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Research article

A fixed-point theorem for generalized strictly nonexpansive mappings on bounded sets in complete metric spaces

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Abstract: This paper offers substantial advances in the theory of fixed points for generalized strictly nonexpansive mappings. We develop a novel proof technique based on nonstandard analysis to establish a new fixed-point theorem. The core result demonstrates that, in a complete metric space, every continuous generalized strictly nonexpansive mapping with a bounded orbit possesses a unique fixed point to which all iterative sequences converge. The significance of this theorem lies in its substantial relaxation of the classical framework: It entirely dispenses with compactness and convexity requirements, which are typically indispensable in the study of nonexpansive mappings (such as in the Browder–Göhde theorem), replacing them solely with a boundedness condition.

Keywords: fixed point; nonstandard analysis; generalized strictly nonexpansive mappings
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1. Introduction

Recent literature on asymptotic contractions has produced numerous significant results. Works such as [1,2] have extensively studied fixed points and iterative fixed points for these mappings, establishing fundamental theorems in the field. For detailed proofs and further insights, readers are directed to the cited references.

In a complete metric space (X, d), a contraction T is defined by the existence of $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{1.1}$$

Classical Banach fixed-point theory establishes that the sequence $\{T^n x\}$ converges to a unique fixed point of T.

Building on this, Rakotch [3] introduced a significant refinement:

$$d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y), \quad \forall x, y \in X, \tag{1.2}$$

where α satisfies certain conditions. Under these conditions, the sequence $\{T^n x\}$ converges to a unique fixed point of T.

In 1969, Boyd and Wong [4] generalized Rakotch's result by considering

$$d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X, \tag{1.3}$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is upper semicontinuous from the right and satisfies $0 \leq \psi(t) < t$ for all $t > 0$. They showed that $\{T^n x\}$ converges to a unique fixed point of T.

In 2003, Kirk [5] introduced an asymptotic version of the Boyd and Wong result. An asymptotic contraction is defined by

$$d(T^n x, T^n y) \leq \psi_n(d(x, y)), \quad \forall x, y \in X, \tag{1.4}$$

where $\psi_n : [0, +\infty) \to [0, +\infty)$ and $\psi_n \to \psi$ uniformly. Kirk proved that $\{T^n x\}$ converges to a unique fixed point of T provided that some orbit of T is bounded and T is continuous.

A significant advancement occurred in 2023 when Lindstrøm and Ross [6] extended Kirk's 2003 result [5], showing that the boundedness condition and the continuity requirements on ψ_n and ψ can be eliminated.

In proving this theorem, Lindstrøm and Ross [6] utilized a refined application of nonstandard analysis, particularly the nonstandard extensions of real numbers and metric space theory.

The Brouwer fixed-point theorem (L.E.J. Brouwer [7], 1912) marks the starting point and cornerstone of this theoretical sequence. Its classical formulation asserts:

Theorem 1.1 (Brouwer fixed-point theorem)**.** *Let K be a nonempty compact convex subset of a finite-dimensional Euclidean space. If $f : K \to K$ is a continuous mapping, then there exists a point $x^* \in K$ such that $f(x^*) = x^*$.*

The Brouwer fixed-point theorem, established by L.E.J. Brouwer in 1912, marks a foundational result in topology and analysis. The theorem requires three key conditions: The domain must be a subset of finite-dimensional Euclidean space, it must be compact and convex, and the mapping must be continuous. These conditions ensure that the topological properties of Euclidean spaces can be effectively utilized in the proof. Brouwer's theorem quickly found applications in game theory and economics, but its restriction to finite dimensions posed a significant limitation for analysis in function spaces, prompting mathematicians to seek extensions to infinite-dimensional settings.

Juliusz Schauder [8] extended Brouwer's result to infinite-dimensional spaces in 1930, significantly broadening the scope of fixed point theory.

Theorem 1.2 (Schauder fixed-point theorem)**.** *Let K be a nonempty compact convex subset of a Banach space X. If $f : K \to K$ is a continuous mapping, then f has at least one fixed point in K.*

Addressing the limitations of Brouwer's theorem, Juliusz Schauder generalized the result in 1930 by extending it to arbitrary Banach spaces. The theorem maintains the requirements of compactness and convexity for the domain, as well as continuity for the mapping, but now allows the domain to reside in an infinite-dimensional complete normed vector space. This advancement encompasses many function spaces arising in differential and integral equations. However, the requirement that the domain be compact becomes more restrictive in infinite dimensions, where closed and bounded sets are not necessarily compact. This compactness condition ensures that the infinite-dimensional problem can be approximated by finite-dimensional ones, bridging back to Brouwer's theorem.

