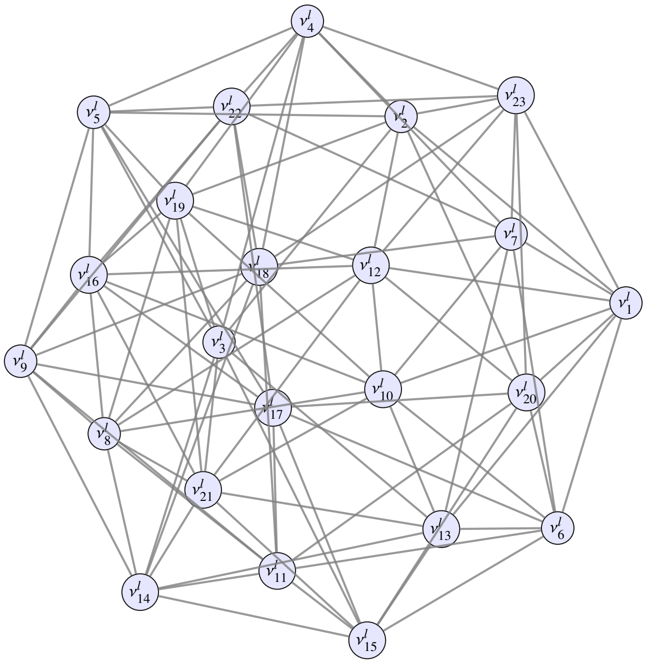

Figure 2. $L(\mathbb{T}^2_9)$.

Definition 2.10. *(Indexing of the vertices of $L(\mathbb{T}^2_m)$) Let $\mathbb{T}^2_m$ be an optimal 2-fault-tolerant token ring of order m with $V(\mathbb{T}^2_m) = \{v_0, v_1, \ldots, v_{m-1}\}$ (indices modulo m). For each $i \in \{0, 1, \ldots, m-1\}$, define the edges*

$$a_i = \{v_i, v_{i+1}\} \qquad \textit{and} \qquad b_i = \{v_i, v_{i+2}\},$$

(indices modulo m). Then $E(\mathbb{T}^2_m) = \{a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}\}$ and $|E(\mathbb{T}^2_m)| = 2m = n$.

We label the vertices of the line graph $L(\mathbb{T}^2_m)$ by

$$v^l_{i+1} = a_i \quad \textit{for } i = 0, 1, \ldots, m-1, \qquad v^l_{m+i+1} = b_i \quad \textit{for } i = 0, 1, \ldots, m-1.$$

Thus $V(L(\mathbb{T}^2_m)) = \{v^l_1, v^l_2, \ldots, v^l_n\}$.

Lemma 2.1. *For $K = 2$, the graph T^2_m is the circulant graph* Circ$(m; \{1, 2\})$*, and hence it is 4-regular. Consequently, its line graph $L(T^2_m)$ is 6-regular.*

Proof. By the definition of T^2_m with $K = 2$, each vertex v_i is adjacent to $v_{i\pm1}$ and $v_{i\pm2}$ (indices modulo m), which matches the definition of the circulant graph Circ$(m; \{1, 2\})$. Therefore, each vertex has degree 4.

In a line graph $L(T^2_m)$, each vertex corresponds to an edge of T^2_m. Since each edge in T^2_m is incident to two vertices, each of degree 4, the corresponding vertex in $L(T^2_m)$ is adjacent to $(4-1) + (4-1) = 6$ vertices. Hence, $L(T^2_m)$ is 6-regular. □

3. Main results

In this section, we determine the metric dimension of the line graph of an optimal 2-fault-tolerant token ring network. The main result is stated in Theorem 3.1, which establishes the exact value of the metric dimension for this class of graphs. The lower bound in Theorem 3.1 follows directly from Lemma 3.1, whereas the proof of the upper bound is divided into four cases according to the structural symmetry of the graph. In each case, we construct appropriate resolving sets and present the corresponding distance representations. These cases are addressed separately in Theorems 3.2–3.5.

Theorem 3.1. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring of order m and size $n := 2m$. Let $L(\mathbb{T}_m^2)$ be its line graph, which has order $n := 2m$. Then*

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Lemma 3.1. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring of order m and size $n := 2m$, and let $L(\mathbb{T}_m^2)$ denote its line graph. Then*

$$n := |V(L(\mathbb{T}_m^2))| = 2m$$

with $m \geq 10$, and the metric dimension of $L(\mathbb{T}_m^2)$ satisfies

$$\dim(L(\mathbb{T}_m^2)) \ \geq \ 4.$$

Proof. We aim to prove $\dim(L(\mathbb{T}_m^2)) \geq 4$. Assume, for the sake of contradiction, that $\dim(L(\mathbb{T}_m^2)) < 4$. By Theorem 1.1, the metric dimension is not equal to 1.

Suppose first that $\dim(L(\mathbb{T}_m^2)) = 2$, and let

$$\mathbb{S}^{l'} = \{v_\eta^l, v_\xi^l\}$$

be a resolving set. Then, for every pair of distinct vertices $v_x^l, v_y^l \in \mathcal{V}(L(\mathbb{T}_m^2))$, we must have

$$r(v_x^l \mid \mathbb{S}^{l'}) \neq r(v_y^l \mid \mathbb{S}^{l'}),$$

i.e., the metric representations with respect to $\mathbb{S}^{l'}$ are distinct.

