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Research article

Ideal topological approaches to hyperconnected and irresolvable spaces

Ibtesam Alshammari*

Department of Mathematics, College of Science, University of Hafr Al Batin, Hafr Al Batin, Saudi Arabia

* **Correspondence:** Email: iealshamri@uhb.edu.sa.

Abstract: In this paper, we presented a systematic study of several new notions in the setting of ideal topological spaces, such as $\widetilde{\omega}$-dense sets, $\widetilde{\omega}$-hyperconnectedness, strongly $\widetilde{\omega}$-hyperconnectedness, and resolvability-type concepts. By employing ω-open sets together with $\widetilde{\omega}$-local functions, we established fundamental properties, explored the relationships among these notions, and demonstrated how they extend and refine classical concepts in general topology.

Keywords: ideal topological space; ω-open set; $\widetilde{\omega}$-dense; $\widetilde{\omega}$-hyperconnected; $\widetilde{\omega}$-resolvable; $\widetilde{\omega}$-irresolvable; ω-irresolvable
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1. Introduction

A topological space Y is called hyperconnected if every pair of nonempty open subsets of Y has a nonempty intersection. This concept was introduced in 1970 by Steen and Seebach [1] and has been extensively studied since then. In this context, Levine [2] showed that hyperconnected spaces are precisely those spaces in which every nonempty open set is dense, referring to them as D-spaces. Hyperconnected spaces are also known as irreducible spaces [3]. Various characterizations, fundamental properties, and extensions of hyperconnectedness, including those in ideal topological spaces such as $*$-hyperconnected spaces, have been investigated in several works; see, for example, [4–7].

Let A be a subset of Y and (Y, σ) be a space. A point $y \in Y$ is referred to be a condensation point of A if the set $U \cap A$ is uncountable for any $U \in \sigma$ with $y \in U$. If a set A includes all of its condensation points, it is said to be ω-closed [8]. The complement of a set that is ω-closed is ω-open. It's common knowledge that a subset W of a space (Y, σ) is ω-open if, and only if, there is $U \in \sigma$ such that $y \in U$ and $U \setminus W$ are countable for every $y \in W$ (see [8]). It is well established that τ_ω is a topology on Y finer than τ. The notion of ω-open sets has been extensively utilized in general topology.

Notably, Lindelöfness has been characterized through ω-open sets [8]; meanwhile, several notions of continuity based on ω-open sets have been introduced and investigated in [9, 10]. Furthermore, various generalizations of paracompactness defined via ω-open sets have been investigated in [11]. A topological space shall be denoted as (Y, σ) or Y throughout this article, and the closure (or interior) of a subset A of Y will be denoted by $Cl(A)$ (or $Int(A)$). With $Cl_\omega(A) = \cap\{F : A \subseteq F, F$ is an ω-closed set$\}$, we set $Int_\omega(A) = \cup\{U : U \subseteq A,\ U$ is an ω-open set$\}$. If $Cl_\omega(A) = Y$, then a subset A of a topological space (Y, σ) is σ_ω-dense. In [12] when $A \subseteq Y$, then

$$(\sigma_A)_\omega = \{A \cap W : W \in \sigma_\omega\} = (\sigma_\omega)_A.$$

An ideal on Y is a nonempty collection $\mathcal{J}$ of subsets of Y [13] if it satisfies two conditions:

(1) If $J_1 \in \mathcal{J}$ and $J_2 \subseteq J_1$, then $J_2 \in \mathcal{J}$ (heredity).
(2) If $J_1 \in \mathcal{J}$ and $J_2 \in \mathcal{J}$, then $J_1 \cup J_2 \in \mathcal{J}$ (finite additivity).

An ideal topological space, represented by $(Y, \sigma, \mathcal{J})$, is a topological space (Y, σ) with an ideal $\mathcal{J}$ on Y. For a subset $A \subseteq Y$, the local function of A with regard to $\mathcal{J}$ and σ is $A^*(\mathcal{J}, \sigma) = \{y \in Y : A \cap U \notin \mathcal{J}$ for any open set U containing $y\}$ (see [13, 14]). If there is no chance for confusion, we simply write A^* instead of $A^*(\mathcal{J}, \sigma)$. In 1943, Edwin Hewitt introduced the notion of a resolvable space [15]. A nonempty topological space (Y, τ) is said to be resolvable if Y can be expressed as the disjoint union of two dense (equivalently, codense) subsets. If this condition fails, the space Y is called irresolvable. Later, in 1999, J. Dontchev and coauthors [16] studied the concepts of $\mathcal{J}$-density and resolvability modulo an ideal, and proved that the density topology is resolvable for more details (see [17]). Some properties pertaining to ω-open sets and ideals have been introduced recently by publications [18, 19]. From here on, an ideal topological space $(Y, \sigma, \mathcal{J})$ will simply be **ITS**.

