



Research article

Degenerate two-variable q -Legendre polynomials via q -operational calculus and degenerate Laplace/Sumudu transforms

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Abstract: We introduce a degenerate (q, λ) -extension of the two-variable q -Legendre-type polynomials by deforming the q -Bessel–Tricomi kernel through the degenerate falling-factorial weights $(1)_{k,\lambda}$. The resulting family $\{\mathcal{L}_{n,q}^{(\lambda)}(x, y)\}_{n \geq 0}$ is defined by a single generating function and interpolates both the recently studied two-variable q -Legendre polynomials and their classical limits as $\lambda \rightarrow 0$ and/or $q \rightarrow 1$. We derive an explicit finite-sum representation, an operational Rodrigues-type formula, and a quasi-monomial structure that yields raising and lowering operators together with a fundamental q -difference equation in the variable y . We further compute the degenerate Laplace and Sumudu transforms of the generating kernel and obtain corresponding transform identities for $\mathcal{L}_{n,q}^{(\lambda)}$. Several reduction formulas, even/odd subsequence decompositions, and a moment-functional interpretation are presented, along with linearization coefficients, low-degree examples, and numerical table.

Keywords: q -calculus; two-variable q -Legendre polynomials; quasi-monomiality; degenerate falling factorial; degenerate Laplace transform; differential equations; process innovation; Mathematical operators; degenerate Sumudu transform

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1. Introduction

Classical Legendre polynomials and their associated orthogonality structure play a fundamental role in approximation theory, spectral methods, and the analysis of boundary-value problems [5]. Their q -analogues arise naturally in basic hypergeometric function theory and in the q -Askey scheme of orthogonal polynomials; see, for example, Gasper and Rahman for basic hypergeometric functions, and Ismail and Koekoek et al. for the q -Askey scheme [9, 10, 22]. From a modern viewpoint, many q -polynomial families can be characterized either by their basic hypergeometric representations or by operator identities in q -calculus, which makes them especially amenable to operational methods [8].

Within this general program, operational techniques in q -calculus provide a flexible framework for constructing and studying multivariate or two-variable polynomial families. A representative example is the two-variable q -Legendre-type polynomial family generated by a zeroth-order q -Bessel–Tricomi kernel; see, for instance, [1, 24, 25]. Such families typically exhibit quasi-monomial behavior in one variable together with a Bessel/Tricomi-type deformation in the other, thereby allowing explicit series expansions and operator equations to be derived by purely algebraic means [1, 2, 25]; related linearization formulas may be found in [23, 27].

In parallel, *degenerate* versions of special functions and transforms have been developed through the use of the degenerate falling factorial $(a)_{n,\lambda}$ [15, 26] and the degenerate exponential $e_\lambda(t) = (1 + \lambda t)^{1/\lambda}$ [12, 20, 21]. This approach yields λ -deformations of classical sequences and has proved effective in the study of degenerate gamma-type functions [14], degenerate Sheffer sequences [15, 26], and Laplace/Sumudu-type transforms in the degenerate setting [4, 12, 13], and the references therein. In addition, λ - q operator frameworks and λ - q -Sheffer sequences provide a systematic language for handling simultaneous q - and λ -deformations [16].

The purpose of this paper is to connect these directions by introducing a deformation parameter λ into the q -Bessel–Tricomi part of the generating function, while preserving the q -Appell behavior in the second variable. More concretely, we deform the kernel coefficients by the weights $(1)_{k,\lambda}$ [15, 26], which produces a new two-variable family $\{\mathcal{L}_{n,q}^{(\lambda)}(x,y)\}_{n \geq 0}$ that admits explicit operational control and stable limit transitions as $\lambda \rightarrow 0$ and/or $q \rightarrow 1$.

Beyond the intrinsic algebraic interest, the extra parameter λ can be viewed as a tunable deformation that may be advantageous in applications where one wishes to interpolate between classical and q -settings or to introduce an additional degree of freedom in moment and transform-based representations. For example, the degenerate Laplace/Sumudu images (in the sense of [7, 12, 26]) derived in Section 7 lead to tractable spectral representations, and the model q -evolution problem in Section 9 illustrates how the deformed kernel can be used to generate families of explicit solutions. Further application-oriented and combinatorial remarks are recorded in Section 12.

The key results established in this work may be summarized as follows:

- A (q, λ) -generating function defining $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ and an explicit finite-sum expansion;
- An operational Rodrigues-type representation and a quasi-monomial structure with raising/lowering operators;
- A fundamental q -difference equation in the y -variable and reduction formulas;
- Degenerate Laplace transform identities for the generating kernel and for $\mathcal{L}_{n,q}^{(\lambda)}$;
- Linearization coefficients and a moment-functional interpretation with even/odd subsequences.

The paper is organized as follows: Section 2 collects the required notions from q -calculus and

degenerate analysis. In Section 3, we introduce the degenerate q -Bessel–Tricomi kernel, and define the new polynomials. Sections 4–6 develop the quasi-monomial operator structure, Rodrigues-type representations, recurrence relations, and the main q - λ operator equation. Section 7 is devoted to degenerate Laplace and Sumudu images. Limit cases are discussed in Section 8, and a model q -evolution problem is treated in Section 9. Sections 10 and 11 address moment functionals, orthogonality considerations, and linearization formulas. Section 12 records comments on potential applications, and Section 13 concludes the paper.

2. Preliminaries

Throughout, we assume $q \in \mathbb{C}$ with $0 < |q| < 1$ and $\lambda \in \mathbb{R}$ unless stated otherwise.

Definition 2.1. For $n \in \mathbb{N}_0$, define (see, e.g., [8])

$$[n]_q := \frac{1 - q^n}{1 - q}, \quad [n]_q! := \prod_{j=1}^n [j]_q \quad ([0]_q! := 1).$$

The (Jackson) q -exponential (see, e.g., [8]) is

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}.$$

Definition 2.2. For integers $0 \leq k \leq n$, the Gaussian coefficient (see, e.g., [8, 9, 11]) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n - k]_q!}.$$

Definition 2.3. For a function f defined near $y \neq 0$, the (Jackson) q -derivative in y (see, e.g., [8]) is

$$(D_{q,y}f)(y) := \frac{f(qy) - f(y)}{(q - 1)y}.$$

Lemma 2.4. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have

$$D_{q,y}(y^n) = [n]_q y^{n-1}, \tag{2.1}$$

$$D_{q,y}(e_q(\alpha y)) = \alpha e_q(\alpha y). \tag{2.2}$$

These identities are standard in q -calculus; see, e.g., [8].

Proof. For (2.1),

$$D_{q,y}(y^n) = \frac{(qy)^n - y^n}{(q - 1)y} = \frac{q^n - 1}{q - 1} y^{n-1} = [n]_q y^{n-1}.$$

For (2.2), use term-by-term differentiation and (2.1):

$$D_{q,y}(e_q(\alpha y)) = \sum_{n=1}^{\infty} \frac{\alpha^n}{[n]_q!} D_{q,y}(y^n) = \sum_{n=1}^{\infty} \frac{\alpha^n}{[n]_q!} [n]_q y^{n-1} = \sum_{n=1}^{\infty} \frac{\alpha^n}{[n - 1]_q!} y^{n-1} = \alpha e_q(\alpha y).$$

□

Definition 2.5. For $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the q -shifted factorial (see, e.g., [8, 9]) is

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j).$$

In particular, $(q; q)_n = \prod_{j=1}^n (1 - q^j)$ and

$$[n]_q! = \frac{(q; q)_n}{(1 - q)^n}. \quad (2.3)$$

Definition 2.6. The basic hypergeometric series ${}_r\phi_s$ (in the normalization used throughout this paper) is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k},$$

whenever the series converges (see [8, 9]). We use the normalization without the additional factor $((-1)^k q^{k(k-1)/2})^{1+s-r}$ that appears in some references.

Definition 2.7. For $\lambda \in \mathbb{R}$, following the degenerate hypergeometric framework in [15, 26] and the basic q -case in [9], we define the degenerate basic hypergeometric series by

$${}_r\phi_s^{(\lambda)} \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) := \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda} (a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k}.$$

Definition 2.8. For $\lambda \in \mathbb{R}$ and $a \in \mathbb{C}$, the degenerate falling factorial is (cf. [15, 26])

$$(a)_{0,\lambda} := 1, \quad (a)_{n,\lambda} := \prod_{j=0}^{n-1} (a - j\lambda) \quad (n \geq 1).$$

The degenerate exponential is $e_\lambda(t) := (1 + \lambda t)^{1/\lambda}$ for $1 + \lambda t > 0$ (see [12, 15]).

Definition 2.9. Let $\lambda > 0$ and let f be defined on $[0, \infty)$. The degenerate Laplace transform of f is

$$(\mathcal{L}_\lambda f)(s) := \int_0^\infty (1 + \lambda t)^{-s/\lambda} f(t) dt,$$

whenever the integral converges. The degenerate Sumudu transform is

$$(\mathcal{S}_\lambda f)(u) := \frac{1}{u} \int_0^\infty (1 + \lambda t)^{-1/(u\lambda)} f(t) dt,$$

whenever the integral converges.

In particular, the degenerate Laplace transform and the degenerate gamma framework used below follow the conventions introduced by Kim and Kim; see [12, 14]. For further related degenerate Sumudu-transform developments, see also [7, 26].

Remark 2.10. Formally, the transforms are related by $(\mathcal{S}_\lambda f)(u) = \frac{1}{u} (\mathcal{L}_\lambda f)(1/u)$.