However, due to the structural symmetry and regularity of $L(\mathbb{T}_m^2)$, there exist two distinct vertices $v_x^l, v_y^l \in N_2(v_\eta^l) \cup N_2(v_\xi^l)$ whose metric representations are identical with respect to $\mathbb{S}^l$, i.e.,

$$r(v_x^l \mid \mathbb{S}^{l'}) = r(v_y^l \mid \mathbb{S}^{l'}),$$

contradicting the assumption that $\mathbb{S}^{l'}$ is a resolving set.

Now suppose that $\dim(L(\mathbb{T}_m^2)) = 3$, with a resolving set

$$\mathbb{S}^{l'} = \{v_\eta^l, v_\xi^l, v_\zeta^l\}.$$

As in the previous case, the graph's symmetry and uniform connectivity imply the existence of two distinct vertices $v_x^l, v_y^l \in N_2(v_\eta^l) \cup N(v_\xi^l) \cup N(v_\zeta^l)$ such that

$$r(v_x^l \mid \mathbb{S}^{l'}) = r(v_y^l \mid \mathbb{S}^{l'}),$$

which again violates the resolving set condition.

In both cases, we arrive at a contradiction. Therefore, no resolving set of cardinality less than 4 exists for $L(\mathbb{T}_m^2)$, and we conclude that

$$\dim(L(\mathbb{T}_m^2)) \geq 4.$$

□

Theorem 3.2. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring of order m, where $m = 4k + 6$ for some integer $k \geq 1$ and size $n := 2m$. Let $L(\mathbb{T}_m^2)$ be its line graph, which has order $n := 2m$. Then*

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Proof. Let $L(\mathbb{T}_m^2)$ be the line graph of $\mathbb{T}_m^2$, and denote its vertices by v_ι^l. Our aim is to show that the metric dimension of $L(\mathbb{T}_m^2)$ equals 4.

First, Lemma 3.1 yields the required lower bound

$$\dim(L(\mathbb{T}_m^2)) \geq 4.$$

Thus, it suffices to construct a resolving set of cardinality 4.

Define

$$\mathbb{S}^l = \{v_1^l, v_4^l, v_{m-1}^l, v_{m+1}^l\}.$$

We claim that $\mathbb{S}^l$ resolves $L(\mathbb{T}_m^2)$. Equivalently, every vertex v_ι^l has a unique metric representation $r\left(v_\iota^l \mid \mathbb{S}^l\right)$.

The metric representations with respect to $\mathbb{S}^l$ are given explicitly as follows:

For $\iota = 1, 2, 3$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{2} \right\rfloor, 1, \frac{n-4}{8} - \left\lfloor \frac{\iota}{2} \right\rfloor + \left\lfloor \frac{\iota}{3} \right\rfloor, \frac{n-4}{8} + \left\lfloor \frac{\iota}{3} \right\rfloor\right).$$

For $\iota = 4, 5, 6$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{4} \right\rfloor, 2\left\lfloor \frac{\iota-3}{2} \right\rfloor, \frac{n-4}{8} - \left\lfloor \frac{\iota-3}{2} \right\rfloor, \frac{n+4}{8} - \iota + 4\right).$$

For $\iota = 7, 8, 9$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{4} \right\rfloor, \left\lfloor \frac{\iota-4}{2} \right\rfloor, \frac{n-4}{8} - \iota + 7, \frac{n-4}{8} - \left\lfloor \frac{\iota-6}{2} \right\rfloor\right).$$

For $\iota = 10, \ldots, m-4$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{4} \right\rfloor, \left\lfloor \frac{\iota+2}{4} \right\rfloor, \frac{n-4}{8} - \left\lfloor \frac{\iota-1}{4} \right\rfloor, \frac{n+4}{8} - \left\lfloor \frac{\iota+1}{4} \right\rfloor\right).$$

For $\iota = m-3, \ldots, m-1$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{4} \right\rfloor, \left\lfloor \frac{\iota+2}{4} \right\rfloor, \frac{n-2}{2} - \iota, \left\lfloor \frac{2\iota-n+10}{4} \right\rfloor\right).$$

For $\iota = m, \ldots, m+2$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{4} \right\rfloor, \left\lfloor \frac{\iota+2}{4} \right\rfloor, \left\lfloor \frac{2\iota-n+6}{4} \right\rfloor - \left\lfloor \frac{2\iota-n+2}{6} \right\rfloor, 2 - \left\lfloor \frac{2\iota-n+6}{4} \right\rfloor - \left\lfloor \frac{2\iota-n+2}{6} \right\rfloor\right).$$

For $\iota = m+3, \ldots, n$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\frac{n+4}{8} - \left\lfloor \frac{2\iota-n-6}{8} \right\rfloor, \frac{n+4}{8} - \left\lfloor \frac{2\iota-n-2}{8} \right\rfloor, \left\lfloor \frac{2\iota-n+8}{8} \right\rfloor, \alpha\right),$$

where

$$\alpha = \begin{cases} 2 - \left\lfloor \frac{2\iota - n - 4}{4} \right\rfloor, & \text{if } \iota = m+3, \ldots, m+5, \\ \left\lfloor \frac{2\iota - n + 4}{8} \right\rfloor, & \text{otherwise.} \end{cases}$$

Since the above formulas yield distinct metric representations for all vertices of $L(\mathbb{T}_m^2)$ when $n := 2m$ and $m = 4k + 6$ for some integer $k \geq 1$, it follows that $\mathbb{S}^l$ is a resolving set. Hence

$$\dim(L(\mathbb{T}_m^2)) \leq 4.$$

Combining this with Lemma 3.1, we conclude that $\dim(L(\mathbb{T}_m^2)) = 4$. □

Theorem 3.3. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring of order m, where $m = 4k + 7$ for some integer $k \geq 1$ and size $n := 2m$. Let $L(\mathbb{T}_m^2)$ denote its line graph, which has order $n := 2m$. Then*

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Proof. Let $L(\mathbb{T}_m^2)$ be the line graph of $\mathbb{T}_m^2$, and denote its vertices by v_ι^l. We claim that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) = 4.$$

By Lemma 3.1, we already have the lower bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \geq 4.$$

Thus, it remains only to establish the upper bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4$$

by exhibiting a resolving set of size 4.