The Browder-Göhde fixed-point theorem, independently proved by Felix E. Browder and Gerhard Göhde [9] in 1965, addresses precisely this question by introducing a geometric condition on the space and relaxing the requirement on the mapping.

Definition 1.3 (Nonexpansive mapping)**.** *Let C be a subset of a normed space X. A mapping $T : C \to C$ is called* nonexpansive *if, for all $x, y \in C$,*

$$\|T(x) - T(y)\| \leq \|x - y\|. \tag{1.5}$$

Definition 1.4 (Uniformly convex Banach space)**.** *A Banach space X is* uniformly convex *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, we have $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.*

With these definitions in place, we can now state the theorem:

Theorem 1.5 (Browder-Göhde fixed-point theorem)**.** *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. If $T : C \to C$ is a nonexpansive mapping, then T has at least one fixed point in C.*

A further breakthrough came in 1965 with the independent work of Felix E. Browder and Gerhard Göhde. Their theorem considers nonexpansive mappings on subsets of uniformly convex Banach spaces, replacing the compactness requirement of Schauder's theorem with the geometric condition of uniform convexity. The domain need only be bounded, closed, and convex, while the mapping must be nonexpansive—a condition stronger than continuity but naturally satisfied in many variational problems. This represents a significant shift, as fixed-point existence is now guaranteed by the geometry of the space rather than the compactness of the domain, allowing the theorem to be applied to a broader class of problems, including those involving unbounded operators.

A mapping T is said to be *strictly nonexpansive* if it satisfies

$$d(Tx, Ty) < d(x, y), \quad \forall x \neq y. \tag{1.6}$$

Boyd and Wong [4] noted that even this condition does not guarantee a fixed point in a complete metric space. This reveals a significant gap: Neither the classical nonexpansiveness of Browder-Gohde nor the stricter condition (1.6) is sufficient, under mere boundedness, to ensure fixed points.

To bridge this gap, we introduce a more refined condition. A mapping T is called *generalized strictly nonexpansive* if it satisfies

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(Ty, y)\}, \quad \forall x \neq y. \tag{1.7}$$

This condition is stronger than (1.6) but non-asymptotic and directly imposed on T itself, differing from the asymptotic contraction framework.

The central aim of this paper is to establish a fixed-point theorem that significantly weakens the foundational assumptions of the classical theory. By strengthening the mapping condition from nonexpansive to generalized strictly nonexpansive and employing nonstandard analysis, we are able to replace the stringent requirements of compactness (Schauder) or uniform convexity (Browder-Gohde) with a simple boundedness condition. Our main result (Theorem 2.1) proves that in a complete metric space, the boundedness of a single orbit is sufficient to guarantee the existence of a unique fixed point and the convergence of all iterative sequences. This represents a novel synthesis, extending the theory to a much broader setting and providing a clear, infinitesimal-based proof via nonstandard analysis.

Inspired by Lindstrøm and Ross's creative application of nonstandard analysis [6], we adopt this tool not to extend their specific results, but to explore its potential for extending classical fixed-point theorems. This motivated a detailed analysis of its role in our context and its subsequent integration into our proof.

2. Definitions and properties

This section presents fundamental concepts in nonstandard analysis.

Let (X,d) be a standard complete metric space. Its nonstandard extension, denoted by $({}^*X, d)$, satisfies the inclusion $X \subseteq {}^*X$. For any finite point $x \in {}^*X$, there exists a unique standard point ${}^o x \in X$ such that $x \approx {}^o x$.

The symbol $\approx$ denotes the relation of being *infinitely close*. In the framework of nonstandard analysis, two points $x, y \in {}^*X$ satisfy $x \approx y$ if and only if their distance $d(x, y)$ is an infinitesimal number, i.e., smaller in absolute value than every positive standard real number. For a finite point x (meaning that $d(x, p)$ is finite for some standard $p \in X$), the point ${}^o x$ is called its *standard part* . The completeness of X guarantees that such a unique standard point exists, and it can be thought of as the "closest" standard point to x, up to an infinitesimal error. Thus, $x \approx {}^o x$ expresses that x is within an infinitesimal distance of its standard part. Comprehensive introductions to nonstandard analysis, covering the rigorous foundation of infinitesimals, the transfer principle, the standard part map, and applications to metric spaces, can be found in Refs. [10, 11].

