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Research article

Some results for a certain class of nonlinear binomial differential equation

Qiyuan Feng and Junfeng Xu*

School of Mathematics and Computational Sciences, Wuyi University, Jiangmen, Guangdong 529000, China

* **Correspondence:** Email: xujunf@gmail.com.

Abstract: In this paper, we study one class of nonlinear binomial differential equation

$$(f(z))^n + a(f^{(k)}(z))^n = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z},$$

where $n, k \in \mathbb{N}^+$, a, p_i, and λ_i are nonzero constants for $i = 1, 2$. Some results answer the previous conjecture given by Linkui Gao and Junyang Gao.

Keywords: binomial differential equation; exponential polynomial; entire function; Wiman–Valiron theorem
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1. Introduction and results

In this paper, all meromorphic functions are defined on the complex $\mathbb{C}$. Nevanlinna theory is the main tool to establish our results. Throughout this paper, we assume that the readers are familiar with the standard notations and basic results in Nevanlinna theory, such as $T(r, f), m(r, f), N(r, f), S(r, f)$, and so on (see [13]). For convenience of the readers, let's recall the definition of an exponential polynomial of the form

$$f(z) = P_1(z)e^{Q_1(z)} + P_2(z)e^{Q_2(z)} + \cdots + P_k(z)e^{Q_k(z)}. \tag{1.1}$$

Let $q = \max\{\deg(Q_j) : Q_j(z) \not\equiv 0\}$, and let $\omega_1, \cdots, \omega_m$ be pairwise different leading coefficients of the polynomials $Q_j(z)$ of maximum degree q. Thus, Eq (1.1) can be rewritten as

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \tag{1.2}$$

where H_j are either exponential polynomials of degree $< q$ or ordinary polynomials in z. Due to the construction, we have $H_j(z) \not\equiv 0$ for $1 \leq j \leq m$.(See. [8, 11])

In [8], Zhitao Wen et al. introduced two special sets of exponential polynomials:

$$\Gamma_0 = \{h(z) = e^{\alpha(z)} : \alpha(z) \text{ is a nonconstant polynomial}\}.$$

$$\Gamma_1 = \{h(z) = e^{\alpha(z)} + d : d \in \mathbb{C} \setminus \{0\} \text{ and } \alpha(z) \text{ is a nonconstant polynomial}\}.$$

Similarly, we define two special sets of exponential polynomials,

$$T_1 = \{h(z) = c_1 e^{\alpha_1(z)} + c_2 e^{\alpha_2(z)}\},$$

$$T_2 = \{h(z) = c_1 e^{\alpha_1(z)} + c_2 e^{\alpha_2(z)} + c_3 e^{\alpha_3(z)}\},$$

where c_1, c_2, c_3 are nonzero constants, and $\alpha_1, \alpha_2, \alpha_3$ are different nonconstant polynomials without a constant term.

Recently, many scholars have focused on the solutions of nonlinear differential equations (See [6, 14, 15]) and the solutions of nonlinear differential-difference equations (See [9, 10, 12]).

In [1], Linkui Gao and Junyang Gao investigated entire solutions of the following equation with finite order,

$$(f(z))^2 + a(f'(z))^2 = p_1 e^{\lambda z} + p_2 e^{-\lambda z},$$

where a, p_1, p_2, and λ are nonzero constants. It is inspired by

$$(\cos z)^2 - ((\cos z)')^2 = \cos 2z = \frac{e^{2iz} + e^{-2iz}}{2}.$$

Theorem A. *Let a, p_1, p_2, and λ be nonzero constants, and let $9a\lambda^2 + 4 \neq 0$. Then, the equation*

$$(f(z))^2 + a(f'(z))^2 = p_1 e^{\lambda z} + p_2 e^{-\lambda z} \tag{1.3}$$

has entire solutions if and only if the condition $a\lambda^2 + 1 = 0$ or $a\lambda^2 - 4 = 0$ holds. Moreover, we have the following

(i) If $a\lambda^2 + 1 = 0$, then the entire solutions of Eq (1.3) are

$$f(z) = r_i e^{\lambda z} + s_i e^{-\lambda z} + t_i,$$

where $t_i (i = 1, 2, 3, 4)$ are the four roots of $t^4 + p_1 p_2 = 0$, $r_i = p_1/2t_i$, and $s_i = p_2/2t_i$.

(ii) If $a\lambda^2 - 4 = 0$, then the entire solutions of Eq (1.3) are

$$f(z) = a_i e^{\lambda z/2} + b_i e^{-\lambda z/2},$$

where $a_i (i = 1, 2)$ are the square roots of $p_1/2$, and $b_i (i = 1, 2)$ are the square roots of $p_2/2$.

Naturally, it is interesting to consider a more general binomial differential equation,

$$(f(z))^2 + a(f'(z))^2 = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}, \tag{1.4}$$

where a, p_i, and λ_i are nonzero constants for $i = 1, 2$. Actually, the situation of $\lambda_1 + \lambda_2 = 0$ in Eq (1.4) was considered in Theorem A. In the following, we consider the situation of $\lambda_1 + \lambda_2 \neq 0$ in Eq (1.4).

Theorem 1.1. *Suppose that a, p_i, and λ_i are nonzero constants for $i = 1, 2$, $\lambda_1 + \lambda_2 \neq 0$. For the entire solution $f(z)$ of Eq (1.4),*
(1) *if $f(z) \in T_1$, then $f(z) = c_1e^{\omega_1 z} + c_2e^{\omega_2 z}$, satisfy the condition (i), (ii), or (iii).*
(i) $1 + a{\omega_2}^2 = 0, \lambda_1 = 2\omega_1, \lambda_2 = \omega_1 + \omega_2, {c_1}^2(1 + a{\omega_1}^2) = p_1, 2c_1c_2(1 + a\omega_1\omega_2) = p_2.$
(ii) $1 + a{\omega_1}^2 = 0, \lambda_1 = 2\omega_2, \lambda_2 = \omega_1 + \omega_2, {c_2}^2(1 + a{\omega_2}^2) = p_1, 2c_1c_2(1 + a\omega_1\omega_2) = p_2.$
(iii) $1 + a\omega_1\omega_2 = 0, \lambda_1 = 2\omega_1, \lambda_2 = 2\omega_2, {c_1}^2(1 + a{\omega_1}^2) = p_1, {c_2}^2(1 + a{\omega_2}^2) = p_2.$
(2) *if $f(z) \in T_2$, then $f(z) = c_1e^{\omega_1 z} + c_2e^{\omega_2 z} + c_3e^{\omega_3 z}$ satisfies the condition (iv).*
(iv) $1 + a{\omega_3}^2 = 1 + a\omega_1\omega_2 = 0, \lambda_1 = 2\omega_1, \lambda_2 = \omega_2 + \omega_3, 2\omega_2 = \omega_1 + \omega_3, {c_1}^2(1 + a{\omega_1}^2) = p_1, 2c_2c_3(1 + a\omega_2\omega_3) = p_2.$

Here, we give three examples to explain the existence of solutions for Eq (1.4).