Consider the set

$$\mathbb{S}^l = \{v_1^l, v_{10}^l, v_{m+6}^l, v_{m+8}^l\}.$$

It can be verified that every vertex of $L(\mathbb{T}_m^2)$ has a distinct metric representation with respect to $\mathbb{S}^l$; hence, $\mathbb{S}^l$ is a resolving set for $L(\mathbb{T}_m^2)$. Since $|\mathbb{S}^l| = 4$, it follows that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4.$$

To derive the distance representations, we compute the distances from each vertex v_ι^l to the vertices of the resolving set

$$\mathbb{S}^l = \{v_1^l, v_{10}^l, v_{m+6}^l, v_{m+8}^l\}$$

in the line graph $L(\mathbb{T}_m^2)$. Recall that each vertex of $L(\mathbb{T}_m^2)$ corresponds to an edge of $\mathbb{T}_m^2$, and two vertices in the line graph are adjacent whenever the corresponding edges in $\mathbb{T}_m^2$ share a common endpoint. Therefore, the distance between two vertices in $L(\mathbb{T}_m^2)$ is obtained by counting the length of the shortest path between the corresponding edges in the original network. Using the cyclic structure and symmetry of the token ring network, these shortest-path distances can be expressed in closed form, which leads to the following representation vectors.

For completeness, the metric representation $r\Big(v_\iota^l \mid \mathbb{S}^l\Big)$ is given explicitly as follows:

For $\iota = 1, 2, 3$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor\frac{\iota}{2}\right\rfloor, \left\lfloor\frac{\iota+3}{2}\right\rfloor, \frac{n+2}{8} - \left\lfloor\frac{\iota}{3}\right\rfloor, \frac{n-6}{8} - \left\lfloor\frac{\iota}{3}\right\rfloor\right).$$

For $\iota = 4, \ldots, m+3$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \begin{cases} \left(\left\lfloor\frac{\iota}{4}\right\rfloor, 7-\iota, \frac{n-6}{8} + \left\lfloor\frac{\iota-3}{2}\right\rfloor - 2\left\lfloor\frac{\iota-3}{4}\right\rfloor, \frac{n-6}{8} + \left\lfloor\frac{\iota-3}{2}\right\rfloor - 2\left\lfloor\frac{\iota-3}{4}\right\rfloor\right), & \text{if } \iota = 4, \ldots, 6, \\ \left(\left\lfloor\frac{\iota}{4}\right\rfloor, 2-\left\lfloor\frac{\iota-6}{2}\right\rfloor, \frac{n-6}{8} + \left\lfloor\frac{\iota-3}{2}\right\rfloor - 2\left\lfloor\frac{\iota-3}{4}\right\rfloor, \frac{n-6}{8} + \left\lfloor\frac{\iota-3}{2}\right\rfloor - 2\left\lfloor\frac{\iota-3}{4}\right\rfloor\right), & \text{if } \iota = 7, \\ \left(\left\lfloor\frac{\iota}{4}\right\rfloor, 2-\left\lfloor\frac{\iota-6}{2}\right\rfloor, \frac{n+2}{8} - \left\lfloor\frac{\iota-5}{4}\right\rfloor, \frac{n+2}{8} - \left\lfloor\frac{\iota-7}{4}\right\rfloor\right), & \text{if } \iota = 8, \ldots, 10, \\ \left(\left\lfloor\frac{\iota}{4}\right\rfloor, \left\lfloor\frac{\iota-6}{4}\right\rfloor, \frac{n+2}{8} - \left\lfloor\frac{\iota-5}{4}\right\rfloor, \frac{n+2}{8} - \left\lfloor\frac{\iota-7}{4}\right\rfloor\right), & \text{otherwise.} \end{cases}$$

For $\iota = m+4, \ldots, n$, let:

$$\Phi(\iota) = \frac{n-6}{8} + \left\lfloor\frac{\iota-3}{2}\right\rfloor - 2\left\lfloor\frac{\iota-3}{4}\right\rfloor, \quad \Psi(\iota, k) = \frac{n+2}{8} - \left\lfloor\frac{\iota-k}{4}\right\rfloor.$$

The values are then given by:

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \begin{cases} \left(\lfloor\frac{\iota}{4}\rfloor, 7-\iota, \Phi(\iota), \Phi(\iota)\right), & \text{if } \iota = 4, 5, 6, \\ \left(\lfloor\frac{\iota}{4}\rfloor, 2-\lfloor\frac{\iota-6}{2}\rfloor, \Phi(\iota), \Phi(\iota)\right), & \text{if } \iota = 7, \\ \left(\lfloor\frac{\iota}{4}\rfloor, 2-\lfloor\frac{\iota-6}{2}\rfloor, \Psi(\iota,5), \Psi(\iota,7)\right), & \text{if } \iota = 8, 9, 10, \\ \left(\lfloor\frac{\iota}{4}\rfloor, \lfloor\frac{\iota-6}{4}\rfloor, \Psi(\iota,5), \Psi(\iota,7)\right), & \text{otherwise.} \end{cases}$$