Proposition 2.1. *For $k \in {}^*N - N$, if $\{x_n\}$ is bounded satisfying $d(x_i, x_j) \approx 0, i, j > k$, then there is unique $x_\infty \in X$ such that $d(x_i, x_\infty) \approx 0, \forall i > k$.*

Proof. By the boundedness, x_i, x_j are all near-standard points. There are ${}^o x_i, {}^o x_j \in X$ such that $d(x_i, {}^o x_i) \approx 0, d(x_j, {}^o x_j) \approx 0$. From $d(x_i, x_j) \approx 0$, we have ${}^o x_i = {}^o x_j = x_\infty$. □

To facilitate the discussion below, we introduce two definitions of convergence.

Definition 2.2. *(X,d) is standard complete metric space, and $({}^*X, d)$ is nonstandard extension. $\{x_n\} \subseteq X, n \in N$, $\{x_i\} \subseteq {}^*X, i \in {}^*N$, $\{x_n\} \subseteq \{x_i\}$. Then we say that $\{x_i\}$ is the extended sequence of $\{x_n\}$.*

(i)x_n converges to $x_\infty \in X$

$\Leftrightarrow \forall i \in {}^*N - N, d(x_i, x_\infty) \approx 0.$

(ii)x_n finally converges to $x_\infty \in X$

$\Leftrightarrow \exists k \in {}^*N - N,$ *for all* $i > k, d(x_i, x_\infty) \approx 0.$

The following proposition is easily obtainable.

Proposition 2.3. *If x_n converges to $x_\infty \in X$, then it finally converges to x_∞.*

Proof. This conclusion is evident from Definition 2.2. □

The inverse proposition may not necessarily hold true. However, we have the following result.

Proposition 2.4. *Suppose x_n finally converges to $x_\infty \in X$ with*

$$\max\{d(x_{n+1}, x_\infty), d(x_{n+1}, x_n)\} \leq \max\{d(x_n, x_\infty), d(x_n, x_{n-1})\}. \tag{2.1}$$

Then x_n converges to x_∞.

Proof. Since

$$0 \leq \max\{d(x_{n+1}, x_\infty), d(x_{n+1}, x_n)\} \leq \max\{d(x_n, x_\infty), d(x_n, x_{n-1})\}, \tag{2.2}$$

there is a real number $a \geq 0$ such that

$$\max\{d(x_n, x_\infty), d(x_n, x_{n-1})\} \to a. \tag{2.3}$$

Suppose, for a contradiction, that $a > 0$. By the descriptions of convergence,

$$\max\{d(x_i, x_\infty), d(x_i, x_{i-1})\} \approx a > 0, \forall i \in {}^*N - N. \tag{2.4}$$

Since x_n finally converges to x_∞, then for sufficiently large infinite i,

$$d(x_i, x_\infty) \approx 0, \quad and \quad d(x_i, x_{i-1}) \approx 0. \tag{2.5}$$

That is a contradiction.

So, $\max\{d(x_n, x_\infty), d(x_n, x_{n-1})\}$ converges to 0. That means x_n converges to x_∞. □

The following contribute to obtain the results of the next section.

Proposition 2.5. *Suppose (X,d) is a metric space, and $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \alpha \cdot \max\{d(x, y), d(x, Tx), d(Ty, y)\}, 0 < \alpha < 1, \forall x, y \in X. \tag{2.6}$$

Then, the following conclusions hold.

(1) For $x \in X, Tx \neq x, m, n \in N^+$,

$$d(T^n x, T^{m+n} x) \leq \alpha^n \cdot \max\{d(Tx, x), d(T^m x, x)\}. \tag{2.7}$$

(2) For $x, y \in X, x \neq y, n \in N^+$,

$$d(T^n x, T^n y) \leq \alpha^n \cdot \max\{d(x, y), d(Tx, x), d(Ty, y)\}. \tag{2.8}$$