Example 1.2. *The function $f(z) = -3e^{2z} + 2e^z$ is the solution of the equation*

$$(f(z))^2 - (f'(z))^2 = -27e^{4z} + 12e^{3z},$$

where $c_1 = -3$, $c_2 = 2$, $\omega_1 = 2$, $\omega_2 = 1$, $a = -1$, $p_1 = -27$, $p_2 = 12$, $\lambda_1 = 4$, and $\lambda_2 = 3$ satisfy the condition (i) of Theorem 1.1.

Example 1.3. *The function $f(z) = e^{-2z} + e^{z/2}$ is the solution of the equation*

$$(f(z))^2 + (f'(z))^2 = 5e^{-4z} + \frac{5}{4}e^z,$$

where $c_1 = 1$, $c_2 = 1$, $\omega_1 = -2$, $\omega_2 = 1/2$, $a = 1$, $p_1 = 5$, $p_2 = 5/4$, $\lambda_1 = -4$, and $\lambda_2 = 1$ satisfy the condition (ii) of Theorem 1.1.

Example 1.4. *The function $f(z) = e^{-4z}/2 + 2ie^{-z} + e^{2z}$ is the solution of the equation*

$$(f(z))^2 - \frac{1}{4}(f'(z))^2 = -\frac{3}{4}e^{-8z} + 6ie^z,$$

where $c_1 = 1/2$, $c_2 = 2i$, $c_3 = 1$, $\omega_1 = -4$, $\omega_2 = -1$, $\omega_3 = 2$, $a = -1/4$, $p_1 = -3/4$, $p_2 = 6i$, $\lambda_1 = -8$, and $\lambda_2 = 1$ satisfy the condition (iv) of Theorem 1.1.

Remark 1. *Equation (1.4) can be rewritten as*

$$(f(z))^2 + a(f'(z))^2 = (p_1 + p_2)e^{\lambda_1 z}, \tag{1.5}$$

where $\lambda_1 = \lambda_2$. Then, by [3], we could know that the entire solution of Eq (1.5) is $f(z) = \sqrt{p_1 + p_2}de^{\lambda_1 z/2}$, where $d \in \mathbb{C}$, $d^2(1 + a\lambda_1^2/4) = 1$.

By Theorem 1.1, it is natural to consider the solutions of the equation

$$(f(z))^n + a(f^{(k)}(z))^n = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}, \tag{1.6}$$

where k and $n(\geq 2)$ are positive integers, and a, p_i, and λ_i are nonzero constants for $i = 1, 2$, $\lambda_1 \neq \lambda_2$. In this paper, we consider this problem and obtain the following result.

Theorem 1.5. *Let* k *and* $n(\geq 2)$ *be positive integers and* a, p_i, *and* λ_i *be nonzero constants for* $i = 1, 2$, $\lambda_1 \neq \lambda_2$. *For the entire solution* $f(z)$ *of Eq (1.6) of the form (1.2), if* $|\omega_1| = |\omega_2| = \cdots = |\omega_m|$ *holds, then* $f(z)$ *reduces to*

$$f(z) = c_0 + c_1 e^{\omega_1 z} + c_2 e^{\omega_2 z} + \cdots + c_m e^{\omega_m z},$$

where c_i *are constants for* $i = 0, 1, 2, \cdots, m$.

Note that the trigonometric identity $(\cos z)^2 - (\sin z)^2 = \cos 2z$ is equivalent to $(\cos z)^4 - (\sin z)^4 = \cos 2z$. As Gao and Gao in [1] studied the special form $f^4 - (f')^4 = (e^{2iz} + e^{-2iz})/2$, it is natural to consider the entire solutions of the equation

$$(f(z))^4 + a(f'(z))^4 = p_1 e^{\lambda z} + p_2 e^{-\lambda z} \tag{1.7}$$

and obtain the following result.

Theorem 1.6. *Let* a, p_1, p_2, *and* λ *be nonzero constants. The entire solution of Eq (1.7) is* $f(z) = c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$, *where* c_1 *and* c_2 *are nonzero constants if and only if the conditions* $1 + a\lambda^4/16 = 0$, $p_1 = 4{c_1}^3 c_2(1 - a\lambda^4/16)$, *and* $p_2 = 4c_1 {c_2}^3(1 - a\lambda^4/16)$ *hold.*

In the following, we give an example which is different with Theorem 7 in [1] to explain the existence of solutions for Eq (1.7).

Example 1.7. *The function* $f(z) = e^z + 2e^{-z}$ *is the solution of the equation*

$$(f(z))^4 - (f'(z))^4 = 16e^{2z} + 64e^{-2z},$$

where $a = -1$, $\lambda = 2$, $c_1 = 1$, $c_2 = 2$, $p_1 = 16$, *and* $p_2 = 64$ *satisfy the condition of Theorem 1.6.*

Motivated by Theorem 1.6, we consider a more general binomial differential equation,

$$(f(z))^{2n} + a(f'(z))^{2n} = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1.8}$$

where a, p_1, p_2, and λ are nonzero constants.

Theorem 1.8. *Let* a, p_1, p_2, *and* λ *be nonzero constants, and let* $n \in \mathbb{N}^+$. *If* $n \geq 3$, *Eq (1.8) has no entire solution.*

Actually, in [1], Gao and Gao proposed the following conjecture.

Conjecture. *Let* n, $k \in \mathbb{N}^+$, a, p_i, *and* $\lambda_i (i = 1, 2)$ *be nonzero constants. If* $n \geq 5$, *then the equation* $(f(z))^n + a(f^{(k)}(z))^n = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}$ *does not have an entire solution.*

Remark 2. *Theorem 1.8 gives some partial answers to the conjecture.*

The paper is organized as follows. In Section 2, we introduce some indispensable lemmas. Section 3 contains the detailed proofs on Theorems 1.1,1.5, 1.6, and 1.8. In Section 4, we discuss the main results obtained in the paper.

2. Some lemmas

In order to prove our result, we need the following lemmas. The first lemma plays an important role in the research the differential equation related to the exponential polynomial. The second lemma is the lemma of logarithmic derivative, which is the basic result in Nevanlinna theory. Lemmas 2.3 and 2.5 are related to Wiman–Valiron theory, which plays a key role in proving Lemma 2.6 and Theorems 1.6 and 1.8.

Lemma 2.1. (See [13]) *Suppose that* $f_1, f_2, \cdots, f_n (n \geq 2)$ *are meromorphic functions and* $g_1, g_2, \cdots, g_n$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j e^{g_i} \equiv 0$*;*

(ii) $g_j - g_k$ *are not constants for* $1 \leq j < k \leq n$*;*

(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$*, where E is a set of* (r, ∞) *with finite linear measure. Then,* $f_j \equiv 0 (j = 1, 2, \cdots, n)$.