Since these formulas yield distinct metric representations for all vertices of $L(\mathbb{T}_m^2)$ when $n := 2m$ and $m = 4k+7$ for some integer $k \geq 1$, we obtain

$$\dim\left(L(\mathbb{T}_m^2)\right) \leq 4.$$

Combining this with the lower bound from Lemma 3.1, we conclude that

$$\dim\left(L(\mathbb{T}_m^2)\right) = 4.$$

This completes the proof. □

Theorem 3.4. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring of order m, where $m = 4k+8$ for some integer $k \geq 1$ and size $n := 2m$. Let $L(\mathbb{T}_m^2)$ denote its line graph, which has order $n := 2m$. Then*

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Proof. Let $L(\mathbb{T}_m^2)$ be the line graph of $\mathbb{T}_m^2$, and label its vertices by v_ι^l. Our goal is to prove that

$$\dim\left(L(\mathbb{T}_m^2)\right) = 4.$$

By Lemma 3.1, we already have the lower bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \geq 4.$$

Therefore, it suffices to establish the upper bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4$$

by constructing a resolving set of size 4.

Consider the set

$$\mathbb{S}^l = \{\nu_1^l,\, \nu_4^l,\, \nu_{m-2}^l,\, \nu_m^l\}.$$

It can be verified that every vertex of $L(\mathbb{T}_m^2)$ has a distinct metric representation with respect to $\mathbb{S}^l$; hence, $\mathbb{S}^l$ is a resolving set for $L(\mathbb{T}_m^2)$. Since $|\mathbb{S}^l| = 4$, it follows that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4.$$

To derive the distance representations, we compute the distances from each vertex ν_ι^l to the vertices of the resolving set

$$\mathbb{S}^l = \{\nu_1^l,\, \nu_4^l,\, \nu_{m-2}^l,\, \nu_m^l\}$$

in the line graph $L(\mathbb{T}_m^2)$. Recall that each vertex of $L(\mathbb{T}_m^2)$ corresponds to an edge of $\mathbb{T}_m^2$, and two vertices in the line graph are adjacent whenever the corresponding edges in $\mathbb{T}_m^2$ share a common endpoint. Therefore, the distance between two vertices in $L(\mathbb{T}_m^2)$ is obtained by counting the length of the shortest path between the corresponding edges in the original network. Using the cyclic structure and symmetry of the token ring network, these shortest-path distances can be expressed in closed form, which leads to the following representation vectors.

For completeness, the metric representation $r\Big(\nu_\iota^l \mid \mathbb{S}^l\Big)$ is given explicitly as follows:

For $\iota = 1, 2, 3$,

$$r\Big(\nu_\iota^l \mid \mathbb{S}^l\Big) = \left(\left\lfloor \frac{\iota}{2} \right\rfloor,\, \left\lfloor \frac{\iota+3}{2} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{\iota-1}{2} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{\iota-1}{2} \right\rfloor\right).$$

For $\iota = 4, \ldots, m+4$,

$$r\Big(\nu_\iota^l \mid \mathbb{S}^l\Big) = \begin{cases} \left(\left\lfloor \frac{\iota}{4} \right\rfloor,\, 7-\iota,\, \frac{n}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - \left\lfloor \frac{\iota-3}{3} \right\rfloor,\, \frac{n-8}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - 2\left\lfloor \frac{\iota-3}{4} \right\rfloor\right), & \text{if } \iota = 4, \ldots, 6, \\ \left(\left\lfloor \frac{\iota}{4} \right\rfloor,\, 2 - \left\lfloor \frac{\iota-6}{2} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{\iota-7}{4} \right\rfloor,\, \frac{n-8}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - 2\left\lfloor \frac{\iota-3}{4} \right\rfloor\right), & \text{if } \iota = 7, \\ \left(\left\lfloor \frac{\iota}{4} \right\rfloor,\, 2 - \left\lfloor \frac{\iota-6}{2} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{\iota-7}{4} \right\rfloor,\, \frac{n+8}{8} - \left\lfloor \frac{\iota-5}{4} \right\rfloor\right), & \text{if } \iota = 8, \ldots, 10, \\ \left(\left\lfloor \frac{\iota}{4} \right\rfloor,\, \left\lfloor \frac{\iota-6}{4} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{\iota-7}{4} \right\rfloor,\, \frac{n+8}{8} - \left\lfloor \frac{\iota-5}{4} \right\rfloor\right), & \text{otherwise.} \end{cases}$$

For $\iota = m+5, \ldots, n$,

$$r\Big(\nu_\iota^l \mid \mathbb{S}^l\Big) = \begin{cases} \left(\frac{n}{8} - \left\lfloor \frac{2\iota-n-10}{8} \right\rfloor,\, \frac{n-8}{8} + \left\lfloor \frac{2\iota-n-4}{8} \right\rfloor,\, \alpha_1,\, \alpha_2\right), & \text{if } \iota = m+5, \ldots, m+10, \\ \left(\frac{n}{8} - \left\lfloor \frac{2\iota-n-10}{8} \right\rfloor,\, \frac{n}{8} - \left\lfloor \frac{2\iota-n-22}{8} \right\rfloor,\, \left\lfloor \frac{2\iota-n-4}{8} \right\rfloor,\, \left\lfloor \frac{2\iota-n-8}{8} \right\rfloor\right), & \text{otherwise,} \end{cases}$$

where

$$\alpha_1 = \begin{cases} \frac{n}{2} - \iota + 6, & \text{if } \iota = m+5, m+6, \\ \left\lfloor \frac{2\iota - n - 4}{8} \right\rfloor, & \text{otherwise,} \end{cases} \qquad \alpha_2 = \begin{cases} 1 - \left\lfloor \frac{2\iota - n - 8}{8} \right\rfloor, & \text{if } \iota = m+5, \ldots, m+8, \\ \left\lfloor \frac{2\iota - n - 8}{8} \right\rfloor, & \text{otherwise.} \end{cases}$$