Proof. (1) From the given condition,

$$
\begin{aligned}
d(T^n x, T^{m+n} x) &\le \alpha \cdot \max\{d(T^{n-1}x, T^{m+n-1}x), d(T^n x, T^{n-1}x), d(T^{n+m}x, T^{n+m-1}x)\} \\
&\le \alpha^2 \cdot \max\Big\{d(T^{n-2}x, T^{m+n-2}x), d(T^{n-1}x, T^{n-2}x), d(T^{n+m-1}x, T^{n+m-2}x) \\
&\quad d(T^{n-1}x, T^{n-2}x), d(T^n x, T^{n-1}x), d(T^{n-1}x, T^{n-2}x), d(T^{n+m-1}x, T^{n+m-2}x) \\
&\quad d(T^{n+m}x, T^{n+m-1}x), d(T^{n+m-1}x, T^{n+m-2}x)\Big\} \\
&\le \alpha^2 \cdot \max\Big\{d(T^{n-2}x, T^{m+n-2}x), d(T^{n-1}x, T^{n-2}x), d(T^{n+m-1}x, T^{n+m-2}x)\Big\} \\
&\le \alpha^3 \cdot \max\Big\{d(T^{n-3}x, T^{m+n-3}x), d(T^{n-2}x, T^{n-3}x), d(T^{n+m-2}x, T^{n+m-3}x)\Big\} \\
&\ \ \vdots \\
&\le \alpha^n \cdot \max\Big\{d(x, T^m x), d(Tx, x), d(T^{m+1}x, T^m x)\Big\} \\
&\le \alpha^n \cdot \max\Big\{d(x, T^m x), d(Tx, x)\Big\}.
\end{aligned} \tag{2.9}
$$

(2) By the similar process, we have

$$
\begin{aligned}
d(T^n x, T^n y) &\le \alpha \cdot \max\{d(T^{n-1}x, T^{n-1}y), d(T^n x, T^{n-1}x), d(T^n y, T^{n-1}y)\} \\
&\le \alpha^2 \cdot \max\Big\{d(T^{n-2}x, T^{n-2}y), d(T^{n-1}x, T^{n-2}x), d(T^{n-1}y, T^{n-2}y) \\
&\quad d(T^{n-1}x, T^{n-2}x), d(T^n x, T^{n-1}x), d(T^{n-1}x, T^{n-2}x), d(T^{n-1}y, T^{n-2}y) \\
&\quad d(T^n y, T^{n-1}y), d(T^{n-1}y, T^{n-2}y)\Big\} \\
&\le \alpha^2 \cdot \max\Big\{d(T^{n-2}x, T^{n-2}y), d(T^{n-1}x, T^{n-2}x), d(T^{n-1}y, T^{n-2}y)\Big\} \\
&\le \alpha^3 \cdot \max\Big\{d(T^{n-3}x, T^{n-3}y), d(T^{n-2}x, T^{n-3}x), d(T^{n-2}y, T^{n-3}y)\Big\} \\
&\ \ \vdots \\
&\le \alpha^n \cdot \max\Big\{d(x, y), d(Tx, x), d(Ty, y)\Big\}.
\end{aligned} \tag{2.10}
$$

□

Following a similar deductive process, it is straightforward to derive the following conclusion.

Proposition 2.6. *Suppose (X,d) is a metric space, and $T : X \to X$ is generalized strictly nonexpansive mapping. Then, the following conclusions hold.*

(1) For $x \in X, Tx \neq x, m, n \in N^+$,

$$d(T^n x, T^{m+n} x) < \max\{d(Tx, x), d(T^m x, x)\}. \tag{2.11}$$

(2) For $x, y \in X, x \neq y, n \in N^+$,

$$d(T^n x, T^n y) < \max\{d(x, y), d(Tx, x), d(Ty, y)\}. \tag{2.12}$$

Proof. The proof is similar to that of Property 2.5 above. □

The following property is inherently valid within the framework of non-standard analysis.

Proposition 2.7. *Suppose* $f : R \to R$ *is continuous, and* $\lim_{x\to 0} f(x) = A$*. If* x_n *converges to 0 with* $n \in N$*, then for all* $i \in {}^*N - N$, $f(x_i) \approx A$*.*

Proof. Since $x_n \to 0$, for any standard $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for all $n > N$. By the transfer principle, this statement also holds in the nonstandard extension. In particular, for any infinite natural number $i \in {}^*\mathbb{N} \setminus \mathbb{N}$, we have $i > N$ (because N is finite), and hence $|x_i| < \varepsilon$. As ε is an arbitrary positive standard real number, we conclude that $x_i \approx 0$.