Lemma 2.2. [4,5] *Let f be a nonconstant meromorphic function,* $k \in N^+$*. Then,*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f).$$

If f is of finite order, then

$$m(r, \frac{f^{(k)}}{f}) = O(\log r).$$

Lemma 2.3. [5] *Let P be a polynomial with the degree p. Then, all nontrivial solutions f of*

$$f'' + Pf = 0$$

have the order of growth $\rho(f) = (p + 2)/2$.

Lemma 2.4. [5] *Let g be an entire function with order* ρ*. Then, we have*

$$\rho(g) = \limsup_{r \to \infty} \frac{\log^+ \nu(r, g)}{\log r},$$

where $\nu(r, g)$ *is the central index of g.*

Lemma 2.5. [5] (Wiman–Valiron theorem) *Let g be a transcendental entire function, and let* $0 < \delta < 1/4$ *and z be such that* $|z| = r$ *and that*

$$|g(z)| > M(r, g) \nu(r, g)^{-\frac{1}{4} + \delta}$$

holds. Then, there exists a set $F \subset \mathbb{R}_+$ *of finite logarithmic measure, this is* $\int_F dt/t < +\infty$*, such that*

$$g^{(m)}(z) = \left(\frac{\nu(r, g)}{z}\right)^m (1 + o(1)) g(z)$$

holds for all $m \geq 0$ *and all* $r \notin F$.

Lemma 2.6. *If k and* $n (\geq 2) \in \mathbb{N}^+$*, a,* p_i*, and* λ_i *are nonzero constants for* $i = 1, 2$*, and f is a transcendental entire function solution of Eq (1.6), then* $\rho(f) = 1$.

Proof. By differentiating Eq (1.6), we have

$$nf^{n-1}f' + na\left(f^{(k)}\right)^{n-1}f^{(k+1)} = p_1\lambda_1e^{\lambda_1z} + p_2\lambda_2e^{\lambda_2z}. \tag{2.1}$$

Differentiating Eq (2.1), we obtain

$$\begin{aligned}&n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' + n(n-1)a\left(f^{(k)}\right)^{n-2}\left(f^{(k+1)}\right)^2 + na\left(f^{(k)}\right)^{n-1}f^{(k+2)}\\ &= p_1{\lambda_1}^2e^{\lambda_1z} + p_2{\lambda_2}^2e^{\lambda_2z}.\end{aligned} \tag{2.2}$$

Multiplying λ_1 on both sides of Eq (2.1), we get

$$n\lambda_1f^{n-1}f' + na\lambda_1\left(f^{(k)}\right)^{n-1}f^{(k+1)} = p_1{\lambda_1}^2e^{\lambda_1z} + p_2\lambda_1\lambda_2e^{\lambda_2z}. \tag{2.3}$$

It follows from Eqs (1.6) and (2.1) that

$$\lambda_1f^n + a\lambda_1\left(f^{(k)}\right)^n - nf^{n-1}f' - na\left(f^{(k)}\right)^{n-1}f^{(k+1)} = p_2(\lambda_1-\lambda_2)e^{\lambda_2z}. \tag{2.4}$$

From Eqs (2.2) and (2.3), we have

$$\begin{aligned}&n\lambda_1f^{n-1}f' + na\lambda_1\left(f^{(k)}\right)^{n-1}f^{(k+1)} - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f''\\ &- n(n-1)a\left(f^{(k)}\right)^{n-2}\left(f^{(k+1)}\right)^2 - na\left(f^{(k)}\right)^{n-1}f^{(k+2)}\\ &= p_2\lambda_2(\lambda_1-\lambda_2)e^{\lambda_2z}.\end{aligned} \tag{2.5}$$

By multiplying λ_2 on both sides of Eq (2.4), we can obtain

$$\lambda_1\lambda_2f^n + a\lambda_1\lambda_2\left(f^{(k)}\right)^n - n\lambda_2f^{n-1}f' - na\lambda_2\left(f^{(k)}\right)^{n-1}f^{(k+1)} = p_2\lambda_2(\lambda_1-\lambda_2)e^{\lambda_2z}. \tag{2.6}$$

By Eqs (2.5) and (2.6), we can get

$$\begin{aligned}&n(\lambda_1+\lambda_2)\left[f^{n-1}f' + a\left(f^{(k)}\right)^{n-1}f^{(k+1)}\right] - \lambda_1\lambda_2\left[f^n + a\left(f^{(k)}\right)^n\right] - n(n-1)f^{n-2}(f')^2\\ &- nf^{n-1}f'' - n(n-1)a\left(f^{(k)}\right)^{n-2}\left(f^{(k+1)}\right)^2 - na\left(f^{(k)}\right)^{n-1}f^{(k+2)}\\ &= 0.\end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned}&n(\lambda_1+\lambda_2)\left[\frac{f'}{f} + a\left(\frac{f^{(k)}}{f}\right)^{n-1}\frac{f^{(k+1)}}{f}\right] - \lambda_1\lambda_2\left[1 + a\left(\frac{f^{(k)}}{f}\right)^n\right] - n(n-1)\left(\frac{f'}{f}\right)^2\\ &- n\frac{f''}{f} - n(n-1)a\left(\frac{f^{(k)}}{f}\right)^{n-2}\left(\frac{f^{(k+1)}}{f}\right)^2 - na\left(\frac{f^{(k)}}{f}\right)^{n-1}\frac{f^{(k+2)}}{f} = 0.\end{aligned} \tag{2.8}$$

Applying Lemma 2.5 to Eq (2.8), we have

$$\begin{aligned}&- n^2a(v(r))^{nk+2}(1+o(1)) + n(\lambda_1+\lambda_2)az(v(r))^{nk+1}(1+o(1))\\ &- a\lambda_1\lambda_2z^2(v(r))^{nk}(1+o(1)) - n^2z^{nk}(v(r))^2(1+o(1))\\ &+ n(\lambda_1+\lambda_2)z^{nk+1}v(r)(1+o(1)) - \lambda_1\lambda_2z^{n+2}\end{aligned} \tag{2.9}$$

$$= 0,$$

where $v(r) \to \infty$ if $r \to \infty$. Because $v(r)$ is a functional solution of its coefficient in Eq (2.9), and $v(r)$ is a function of its coefficient, the solution of Eq (2.9) is asymptotically equal to the solution of the following equation:

$$\begin{aligned}
&- n^2 a(v(r))^{nk+2} + n(\lambda_1 + \lambda_2)az(v(r))^{nk+1} - a\lambda_1\lambda_2 z^2(v(r))^{nk} \\
&- n^2 z^{nk}(v(r))^2 + n(\lambda_1 + \lambda_2)z^{nk+1}v(r) - \lambda_1\lambda_2 z^{n+2} \\
&= 0.
\end{aligned} \tag{2.10}$$