Since these formulas yield distinct metric representations for all vertices of $L(\mathbb{T}_m^2)$ when $n := 2m$ and $m = 4k + 8$ for some integer $k \geq 1$, we conclude that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4.$$

Combining this with the lower bound from Lemma 3.1, we obtain

$$\dim\Big(L(\mathbb{T}_m^2)\Big) = 4.$$

This completes the proof. □

Theorem 3.5. *Let $\mathbb{T}_m^2$ be an optimal 2-fault-tolerant token ring network of order m, where $m = 4k + 9$ for some integer $k \geq 1$ and size $n := 2m$. Let $L(\mathbb{T}_m^2)$ denote its line graph, which has order $n := 2m$. Then*

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Proof. Let $L(\mathbb{T}_m^2)$ be the line graph of $\mathbb{T}_m^2$, and label its vertices by v_ι^l. We claim that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) = 4.$$

By Lemma 3.1, we already have the lower bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \geq 4.$$

Hence, it remains only to prove the upper bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4$$

by constructing a resolving set of size 4.

Consider the set

$$\mathbb{S}^l = \{v_1^l,\ v_4^l,\ v_{m-2}^l,\ v_m^l\}.$$

It can be verified that every vertex of $L(\mathbb{T}_m^2)$ has a distinct metric representation with respect to $\mathbb{S}^l$; therefore, $\mathbb{S}^l$ is a resolving set for $L(\mathbb{T}_m^2)$. Since $|\mathbb{S}^l| = 4$, it follows that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \leq 4.$$

To derive the distance representations, we compute the distances from each vertex v_ι^l to the vertices of the resolving set

$$\mathbb{S}^l = \{v_1^l,\ v_4^l,\ v_{m-2}^l,\ v_m^l\}$$

in the line graph $L(\mathbb{T}_m^2)$. Recall that each vertex of $L(\mathbb{T}_m^2)$ corresponds to an edge of $\mathbb{T}_m^2$, and two vertices in the line graph are adjacent whenever the corresponding edges in $\mathbb{T}_m^2$ share a common endpoint. Therefore, the distance between two vertices in $L(\mathbb{T}_m^2)$ is obtained by counting the length of the shortest path between the corresponding edges in the original network. Using the cyclic structure and symmetry of the token ring network, these shortest-path distances can be expressed in closed form, which leads to the following representation vectors.

For completeness, the metric representation $r\left(v_\iota^l \mid \mathbb{S}^l\right)$ is given explicitly as follows:

For $\iota = 1, 2, 3$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \left(\left\lfloor \frac{\iota}{2} \right\rfloor, \left\lfloor \frac{\iota+3}{2} \right\rfloor, \frac{n-2}{8} - \left\lfloor \frac{\iota-1}{2} \right\rfloor, \frac{n-2}{8} - \left\lfloor \frac{\iota-1}{2} \right\rfloor\right).$$

For $\iota = 4, \dots, m+2$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \begin{cases} \left(\left\lfloor \frac{\iota}{4} \right\rfloor, 7-\iota, \frac{n-2}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - \left\lfloor \frac{\iota-3}{3} \right\rfloor, \frac{n-10}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - 2\left\lfloor \frac{\iota-3}{4} \right\rfloor\right), & \text{if } \iota = 4, \dots, 6, \\ \left(\left\lfloor \frac{\iota}{4} \right\rfloor, \alpha_1, \frac{n-2}{8} - \left\lfloor \frac{\iota-7}{4} \right\rfloor, \alpha_2\right), & \text{otherwise,} \end{cases}$$

where

$$\alpha_1 = \begin{cases} 2 - \left\lfloor \frac{\iota-6}{2} \right\rfloor, & \text{if } \iota = 4, \dots, 10, \\ \left\lfloor \frac{\iota-6}{4} \right\rfloor, & \text{otherwise,} \end{cases} \qquad \alpha_2 = \begin{cases} \frac{n-10}{8} + \left\lfloor \frac{\iota-3}{2} \right\rfloor - 2\left\lfloor \frac{\iota-3}{4} \right\rfloor, & \text{if } \iota = 4, \\ \frac{n+6}{8} - \left\lfloor \frac{\iota-5}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$

For $\iota = m+3, \dots, n$,

$$r\left(v_\iota^l \mid \mathbb{S}^l\right) = \begin{cases} \left(\frac{n+6}{8} - \left\lfloor \frac{2\iota-n-4}{8} \right\rfloor, \frac{n-10}{8} + \left\lfloor \frac{2\iota-n-1}{8} \right\rfloor, \beta_1, \beta_2\right), & \text{if } \iota = m+3, \dots, m+10, \\ \left(\frac{n+6}{8} - \left\lfloor \frac{2\iota-n-4}{8} \right\rfloor, \frac{n+6}{8} - \left\lfloor \frac{2\iota-n-16}{8} \right\rfloor, \left\lfloor \frac{2\iota-n-6}{8} \right\rfloor, \left\lfloor \frac{2\iota-n-10}{8} \right\rfloor\right), & \text{otherwise,} \end{cases}$$

where

$$\beta_1 = \begin{cases} \left\lfloor \frac{2\iota-n}{4} \right\rfloor - \left\lfloor \frac{2\iota-n-4}{6} \right\rfloor, & \text{if } \iota = m+3, \dots, m+5, \\ \iota - \frac{n}{2} - 6, & \text{if } \iota = m+6, \dots, m+8, \\ \left\lfloor \frac{2\iota-n-6}{8} \right\rfloor, & \text{otherwise,} \end{cases}$$

and

$$\beta_2 = \begin{cases} 2 - \left\lfloor \frac{2\iota-n-4}{4} \right\rfloor + 2\left\lfloor \frac{2\iota-n-4}{8} \right\rfloor, & \text{if } \iota = m+3, \dots, m+6, \\ 1 - \left\lfloor \frac{2\iota-n-12}{4} \right\rfloor + \left\lfloor \frac{2\iota-n-12}{6} \right\rfloor + 2\left\lfloor \frac{2\iota-n-12}{8} \right\rfloor, & \text{otherwise.} \end{cases}$$