This work is motivated by the need for a more refined framework to study density and connectedness in topological spaces. By using ω-open sets and the $\widetilde{\omega}$-local function, we can capture finer structural properties and unify several existing notions that are not fully addressed by classical approaches.

Definition 1. [20] Consider the **ITS** $(Y, \sigma, \mathcal{J})$. For an arbitrary subset $A \subseteq Y$, we define the following set: $\widetilde{\omega}(A)(\mathcal{J}, \sigma) = \{y \in Y : A \cap U \notin \mathcal{J}$ for every $U \in \sigma_\omega(y)\}$, where $\sigma_\omega(y)$ is the set of all ω-open sets in Y containing y. We write $\widetilde{\omega}(A)$ instead of $\widetilde{\omega}(A)(\mathcal{J}, \sigma)$ in case there is no confusion and it is known as the $\widetilde{\omega}$-local function of A with respect to $\mathcal{J}$ and σ.

Remark 2. In an **ITS** $(Y, \sigma, \mathcal{J})$, a subset A is $\widetilde{\omega}$-dense (resp., $\mathcal{J}$-dense) if every point of Y is in $\widetilde{\omega}(A)$ (resp., A^*), i.e., if $\widetilde{\omega}(A) = Y$ (resp., $A^* = Y$). If $D \subseteq Y$ is $\widetilde{\omega}$-dense, then Y is also $\widetilde{\omega}$-dense, i.e., $\widetilde{\omega}(Y) = Y$.

This article introduces and explores the concepts of $\widetilde{\omega}$-dense sets, $\widetilde{\omega}$-hyperconnectedness (briefly, $\widetilde{\omega}$-$\mathcal{HYC}$), strongly $\widetilde{\omega}$-hyperconnectedness (briefly, strongly $\widetilde{\omega}$-$\mathcal{HYC}$), and ω-irresolvability (briefly, ω-$\mathcal{IRR}$) utilizing ω-open sets and $\widetilde{\omega}$-local functions in ideal topological spaces,

2. On ω-hyperconnected spaces in ideal topological spaces

In this section, we investigate a particular class of topological spaces known as ω–hyperconnected spaces. We first present the basic definitions and fundamental properties associated with this notion and discuss its relationship with other forms of connectedness studied in the literature. We then focus on ω-hyperconnected spaces satisfying the condition $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, highlighting its significance in simplifying the structure and unifying several related results.

2.1. ω-hyperconnected spaces

Definition 3. An **ITS** $(Y, \sigma, \mathcal{J})$ is described as

(1) ω-$\mathcal{HYC}$ space if there are nonempty intersections between each pair of nonempty ω-open sets of Y.
(2) ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$ if each nonempty ω-open set's intersection does not lie inside $\mathcal{J}$.
(3) $\widetilde{\omega}$-$\mathcal{HYC}$ space if all nonempty ω-open sets in Y are $\widetilde{\omega}$-dense.

Example 4. Let $\mathbb{R}$ be real numbers with left ray topological

$$\tau = \{(\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset\} \cup \mathbb{R}.$$

Then, $\tau_\omega = \{(\infty, a) \setminus C : a \in \mathbb{R} \text{ and } C \text{ is a countable set}\} \cup \{\emptyset\} \cup \mathbb{R}$. Let $H, K \in \tau_\omega$ be any nonempty ω-open sets. It is clear that $H \cap k \neq \emptyset$, then $(\mathbb{R}, \tau)$ is a ω-$\mathcal{HYC}$ space.

Example 5. Let $\mathbb{R}$ represent the co-finite topological real numbers $\tau = \{A \subseteq \mathbb{R} : A^c \text{ is finite}\} \cup \{\emptyset\}$. Let $\mathcal{J}_f$ be the ideal of every finite subset. Then, $\tau_\omega = \{A \subseteq \mathbb{R} : A^c \text{ is countable}\} \cup \{\emptyset\}$. It is clear that $\tau_\omega \cap \mathcal{J} = \{\emptyset\}$. Let $H, K \in \tau_\omega$ be nonempty ω-open sets, then $\mathbb{R} \setminus H$ and $\mathbb{R} \setminus K$ are countable. Thus,

$$(\mathbb{R} \setminus H) \cup (\mathbb{R} \setminus K) = \mathbb{R} \setminus (H \cap K)$$

is countable, so $H \cap K$ is not finite. Hence, $H \cap K \notin \mathcal{J}_f$. Therefore, $(\mathbb{R}, \tau, \mathcal{J}_f)$ is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$.