Let $\rho(f) = \rho < \infty$. When $r \to \infty (r \notin F)$, $v(r) \sim \alpha z^\rho$, where α is a nonzero constant, and ρ is a nonzero finite number, Eq (2.10) becomes

$$\begin{aligned}
&- n^2 a\alpha^{nk+2} r^{(nk+2)\rho} + n(\lambda_1 + \lambda_2)a\alpha^{nk+1} r^{(nk+1)\rho+1} - a\lambda_1\lambda_2\alpha^{nk} r^{nk\rho+2} \\
&- n^2\alpha^2 r^{2\rho+nk} + n(\lambda_1 + \lambda_2)\alpha r^{\rho+nk+1} - \lambda_1\lambda_2 r^{nk+2} \\
&= 0.
\end{aligned} \tag{2.11}$$

Therefore, we can obtain $(nk + 2)\rho = (nk + 1)\rho + 1$ or $(nk + 2)\rho = nk\rho + 2$ or $(nk + 2)\rho = 2\rho + nk$ or $(nk + 2)\rho = \rho + nk + 1$ or $(nk + 2)\rho = nk + 2$, which implies $\rho = 1$.

If $\rho(f) = \infty$, then for any large $N(> 0)$, we have for sufficiently large r_m, $v(r_m) > r_m^N$ (see Lemma 2.5 in [7]). By (2.9), we obtain that

$$n^2|a|v(r_m)(1 + o(1)) \le Mr_m(1 + o(1)),$$

where $M(> 0)$ is some constant. This gives that

$$n^2|a|r_m^N \le n^2|a|v(r_m)(1 + o(1)) \le Mr_m(1 + o(1)),$$

which contradicts with any large number N. This completes the proof.

3. The proofs of theorems

In the following, we will give the detail of the our main results. Some ideas of the paper come from [1] and [7].

3.1. The proof of Theorem 1.1

Case 1. If $f(z) \in T_1$, by Lemma 2.6, we have $f(z) = c_1 e^{\omega_1 z} + c_2 e^{\omega_2 z}$, where ω_1, ω_2 are different nonzero constants, and c_1, c_2 are nonzero constants, then

$$\begin{aligned}
(f(z))^2 + a(f'(z))^2 = {} & c_1{}^2(1 + a\omega_1{}^2)e^{2\omega_1 z} + c_2{}^2(1 + a\omega_2{}^2)e^{2\omega_2 z} \\
& + 2c_1c_2(1 + a\omega_1\omega_2)e^{(\omega_1+\omega_2)z}.
\end{aligned} \tag{3.1}$$

By Eqs (1.4) and (3.1), without losing generality, we consider the following three subcases.

Case 1.1. If $p_1 e^{\lambda_1 z} = c_1{}^2(1 + a\omega_1{}^2)e^{2\omega_1 z}$ and $p_2 e^{\lambda_2 z} = c_2{}^2(1 + a\omega_2{}^2)e^{2\omega_2 z}$, by Lemma 2.1, then we have $1 + a\omega_1\omega_2 = 0$.

Case 1.2. If $p_1e^{\lambda_1 z} = c_1{}^2(1+a\omega_1{}^2)e^{2\omega_1 z}$ and $p_2e^{\lambda_2 z} = 2c_1c_2(1+a\omega_1\omega_2)e^{(\omega_1+\omega_2)z}$, by Lemma 2.1, then we obtain $1 + a\omega_2{}^2 = 0$.

Case 1.3. If $p_1e^{\lambda_1 z} = c_2{}^2(1+a\omega_2{}^2)e^{2\omega_2 z}$ and $p_2e^{\lambda_2 z} = 2c_1c_2(1+a\omega_1\omega_2)e^{(\omega_1+\omega_2)z}$, by Lemma 2.1, then we obtain $1 + a\omega_1{}^2 = 0$.

Case 2. If $f(z) \in T_2$, by Lemma 2.6, we have $f(z) = c_1e^{\omega_1 z} + c_2e^{\omega_2 z} + c_3e^{\omega_3 z}$, where $\omega_1, \omega_2, \omega_3$ are different nonzero constants, and c_1, c_2, c_3 are nonzero constants, then

$$\begin{aligned}(f(z))^2 + a(f'(z))^2 &= c_1{}^2(1+a\omega_1{}^2)e^{2\omega_1 z} + c_2{}^2(1+a\omega_2{}^2)e^{2\omega_2 z} + c_3{}^2(1+a\omega_3{}^2)e^{2\omega_3 z}\\ &\quad + 2c_1c_2(1+a\omega_1\omega_2)e^{(\omega_1+\omega_2)z} + 2c_1c_3(1+a\omega_1\omega_3)e^{(\omega_1+\omega_3)z}\\ &\quad + 2c_2c_3(1+a\omega_2\omega_3)e^{(\omega_2+\omega_3)z}.\end{aligned} \tag{3.2}$$

It is easy to know that among

$$2\omega_1, 2\omega_2, 2\omega_3, \omega_1+\omega_2, \omega_1+\omega_3, \omega_2+\omega_3, \tag{3.3}$$

they are either different, or there exist only one of the following three items:

$$2\omega_1 = \omega_2+\omega_3, 2\omega_2 \neq \omega_1+\omega_3, 2\omega_3 \neq \omega_1+\omega_2, \tag{3.4}$$

$$2\omega_2 = \omega_1+\omega_3, 2\omega_1 \neq \omega_2+\omega_3, 2\omega_3 \neq \omega_1+\omega_2, \tag{3.5}$$

$$2\omega_3 = \omega_1+\omega_2, 2\omega_1 \neq \omega_2+\omega_3, 2\omega_2 \neq \omega_1+\omega_3. \tag{3.6}$$

As above, by Eqs (1.4) and (3.2), without losing generality, we consider the following cases.

Case 2.1. $\lambda_1 = 2\omega_1$ and $\lambda_2 = 2\omega_2$. If the six constants of (3.3) are different, by Eqs (1.4) and (3.2) and Lemma 2.1, we have

$$1 + a\omega_3{}^2 = 1 + a\omega_1\omega_2 = 1 + a\omega_1\omega_3 = 1 + a\omega_2\omega_3 = 0,$$

which leads to $\omega_1 = \omega_2 = \omega_3$, a contradiction. If (3.4) holds, by the same way, we have

$$1 + a\omega_3{}^2 = 1 + a\omega_1\omega_2 = 1 + a\omega_2\omega_3 = 0,$$

so there is $\omega_1 = \omega_2 = \omega_3$, a contradiction. If (3.5) holds, by a similar discussion, we obtain

$$1 + a\omega_3{}^2 = 1 + a\omega_1\omega_2 = 1 + a\omega_1\omega_3 = 0,$$

and it yields $\omega_1 = \omega_2 = \omega_3$, a contradiction. If (3.6) holds, as above, we can get

$$1 + a\omega_1\omega_3 = 1 + a\omega_2\omega_3 = 0;$$

therefore, we obtain $\omega_1 = \omega_2$, a contradiction.