Since these formulas yield distinct metric representations for all vertices of $L(\mathbb{T}_m^2)$ when $n := 2m$ and $m = 4k + 9$ for some integer $k \geq 1$, it follows that

$$\dim\left(L(\mathbb{T}_m^2)\right) \leq 4.$$

Combining this with the lower bound from Lemma 3.1, we conclude that

$$\dim\left(L(\mathbb{T}_m^2)\right) = 4.$$

This completes the proof. □

Example 3.1. *Let $\mathbb{T}_{11}^2$ be an optimal 2-fault-tolerant token ring of order 11 and size 22. Let $L(\mathbb{T}_{11}^2)$ be its line graph, which has order 22. Then*

$$\dim(L(\mathbb{T}_{11}^2)) = 4.$$

Table 1 shows the representation of all vertices of $L(\mathbb{T}_{11}^2)$ with respect to

$$\mathbb{S}^l = \{v_1^l,\ v_{10}^l,\ v_{17}^l,\ v_{19}^l\}.$$

Table 1. Representation of the vertices of $L(\mathbb{T}_1 1^2)$ with respect to $\mathbb{S}^l$.

v_v^l	$r(v_\iota^l \mid \mathbb{S}^l)$	v_v^l	$r(v_\iota^l \mid \mathbb{S}^l)$
v_1^l	(0, 2, 3, 2)	v_{12}^l	(3, 1, 2, 2)
v_2^l	(1, 2, 3, 2)	v_{13}^l	(3, 1, 1, 2)
v_3^l	(1, 3, 2, 1)	v_{14}^l	(3, 2, 1, 2)
v_4^l	(1, 3, 2, 2)	v_{15}^l	(3, 2, 2, 1)
v_5^l	(1, 2, 3, 3)	v_{16}^l	(3, 2, 1, 1)
v_6^l	(1, 1, 3, 3)	v_{17}^l	(3, 2, 0, 2)
v_7^l	(1, 2, 2, 2)	v_{18}^l	(3, 3, 1, 1)
v_8^l	(2, 1, 3, 3)	v_{19}^l	(2, 3, 2, 0)
v_9^l	(2, 1, 2, 3)	v_{20}^l	(2, 3, 1, 1)
v_{10}^l	(2, 0, 2, 3)	v_{21}^l	(2, 3, 1, 2)
v_{11}^l	(2, 1, 2, 2)	v_{22}^l	(2, 3, 2, 1)

4. Comparative analysis with the ordinary token ring

In order to illustrate the influence of fault-tolerant links on the metric dimension, we compare our results with the ordinary token ring network. The ordinary token ring corresponds to the cycle graph C_m, which contains only adjacent connections between consecutive vertices.

It is well known that the metric dimension of a cycle satisfies

$$\dim(C_m) = 2, \quad m \geq 3.$$

Moreover, the line graph of a cycle is again a cycle, that is, $L(C_m) \cong C_m$. Consequently,

$$\dim(L(C_m)) = 2.$$

In contrast, the optimal 2-fault-tolerant token ring network $\mathbb{T}_m^2$ contains additional links joining vertices at distance two in the basic ring. These extra edges increase the connectivity and symmetry of the network and introduce more adjacency relations in the corresponding line graph. As established in Theorems 3.2–3.5, the metric dimension of the line graph $L(\mathbb{T}_m^2)$ satisfies

$$\dim(L(\mathbb{T}_m^2)) = 4.$$

Hence, compared with the ordinary token ring, the introduction of fault-tolerant links increases the metric dimension from 2 to 4. This demonstrates that the additional fault-tolerant edges make the structure of the line graph more complex, requiring a larger resolving set to uniquely distinguish all vertices.

5. Conclusions

In this work, we investigated the distance-based resolvability of line graphs arising from fault-tolerant token ring topologies. Specifically, for the optimal 2-fault-tolerant token ring network $\mathbb{T}_m^2$ of order m, we studied the metric dimension of its line graph $L(\mathbb{T}_m^2)$ (of order $n := 2m$), where vertices represent links of $\mathbb{T}_m^2$ and distances in the line graph capture compatibility through shared incidences. Since metric dimension (locating number) provides the smallest number of landmarks needed to uniquely identify vertices through distance profiles, our results admit a natural interpretation in network monitoring: a small set of carefully placed link-monitors can uniquely localize every link-event in the underlying fault-tolerant ring.

Our main contribution is twofold. First, we established a uniform lower bound

$$\dim\Big(L(\mathbb{T}_m^2)\Big) \geq 4$$

for $m \geq 10$, showing that at least four landmarks are necessary for unique identification in the line-graph setting. Second, we constructed explicit resolving sets of cardinality 4 for the principal congruence classes $m = 4k + 6$, $m = 4k + 7$, $m = 4k + 8$, and $m = 4k + 9$ ($k \geq 1$), thereby proving that

$$\dim\Big(L(\mathbb{T}_m^2)\Big) = 4$$

for these families. In particular, the metric dimension remains constant, independent of the network size, highlighting a strong stability phenomenon for resolvability under the combined operations of optimal fault-tolerant augmentation and line-graph transformation.