Lemma 2.11. Let $\lambda > 0$ and $n \in \mathbb{N}_0$. For $s > (n + 1)\lambda$, we have

$$\mathcal{L}_\lambda(t^n)(s) = \frac{n!}{\prod_{j=1}^{n+1}(s - j\lambda)}. \quad (2.4)$$

For $u \in \mathbb{R}$ such that $1 - j\lambda u > 0$ for $j = 1, \dots, n + 1$, we have

$$\mathcal{S}_\lambda(t^n)(u) = \frac{n! u^n}{\prod_{j=1}^{n+1}(1 - j\lambda u)}. \quad (2.5)$$

Proof. We prove (2.4). Let $u = 1 + \lambda t$, so $t = (u - 1)/\lambda$ and $dt = du/\lambda$. Then,

$$\mathcal{L}_\lambda(t^n)(s) = \int_1^\infty u^{-s/\lambda} \left(\frac{u-1}{\lambda}\right)^n \frac{du}{\lambda} = \lambda^{-(n+1)} \int_1^\infty u^{-s/\lambda} (u-1)^n du.$$

Next, set $v = 1/u$ so that $u = 1/v$ and $du = -v^{-2}dv$. The integral becomes

$$\lambda^{-(n+1)} \int_0^1 v^{s/\lambda - n - 2} (1-v)^n dv = \lambda^{-(n+1)} \mathbf{B}\left(\frac{s}{\lambda} - n - 1, n + 1\right),$$

where \mathbf{B} is the Beta function. Using $\mathbf{B}(A, B) = \Gamma(A)\Gamma(B)/\Gamma(A + B)$ and $\Gamma(n + 1) = n!$,

$$\mathcal{L}_\lambda(t^n)(s) = \lambda^{-(n+1)} \frac{\Gamma(\frac{s}{\lambda} - n - 1) n!}{\Gamma(\frac{s}{\lambda})}.$$

Finally, apply $\Gamma(z) = \Gamma(z - n - 1) \prod_{j=1}^{n+1}(z - j)$ with $z = \frac{s}{\lambda}$ to obtain (2.4). The Sumudu identity (2.5) follows from $(\mathcal{S}_\lambda f)(u) = \frac{1}{u}(\mathcal{L}_\lambda f)(1/u)$ and (2.4). \square

For convenience, the standing assumptions and admissible parameter ranges used throughout the paper are summarized in Table 1.

Table 1. Parameter conventions used throughout the paper.

| Object | Assumptions / admissible ranges |
|---------------------------------|---|
| q -Calculus conventions | $0 < q < 1$ (unless $q \rightarrow 1^-$ in Section 8) |
| Degeneracy parameter λ | $\lambda \in \mathbb{R}; \lambda > 0$ for Laplace/Sumudu and gamma moments |
| Laplace parameter s | $s > (n + 1)\lambda$ for $\mathcal{L}_\lambda(t^n)$; s in the absolute convergence domain of (7.1) |
| Sumudu parameter u | $1 - j\lambda u > 0$ ($j = 1, \dots, n + 1$); u in the absolute convergence domain of (7.2) |
| Gamma/moment parameter α | $0 < \Re(\alpha) < 1/\lambda$ (Definition 10.1); $\alpha > 0, \alpha < 1/\lambda$ for moments |

3. Degenerate q -Bessel–Tricomi kernel and new polynomials

This section introduces the degenerate q -Bessel–Tricomi kernel that underpins our construction and then defines the associated two-variable polynomial family via a generating function. We also provide an explicit finite-sum expansion and record elementary reduction properties needed in subsequent sections.

Definition 3.1. For $\lambda \in \mathbb{R}$ and $x \in \mathbb{C}$, define

$$C_{0,q}^{(\lambda)}(x) := \sum_{k=0}^{\infty} \frac{(-1)^k (1)_{k,\lambda}}{([k]_q!)^2} x^k. \quad (3.1)$$

Remark 3.2. As $\lambda \rightarrow 0$, $(1)_{k,\lambda} \rightarrow 1$ for each fixed k , hence $C_{0,q}^{(\lambda)}(x) \rightarrow C_{0,q}(x)$, the standard q -Bessel-Tricomi function used in [25].

Remark 3.3. If $\lambda = 1/M$ with $M \in \mathbb{N}$, then $(1)_{k,\lambda} = 0$ for $k \geq M + 1$, so the series (3.1) truncates and $C_{0,q}^{(\lambda)}$ is a polynomial. For general $\lambda \neq 0$, the expansion (3.1) is most naturally interpreted as a formal power series in x (or in t after the substitution $x \mapsto -xt^2$). This is sufficient for defining $\mathcal{L}_{n,q}^{(\lambda)}$ via (3.5), because only finitely many kernel coefficients contribute to each fixed degree n .

The series (3.1) can be rephrased using the q -shifted factorial from Definition 2.5. Indeed, by (2.3), $([k]_q!)^{-2} = (1 - q)^{2k} / (q; q)_k^2$.

Proposition 3.4. For $\lambda \in \mathbb{R}$ and $x \in \mathbb{C}$,

$$C_{0,q}^{(\lambda)}(x) = {}_0\phi_1^{(\lambda)}\left(\begin{matrix} - \\ q \end{matrix}; q, -(1 - q)^2 x\right) := \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{(q; q)_k^2} (-(1 - q)^2 x)^k. \quad (3.2)$$

In particular, the kernel is a degenerate basic hypergeometric series in the sense of Definition 2.7.

Proof. Starting from (3.1) and using (2.3), we obtain

$$C_{0,q}^{(\lambda)}(x) = \sum_{k=0}^{\infty} (-1)^k (1)_{k,\lambda} \frac{(1 - q)^{2k}}{(q; q)_k^2} x^k = \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{(q; q)_k^2} (-(1 - q)^2 x)^k,$$

which is exactly (3.2). □

Write

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k}^{(\lambda)}(q) x^k y^{n-2k}. \quad (3.3)$$

By (3.6), $a_{n,k}^{(\lambda)}(q) = [n]_q! (1)_{k,\lambda} / (([k]_q!)^2 [n - 2k]_q!)$.

Proposition 3.5. For $n \geq 2$ and $1 \leq k \leq \lfloor n/2 \rfloor$, the coefficients in (3.3) satisfy

$$a_{n,k}^{(\lambda)}(q) = \frac{[n]_q [n - 1]_q}{[k]_q^2} (1 - (k - 1)\lambda) a_{n-2,k-1}^{(\lambda)}(q), \quad a_{n,0}^{(\lambda)}(q) = 1. \quad (3.4)$$

Proof. From the closed form $a_{n,k}^{(\lambda)}(q) = [n]_q! (1)_{k,\lambda} / (([k]_q!)^2 [n - 2k]_q!)$, we compute

$$\frac{a_{n,k}^{(\lambda)}(q)}{a_{n-2,k-1}^{(\lambda)}(q)} = \frac{[n]_q!}{[n - 2]_q!} \cdot \frac{(1)_{k,\lambda}}{(1)_{k-1,\lambda}} \cdot \frac{([k - 1]_q!)^2}{([k]_q!)^2}.$$

Since $[n]_q! / [n - 2]_q! = [n]_q [n - 1]_q$, $(1)_{k,\lambda} / (1)_{k-1,\lambda} = 1 - (k - 1)\lambda$, and $([k - 1]_q!)^2 / ([k]_q!)^2 = 1 / [k]_q^2$, we obtain (3.4). □

Remark 3.6. The recursion (3.4) provides a stable way to generate the whole family from the initial conditions $a_{n,0}^{(\lambda)}(q) = 1$ and $a_{2,1}^{(\lambda)}(q) = [2]_q [1]_q$.

Definition 3.7. Define $\{\mathcal{L}_{n,q}^{(\lambda)}(x, y)\}_{n \in \mathbb{N}_0}$ by the generating function

$$e_q(yt) C_{0,q}^{(\lambda)}(-xt^2) = \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{t^n}{[n]_q!}. \quad (3.5)$$

Theorem 3.8. For $n \in \mathbb{N}_0$,

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k y^{n-2k}. \quad (3.6)$$

Proof. Expand both series in (3.5). By (3.1),

$$C_{0,q}^{(\lambda)}(-xt^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (1)_{k,\lambda}}{([k]_q!)^2} (-x)^k t^{2k} = \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k t^{2k}.$$

Also, $e_q(yt) = \sum_{m=0}^{\infty} \frac{(yt)^m}{[m]_q!}$. Hence,

$$e_q(yt) C_{0,q}^{(\lambda)}(-xt^2) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [m]_q!} x^k y^m t^{m+2k}.$$

Fix $n \in \mathbb{N}_0$ and collect the coefficient of t^n , which requires $m = n - 2k \geq 0$, i.e., $0 \leq k \leq \lfloor n/2 \rfloor$:

$$[t^n](e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k y^{n-2k}.$$

Comparing with (3.5) gives (3.6). \square

To complement the explicit formula (3.6), we visualize the polynomials $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ for representative parameter values. Figure 1 shows one-dimensional sections $y \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x_0, y)$ for several degrees n , while Figure 2 displays a surface plot in the (x, y) -plane for fixed n . In addition, Figures 3 and 4 illustrate the deformation effect as $q \rightarrow 1$, and Figures 5 and 6 illustrate the degenerate limit as $\lambda \rightarrow 0$.

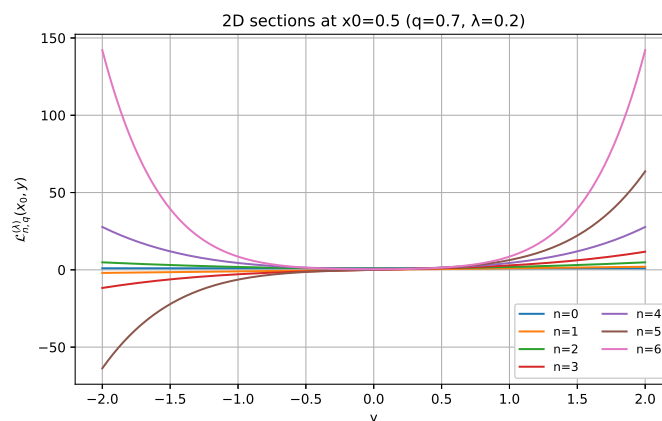


Figure 1. Two-dimensional sections $y \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x_0, y)$ for $n = 0, \dots, 6$ with $x_0 = 0.5$, $q = 0.7$, and $\lambda = 0.2$.

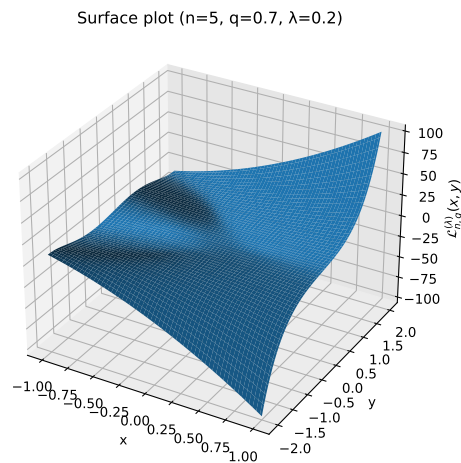


Figure 2. Surface plot of $(x, y) \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x, y)$ for $n = 5$ with $q = 0.7$ and $\lambda = 0.2$.