Case 2.2. $\lambda_1 = 2\omega_1$ and $\lambda_2 = \omega_1 + \omega_2$. If the six constants of (3.3) are different, by Eqs (1.4) and (3.2) and Lemma 2.1, we have

$$1 + a\omega_2{}^2 = 1 + a\omega_3{}^2 = 1 + a\omega_1\omega_3 = 1 + a\omega_2\omega_3 = 0,$$

which leads to $\omega_1 = \omega_2 = \omega_3$, a contradiction. If (3.4) holds, by the same way, we have

$$1 + a\omega_2{}^2 = 1 + a\omega_3{}^2 = 1 + a\omega_1\omega_3 = 0,$$

so there is $\omega_1 = \omega_3$, a contradiction. If (3.5) holds, by a similar discussion, we obtain

$$1 + a\omega_2{}^2 = 1 + a\omega_2\omega_3 = 0,$$

and it yields $\omega_2 = \omega_3$, a contradiction. If (3.6) holds, as above, we can get

$$1 + a\omega_2{}^2 = 1 + a\omega_1\omega_3 = 1 + a\omega_2\omega_3 = 0;$$

therefore, we obtain $\omega_1 = \omega_2 = \omega_3$, a contradiction.

Case 2.3. $\lambda_1 = 2\omega_1$ and $\lambda_2 = \omega_2 + \omega_3$. If the six constants of (3.3) are different, by Eqs (1.4) and (3.2) and Lemma 2.1, we have

$$1 + a\omega_2{}^2 = 1 + a\omega_3{}^2 = 1 + a\omega_1\omega_2 = 1 + a\omega_1\omega_3 = 0,$$

which leads to $\omega_1 = \omega_2 = \omega_3$, a contradiction. If (3.4) holds, we have

$$\lambda_1 = \omega_2 + \omega_3 = \lambda_2,$$

a contradiction. If (3.5) holds, by a similar discussion, we have

$$1 + a\omega_3{}^2 = 1 + a\omega_1\omega_2 = 0.$$

If (3.6) holds, as above, we can get

$$1 + a\omega_2{}^2 = 1 + a\omega_1\omega_3 = 0,$$

which is equivalent to the condition we obtain above.

Case 2.4. $\lambda_1 = \omega_1 + \omega_2$ and $\lambda_2 = \omega_1 + \omega_3$. If the six constants of (3.3) are different, by Eqs (1.4) and (3.2) and Lemma 2.1, we have

$$1 + a\omega_1{}^2 = 1 + a\omega_2{}^2 = 1 + a\omega_3{}^2 = 1 + a\omega_2\omega_3 = 0,$$

which leads to $\omega_2 = \omega_3$, a contradiction. If (3.4) holds, we have

$$1 + a\omega_2{}^2 = 1 + a\omega_3{}^2 = 0.$$

For $\omega_2 \neq \omega_3$, we can get $\omega_2 = -\omega_3$ and then $2\omega_1 = \omega_2 + \omega_3 = 0$, a contradiction. If (3.5) holds, by a similar discussion, we obtain

$$1 + a\omega_1{}^2 = 1 + a\omega_3{}^2 = 1 + a\omega_2\omega_3 = 0,$$

which yields $\omega_2 = \omega_3$, a contradiction. If (3.6) holds, as above, we can get

$$1 + a\omega_1{}^2 = 1 + a\omega_2{}^2 = 1 + a\omega_2\omega_3 = 0;$$

therefore, we obtain $\omega_2 = \omega_3$, a contradiction.

This completes the proof of Theorem 1.1.

3.2. The proof of Theorem 1.5

Suppose $f(z)$ is the entire solution of Eq (1.6) of the form (1.2). By Lemma 2.6, we get $f(z) = H_0(z) + H_1(z)e^{\omega_1 z} + H_2(z)e^{\omega_2 z} + \cdots + H_m(z)e^{\omega_m z}$, so then we have

$$\begin{aligned}(f(z))^n + a(f^{(k)}(z))^n = &\sum_{t_0+t_1+\cdots+t_m=n} \frac{n!}{t_0!t_1!\cdots t_m!} \prod_{j=0}^{m} (H_j(z)e^{\omega_j z})^{t_j} \\ &+ a \sum_{t_0+t_1+\cdots+t_m=n} \frac{n!}{t_0!t_1!\cdots t_m!} \prod_{j=1}^{m} (G_j(z)e^{\omega_j z})^{t_j},\end{aligned} \tag{3.7}$$

where $\omega_0 = 0$, $t_0, t_1, \cdots, t_m$ are natural numbers,

$$G_j(z) = \sum_{i=0}^{k} \binom{k}{i} H_j^{(k-i)}(z)\omega_j^i.$$

It is easy to know $G_0(z) = H_0^{(k)}(z)$ and $\deg G_j(z) = \deg H_j(z)$ for $j = 1, 2, \cdots, m$. Because $|\omega_1| = |\omega_2| = \cdots = |\omega_m|$, we can write $|\omega_1| = |\omega_2| = \cdots = |\omega_m| = r$. In the following, we first claim that for $\forall i \in \{1, 2, \cdots, m\}$, we have $n\omega_i \neq s_1\omega_1 + s_2\omega_2 + \cdots + s_m\omega_m$, where $s_1, s_2, \cdots, s_m (m \geq 2)$ are natural numbers, and $s_1 + s_2 + \cdots + s_m = n$.

If not, suppose $\exists j \in \{1, 2, \cdots, m\}$, $s_u, s_v \in \{1, 2, \cdots, n\}$ and natural numbers s_1, s_2, $\cdots$, s_{u-1},s_{u+1}, $\cdots$, s_{v-1}, s_{v+1}, $\cdots$, s_m, so that

$$n\omega_j = s_1\omega_1 + s_2\omega_2 + \cdots + s_m\omega_m$$

satisfy $s_1 + s_2 + \cdots + s_m = n$. Because ω_u, ω_v are two different constants, and $|\omega_u| = |\omega_v|$, we have $|s_u\omega_u + s_v\omega_v| < |s_u\omega_u| + |s_v\omega_v| = (s_u + s_v)r$, so that we get

$$nr = |n\omega_j| = |s_1\omega_1 + s_2\omega_2 + \cdots + s_m\omega_m| < |s_1\omega_1| + |s_2\omega_2| + \cdots + |s_m\omega_m| = nr,$$

a contradiction.

Next, we know that for $\forall u, v \in \{1, 2, \cdots, m\} (u \neq v)$, $n\omega_u \neq n\omega_v$.