Several directions naturally arise from this study. A first extension is to investigate other resolvability parameters on the same structures, such as edge metric dimension, mixed metric dimension, and partition dimension, which are often better aligned with monitoring tasks that involve both nodes and links. A second direction is robustness: determining the fault-tolerant metric dimension of $L(\mathbb{T}_m^2)$ would quantify the additional landmark cost required to maintain unique identification under monitor failures. Finally, the framework may be generalized to optimal $\mathbb{K}$-fault-tolerant token ring augmentations $\mathbb{T}_n^{\mathbb{K}}$ and to configuration-based constructions (such as token graphs) built over ring-derived topologies. These problems remain open and are expected to further strengthen the connection between resolvability theory and practical monitoring of fault-tolerant interconnection networks.

Author contributions

Asma Alasmri: conceptualization, methodology, formal analysis, investigation, writing—original draft preparation, writing—review and editing; Nor Muhainiah Mohd Ali: conceptualization, validation, formal analysis, investigation, writing—review and editing; Ali Ahmad:

conceptualization, methodology, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing, funding acquisition; Muhammad Faisal Nadeem: conceptualization, methodology, validation, investigation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors gratefully acknowledge the funding of the Deanship of Graduate Studies and Scientific Research, Jazan University, Saudi Arabia, through project number: JU-20250276-DGSSR-ORA-2025.

Conflict of interest

The authors declare that there are no conflicts of interest associated with this work.

References

1. J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, Macmillan, 1976.
2. D. B. West, *Introduction to graph theory*, Prentice Hall, 2001.
3. IEEE, *IEEE standard for information technology—token ring access method and physical layer specifications*, IEEE Std 802.5, 1989.
4. R. Metcalfe, D. R. Boggs, Ethernet: distributed packet switching for local computer networks, *Commun. ACM*, **19** (1976), 395–404. https://doi.org/10.1145/360248.360253
5. J. Duato, S. Yalamanchili, L. Ni, *Interconnection networks: an engineering approach*, Morgan Kaufmann Publishers Inc., 2003. https://doi.org/10.5555/572400
6. T. Haynes, S. Hedetniemi, P. Slater, *Foundations of interconnection networks*, Wiley, 1999.
7. P. J. Slater, Leaves of trees, *Congressus Numerantium*, **14** (1975), 549–559.
8. P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.*, **22** (1988), 445–455.
9. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Comb.*, **1976** (1976), 191–195.
10. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Comput. Vision Graphics Image Process.*, **25** (1984), 113–121. https://doi.org/10.1016/0734-189X(84)90051-3
11. S. Khuller, B. Raghavachari, A. Rosenfeld, Localization in graphs, *Technical Report UMIACS-TR-94-92*, University of Maryland, Institute for Advanced Computer Studies, 1994.
12. J. Kratica, V. Kovačević-Vujčić, M. Čangalović, M. Stojanović, Minimal doubly resolving sets and the strong metric dimension of some convex polytopes, *Appl. Math. Comput.*, **218** (2012), 9790–9801. https://doi.org/10.1016/j.amc.2012.03.047

13. G. Chartrand, L. Eroh, M. A Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99–113. https://doi.org/10.1016/S0166-218X(00)00198-0

14. M. Imran, A. Q. Baig, S. A. U. H. Bokhary, I. Javaid, On the metric dimension of circulant graphs, *Appl. Math. Lett.*, **25** (2012), 320–325. https://doi.org/10.1016/j.aml.2011.09.008

15. M. Imran, M. K. Siddiqui, R. Naeem, On the metric dimension of generalized petersen multigraphs, *IEEE Access*, **6** (2018), 74328–74338. https://doi.org/10.1109/ACCESS.2018.2883556

16. I. G. Yero, D. Kuziak, J. A. Rodríguez-Velázquez, On the metric dimension of corona product graphs, *Comput. Math. Appl.*, **61** (2011), 2793–2798. https://doi.org/10.1016/j.camwa.2011.03.046

17. A. Szolnoki, M. Perc, Correlation of positive and negative reciprocity fails to confer an evolutionary advantage: phase transitions to elementary strategies, *Phys. Rev. X*, **3** (2013), 041021. https://doi.org/10.1103/PhysRevX.3.041021

18. R. Naeem, M. Imran, On resolvability and exchange property in antiweb-wheels, *Utilitas Math.*, **104** (2017), 187–200.

19. M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: a review, *J. R. Soc. Interface*, **10** (2013), 0997. https://doi.org/10.1098/rsif.2012.0997

20. M. Ali, M. T. Rahim, G. Ali, On path related graphs with constant metric dimension, *Utilitas Math.*, **88** (2012), 203–209. https://doi.org/10.1155/2021/2085778

21. M. M. Alholi, O. A. Abughneim, H. A. Ezeh, Metric dimension of some path related graphs, *Global J. Pure Appl. Math.*, **3** (2017), 149–157.

22. A. T. Shahida, M. S. Sunitha, On the metric dimension of joins of two graphs, *Global J. Pure Appl. Math.*, **5** (2014), 33–38.

23. H. Pan, M. Ali, G. Ali, M. T. Ali, X. Yang, On the families of graphs with unbounded metric dimension, *IEEE Access*, **7** (2019), 165060–165064. https://doi.org/10.1109/ACCESS.2019.2952192

24. H. Peng, X. Wang, J. Wang, J. Chen, On the metric dimension of some kneser graphs, *J. Comput. Theor. Nanosci.*, **13** (2016), 3013–3018.