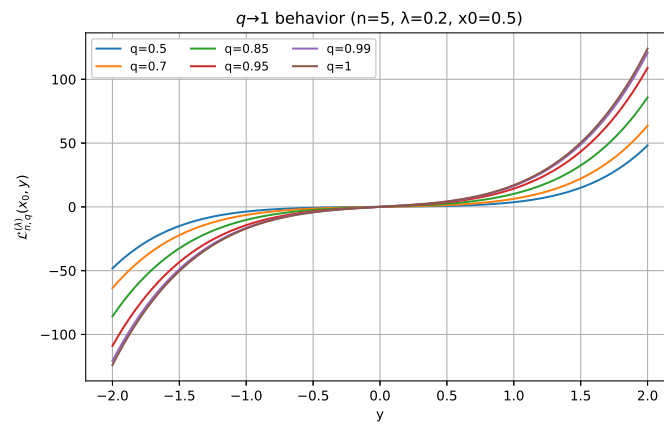


Figure 3. Overlay of the sections $y \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x_0, y)$ for $n = 5$, $\lambda = 0.2$, $x_0 = 0.5$, and several q values approaching 1.

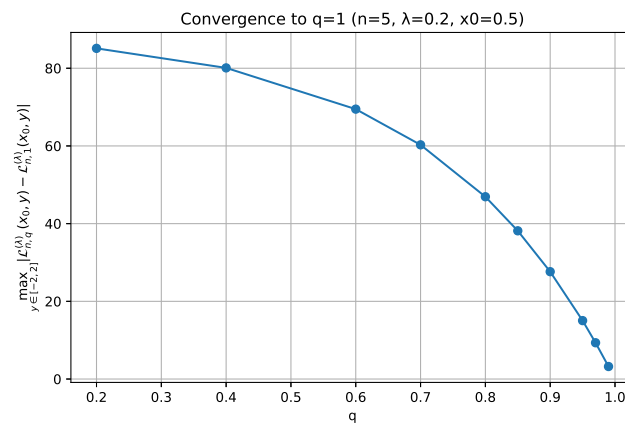


Figure 4. Maximum absolute deviation on $y \in [-2, 2]$ between q -deformed and classical sections, $\max_{y \in [-2, 2]} |\mathcal{L}_{n,q}^{(\lambda)}(x_0, y) - \mathcal{L}_{n,1}^{(\lambda)}(x_0, y)|$, for $n = 5$, $\lambda = 0.2$, and $x_0 = 0.5$.

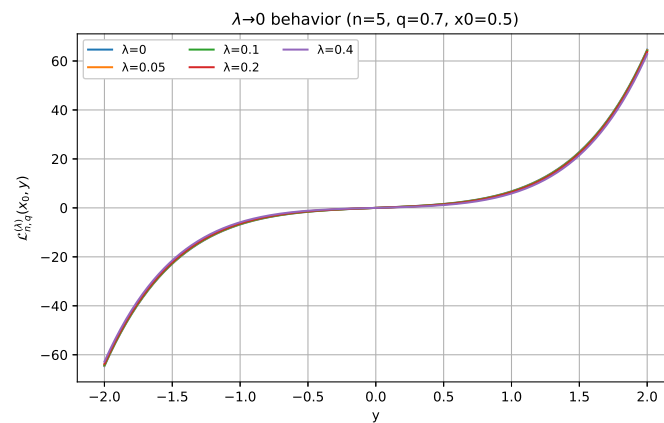


Figure 5. Overlay of the sections $y \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x_0, y)$ for $n = 5$, $q = 0.7$, $x_0 = 0.5$, and several λ values approaching 0.

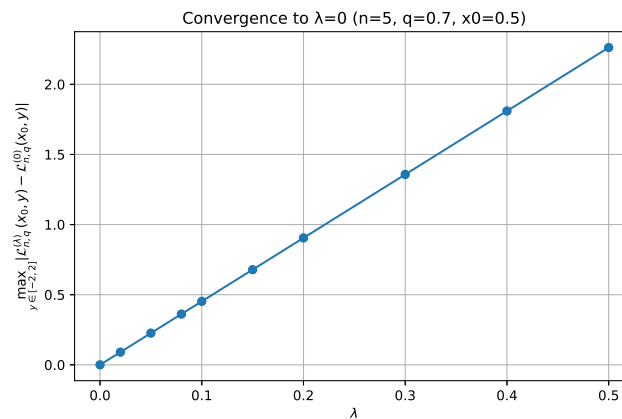


Figure 6. Maximum absolute deviation on $y \in [-2, 2]$ between the degenerate and non-degenerate cases, $\max_{y \in [-2, 2]} |\mathcal{L}_{n,q}^{(\lambda)}(x_0, y) - \mathcal{L}_{n,q}^{(0)}(x_0, y)|$, for $n = 5$, $q = 0.7$, and $x_0 = 0.5$.

The generating function (3.5) has the Sheffer–Appell structure $A(t)e_q(yt)$ with respect to the variable y . This viewpoint yields compact operational identities.

Proposition 3.9. For integers $r, n \in \mathbb{N}_0$ with $r \leq n$,

$$D_{q,y}^r \mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q [n-1]_q \cdots [n-r+1]_q \mathcal{L}_{n-r,q}^{(\lambda)}(x, y), \quad (3.7)$$

and $D_{q,y}^r \mathcal{L}_{n,q}^{(\lambda)}(x, y) = 0$ for $r > n$.

Proof. The case $r = 1$ is Proposition 4.1. Assuming (3.7) holds for r , apply $D_{q,y}$ once more and use (4.2) with n replaced by $n - r$ to obtain the case $r + 1$. The vanishing for $r > n$ follows from iterating until the index becomes negative. \square

Proposition 3.10. *Define the formal operator*

$$A(D_{q,y}) := \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k D_{q,y}^{2k}. \quad (3.8)$$

Then, for every $n \in \mathbb{N}_0$,

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = A(D_{q,y}) y^n. \quad (3.9)$$

Proof. We use the basic identity $D_{q,y}^m y^n = [n]_q! y^{n-m} / [n-m]_q!$ for $0 \leq m \leq n$, and $D_{q,y}^m y^n = 0$ for $m > n$, which follows by induction from $D_{q,y}(y^n) = [n]_q y^{n-1}$. Applying $A(D_{q,y})$ in (3.8) to y^n gives

$$A(D_{q,y}) y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k \frac{[n]_q!}{[n-2k]_q!} y^{n-2k},$$

which coincides with (3.6). \square

Corollary 3.11. *For $n \in \mathbb{N}_0$,*

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} (1)_{k,\lambda} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q x^k y^{n-2k}. \quad (3.10)$$

Proof. Using $\begin{bmatrix} n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \frac{[n]_q!}{[2k]_q! [n-2k]_q!} \cdot \frac{[2k]_q!}{([k]_q!)^2} = \frac{[n]_q!}{([k]_q!)^2 [n-2k]_q!}$, the identity (3.10) follows from (3.6). \square

4. Quasi-monomiality and operator structure

In this section, we establish the quasi-monomial character of $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ by deriving the lowering and raising operators associated with the generating function; compare also the recent q -monomiality developments in [1, 24]. These operators provide a compact operational framework and lead directly to q -difference relations for the polynomial sequence.

Let

$$A(t) := C_{0,q}^{(\lambda)}(-xt^2). \quad (4.1)$$

Then, (3.5) reads as $G(t) := A(t)e_q(yt) = \sum_{n \geq 0} \mathcal{L}_{n,q}^{(\lambda)}(x, y) t^n / [n]_q!$.

Proposition 4.1. *For $n \geq 1$,*

$$D_{q,y} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q \mathcal{L}_{n-1,q}^{(\lambda)}(x, y), \quad D_{q,y} \mathcal{L}_{0,q}^{(\lambda)}(x, y) = 0. \quad (4.2)$$

Proof. Apply $D_{q,y}$ to (3.5). Since $A(t)$ does not depend on y ,

$$D_{q,y}(e_q(yt) A(t)) = A(t) D_{q,y}(e_q(yt)).$$

Using Lemma 2.4 with $\alpha = t$ gives $D_{q,y}(e_q(yt)) = t e_q(yt)$, hence,

$$D_{q,y}(e_q(yt) A(t)) = t e_q(yt) A(t).$$

Now, expand both sides using (3.5). The left-hand side is

$$\sum_{n=0}^{\infty} D_{q,y} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{t^n}{[n]_q!}.$$

The right-hand side is

$$t \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty} \mathcal{L}_{n-1,q}^{(\lambda)}(x, y) \frac{t^n}{[n-1]_q!} = \sum_{n=1}^{\infty} [n]_q \mathcal{L}_{n-1,q}^{(\lambda)}(x, y) \frac{t^n}{[n]_q!}.$$

Equating coefficients yields (4.2). \square

Lemma 4.2. Let $\Phi(t) = \sum_{m=0}^{\infty} a_m t^m$ be a formal power series. Then, with $G(t) = \sum_{n \geq 0} \mathcal{L}_{n,q}^{(\lambda)}(x, y) t^n / [n]_q!$,

$$\Phi(t) G(t) = \sum_{n=0}^{\infty} (\Phi(D_{q,y}) \mathcal{L}_{n,q}^{(\lambda)}(x, y)) \frac{t^n}{[n]_q!}, \quad (4.3)$$

where $\Phi(D_{q,y}) := \sum_{m=0}^{\infty} a_m D_{q,y}^m$.