Therefore, we can know that there are m terms which are different from each other on the right of Eq (3.7); they are

$$e^{n\omega_j z}\left[(H_j(z))^n + a\left(\sum_{p=0}^{k} \binom{k}{p} {\omega_j}^{k-p} H_j^{(p)}(z)\right)^n\right], \quad (j = 1, 2, \cdots, m). \tag{3.8}$$

By Eqs (1.6) and (3.7), we consider the following three cases.

Case 1. If $p_1e^{\omega_1 z}$ and $p_2e^{\omega_2 z}$ come from (3.8) without losing generality, suppose that

$$p_1e^{\omega_1 z} = e^{n\omega_1 z}\left[(H_1(z))^n + a\left(\sum_{p=0}^{k} \binom{k}{p} {\omega_1}^{k-p} H_1^{(p)}(z)\right)^n\right],$$

$$p_2e^{\omega_2 z} = e^{n\omega_2 z}\left[(H_2(z))^n + a\left(\sum_{p=0}^{k} \binom{k}{p} {\omega_2}^{k-p} H_2^{(p)}(z)\right)^n\right].$$

By Eqs (1.6) and (3.7) and Lemma 2.1, we have

$$(H_j(z))^n + a\left(\sum_{p=0}^{k}\binom{k}{p}\omega_j^{\,k-p}H_j^{(p)}(z)\right)^n \equiv 0.(j = 3, 4, \cdots, m).$$

Therefore, we have

$$\deg\left[(H_j(z))^n + a\left(\sum_{p=0}^{k}\binom{k}{p}\omega_j^{\,k-p}H_j^{(p)}(z)\right)^n\right] = 0.(j = 1, 2, \cdots, m).$$

If $\deg H_j(z) \geq 1 (j = 1, 2, \cdots, m)$, we can get

$$\deg\left[(H_j(z))^n + a\left(\sum_{p=0}^{k}\binom{k}{p}\omega_j^{\,k-p}H_j^{(p)}(z)\right)^n\right] \geq n-1, (j = 1, 2, \cdots, m),$$

a contradiction. Thus, we have $\deg H_j(z) = 0 (j = 1, 2, \cdots, m)$.

Case 2. If only one of $p_1e^{\omega_1 z}$ and $p_2e^{\omega_2 z}$ comes from (3.8), by a similar discussion as Case 1, we have $\deg H_j(z) = 0 (j = 1, 2, \cdots, m)$.

Case 3. If $p_1e^{\omega_1 z}$ and $p_2e^{\omega_2 z}$ do not come from (3.8), by the same way as Case 1, we have $\deg H_j(z) = 0 (j = 1, 2, \cdots, m)$.

Then, we have

$$f(z) = H_0(z) + c_1e^{\omega_1 z} + c_2e^{\omega_2 z} + \cdots + c_me^{\omega_m z},$$

where $c_1, c_2, \cdots, c_m$ are constants. Substitute this expression into Eq (3.7) to obtain the polynomial

$$(H_0(z))^n + G(H_0(z), H_0^{(k)}(z))$$

about $H_0(z)$ and $H_0^{(k)}(z))$ in the right of Eq (3.7), which satisfy $\deg\,[(H_0(z))^n] > \deg G(H_0(z), H_0^{(k)}(z))$. By Eqs (1.6) and (3.7) and Lemma 2.1, we have

$$(H_0(z))^n + G(H_0(z), H_0^{(k)}(z)) \equiv 0.$$

If $\deg H_0(z) \geq 1$, then

$$\deg\left[(H_0(z))^n + G(H_0(z), H_0^{(k)}(z))\right] \geq n,$$

a contradiction. Therefore, we have $\deg H_0(z) = 0$. Let $H_0(z) = c_0$, leading to

$$f(z) = c_0 + c_1e^{\omega_1 z} + c_2e^{\omega_2 z} + \cdots + c_me^{\omega_m z},$$

where $c_0, c_1, \cdots, c_m$ are constants.

This completes the proof of Theorem 1.5.

3.3. The proof of Theorem 1.6

Let f be a transcendental entire solution. By differentiating Eq (1.7), we have

$$4f^3f' + 4a(f')^3f'' = p_1\lambda e^{\lambda z} - p_2\lambda e^{-\lambda z}. \tag{3.9}$$

It follows from Eqs (1.7) and (3.9) that

$$\lambda f^4 + a\lambda(f')^4 + 4f^3f' + 4a(f')^3f'' = 2p_1\lambda e^{\lambda z}, \tag{3.10}$$

and

$$\lambda f^4 + a\lambda(f')^4 - 4f^3f' - 4a(f')^3f'' = 2p_2\lambda e^{-\lambda z}. \tag{3.11}$$

Differentiating Eq (3.9) and eliminating $e^{\lambda z}$ and $e^{-\lambda z}$, we obtain

$$\lambda^2f^4 + a\lambda^2(f')^4 - 12f^2(f')^2 - 4f^3f'' - 12a(f')^2(f'')^2 - 4a(f')^3f''' = 0. \tag{3.12}$$

From Eq (3.12), we obtain

$$\begin{aligned} &4\lambda^2f^3f' + 4a\lambda^2(f')^3f'' - 24f(f')^3 - 36f^2f'f'' - 4f^3f''' \\ &- 24af'(f'')^3 - 36a(f')^2f''f''' - 4a(f')^3f^{(4)} = 0. \end{aligned} \tag{3.13}$$

We simplify Eq (3.13) and get

$$\begin{aligned} &f'[4\lambda^2f^3 + 4a\lambda^2(f')^2f'' - 24f(f')^2 - 36f^2f'' \\ &- 24a(f'')^3 - 36af'f''f''' - 4a(f')^2f^{(4)}] = 4f^3f'''. \end{aligned} \tag{3.14}$$

By Eq (3.14) and Lemmas 2.4 and 2.5, using the same method as in the proof of Lemma 2.6, we obtain

$$\rho(f) = \rho(f') = \limsup_{r\to\infty} \frac{\log^+ \nu(r)}{\log r} = 1.$$

It is easy to see that the zeros of $f'(z)$ are zeros of $f(z)$ or $f'''(z)$ by Eq (3.14). From (3.10), it is easy to see that $f(z)$ and $f'(z)$ do not have the same zero. Therefore, $f'''(z)/f'(z)$ is an entire function. Now, we set $b(z) = f'''(z)/f'(z)$, and then by Lemma 2.2, we have

$$T(r, b) = m(r, b) = S(r, f') = O(\log r);$$

hence, $b(z)$ is a polynomial. Note that f is a solution of

$$f'''(z) - b(z)f'(z) = 0. \tag{3.15}$$

By Lemma 2.3, we obtain that $\rho(f) = \rho(f') = (p + 2)/2 = 1$, where p is the degree of $b(z)$. Consequently, $p = 0$. This means $b(z)$ is a constant. It follows from Eq (3.15) that there exists a constant d such that

$$f''(z) = bf(z) + d. \tag{3.16}$$

We substitute $f''' = bf'$ and $f^{(4)} = bf''$ into (3.12) and (3.13), respectively, and get

$$\lambda^2f^4 + (a\lambda^2 - 4ab)(f')^4 - 12f^2(f')^2 - 4f^3f'' - 12a(f')^2(f'')^2 = 0 \tag{3.17}$$

and

$$(4\lambda^2 - 4b)f^3 f' + (4a\lambda^2 - 40ab)(f')^3 f'' - 24f(f')^3 - 36f^2 f' f'' - 24af'(f'')^3 = 0. \tag{3.18}$$