Now, because $\lim_{x\to 0} f(x) = A$, for any standard $\varepsilon' > 0$, there exists a standard $\delta > 0$ such that $|f(x) - A| < \varepsilon'$ whenever $|x| < \delta$. Since $x_i \approx 0$, we have $|x_i| < \delta$ (as δ is a positive standard real number). Applying the transfer principle to the implication "if $|x| < \delta$, then $|f(x) - A| < \varepsilon'$", we obtain $|f(x_i) - A| < \varepsilon'$. Because ε' is an arbitrary standard positive number, it follows that $f(x_i) \approx A$. □

3. Main results

This section presents our principal theoretical contributions along with their important corollaries. Most significantly, we have successfully established fixed-point theorems for strictly nonexpansive mappings and their generalized counterparts under boundedness conditions, substantially weakening the conventional compactness requirements.

Theorem 3.1 (Main theorem)**.** *Suppose (X,d) is a complete metric space, and* $T : X \to X$ *is continuous and generalized strictly nonexpansive mapping. If there exists a bounded orbit of* T*, then* T *has a unique fixed point* x_∞*, and for every* $x \in X$*, the iterative sequence* $\{T^n x\}$ *converges to* x_∞*.*

Proof. Suppose $\{T^n x_0\}$ is a bounded orbit. For orbit $\{T^n x\}$, by Proposition 2.6, we have

$$\begin{aligned} d(T^n x, x) &\le d(T^n x, T^n x_0) + d(T^n x_0, x_0) + d(x_0, x) \\ &\le \max\{d(x, x_0), d(Tx, x), d(Tx_0, x_0)\} + d(T^n x_0, x_0) + d(x_0, x). \end{aligned} \tag{3.1}$$

That means $\{T^n x\}$ is bounded orbit.

By the definition of generalized strictly nonexpansive mapping, for $x \neq y$,

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(Ty, y)\}. \tag{3.2}$$

Then

$$1 - \frac{d(Tx, Ty)}{\max\{d(x, y), d(x, Tx), d(Ty, y)\}} > 0. \tag{3.3}$$

Since inequality (3.3) holds, for every pair of distinct points $x \neq y$, the quantity

$$\eta(x, y) := 1 - \frac{d(Tx, Ty)}{\max\{d(x, y), d(x, Tx), d(Ty, y)\}}$$

is a positive standard real number for all $x \neq y$. Thus, the set $\{\eta(x, y) : x, y \in X, x \neq y\}$ consists of positive standard numbers. Now, fix any positive infinitesimal $q \in *\mathbb{R}^+$. By the definition of an infinitesimal, we have $0 < q < r$ for every positive standard real number r. In particular, for all $x \neq y$,

$$0 < q < \eta(x, y).$$

This q is universal and independent of the specific choice of x and y. Consequently, we have

$$1 - \frac{d(Tx, Ty)}{\max\{d(x, y), d(x, Tx), d(Ty, y)\}} > q > 0. \tag{3.4}$$

That is,

$$d(Tx, Ty) < (1 - q) \max\{d(x, y), d(x, Tx), d(Ty, y)\}. \tag{3.5}$$

From Proposition 2.5, for all $i, j \in {}^*N$,

$$d(T^i x, T^{i+j} x) < (1 - q)^i \max\{d(x, Tx), d(T^j x, x)\}. \tag{3.6}$$

By the boundedness of $\{T^n x\}$, there exists $M > 0$ such that

$$d(T^n x, x) \le M, \forall n \in N^+. \tag{3.7}$$

It can be naturally extended to non-standard domains, which means

$$d(T^i x, T^{i+j} x) < (1 - q)^i M. \tag{3.8}$$

Let $i > \frac{1}{q^2}$, then

$$d(T^i x, T^{i+j} x) \le (1 - q)^{\frac{1}{q^2}} M. \tag{3.9}$$

Since $\lim_{t \to 0^+} (1 - t)^{\frac{1}{t^2}} = 0$ with $t \in R^+$, we have $(1 - q)^{\frac{1}{q^2}} \approx 0$ for $q \approx 0$. That is,

$$d(T^i x, T^{i+j} x) \approx 0, \forall i > \frac{1}{q^2}, j \in {}^*N - N, x \in X. \tag{3.10}$$

By the boundedness of $T^n x$, there is $x_\infty \in X$, and

$$d(T^i x, x_\infty) \approx 0, \ for\ all\ i > \frac{1}{q^2}. \tag{3.11}$$

By continuity of T,

$$x_\infty \approx T^{i+1} x_\infty = T(T^i x_\infty) \approx T(x_\infty). \tag{3.12}$$

Hence, $T(x_\infty) = x_\infty$.