Proof. By Proposition 4.1, we have $D_{q,y} G(t) = tG(t)$, because

$$D_{q,y} G(t) = \sum_{n=0}^{\infty} D_{q,y} \mathcal{L}_{n,q}^{(\lambda)} \frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty} [n]_q \mathcal{L}_{n-1,q}^{(\lambda)} \frac{t^n}{[n]_q!} = t \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)} \frac{t^n}{[n]_q!} = tG(t).$$

Iterating yields $D_{q,y}^m G(t) = t^m G(t)$ for all $m \in \mathbb{N}_0$. Therefore,

$$\Phi(D_{q,y}) G(t) = \sum_{m=0}^{\infty} a_m D_{q,y}^m G(t) = \sum_{m=0}^{\infty} a_m t^m G(t) = \Phi(t) G(t),$$

which is equivalent to (4.3) by comparing coefficients in the t -expansions. \square

Theorem 4.3. Let $A(t)$ be as in (4.1), and define

$$\psi(t) := \frac{D_{q,t} A(t)}{A(t)}. \quad (4.4)$$

Further, set

$$\rho(t) := \frac{A(qt)}{A(t)}. \quad (4.5)$$

Define the operators

$$\widehat{M}_{q,\lambda} := y\rho(D_{q,y}) + \psi(D_{q,y}), \quad \widehat{P}_{q,\lambda} := D_{q,y}. \quad (4.6)$$

Then, for all $n \in \mathbb{N}_0$,

$$\widehat{M}_{q,\lambda} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = \mathcal{L}_{n+1,q}^{(\lambda)}(x, y), \quad (4.7)$$

and the quasi-monomiality relation holds:

$$\widehat{M}_{q,\lambda} \widehat{P}_{q,\lambda} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q \mathcal{L}_{n,q}^{(\lambda)}(x, y). \quad (4.8)$$

Proof. Differentiate $G(t) = A(t)e_q(yt)$ with respect to t in the q -sense. Using the q -product rule and Lemma 2.4 gives

$$D_{q,t}G(t) = (D_{q,t}A(t)) e_q(yt) + A(qt) D_{q,t}(e_q(yt)) = (D_{q,t}A(t)) e_q(yt) + y A(qt) e_q(yt).$$

Dividing by $A(t)$ and using (4.4) and (4.5) yields

$$D_{q,t}G(t) = (\psi(t) + y\rho(t))G(t).$$

On the other hand, from $G(t) = \sum_{n \geq 0} \mathcal{L}_{n,q}^{(\lambda)} t^n / [n]_q!$ and $D_{q,t}(t^n) = [n]_q t^{n-1}$, we have

$$D_{q,t}G(t) = \sum_{n=1}^{\infty} \mathcal{L}_{n,q}^{(\lambda)} \frac{[n]_q t^{n-1}}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{L}_{n+1,q}^{(\lambda)} \frac{t^n}{[n]_q!}.$$

Therefore,

$$(\psi(t) + y\rho(t))G(t) = \sum_{n=0}^{\infty} \mathcal{L}_{n+1,q}^{(\lambda)} \frac{t^n}{[n]_q!}.$$

Now, apply Lemma 4.2 to both ψ and ρ and use that multiplication by y acts coefficientwise:

$$(\psi(t) + y\rho(t))G(t) = \sum_{n=0}^{\infty} (\psi(D_{q,y}) + y\rho(D_{q,y})) \mathcal{L}_{n,q}^{(\lambda)} \frac{t^n}{[n]_q!}.$$

Comparing coefficients yields (4.7). Finally, (4.8) follows by applying $\widehat{P}_{q,\lambda} = D_{q,y}$ to (4.7) and using Proposition 4.1. \square

Remark 4.4. From $A(t) = \sum_{k \geq 0} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k t^{2k}$, we obtain

$$D_{q,t}A(t) = \sum_{k \geq 1} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k [2k]_q t^{2k-1},$$

hence, $\psi(t)$ is an odd formal series. Moreover, $\rho(t) = A(qt)/A(t)$ is an even series with $\rho(0) = 1$. For raising identities, the closed ratio forms (4.4) and (4.5) are typically the most stable representations.

5. Rodrigues-type formula and recurrence relations

This section complements the quasi-monomiality approach by providing an explicit operational representation and a finite recurrence in the polynomial index.

Theorem 5.1. For every $n \in \mathbb{N}_0$,

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = C_{0,q}^{(\lambda)}(-x D_{q,y}^2) y^n, \quad (5.1)$$

where the series $C_{0,q}^{(\lambda)}(z) = \sum_{k \geq 0} \frac{(-1)^k (1)_{k,\lambda}}{([k]_q!)^2} z^k$ is interpreted as a formal power series in the operator $D_{q,y}^2$. Equivalently,

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k y^{n-2k}. \quad (5.2)$$

Proof. Expanding (5.1) into the defining series of $C_{0,q}^{(\lambda)}$ yields

$$C_{0,q}^{(\lambda)}(-x D_{q,y}^2) y^n = \sum_{k=0}^{\infty} \frac{(1)_{k,\lambda}}{([k]_q!)^2} x^k D_{q,y}^{2k} y^n.$$

Since $D_{q,y}^{2k} y^n = [n]_q! y^{n-2k} / [n-2k]_q!$ for $2k \leq n$ and vanishes otherwise, the sum truncates at $k = \lfloor n/2 \rfloor$, and we obtain (5.2), which coincides with the explicit representation derived from the generating function. \square

Corollary 5.2. For every $n \geq 2$,

$$D_{q,y}^2 \mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q [n-1]_q \mathcal{L}_{n-2,q}^{(\lambda)}(x, y).$$

Proof. Differentiate (5.1) twice and use $D_{q,y}^2 y^n = [n]_q [n-1]_q y^{n-2}$. Because $C_{0,q}^{(\lambda)}(-x D_{q,y}^2)$ commutes with $D_{q,y}^2$, the claim follows. \square

Theorem 5.3. Let

$$A(t) = C_{0,q}^{(\lambda)}(-x t^2) = \sum_{m=0}^{\infty} a_m t^{2m}, \quad a_m = \frac{(1)_{m,\lambda}}{([m]_q!)^2} x^m,$$

and define the even and odd series

$$\rho(t) = \frac{A(qt)}{A(t)} = \sum_{m=0}^{\infty} r_m t^{2m}, \quad r_0 = 1, \quad (5.3)$$

and

$$\Phi(t) = \frac{D_{q,t} A(t)}{A(t)} = \sum_{m=0}^{\infty} \kappa_m t^{2m+1}. \quad (5.4)$$

Then, for every $n \in \mathbb{N}_0$,

$$\mathcal{L}_{n+1,q}^{(\lambda)}(x, y) = y \sum_{m=0}^{\lfloor n/2 \rfloor} r_m D_{q,y}^{2m} \mathcal{L}_{n,q}^{(\lambda)}(x, y) + \sum_{m=0}^{\lfloor n/2 \rfloor} \kappa_m D_{q,y}^{2m+1} \mathcal{L}_{n,q}^{(\lambda)}(x, y). \quad (5.5)$$

Equivalently, with the empty product interpreted as 1,

$$\begin{aligned} \mathcal{L}_{n+1,q}^{(\lambda)}(x, y) &= y \sum_{m=0}^{\lfloor n/2 \rfloor} r_m \prod_{j=0}^{2m-1} [n-j]_q \mathcal{L}_{n-2m,q}^{(\lambda)}(x, y) \\ &\quad + \sum_{m=0}^{\lfloor n/2 \rfloor} \kappa_m \prod_{j=0}^{2m} [n-j]_q \mathcal{L}_{n-2m-1,q}^{(\lambda)}(x, y). \end{aligned} \quad (5.6)$$

Moreover, the coefficients r_m and κ_m are determined recursively by

$$r_0 = 1, \quad r_m = q^{2m} a_m - \sum_{j=1}^m a_j r_{m-j} \quad (m \geq 1), \quad (5.7)$$

$$\kappa_0 = b_0, \quad \kappa_m = b_m - \sum_{j=0}^{m-1} a_{m-j} \kappa_j \quad (m \geq 1), \quad b_m = [2m+2]_q a_{m+1}, \quad (5.8)$$

and in particular $r_1 = (q^2 - 1)x$ and $\kappa_0 = [2]_q x$.

Proof. By Theorem 4.3, the raising operator is $\widehat{M}_{q,\lambda} = y\rho(D_{q,y}) + \Phi(D_{q,y})$, where ρ and Φ are defined by (5.3) and (5.4). Therefore,

$$\mathcal{L}_{n+1,q}^{(\lambda)}(x,y) = \widehat{M}_{q,\lambda}\{\mathcal{L}_{n,q}^{(\lambda)}(x,y)\} = y\rho(D_{q,y})\mathcal{L}_{n,q}^{(\lambda)}(x,y) + \Phi(D_{q,y})\mathcal{L}_{n,q}^{(\lambda)}(x,y),$$

which gives (5.5) after truncating (since $D_{q,y}^r \mathcal{L}_{n,q}^{(\lambda)} = 0$ for $r > n$). Using Proposition 3.9 yields (5.6). The recursions (5.7) and (5.8) follow by multiplying (5.3) and (5.4) by $A(t)$ and comparing coefficients in the identities $A(t)\rho(t) = A(qt)$ and $A(t)\Phi(t) = D_{q,t}A(t)$. \square

Remark 5.4. Since $r_m = O(x^m)$ and $\kappa_m = O(x^{m+1})$, truncating (5.6) at first order in x gives

$$\mathcal{L}_{n+1,q}^{(\lambda)}(x,y) = y\mathcal{L}_{n,q}^{(\lambda)}(x,y) + [2]_q x [n]_q \mathcal{L}_{n-1,q}^{(\lambda)}(x,y) + (q^2 - 1)xy [n]_q [n-1]_q \mathcal{L}_{n-2,q}^{(\lambda)}(x,y) + O(x^2),$$

which already shows that, for fixed $q \neq 1$, the q -raising step is not purely three-term at first order in x .

6. q - λ Operator equation

Here, we recast the quasi-monomiality relations into an explicit q - λ operator equation. We then unfold this relation into a concrete q -difference equation in the variable y , which may be viewed as a governing equation for the family.

Corollary 6.1. *For $n \in \mathbb{N}_0$, the polynomials $\mathcal{L}_{n,q}^{(\lambda)}(x,y)$ satisfy the operator equation*

$$(\widehat{M}_{q,\lambda}\widehat{P}_{q,\lambda} - [n]_q)\mathcal{L}_{n,q}^{(\lambda)}(x,y) = 0, \quad (6.1)$$

with $\widehat{M}_{q,\lambda}, \widehat{P}_{q,\lambda}$ given in (4.6).

Proof. This is exactly (4.8) in Theorem 4.3. \square

Remark 6.2. The operator identity (6.1) should be read in the broader context of the λ - q -operator / λ - q -Sheffer framework developed in [16]. Our contribution here is a concrete specialization generated by the even kernel $A(t) = C_{0,q}^{(\lambda)}(-xt^2)$, together with explicit coefficient formulas and transform identities for the resulting two-variable family. In particular, we do not claim a new general λ - q operator calculus; rather, we derive a new example within that established framework.