Now, we claim that $d = 0$. If not, let z_1 be a zero of $f(z)$. Then, $f'' = bf + d$ and Eq (1.7) show that $f''(z_1) \neq 0$, and $f'(z_1) \neq 0$. By Eqs (3.17), and (3.18), we have

$$(\lambda^2 - 4b)(f')^2(z_1) - 12(f'')^2(z_1) = 0 \tag{3.19}$$

and

$$(\lambda^2 - 10b)(f')^2(z_1) - 6(f'')^2(z_1) = 0. \tag{3.20}$$

Combining Eq (3.19) and (3.20), we have $b = \lambda^2/16$. Substituting it into $f'' = bf + d$, we can get

$$f(z) = c_1 e^{\frac{1}{4}\lambda z} + c_2 e^{-\frac{1}{4}\lambda z} - \frac{16d}{\lambda^2}. \tag{3.21}$$

By substituting this expression into Eq (1.7), we obtain a contradiction.

Hence, $f'' = bf$. Rewrite Eq (3.17) as

$$f(\lambda^2 f^3 - 12f(f')^2 - 4f^2 f'' - 12ab(f')^2) + (a\lambda^2 - 4ab)(f')^4 = 0. \tag{3.22}$$

It can be verified that $N(r, 1/f) \neq S(r, f)$ by Eq (1.7). If not, suppose $f(z) = p(z)e^{\alpha z + \beta}$, where $\lambda(p) = \lambda(f) < \rho(f)$, $\alpha(\neq 0)$, and β are constants, so that we get $f^4 + (f')^4 = [p + a(p' + \alpha p)^4]e^{4(\alpha z + \beta)}$, a contradiction.

If $f = 0$, analyzing the multiplicity of zeros of $(f(z))^4 + a(f'(z))^4$ and $p_1 e^{\lambda z} + p_2 e^{-\lambda z}$ in Eq (1.7), then we have $f' \neq 0$; hence, from (3.22), we have

$$a\lambda^2 - 4ab = 0.$$

As a result, $b = \lambda^2/4$. Hence, $f''(z) - (\lambda^2/4)f(z) = 0$, and its general solution is $f(z) = c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$, where $c_1 \neq 0$, $c_2 \neq 0$. Substituting $f(z) = c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}$ into Eq (1.7), we have

$$\begin{aligned} &{c_1}^4(1 + a\frac{\lambda^4}{16})e^{2\lambda z} + [4{c_1}^3 c_2(1 - a\frac{\lambda^4}{16}) - p_1]e^{\lambda z} + [4c_1{c_2}^3(1 - a\frac{\lambda^4}{16}) - p_2]e^{-\lambda z} \\ &+ {c_2}^4(1 + a\frac{\lambda^4}{16})e^{-2\lambda z} + 6{c_1}^2{c_2}^2(1 + a\frac{\lambda^4}{16}) = 0, \end{aligned} \tag{3.23}$$

and by Lemma 2.1, we get $(1 + a\lambda^4/16) = 0$, $4c_1^3 c_2(1 - a\lambda^4/16) = p_1$, and $4c_1 c_2^3(1 - a\lambda^4/16) = p_2$.

This completes the proof of Theorem 1.6.

3.4. The proof of Theorem 1.8

Let f be a transcendental entire solution. By differentiating Eq (1.8), we have

$$2nf^{2n-1}f' + 2an(f')^{2n-1}f'' = p_1\lambda e^{\lambda z} - p_2\lambda e^{-\lambda z}. \tag{3.24}$$

It follows from Eqs (1.8) and (3.24) that

$$\lambda f^{2n} + a\lambda(f')^{2n} + 2nf^{2n-1}f' + 2an(f')^{2n-1}f'' = 2p_1\lambda e^{\lambda z}, \tag{3.25}$$

and

$$\lambda f^{2n} + a\lambda (f')^{2n} - 2nf^{2n-1}f' - 2an(f')^{2n-1}f'' = 2p_2\lambda e^{\lambda z}. \tag{3.26}$$

Differentiating Eq (3.25) and eliminating $e^{\lambda z}$, we obtain

$$\begin{aligned} &\lambda^2 f^{2n} + a\lambda^2 (f')^{2n} - 2n(2n-1)f^{2n-2}(f')^2 - 2nf^{2n-1}f'' \\ &- 2an(2n-1)(f')^{2n-2}(f'')^2 - 2an(f')^{2n-1}f''' = 0. \end{aligned} \tag{3.27}$$

From (3.27), we have

$$\begin{aligned} &2n\lambda^2 f^{2n-1}f' + 2an\lambda^2(f')^{2n-1}f'' - 2n(2n-1)(2n-2)f^{2n-3}(f')^3 \\ &- 6n(2n-1)f^{2n-2}f'f'' - 2nf^{2n-1}f''' - 2an(2n-1)(2n-2)(f')^{2n-3}(f'')^3 \\ &- 6an(2n-1)(f')^{2n-2}f''f''' - 2an(f')^{2n-1}f^{(4)} = 0. \end{aligned} \tag{3.28}$$

We simplify Eq (3.28) as

$$\begin{aligned} &f'[2n\lambda^2 f^{2n-1} + 2an\lambda^2(f')^{2n-2}f'' - 2n(2n-1)(2n-2)f^{2n-3}(f')^2 \\ &- 6n(2n-1)f^{2n-2}f'' - 2an(2n-1)(2n-2)(f')^{2n-4}(f'')^3 \\ &- 6an(2n-1)(f')^{2n-3}f''f''' - 2an(f')^{2n-2}f^{(4)}] = 2nf^{2n-1}f'''. \end{aligned} \tag{3.29}$$