25. M. F. Nadeem, W. Ali, H. M. A. Siddiqui, Locating number of biswapped networks, *Int. J. Found. Comput. Sci.*, **33** (2022), 667–690. https://doi.org/10.1142/S0129054122420096

26. M. Arulperumjothi, S. Klavžar, S. Prabhu, Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension, *Appl. Math. Comput.*, **452** (2023), 128037. https://doi.org/10.1016/j.amc.2023.128037

27. T. Vetrík, M. Imran, M. Knor, R. Škrekovski, The metric dimension of the circulant graph with $2k$ generators can be less than k, *J. King Saud Univ. Sci.*, **35** (2023), 102834. https://doi.org/10.1016/j.jksus.2023.102834

28. A. Hakanen, V. Junnila, T. Laihonen, I. G. Yero, On the unicyclic graphs having vertices that belong to all their (strong) metric bases, *Discrete Appl. Math.*, **353** (2024), 191–207. https://doi.org/10.1016/j.dam.2024.04.020

29. B. Assiri, M. F. Nadeem, W. Ali, A. Ahmad, Fault-tolerance in biswapped multiprocessor interconnection networks, *J. Parallel Distrib. Comput.*, **196** (2025), 105009. https://doi.org/10.1016/j.jpdc.2024.105009

30. K. Nie, K. Xu, The doubly metric dimensions of cactus graphs and block graphs, *J. Comb. Optim.*, **47** (2024), 120–134. https://doi.org/10.1007/s10878-024-01168-0

31. L. S. Mhagama, M. N. Husin, M. F. Nadeem, W. Ali, Computing the edge metric dimension of the line graph of the molec-ular graphs of some classes of antiviral drugs, with focus on mpox treatments, *Contemp. Math.*, **6** (2025), 1784–1802. https://doi.org/10.37256/cm.6220255898

32. D. Dolžan, M. Koš, Metric dimension in semiring total graphs, *Mediterr. J. Math.*, **22** (2025), 112.

33. S. Nazeer, M. Hussain, A. ur Rehman, A. H. Ali, On metric dimension of symmetrical planar pyramid related graphs, *Afr. Mat.*, **36** (2025), 54. https://doi.org/10.1007/s13370-025-01253-5

34. A. Khan, S. Ali, S. Hayat, M. Azeem, Y. Zhong, M. A. Zahid, et al., Fault-tolerance and unique identification of vertices and edges in a graph: the fault-tolerant mixed metric dimension, *J. Parallel Distrib. Comput.*, **197** (2025), 105024. https://doi.org/10.1016/j.jpdc.2024.105024

35. K. Azhar, A. Nadeem, Y. Shang, Fault-tolerant metric dimension in carbon networks, *Foundations*, **5** (2025), 13. https://doi.org/10.3390/foundations5020013

36. M. Knor, J. Sedlar, R. Škrekovski, Fault tolerance of metric basis can be expensive, *Mediterr. J. Math.*, **22** (2025), 148. https://doi.org/10.1007/s00009-025-02894-3

37. B. Mohamed, Metric dimension of graphs and its application to robotic navigation, *Int. J. Comput. Appl.*, **184** (2022), 15. https://doi.org/10.5120/ijca2022922090

38. S. Hayat, R. Naeem, H. M. A. Siddiqui, A. Riaz, M. Imran, M. B. Belay, et al., Metric, edge-metric, mixed-metric, and fault-tolerant metric dimensions of geometric networks with potential applications, *Sci. Rep.*, **15** (2025), 41035. https://doi.org/10.1038/s41598-025-24939-z

39. M. Feng, M. Xu, K. Wang, On the metric dimension of line graphs, *Discrete Appl. Math.*, **161** (2013), 802–805. https://doi.org/10.1016/j.dam.2012.10.018

40. L. Eroh, S. Kang, H. Yi, Metric dimension and zero forcing number of line graphs of wheel graphs and bouquet graphs, *AKCE Int. J. Graphs Comb.*, **15** (2018), 267–273. https://doi.org/10.21136/MB.2014.143937

41. M. U. Farooq, A. Rehman, T. Q. Ibrahim, M. Hussain, A. H. Ali, B. Rashwani, Metric dimension of line graphs of Bakelite and subdivided Bakelite network, *Discrete Dyn. Nat. Soc.*, **2023** (2023), 7656214. https://doi.org/10.1155/2023/7656214

42. J. Wu, L. Wang, W. Yang, Learning to compute the metric dimension of graphs, *Appl. Math. Comput.*, **432** (2022), 127350. https://doi.org/10.1016/j.amc.2022.127350

43. A. Khan, G. Haidar, N. Abbas, M. U. I. Khan, A. U. K. Niazi, A. U. I. Khan, Metric dimensions of bicyclic graphs, *Mathematics*, **11** (2023), 869. https://doi.org/10.3390/math11040869

44. Pranjali, A. Kumar, R. Sharma, Metric dimension of unit graphs, *Discrete Math. Algorithms Appl.*, **17** (2025), 2450057. https://doi.org/10.1142/S1793830924500575

45. L. S. Mhagama, M. N. Husin, M. F. Nadeem, W. Ali, On the edge metric dimension of some families of splitting graphs, *Discrete Math. Algorithms Appl.*, **9** (2025), 2550061. https://doi.org/10.1142/S1793830925500612

46. S. Vidya, S. K. Sharma, P. Poojary, G. R. V. Bhatta, Metric basis and metric dimension of some infinite planar graphs, *Discrete Math. Algorithms Appl.*, **16** (2024), 2450076. https://doi.org/10.1142/S1793830924500769

47. T. Y. Sung, T. Y. Ho, C. P. Chang, L. H. Hsu, Optimal k-fault-tolerant networks for token rings, *J. Inf. Sci. Eng.*, **16** (2000), 381–390.