Now, we rewrite the quasi-monomiality relation (4.8) as an explicit q -difference equation in the variable y , in terms of the q -derivative $D_{q,y}$.

Proposition 6.3. *Let $\psi(t)$ and $\rho(t)$ be defined in (4.4) and (4.5). Then, the family $\mathcal{L}_{n,q}^{(\lambda)}(x,y)$ satisfies*

$$(y\rho(D_{q,y})D_{q,y} + \psi(D_{q,y})D_{q,y} - [n]_q)\mathcal{L}_{n,q}^{(\lambda)}(x,y) = 0, \quad n \in \mathbb{N}_0. \quad (6.2)$$

Equivalently, if we expand ρ and ψ as

$$\rho(t) = \sum_{m=0}^{\infty} \gamma_m(x, \lambda; q) t^{2m}, \quad \psi(t) = \sum_{m=0}^{\infty} \beta_m(x, \lambda; q) t^{2m+1}, \quad (6.3)$$

then (6.2) can be written as

$$\left(y \sum_{m=0}^{\infty} \gamma_m(x, \lambda; q) D_{q,y}^{2m+1} + \sum_{m=0}^{\infty} \beta_m(x, \lambda; q) D_{q,y}^{2m+2} - [n]_q \right) \mathcal{L}_{n,q}^{(\lambda)}(x, y) = 0. \quad (6.4)$$

The first coefficients are

$$\gamma_0 = 1, \quad \gamma_1 = (q^2 - 1)x, \quad \beta_0 = x [2]_q, \quad \beta_1 = x^2 \left((1 - \lambda) \frac{[4]_q}{[2]_q!^2} - [2]_q \right). \quad (6.5)$$

Proof. Starting from (4.8) and using $\widehat{P}_{q,\lambda} = D_{q,y}$ and $\widehat{M}_{q,\lambda} = y\rho(D_{q,y}) + \psi(D_{q,y})$ from (4.6), we obtain

$$(y\rho(D_{q,y}) + \psi(D_{q,y}))D_{q,y} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = [n]_q \mathcal{L}_{n,q}^{(\lambda)}(x, y),$$

which is exactly (6.2). Expanding ρ and ψ as in (6.3) gives (6.4).

To compute (6.5), we use $A(t) = C_{0,q}^{(\lambda)}(-xt^2)$ and the expansions

$$A(t) = 1 + xt^2 + \frac{(1-\lambda)x^2}{[2]_q!^2} t^4 + O(t^6), \quad A(qt) = 1 + xq^2t^2 + \frac{(1-\lambda)x^2q^4}{[2]_q!^2} t^4 + O(t^6).$$

Therefore,

$$\rho(t) = \frac{A(qt)}{A(t)} = 1 + (q^2 - 1)xt^2 + O(t^4),$$

which gives $\gamma_0 = 1$ and $\gamma_1 = (q^2 - 1)x$. Further, since $D_{q,t}(t^{2k}) = [2k]_q t^{2k-1}$,

$$D_{q,t}A(t) = x [2]_q t + \frac{(1-\lambda)x^2 [4]_q}{[2]_q!^2} t^3 + O(t^5),$$

and hence,

$$\psi(t) = \frac{D_{q,t}A(t)}{A(t)} = x [2]_q t + \left(\frac{(1-\lambda)x^2 [4]_q}{[2]_q!^2} - x^2 [2]_q \right) t^3 + O(t^5),$$

which yields the stated β_0, β_1 . \square

Remark 6.4. For small $|x|$, keeping only the leading coefficients $\gamma_0 = 1$ and $\beta_0 = x [2]_q$ in (6.4) yields the first-order truncation

$$(y D_{q,y} + x [2]_q D_{q,y}^2 + (q^2 - 1)xy D_{q,y}^3 - [n]_q) \mathcal{L}_{n,q}^{(\lambda)}(x, y) = O(x^2),$$

which can be useful for perturbative estimates.

7. Degenerate Laplace and Sumudu images

Next, we derive degenerate Laplace and Sumudu transform identities for the generating kernel and, consequently, for the polynomials themselves. These formulas complement the operational representation by providing closed-form expressions in the transform variables and by connecting the family to degenerate gamma-type factors.

We note that these identities should also be read in the context of the existing q -Laplace transform literature, including [3, 6]; in the present paper, the novelty lies in the degenerate setting and in the specific generating kernel producing the family $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$.

Theorem 7.1. Assume $\lambda > 0$. For values of s such that the series below converges absolutely,

$$\mathcal{L}_\lambda[e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)](s) = \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{1}{[n]_q!} \frac{n!}{\prod_{j=1}^{n+1}(s - j\lambda)}. \quad (7.1)$$

Similarly, for u in the domain of absolute convergence,

$$\mathcal{S}_\lambda[e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)](u) = \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{1}{[n]_q!} \frac{n! u^n}{\prod_{j=1}^{n+1}(1 - j\lambda u)}. \quad (7.2)$$

Proof. Insert the expansion (3.5) into the transform integral and justify term-by-term integration by absolute convergence in the stated domain. Then, apply Lemma 2.11 to each monomial term. The Sumudu identity follows either by the same argument or by $\mathcal{S}_\lambda f(u) = \frac{1}{u} \mathcal{L}_\lambda f(1/u)$. \square

Combining Theorems 3.8 and 7.1 yields a fully explicit double series. We record the Laplace-side identity.

Corollary 7.2. Assume $\lambda > 0$ and absolute convergence. Then,

$$\mathcal{L}_\lambda[e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)](s) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k y^{n-2k} \frac{n!}{\prod_{j=1}^{n+1}(s - j\lambda)}.$$

8. Limit cases

This section collects reduction limits with respect to the deformation parameters λ and q . We show that the polynomials reduce to the previously studied two-variable q -Legendre family as $\lambda \rightarrow 0$, and to a degenerate classical family as $q \rightarrow 1^-$, and we include a uniform convergence statement on compact subsets.

Proposition 8.1. For each fixed $n \in \mathbb{N}_0$ and fixed x, y, q , we have

$$\lim_{\lambda \rightarrow 0} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = \mathcal{L}_{n,q}(x, y),$$

where $\mathcal{L}_{n,q}(x, y)$ is the two-variable q -Legendre polynomial family generated by $e_q(yt) C_{0,q}(-xt^2)$.

Proof. By Theorem 3.8, the polynomial $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ is a finite sum over k . For each k , $(1)_{k,\lambda} \rightarrow 1$ as $\lambda \rightarrow 0$. Therefore, the coefficient-wise limit exists and equals the corresponding finite sum with $(1)_{k,\lambda}$ replaced by 1, i.e., the $\lambda = 0$ family. \square

Proposition 8.2. For each fixed $n \in \mathbb{N}_0$ and fixed x, y, λ , we have

$$\lim_{q \rightarrow 1^-} \mathcal{L}_{n,q}^{(\lambda)}(x, y) = \mathcal{L}_n^{(\lambda)}(x, y),$$

where

$$\mathcal{L}_n^{(\lambda)}(x, y) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{(k!)^2 (n-2k)!} x^k y^{n-2k}.$$

Proof. Use $[m]_q! \rightarrow m!$ and $[m]_q \rightarrow m$ as $q \rightarrow 1^-$. In (3.6), all sums are finite and the coefficients converge term-by-term. \square

Theorem 8.3. Fix $n \in \mathbb{N}_0$ and a compact set $K \subset \mathbb{C}^2$ in the (x, y) -plane. Then, for each fixed $q \in (0, 1)$,

$$\lim_{\lambda \rightarrow 0} \sup_{(x,y) \in K} |\mathcal{L}_{n,q}^{(\lambda)}(x, y) - \mathcal{L}_{n,q}(x, y)| = 0,$$

and, for each fixed λ ,

$$\lim_{q \rightarrow 1^-} \sup_{(x,y) \in K} |\mathcal{L}_{n,q}^{(\lambda)}(x, y) - \mathcal{L}_n^{(\lambda)}(x, y)| = 0.$$

In particular, both limits hold coefficient-wise in the finite expansion (3.6).

Proof. For each fixed n the representation (3.6) is a finite sum over $0 \leq k \leq \lfloor n/2 \rfloor$. Hence, the limits $\lambda \rightarrow 0$ and $q \rightarrow 1^-$ can be passed term-by-term. Uniformity on compact sets follows because each coefficient in (3.6) depends continuously on (x, y) and the parameter, and a finite sum of uniformly convergent terms is uniformly convergent. \square

Proposition 8.4. Fix $0 < q < 1$ and let $G_q^{(\lambda)}(t) := e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)$, interpreted as a formal power series in t with coefficients in $\mathbb{C}[x, y]$. Then, in $\mathbb{C}[x, y][[t]]$,

$$G_q^{(\lambda)}(t) \xrightarrow{\lambda \rightarrow 0} G_q^{(0)}(t) \quad \text{and} \quad G_q^{(\lambda)}(t) \xrightarrow{q \rightarrow 1^-} G_1^{(\lambda)}(t)$$

coefficient-wise, where $G_1^{(\lambda)}(t) := \exp(yt) \sum_{k \geq 0} \frac{(-1)^k (1)_{k,\lambda}}{(k!)^2} (xt^2)^k$. Moreover, if $\lambda = 0$ (or more generally if $\lambda = 1/M$ with $M \in \mathbb{N}$, so that $C_{0,q}^{(\lambda)}$ truncates), then these limits hold uniformly on sets $\{(x, y, t) : (x, y) \in K, |t| \leq R\}$ whenever $K \subset \mathbb{R}^2$ is compact and $R > 0$ is such that $(1 - q)^2 \sup_{(x,y) \in K} |x| R^2 < 1$.

Proof. The coefficient of t^n in $G_q^{(\lambda)}(t)$ is $\mathcal{L}_{n,q}^{(\lambda)}(x, y) / [n]_q!$ by Definition 3.7. The coefficient-wise limits as $\lambda \rightarrow 0$ and $q \rightarrow 1^-$ therefore follow from Theorem 3.8 together with Propositions 8.1 and 8.2.