By Eq (3.28) and Lemmas 2.1 and 2.5, using the same method as in the proof of Lemma 2.6, we obtain

$$\rho(f) = \rho(f') = \limsup_{r\to\infty} \frac{\log^+ \nu(r)}{\log r} = 1.$$

It is easy to see that the zeros of $f'(z)$ are zeros of $f(z)$ or $f'''(z)$ by Eq (3.29). From (3.25), we know that $f(z)$ and $f'(z)$ do not have the same zero. Therefore, $f'''(z)/f'(z)$ is an entire function. Now, we set $b(z) = f'''(z)/f'(z)$, and then by Lemma 2.2, we have

$$T(r, b) = m(r, b) = S(r, f') = O(\log r).$$

Hence, $b(z)$ is a polynomial. Note that f is a solution of

$$f'''(z) - b(z)f'(z) = 0. \tag{3.30}$$

By Lemma 2.3, we get that $\rho(f) = \rho(f') = (p + 2)/2 = 1$, where p is the degree of $b(z)$. Consequently, $p = 0$, which means $b(z)$ is a nonzero constant. It follows from Eq (3.30) that there exists a constant d such that

$$f''(z) = bf(z) + d. \tag{3.31}$$

Substituting $f''' = bf'$ and $f^{(4)} = bf''$ into Eqs (3.27) and (3.28), we get

$$\begin{aligned} &\lambda^2 f^{2n} + (a\lambda^2 - 2abn)(f')^{2n} - 2n(2n-1)f^{2n-2}(f')^2 \\ &- 2nf^{2n-1}f'' - 2an(2n-1)(f')^{2n-2}(f'')^2 = 0 \end{aligned} \tag{3.32}$$

and

$$(2n\lambda^2 - 2bn)f^{2n-1}f' + (2an\lambda^2 - 6abn(2n-1) - 2abn)(f')^{2n-1}f''$$

$$\begin{aligned}&- 2n(2n-1)(2n-2)f^{2n-3}(f')^3 - 6n(2n-1)f^{2n-2}f'f'' \\ &- 2an(2n-1)(2n-2)(f')^{2n-3}(f'')^3 = 0.\end{aligned} \tag{3.33}$$

Now, we claim that $d = 0$. If not, let z_1 be a zero of $f(z)$. Then, $f'' = b(z)f + d$ and Eq (1.8) show that $f''(z_1) \neq 0$, and $f'(z_1) \neq 0$. By Eqs (3.32) and (3.33), we have

$$(\lambda^2 - 2bn)(f')^2 - 2n(2n-1)(f'')^2 = 0 \tag{3.34}$$

and

$$(\lambda^2 - (6n-2)b)(f')^2 - (2n-1)(2n-2)(f'')^2 = 0. \tag{3.35}$$

Combining Eqs (3.34) and (3.35), we have $b = \lambda^2/4n^2$. Substituting it into $f'' = bf + d$, we can get

$$f(z) = c_1 e^{\frac{1}{2n}\lambda z} + c_2 e^{-\frac{1}{2n}\lambda z} - \frac{4n^2 d}{\lambda^2}. \tag{3.36}$$

By substituting this expression into Eq (1.8), we get a contradiction.

Hence, $f'' = bf$. Substituting it into Eq (3.32), we have

$$\begin{aligned}&f(\lambda^2 f^{2n-1} - 2n(2n-1)f^{2n-3}(f')^2 - 2nf^{2n-2}f'' \\ &- 2ab^2n(2n-1)f(f')^{2n-2}) + (a\lambda^2 - 2abn)(f')^{2n} = 0.\end{aligned} \tag{3.37}$$

Similarly, it can be verified that $N(r, 1/f) \neq S(r, f)$ by Eq (1.8). If $f = 0$, analyzing the multiplicity of zeros of $(f(z))^{2n} + a(f'(z))^{2n}$ and $p_1 e^{\lambda z} + p_2 e^{-\lambda z}$ in Eq (1.8), we have $f' \neq 0$; hence,

$$a\lambda^2 - 2abn = 0,$$

that is, $b = \lambda^2/2n$. Using the same method as in the proof of Theorem 1.6, we get

$$f(z) = c_1 e^{\frac{1}{\sqrt{2n}}\lambda z} + c_2 e^{-\frac{1}{\sqrt{2n}}\lambda z},$$

where $c_1 \neq 0$, $c_2 \neq 0$. Then

$$\begin{aligned}f^{2n} + a(f')^{2n} &= \left(1 + a\left(\frac{\lambda}{\sqrt{2n}}\right)^{2n}\right) \sum_{k=0}^{2n} \binom{2n}{k} {c_1}^k {c_2}^{2n-k} e^{2(k-n)\frac{\lambda}{\sqrt{2n}}z} \\ &+ \left(1 - a\left(\frac{\lambda}{\sqrt{2n}}\right)^{2n}\right) \sum_{j=1}^{2n-1} \binom{2n}{j} {c_1}^j {c_2}^{2n-j} e^{2(j-n)\frac{\lambda}{\sqrt{2n}}z},\end{aligned} \tag{3.38}$$

where k is an even number, and j is an odd number. Because $n \geq 3$, and $k \neq j$, we have that $2(k-n)\lambda z/\sqrt{2n}$, $2(j-n)\lambda z/\sqrt{2n}$, λz, and $-\lambda z$ are different from each other. By Eqs (1.8) and (3.38) and Lemma 2.1, we know

$$1 + a\left(\frac{\lambda}{\sqrt{2n}}\right)^{2n} = 0,$$

and

$$1 - a\left(\frac{\lambda}{\sqrt{2n}}\right)^{2n} = 0.$$

Therefore, we get

$$f^{2n} + a(f')^{2n} \equiv 0,$$

which is a contradiction.

This completes the proof of Theorem 1.8.

4. Conclusions

Gundersen et al. in [2] studied the equation $f^2(z) - f'(z)^2 = \cos 2z$ and proved that the equation has only four entire solutions: $f(z) = \pm\cos z, \pm i\sin z$, which are linked to the classic trigonometric identities $\cos^2 z - \sin^2 z = \cos 2z$. In [1], Gao and Gao studied the entire solution of the more general form $f^2(z) + af'^2(z) = p_1e^{\lambda z} + p_2e^{-\lambda z}$, where a, p_1, p_2, and λ are nonzero constants. Naturally, one can consider the more general form $f^n(z)+a(f^{(k)})^n(z) = p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}$, where a, p_1, p_2, λ_1, and $\lambda_2(\neq \lambda_1)$ are nonzero constants. It is very difficult to give the entire solution of the general equation. In this paper, we first consider the exponential polynomial solution of the general equation under set condition. Next we consider two special cases of the general equation (i) $f^4(z)+af'^4(z) = p_1e^{\lambda z}+p_2e^{-\lambda z}$, and we obtain the entire solution is $f(z) = c_1e^{\lambda z/2} + c_2e^{-\lambda z/2}$. (ii) Given $f^{2n}(z) + af'^{2n}(z) = p_1e^{\lambda z} + p_2e^{-\lambda z}$, we prove that the equation has no entire solution when $n \geq 3$. The conclusion gives some partial answers to the conjecture of Gao and Gao in [1].

Author contributions

Qiyuan Feng: Conceptualization, Methodology, Writing–original draft; Junfeng Xu: Supervision, Writing–review and editing. Both of the authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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