Since

$$\max\{d(T^{n+1} x, x_\infty), d(T^{n+1} x, T^n x)\} < \max\{d(T^n x, x_\infty), d(T^n x, T^{n-1} x)\}, \tag{3.13}$$

by Proposition 2.4, we have that $T^n x$ converges to x_∞. The inequality

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\} \tag{3.14}$$

means that fixed point x_∞ is unique. □

Considering the key condition of boundedness in the above theorem, we can give the following corollaries.

Corollary 3.2. *Suppose (X,d) is a complete metric space, and $T : X \to X$ is continuous and generalized strictly nonexpansive mapping. If there is a fixed point of T, then, for all $x \in X$, $T^n x$ converges to the unique fixed point.*

This finding significantly enhances the scope of fixed-point theory, as it demonstrates convergence under the generalized condition of strict nonexpansiveness whenever a fixed point exists. This effectively relaxes the classical contraction requirement in Banach's theorem, thereby extending the applicability of fixed-point analysis to a wider class of operators in complete metric spaces.

Corollary 3.3. *Suppose M is a bounded closed subset of complete metric space (X,d), and $T : M \to M$ is continuous and generalized strictly nonexpansive mapping. Then, for all $x \in M$, $T^n x$ converges to the unique fixed point.*

As a special case of it, there are also corresponding results for strictly nonexpansive mappings.

Corollary 3.4. *Suppose M is a bounded closed subset of complete metric space (X,d), and $T : M \to M$ is strictly nonexpansive mapping. Then, for all $x \in M$, $T^n x$ converges to the unique fixed point.*

The particular significance of this finding stems from its elimination of the prerequisite for fixed-point existence. By capitalizing on the boundedness and closedness of subset M, this corollary enables direct convergence analysis based solely on structural properties. This advancement proves instrumental in computational applications, as it ensures iteration schemes converge to the unique fixed point using only the nonexpansive property of the mapping, thereby providing a more streamlined theoretical foundation for solving functional equations.

The following example, which is set in a non-uniformly convex space, demonstrates that the hypotheses of Theorem 3.1 are satisfied by a concrete mapping that is not nonexpansive, and it shows the convergence of its iterates to the unique fixed point.

Example 3.1. *Let $X = \ell^1 = \{(x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i| < +\infty\}$ with the norm $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$. Define a mapping $f : X \to X$ as follows. First, define $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(t) = \begin{cases} 1 + \sqrt{t+1}, & t \geq -1, \\ 1, & t < -1. \end{cases}$$

For any $x = (x_1, x_2, \dots) \in \ell^1$, set

$$f(x) = \left(g(x_1), \frac{x_2}{2}, \frac{x_3}{3}, \dots \right).$$

Clearly, f is continuous. We now analyze its properties in detail.

(1) Generalized strictly nonexpansive. *For any $x \neq y$, we need to show*

$$\|f(x) - f(y)\|_1 < \max\{\|x - y\|_1, \|x - f(x)\|_1, \|y - f(y)\|_1\}.$$

Denote

$$A = \|f(x) - f(y)\|_1, \quad B = \|x - y\|_1, \quad C = \|x - f(x)\|_1, \quad D = \|y - f(y)\|_1.$$

Observe that for $i \geq 2$,

$$|f_i(x) - f_i(y)| = \frac{|x_i - y_i|}{i} \leq \frac{1}{2}|x_i - y_i|.$$

Thus, the contribution of coordinates $i \geq 2$ to A is at most half of their contribution to B.

For the first coordinate, the function g satisfies the following one-dimensional inequality: For any real numbers $a \neq b$,

$$|g(a) - g(b)| < \max\{|a-b|, |a-g(a)|, |b-g(b)|\}.$$

This can be verified by analyzing g and $\phi(t) = g(t) - t$. A straightforward case analysis (dividing into $a < b \leq 3$, $3 \leq a < b$, and $a < 3 < b$) confirms the inequality.

Now consider two cases for x and y.