For the uniform statement in the analytic case $\lambda = 0$, note that $C_{0,q}^{(0)}(-xt^2) = \sum_{k \geq 0} \frac{(1-q)^{2k}}{(q;q)_k^2} (-xt^2)^k$ is absolutely and uniformly convergent on $\{(x, t) : |t| \leq R, (x, y) \in K\}$ provided $(1 - q)^2 \sup_{(x,y) \in K} |x| R^2 < 1$. Uniform convergence of $e_q(yt)$ on compact y -ranges is standard, and the claim follows by the Weierstrass M-test. If $\lambda = 1/M$, then $C_{0,q}^{(\lambda)}$ is a polynomial by Remark 3.3, so the same argument applies on any bounded t -range, and the limit as $M \rightarrow \infty$ reduces to the previous $\lambda \rightarrow 0$ case. \square

Proposition 8.5. For $m \in \mathbb{N}_0$,

$$\mathcal{L}_{2m,q}^{(\lambda)}(x, 0) = [2m]_q! \frac{(1)_{m,\lambda}}{([m]_q!)^2} x^m, \quad \mathcal{L}_{2m+1,q}^{(\lambda)}(x, 0) = 0.$$

Proof. In (3.6), set $y = 0$. The term y^{n-2k} is zero unless $n - 2k = 0$. Thus, n must be even, say $n = 2m$, and then the only surviving index is $k = m$. \square

Proposition 8.6. For all $n \in \mathbb{N}_0$,

$$\mathcal{L}_{n,q}^{(\lambda)}(0, y) = y^n.$$

Proof. Setting $x = 0$ in the generating function (3.5) gives

$$\sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(0, y) \frac{t^n}{[n]_q!} = e_q(yt) C_{0,q}^{(\lambda)}(0) = e_q(yt) = \sum_{n=0}^{\infty} y^n \frac{t^n}{[n]_q!}.$$

Matching coefficients yields the claim. \square

For quick reference and for numerical verification, we list $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ for $0 \leq n \leq 15$. These expressions follow directly from the finite expansion (3.6). A numerical sample corresponding to the parameter choice $q = \frac{1}{2}$, $\lambda = 0.1$, $x = 1$, and $y \in \{0, 1, 2\}$ is reported in Table 2, which illustrates in particular the parity pattern at $y = 0$ from Proposition 8.5.

$$\mathcal{L}_{0,q}^{(\lambda)}(x, y) = 1,$$

$$\mathcal{L}_{1,q}^{(\lambda)}(x, y) = y,$$

$$\mathcal{L}_{2,q}^{(\lambda)}(x, y) = y^2 + [2]_q x,$$

$$\mathcal{L}_{3,q}^{(\lambda)}(x, y) = y^3 + [3]_q [2]_q xy,$$

$$\mathcal{L}_{4,q}^{(\lambda)}(x, y) = y^4 + [4]_q [3]_q xy^2 + (1 - \lambda) \frac{[4]_q!}{([2]_q!)^2} x^2,$$

$$\mathcal{L}_{5,q}^{(\lambda)}(x, y) = y^5 + [5]_q [4]_q xy^3 + (1 - \lambda) \frac{[5]_q!}{([2]_q!)^2} x^2 y,$$

$$\mathcal{L}_{6,q}^{(\lambda)}(x, y) = y^6 + [6]_q [5]_q xy^4 + (1 - \lambda) \frac{[6]_q!}{([2]_q!)^2 [2]_q!} x^2 y^2 + (1 - \lambda)(1 - 2\lambda) \frac{[6]_q!}{([3]_q!)^2} x^3,$$

$$\mathcal{L}_{7,q}^{(\lambda)}(x, y) = y^7 + [7]_q [6]_q xy^5 + (1 - \lambda) \frac{[7]_q!}{([2]_q!)^2 [3]_q!} x^2 y^3 + (1 - \lambda)(1 - 2\lambda) \frac{[7]_q!}{([3]_q!)^2} x^3 y,$$

$$\begin{aligned} \mathcal{L}_{8,q}^{(\lambda)}(x, y) &= y^8 + [8]_q [7]_q xy^6 + (1 - \lambda) \frac{[8]_q!}{([2]_q!)^2 [4]_q!} x^2 y^4 + (1 - \lambda)(1 - 2\lambda) \frac{[8]_q!}{([3]_q!)^2 [2]_q!} x^3 y^2 \\ &\quad + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[8]_q!}{([4]_q!)^2} x^4, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{9,q}^{(\lambda)}(x, y) &= y^9 + [9]_q [8]_q xy^7 + (1 - \lambda) \frac{[9]_q!}{([2]_q!)^2 [5]_q!} x^2 y^5 + (1 - \lambda)(1 - 2\lambda) \frac{[9]_q!}{([3]_q!)^2 [3]_q!} x^3 y^3 \\ &\quad + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[9]_q!}{([4]_q!)^2} x^4 y, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{10,q}^{(\lambda)}(x, y) &= y^{10} + [10]_q [9]_q xy^8 + (1 - \lambda) \frac{[10]_q!}{([2]_q!)^2 [6]_q!} x^2 y^6 + (1 - \lambda)(1 - 2\lambda) \frac{[10]_q!}{([3]_q!)^2 [4]_q!} x^3 y^4 \\ &\quad + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[10]_q!}{([4]_q!)^2 [2]_q!} x^4 y^2 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[10]_q!}{([5]_q!)^2} x^5, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{11,q}^{(\lambda)}(x, y) &= y^{11} + [11]_q [10]_q xy^9 + (1 - \lambda) \frac{[11]_q!}{([2]_q!)^2 [7]_q!} x^2 y^7 + (1 - \lambda)(1 - 2\lambda) \frac{[11]_q!}{([3]_q!)^2 [5]_q!} x^3 y^5 \\ &\quad + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[11]_q!}{([4]_q!)^2 [3]_q!} x^4 y^3 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[11]_q!}{([5]_q!)^2} x^5 y, \end{aligned}$$

$$\mathcal{L}_{12,q}^{(\lambda)}(x, y) = y^{12} + [12]_q [11]_q xy^{10} + (1 - \lambda) \frac{[12]_q!}{([2]_q!)^2 [8]_q!} x^2 y^8 + (1 - \lambda)(1 - 2\lambda) \frac{[12]_q!}{([3]_q!)^2 [6]_q!} x^3 y^6$$

$$\begin{aligned}
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[12]_q!}{([4]_q!)^2 [4]_q!} x^4 y^4 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[12]_q!}{([5]_q!)^2 [2]_q!} x^5 y^2 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda) \frac{[12]_q!}{([6]_q!)^2} x^6, \\
\mathcal{L}_{13,q}^{(\lambda)}(x, y) & = y^{13} + [13]_q [12]_q xy^{11} + (1 - \lambda) \frac{[13]_q!}{([2]_q!)^2 [9]_q!} x^2 y^9 + (1 - \lambda)(1 - 2\lambda) \frac{[13]_q!}{([3]_q!)^2 [7]_q!} x^3 y^7 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[13]_q!}{([4]_q!)^2 [5]_q!} x^4 y^5 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[13]_q!}{([5]_q!)^2 [3]_q!} x^5 y^3 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda) \frac{[13]_q!}{([6]_q!)^2} x^6 y, \\
\mathcal{L}_{14,q}^{(\lambda)}(x, y) & = y^{14} + [14]_q [13]_q xy^{12} + (1 - \lambda) \frac{[14]_q!}{([2]_q!)^2 [10]_q!} x^2 y^{10} + (1 - \lambda)(1 - 2\lambda) \frac{[14]_q!}{([3]_q!)^2 [8]_q!} x^3 y^8 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[14]_q!}{([4]_q!)^2 [6]_q!} x^4 y^6 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[14]_q!}{([5]_q!)^2 [4]_q!} x^5 y^4 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda) \frac{[14]_q!}{([6]_q!)^2 [2]_q!} x^6 y^2 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda)(1 - 6\lambda) \frac{[14]_q!}{([7]_q!)^2} x^7, \\
\mathcal{L}_{15,q}^{(\lambda)}(x, y) & = y^{15} + [15]_q [14]_q xy^{13} + (1 - \lambda) \frac{[15]_q!}{([2]_q!)^2 [11]_q!} x^2 y^{11} \\
& + (1 - \lambda)(1 - 2\lambda) \frac{[15]_q!}{([3]_q!)^2 [9]_q!} x^3 y^9 + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda) \frac{[15]_q!}{([4]_q!)^2 [7]_q!} x^4 y^7 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda) \frac{[15]_q!}{([5]_q!)^2 [5]_q!} x^5 y^5 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda) \frac{[15]_q!}{([6]_q!)^2 [3]_q!} x^6 y^3 \\
& + (1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)(1 - 4\lambda)(1 - 5\lambda)(1 - 6\lambda) \frac{[15]_q!}{([7]_q!)^2} x^7 y.
\end{aligned}$$

Table 2. Numerical values for $q = \frac{1}{2}$ and $\lambda = 0.1$ at $x = 1$ and $y \in \{0, 1, 2\}$ for $0 \leq n \leq 15$.

| n | $\mathcal{L}_{n,q}^{(\lambda)}(1, 0)$ | $\mathcal{L}_{n,q}^{(\lambda)}(1, 1)$ | $\mathcal{L}_{n,q}^{(\lambda)}(1, 2)$ |
|-----|---------------------------------------|---------------------------------------|---------------------------------------|
| 0 | 1.000000 | 1.000000 | 1.000000 |
| 1 | 0.000000 | 1.000000 | 2.000000 |
| 2 | 1.500000 | 2.500000 | 5.500000 |
| 3 | 0.000000 | 3.625000 | 13.250000 |
| 4 | 1.968750 | 6.250000 | 31.093750 |
| 5 | 0.000000 | 8.447266 | 68.691406 |
| 6 | 1.961719 | 11.782642 | 147.018848 |
| 7 | 0.000000 | 14.476503 | 306.217029 |
| 8 | 1.544138 | 17.699282 | 627.741019 |
| 9 | 0.000000 | 20.170193 | 1273.200117 |
| 10 | 0.984330 | 22.669094 | 2566.582610 |
| 11 | 0.000000 | 24.488649 | 5155.147440 |
| 12 | 0.507541 | 26.090479 | 10333.868197 |
| 13 | 0.000000 | 27.197145 | 20692.410622 |
| 14 | 0.206188 | 28.055016 | 41410.350330 |
| 15 | 0.000000 | 28.614739 | 82846.787911 |

Remark 8.7. The following limits are immediate from Propositions 8.1 and 8.2:

| | | |
|-------------------------|---|---------------------------------------|
| limit | $C_{0,q}^{(\lambda)}(x)$ | $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ |
| $\lambda \rightarrow 0$ | $C_{0,q}(x)$ | $\mathcal{L}_{n,q}(x, y)$ |
| $q \rightarrow 1^-$ | $\sum_{k \geq 0} \frac{(-1)^k (1)_{k,\lambda}}{(k!)^2} x^k$ | $\mathcal{L}_n^{(\lambda)}(x, y)$ |

9. A model q -evolution problem

Fix x , λ , and q and let t be a spectral parameter. Consider the q -evolution equation in y :

$$D_{q,y}U(y; t) = tU(y; t), \quad U(0; t) = C_{0,q}^{(\lambda)}(-xt^2). \quad (9.1)$$

Proposition 9.1. *The unique formal power series solution of (9.1) is*

$$U(y; t) = e_q(yt) C_{0,q}^{(\lambda)}(-xt^2) = \sum_{n=0}^{\infty} \mathcal{L}_{n,q}^{(\lambda)}(x, y) \frac{t^n}{[n]_q!}.$$

Proof. By Lemma 2.4, $D_{q,y}(e_q(yt)) = te_q(yt)$ and $D_{q,y}(C_{0,q}^{(\lambda)}(-xt^2)) = 0$. Thus, $U(y; t) = e_q(yt)C_{0,q}^{(\lambda)}(-xt^2)$ satisfies $D_{q,y}U = tU$. The initial condition follows from $e_q(0) = 1$. Uniqueness in the class of formal q -analytic solutions follows by expanding $U(y; t) = \sum_{m \geq 0} a_m(t)y^m$ and using $D_{q,y}(y^m) = [m]_q y^{m-1}$ to obtain a unique recurrence for $a_m(t)$. \square

Remark 9.2. When $\lambda = 0$, the initial condition in (9.1) reduces to the standard q -Bessel–Tricomi kernel $C_{0,q}(-xt^2)$, and the solution expands in the previously studied two-variable q -Legendre family.

For $\lambda \neq 0$, the weights $(1)_{k,\lambda}$ deform the kernel coefficients and therefore modify the solution space through a tunable parameter, while preserving the same q -evolution equation in y . This provides a simple setting where the new family gives an explicit interpolation between $\lambda = 0$ and genuinely degenerate regimes.

10. A moment functional and an orthogonality outlook

One natural way to connect $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ with analysis and probability is through the degenerate gamma function (see [12, 14]).

Definition 10.1. For $\lambda > 0$ and $\alpha \in \mathbb{C}$ with $0 < \Re(\alpha) < 1/\lambda$, the *degenerate gamma function* is

$$\Gamma_\lambda(\alpha) := \int_0^\infty t^{\alpha-1} (1 + \lambda t)^{-1/\lambda} dt = \int_0^\infty t^{\alpha-1} e_\lambda(-t) dt.$$

Definition 10.2. Fix $\lambda > 0$ and $\alpha > 0$ with $\alpha < 1/\lambda$. For polynomials p in one variable, define the linear functional

$$\mathcal{M}_{\alpha,\lambda}[p] := \frac{1}{\Gamma_\lambda(\alpha)} \int_0^\infty p(y) y^{\alpha-1} (1 + \lambda y)^{-1/\lambda} dy.$$

Proposition 10.3. For $m \in \mathbb{N}_0$,

$$\mathcal{M}_{\alpha,\lambda}[y^m] = \frac{\Gamma_\lambda(\alpha + m)}{\Gamma_\lambda(\alpha)}. \quad (10.1)$$

Proof. By Definition 10.2,

$$\mathcal{M}_{\alpha,\lambda}[y^m] = \frac{1}{\Gamma_\lambda(\alpha)} \int_0^\infty y^{m+\alpha-1} (1 + \lambda y)^{-1/\lambda} dy = \frac{\Gamma_\lambda(\alpha + m)}{\Gamma_\lambda(\alpha)},$$

where the last identity follows from Definition 10.1. \square

Proposition 10.4. For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$,

$$\mathcal{M}_{\alpha,\lambda}[\mathcal{L}_{n,q}^{(\lambda)}(x, \cdot)] = [n]_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k \frac{\Gamma_\lambda(\alpha + n - 2k)}{\Gamma_\lambda(\alpha)}. \quad (10.2)$$

Proof. Insert the explicit series form (3.6) into Definition 10.2 and apply Proposition 10.3 term-by-term:

$$\mathcal{M}_{\alpha,\lambda}[\mathcal{L}_{n,q}^{(\lambda)}(x, \cdot)] = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k}^{(\lambda)}(q) x^k \mathcal{M}_{\alpha,\lambda}[y^{n-2k}],$$

which yields (10.2). \square

Let

$$E_{m,q}^{(\lambda)}(x, y) := \mathcal{L}_{2m,q}^{(\lambda)}(x, y), \quad O_{m,q}^{(\lambda)}(x, y) := \mathcal{L}_{2m+1,q}^{(\lambda)}(x, y), \quad m \in \mathbb{N}_0.$$

The generating function $G(t) = e_q(yt) C_{0,q}^{(\lambda)}(-xt^2)$ naturally decomposes into its even and odd parts.

Proposition 10.5. *We have*

$$\sum_{m=0}^{\infty} E_{m,q}^{(\lambda)}(x, y) \frac{t^{2m}}{[2m]_q!} = \frac{G(t) + G(-t)}{2}, \quad (10.3)$$

$$\sum_{m=0}^{\infty} O_{m,q}^{(\lambda)}(x, y) \frac{t^{2m+1}}{[2m+1]_q!} = \frac{G(t) - G(-t)}{2}. \quad (10.4)$$

Proof. Both identities follow by separating the even and odd powers of t in the series $G(t) = \sum_{n \geq 0} \mathcal{L}_{n,q}^{(\lambda)}(x, y) t^n / [n]_q!$. \square

The moment functional $\mathcal{M}_{\alpha,\lambda}$ induces Hankel matrices

$$H_n := (\mu_{i+j})_{i,j=0}^n, \quad \mu_m := \mathcal{M}_{\alpha,\lambda}[y^m] = \frac{\Gamma_\lambda(\alpha + m)}{\Gamma_\lambda(\alpha)}.$$

In particular, one may consider even and odd Hankel determinants

$$\Delta_n^{(e)} := \det(\mu_{2(i+j)})_{i,j=0}^n, \quad \Delta_n^{(o)} := \det(\mu_{2(i+j)+1})_{i,j=0}^n.$$

In particular, the first nontrivial cases are

$$\Delta_0^{(e)} = \mu_0 = 1, \quad \Delta_1^{(e)} = \det \begin{pmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} = \mu_4 - \mu_2^2,$$

and

$$\Delta_0^{(o)} = \mu_1 = \frac{\Gamma_\lambda(\alpha + 1)}{\Gamma_\lambda(\alpha)}, \quad \Delta_1^{(o)} = \det \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_5 \end{pmatrix} = \mu_1 \mu_5 - \mu_3^2,$$

where $\mu_m = \Gamma_\lambda(\alpha + m) / \Gamma_\lambda(\alpha)$. These determinants control the existence of orthogonal polynomial families associated with the restriction of $\mathcal{M}_{\alpha,\lambda}$ to even/odd subspaces; see, e.g., [5, Ch. 1] and [10, Ch. 2]. In our setting, $\mu_m > 0$ for $\alpha > 0$ and $0 < \alpha < 1/\lambda$, and therefore $\Delta_n^{(e)}, \Delta_n^{(o)}$ provide natural non-degeneracy tests for potential orthogonality measures.

Remark 10.6. For fixed x , the mapping $y \mapsto \mathcal{L}_{n,q}^{(\lambda)}(x, y)$ is a polynomial sequence. It is therefore natural to ask whether the family admits an orthogonality relation with respect to a (degenerate) positive measure, possibly after a normalization. Proposition 10.4 provides a moment-type link between the family and the degenerate gamma law, which can be used as a starting point for a future Favard-type analysis.

Example 10.7. Fix $\lambda = 0.1$ and $\alpha = 1$, so that $0 < \alpha < 1/\lambda$. Using the closed form $\Gamma_\lambda(a) = \lambda^{-a} \Gamma(a) \Gamma(1/\lambda - a) / \Gamma(1/\lambda)$, one obtains the moments $\mu_m = \Gamma_\lambda(\alpha + m) / \Gamma_\lambda(\alpha)$ explicitly, and hence the Hankel determinants

$$\Delta_0^{(e)} = 1, \quad \Delta_1^{(e)} \approx 1.301020 \times 10^2, \quad \Delta_2^{(e)} \approx 1.176822 \times 10^{10},$$

and

$$\Delta_0^{(o)} \approx 1.25, \quad \Delta_1^{(o)} \approx 1.913265 \times 10^3.$$

In particular, these values are strictly positive, providing a basic non-degeneracy indication for an even/odd orthogonality program.

11. Linearization and moment functionals

In this section, we record two auxiliary results for the present family, namely a linearization formula and a moment functional representation.

For $n \in \mathbb{N}_0$, we write the explicit expansion

$$\mathcal{L}_{n,q}^{(\lambda)}(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}^{(\lambda)}(q) x^k y^{n-2k}, \quad c_{n,k}^{(\lambda)}(q) = \frac{[n]_q!}{([k]_q!)^2 [n-2k]_q!} (1)_{k,\lambda}. \quad (11.1)$$

Theorem 11.1. *Let $m, n \in \mathbb{N}_0$ and set $N = m + n$. There exist unique coefficients $\Lambda_{m,n}^{(\lambda)}(j; q) \in \mathbb{C}$ ($0 \leq j \leq \lfloor N/2 \rfloor$) such that*

$$\mathcal{L}_{m,q}^{(\lambda)}(x, y) \mathcal{L}_{n,q}^{(\lambda)}(x, y) = \sum_{j=0}^{\lfloor N/2 \rfloor} \Lambda_{m,n}^{(\lambda)}(j; q) x^j \mathcal{L}_{N-2j,q}^{(\lambda)}(x, y). \quad (11.2)$$

Moreover, the coefficients are determined recursively by

$$\Lambda_{m,n}^{(\lambda)}(0; q) = 1, \quad (11.3)$$

$$\Lambda_{m,n}^{(\lambda)}(K; q) = \sum_{i=0}^K c_{m,i}^{(\lambda)}(q) c_{n,K-i}^{(\lambda)}(q) - \sum_{j=0}^{K-1} \Lambda_{m,n}^{(\lambda)}(j; q) c_{N-2j,K-j}^{(\lambda)}(q), \quad 1 \leq K \leq \left\lfloor \frac{N}{2} \right\rfloor. \quad (11.4)$$