- ***Case 1:** $x_i = y_i$ for all $i \geq 2$. Then*

$$A = |g(x_1) - g(y_1)|, \quad B = |x_1 - y_1|, \quad C \geq |x_1 - g(x_1)|, \quad D \geq |y_1 - g(y_1)|.$$

  The one-dimensional inequality gives $A < \max\{B, C, D\}$.
- ***Case 2:** There exists some $i \geq 2$ with $x_i \neq y_i$. Then the sum over $i \geq 2$ in A is strictly less than half of that in B. Let $M = \max\{B, C, D\}$.*

  If $M = B$, we consider two subcases. If $|g(x_1) - g(y_1)| \leq |x_1 - y_1|$, then

$$A \leq |x_1 - y_1| + \frac{1}{2}\sum_{i \geq 2}|x_i - y_i| < B.$$

  If $|g(x_1) - g(y_1)| > |x_1 - y_1|$, then by the one-dimensional inequality, $|g(x_1) - g(y_1)| < \max\{|x_1 - g(x_1)|, |y_1 - g(y_1)|\}$. Without loss, assume it is $< |x_1 - g(x_1)|$. Since $B \geq C \geq |x_1 - g(x_1)|$, we have $|g(x_1) - g(y_1)| < B$. Moreover, the compression in coordinates $i \geq 2$ ensures that the total contribution from these coordinates to A is less than their contribution to B. Combining these facts yields $A < B$.

  If $M = C$ or $M = D$, a similar argument (using the one-dimensional inequality and coordinate-wise compression) gives $A < M$.

Hence, in all cases, $A < \max\{B, C, D\}$, so f is generalized strictly nonexpansive.

***(2) Non-nonexpansiveness.** The mapping f is not nonexpansive: Take $x = (-0.99, 0, 0, \dots)$ and $y = (-0.98, 0, 0, \dots)$. Then $\|x - y\|_1 = 0.01$, but*

$$|g(-0.99) - g(-0.98)| \approx 0.0414 > 0.01,$$

so $\|f(x) - f(y)\|_1 > \|x - y\|_1$.

(3) Iteration and unique fixed point.

For any $x \in \ell^1$, $f^n(x) = (g^n(x_1), x_2/2^n, x_3/3^n, \dots)$. The function g has a unique fixed point $t_0 = 3$, and $g^n(t) \to 3$ for every $t \in \mathbb{R}$. The factors $1/i^n \to 0$ for $i \geq 2$. Hence, $f^n(x)$ converges pointwise (and in ℓ^1-norm) to $(3, 0, 0, \dots)$, which is the unique fixed point of f. Every orbit converges to it.

In summary, f is continuous, and generalized strictly nonexpansive but not nonexpansive, and all its iterates converge to the unique fixed point $(3, 0, 0, \dots)$.

4. Comparison with existing fixed point theorems

Table 1 summarizes the key fixed-point theorems discussed in the introduction alongside our main result, highlighting the differences in mapping conditions and space assumptions.

Table 1. Comparison of fixed point theorems.

Reference	Mapping condition	Space / domain assumptions
Brouwer (1912)	Continuous	Nonempty compact convex subset of $\mathbb{R}^n$
Schauder (1930)	Continuous	Nonempty compact convex subset of a Banach space
Banach (1922)	$d(Tx, Ty) \leq \alpha d(x, y)$, $0 \leq \alpha < 1$	Complete metric space
Rakotch (1962)	$d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y)$	Complete metric space
Boyd–Wong (1969)	$d(Tx, Ty) \leq \psi(d(x, y))$, $\psi(t) < t$	Complete metric space
Kirk (2003)	$d(T^n x, T^n y) \leq \psi_n(d(x, y))$, $\psi_n \to \psi, \psi(t) < t$	Complete metric space, T continuous, existence of a bounded orbit
Lindstrøm–Ross (2023)	$d(T^n x, T^n y) \leq \psi_n(d(x, y))$, $\psi_n \to \psi, \psi(t) < t$	Complete metric space (no boundedness/continuity required)
Browder–de Figueiredo (1965)	Nonexpansive: $\|Tx - Ty\| \leq \|x - y\|$	Nonempty bounded closed convex subset of a uniformly convex Banach space
This paper (Theorem 3.1)	$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(Ty, y)\}$	Complete metric space, T continuous, existence of a bounded orbit
This paper (Corollary 3.4)	$d(Tx, Ty) < d(x, y)$, $x \neq y$	bounded closed subset of complete metric space

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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