Proof. Expanding the left-hand side of (11.2) using (3.3) gives

$$\mathcal{L}_{m,q}^{(\lambda)}(x, y) \mathcal{L}_{n,q}^{(\lambda)}(x, y) = \sum_{K=0}^{\lfloor N/2 \rfloor} \left(\sum_{i=0}^K c_{m,i}^{(\lambda)}(q) c_{n,K-i}^{(\lambda)}(q) \right) x^K y^{N-2K}.$$

Likewise, expanding the right-hand side of (11.2) yields

$$\begin{aligned} \sum_{j=0}^{\lfloor N/2 \rfloor} \Lambda_{m,n}^{(\lambda)}(j; q) x^j \sum_{k=0}^{\lfloor (N-2j)/2 \rfloor} c_{N-2j,k}^{(\lambda)}(q) x^k y^{N-2j-2k} \\ = \sum_{K=0}^{\lfloor N/2 \rfloor} \left(\sum_{j=0}^K \Lambda_{m,n}^{(\lambda)}(j; q) c_{N-2j,K-j}^{(\lambda)}(q) \right) x^K y^{N-2K}. \end{aligned}$$

Since $\{x^K y^{N-2K}\}_{0 \leq K \leq \lfloor N/2 \rfloor}$ is a basis of the relevant finite-dimensional space, equating coefficients of $x^K y^{N-2K}$ gives the triangular system

$$\sum_{j=0}^K \Lambda_{m,n}^{(\lambda)}(j; q) c_{N-2j,K-j}^{(\lambda)}(q) = \sum_{i=0}^K c_{m,i}^{(\lambda)}(q) c_{n,K-i}^{(\lambda)}(q).$$

Because $c_{N-2K,0}^{(\lambda)}(q) = 1$ for every K , the system has a unique solution, and (11.3) and (11.4) follow by forward substitution. \square

Remark 11.2. The linearization coefficients $\Lambda_{m,n}^{(\lambda)}(j; q)$ are finite sums of products of $(1)_{k,\lambda}$ and hence are polynomials in λ of degree at most j . In particular, (11.2) yields an explicit deformation of the product formula for the nondegenerate case $\lambda = 0$.

Example 11.3. For $(m, n) = (2, 2)$, we have $\mathcal{L}_{2,q}^{(\lambda)}(x, y) = y^2 + [2]_q x$ and $N = m + n = 4$. The product identity (11.2) reads

$$\mathcal{L}_{2,q}^{(\lambda)}(x, y)^2 = \mathcal{L}_{4,q}^{(\lambda)}(x, y) + \Lambda_{2,2}^{(\lambda)}(1; q) x \mathcal{L}_{2,q}^{(\lambda)}(x, y) + \Lambda_{2,2}^{(\lambda)}(2; q) x^2,$$

where the first coefficient follows by comparing the xy^2 terms:

$$\Lambda_{2,2}^{(\lambda)}(1; q) = 2 [2]_q - [4]_q [3]_q.$$

Using the x^2 coefficient of $\mathcal{L}_{4,q}^{(\lambda)}(x, y)$ from the list in Section 8, we further obtain

$$\Lambda_{2,2}^{(\lambda)}(2; q) = [2]_q^2 - (1 - \lambda) \frac{[4]_q!}{([2]_q!)^2} - [2]_q \Lambda_{2,2}^{(\lambda)}(1; q).$$

This illustrates how the triangular recurrence (11.4) yields explicit low-degree coefficients.

Let Γ_λ be the degenerate gamma function introduced in Definition 10.1. Fix $\alpha > 0$ and define the linear functional $\mathcal{M}_{\alpha,\lambda}$ on $\mathbb{C}[y]$ by

$$\mathcal{M}_{\alpha,\lambda}[y^n] = \frac{\Gamma_\lambda(\alpha + n)}{\Gamma_\lambda(\alpha)}, \quad n \in \mathbb{N}_0. \quad (11.5)$$

This functional induces the bilinear form $\langle f, g \rangle_{\alpha,\lambda} = \mathcal{M}_{\alpha,\lambda}[f(y)g(y)]$ on $\mathbb{C}[y]$. Such moment functionals are standard tools in the study of orthogonality and Favard-type characterizations of polynomial systems; see, e.g., [5, 10].

Proposition 11.4. *For $m, n \in \mathbb{N}_0$, we have*

$$\mathcal{M}_{\alpha,\lambda}[y^m \mathcal{L}_{n,q}^{(\lambda)}(x, y)] = [n]_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1)_{k,\lambda}}{([k]_q!)^2 [n-2k]_q!} x^k \frac{\Gamma_\lambda(\alpha + m + n - 2k)}{\Gamma_\lambda(\alpha)}. \quad (11.6)$$

Proof. Insert the explicit expansion (3.3) into the left-hand side and apply (11.5) term-by-term:

$$\mathcal{M}_{\alpha,\lambda}[y^m \mathcal{L}_{n,q}^{(\lambda)}(x, y)] = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}^{(\lambda)}(q) x^k \mathcal{M}_{\alpha,\lambda}[y^{m+n-2k}] = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}^{(\lambda)}(q) x^k \frac{\Gamma_\lambda(\alpha + m + n - 2k)}{\Gamma_\lambda(\alpha)}.$$

Finally, substitute the closed form of $c_{n,k}^{(\lambda)}(q)$ from (3.3). □

Remark 11.5. Proposition 11.4 supplies a concrete Hankel-type moment structure for the sequence $\{\mathcal{L}_{n,q}^{(\lambda)}(x, \cdot)\}_{n \geq 0}$. Determining a positive measure that yields an orthogonality relation for this family, or for its even/odd subsequences, is an interesting problem that we leave for future work.

12. Comments on applications and related combinatorial directions

In particular, the deformation parameter λ supplies an additional handle beyond the standard q -parameter. At the level of the generating kernel, $(1)_{k,\lambda}$ reweights the even-indexed t^{2k} contributions, so that λ can be used to interpolate between families and to tune the growth of coefficients and moments. This flexibility can be useful, for instance, in moment/Hankel determinant problems, in transform-based solution representations for q -difference models, and in combinatorial specializations where degenerate factorial weights arise.

Remark 12.1 (Normal ordering and bosonic operator models). The quasi-monomial pair $(\widehat{M}_{q,\lambda}, \widehat{P}_{q,\lambda})$ and the generating kernel $e_q(yt)C_{0,q}^{(\lambda)}(-xt^2)$ suggest a natural link with normal ordering problems in mathematical physics. Indeed, Sheffer-type and Appell-type polynomial systems are frequently realized through boson creation/annihilation operators, and the operator Eq (6.1) provides a convenient starting point for such a realization in the present q -degenerate setting. A full bosonic representation is beyond the scope of this paper, but the operational formulas derived in Sections 4–6 are designed to support that direction.

Remark 12.2 (Heterogeneous polynomial viewpoint and specializations). Recent work on heterogeneous Stirling numbers and heterogeneous Bell polynomials [17, 18] indicates that degenerate parameters can organize families that interpolate between distinct combinatorial and classical polynomial structures. From this perspective, the parameter λ in $\mathcal{L}_{n,q}^{(\lambda)}(x, y)$ may be used as a deformation parameter to study transitions and specializations. In particular, the explicit coefficient formula (3.6) and the limit results of Section 8 provide a basis for investigating reductions to classical-type families and to combinatorial arrays related to Lah-type numbers after suitable choices of variables/normalizations. We leave the systematic classification of these specializations to a separate study.

Remark 12.3 (Spivey-type recurrences). The linearization formulas in Section 11, together with the operational identities and generating functions established earlier, also suggest the possibility of deriving Spivey-type recurrence relations for deformed Bell-/Dowling-type objects associated with the present family. This direction is motivated by recent Spivey-type formulas for degenerate Bell and Dowling polynomials in [19]. In addition, following the reviewer's suggestion, a Spivey differentiation-formula approach in the degenerate/ q -degenerate setting may provide a systematic route to such identities. Developing an analogous Spivey-type scheme for $\mathcal{L}_{n,q}^{(\lambda)}$ (or for polynomial sequences induced from it) would further strengthen the applicability of the present framework.

13. Conclusions

We constructed a degenerate extension of the two-variable q -Legendre-type polynomials by inserting a degenerate falling-factorial weight into the q -Bessel–Tricomi kernel. The resulting family admits (i) an explicit finite series representation with Gaussian q -binomial structure, (ii) an operational Rodrigues-type formula $\mathcal{L}_{n,q}^{(\lambda)}(x, y) = C_{0,q}^{(\lambda)}(-xD_{q,y}^2)y^n$, and (iii) a quasi-monomial pair of lowering/raising operators producing a compact q - λ operator equation. We have also clarified the relation of this operator equation to the established λ - q -Sheffer/operator framework [16] and explicitly acknowledged that the degenerate Laplace/gamma tools used in the paper follow the earlier work of Kim and Kim [12, 14].

On the transform side, we obtained degenerate Laplace and Sumudu images by combining the monomial transform rules with the generating kernel, which yields effective representations in the spectral variable. We also established robust reduction limits together with a term-by-term convergence theorem at the level of generating functions. Further, we recorded linearization expansions and an even/odd decomposition that interfaces naturally with moment matrices and Hankel determinants under degenerate gamma-type functionals.

Two natural continuations appear particularly promising. First, identifying positive measures and precise Favard-type conditions for orthogonality in the degenerate setting would elevate the analytic theory of the family. Second, connection and linearization problems with other two-variable q -families (e.g., q -Hermite/Laguerre-type systems), together with applications to q -difference and degenerate integral models, should provide fertile ground for follow-up work. Relatedly, the application-oriented remarks in Section 12 point toward normal-ordering models, heterogeneous polynomial specializations, and Spivey-type recurrences as concrete next steps.

Author contributions

Oğuz Yağcı: Conceptualization, methodology, formal analysis, investigation, writing—original draft preparation; Waseem Ahmad Khan: Methodology, formal analysis, validation, writing—review & editing; Khidir Shaib Mohamed: Conceptualization, supervision, project administration, funding acquisition, validation, writing—review & editing; Azhar Iqbal: Investigation, formal analysis, visualization, writing—review & editing; Wei Sin Koh: Investigation, validation, visualization, writing—review & editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the development of this manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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