Remark 6. (1) Let Y be a finite set with $|Y| \geq 2$ and σ any topology on Y. Then, σ_ω coincides with the discrete topology on Y. Let $\mathcal{J}$ be any finite ideal on Y. It follows that $(Y, \sigma, \mathcal{J})$ is neither an ω-$\mathcal{HYC}$ space, nor an ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$, nor a $\widetilde{\omega}$-$\mathcal{HYC}$ space.
(2) Let $\mathbb{R}$ be the real numbers with topology

$$\sigma = \{\emptyset, \mathbb{R}, Q, Q^c\}.$$

Let $\mathcal{J}_f$ be the ideal of all finite subsets of $\mathbb{R}$. Then, $\sigma_\omega = \{A \subseteq \mathbb{R} : A = U \setminus C \text{ for every open set } U \in \sigma \text{ and } C \text{ is a countable set}\}$. It is clear that $(\mathbb{R}, \sigma, \mathcal{J})$ is neither an ω-$\mathcal{HYC}$ space, nor an ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$, nor a $\widetilde{\omega}$-$\mathcal{HYC}$ space.

Lemma 7. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then, Y is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$ if, and only if, there are no ω-closed sets U_1 and U_2 that are proper with $Y \setminus (U_1 \cup U_2) \in \mathcal{J}$.

Proof. Assume that there are ω-closed sets U_1 and U_2 that are proper such that $Y \setminus (U_1 \cup U_2) \in \mathcal{J}$. If $U_2 = \emptyset$, then $Y \setminus U_1 \in \mathcal{J}$, since $Y \setminus U_1$ and Y are nonempty ω-open sets with

$$Y \cap (Y \setminus U_1) = (Y \setminus U_1) \in \mathcal{J}.$$

This is in conflict with Y which is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$. For this reason, U_1 and U_2 are both nonempty proper ω-closed sets. Then, $Y \setminus U_1$ and $Y \setminus U_2$ are nonempty ω-open sets. Also,

$$(Y \setminus U_1) \cap (Y \setminus U_2) = Y \setminus (U_1 \cup U_2) \in \mathcal{J}.$$

This contradicts Y being ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$ as $(Y \setminus U_1) \cap (Y \setminus U_2) \in \mathcal{J}$. Hence, there are no ω-closed sets U_1 and U_2 that are proper with $Y \setminus (U_1 \cup U_2) \in \mathcal{J}$.

Conversely, let V_1 and V_2 be any nonempty ω-open sets in Y. Then, $Y \setminus V_1$ and $Y \setminus V_2$ are proper ω-closed sets in Y and $Y \setminus [(Y \setminus V_1) \cup (Y \setminus V_2)] \notin \mathcal{J}$. From this it follows that $Y \setminus [Y \setminus (V_1 \cap V_2)] \notin \mathcal{J}$. Therefore, $(V_1 \cap V_2) \notin \mathcal{J}$. Hence, Y is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$. □

Theorem 8. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then, Y is ω-$\mathcal{HYC}$ space if, and only if, the union of two non σ_ω-dense sets is a non σ_ω-dense set.

Proof. Let U_1, U_2 be two non σ_ω-dense sets in ω-$\mathcal{HYC}$ space Y. Then, there exist two ω-open sets V_1 and V_2 nonempty such that

$$V_1 \cap U_1 = \emptyset \text{ and } V_2 \cap U_2 = \emptyset.$$

Since Y is a ω-$\mathcal{HYC}$ space, $V_1 \cap V_2 \neq \emptyset$, then

$$(V_1 \cap V_2) \cap (U_1 \cup U_2) = \emptyset$$

and, thus, $U_1 \cup U_2$ is non σ_ω-dense in Y. Since Y is a ω-$\mathcal{HYC}$ space, we obtain $V_1 \cap V_2$ as nonempty ω-open sets. Therefore, $U_1 \cup U_2$ is non σ_ω-dense in Y since

$$(V_1 \cap V_2) \cap (U_1 \cup U_2) = \emptyset.$$

On the other hand, suppose Y is not a ω-$\mathcal{HYC}$ space, and the union of two non σ_ω-dense sets is a non σ_ω-dense set. After that, there are two nonempty ω-open sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$. Therefore, $U_1 \subseteq Y \setminus U_2$ and $U_2 \subseteq Y \setminus U_1$. Hence, $Y \setminus U_1$ and $Y \setminus U_2$ are ω-closed sets, thus, it is non σ_ω-dense in Y, but

$$(Y \setminus U_1) \cup (Y \setminus U_2) = Y$$

which is in opposition to the reality that the union of two non σ_ω-dense sets is a non σ_ω-dense set. Thus, the theorem has been proven.

This completes the proof. □

2.2. *ω-hyperconnected spaces with $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$*

Next, we study ω-hyperconnected spaces under the condition $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, and show how this assumption streamlines the structure and brings together a number of related results.

Theorem 9. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**, and if $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, then Y is a ω-$\mathcal{HYC}$ space if, and only if, Y is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$.

Proof. ($\Longrightarrow$) Let Y be ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$. Let U_1 and U_2 be any nonempty ω-open sets, then $U_1 \cap U_2 \notin \mathcal{J}$. Since $\emptyset \in \mathcal{J}$ for any ideal, we get $U_1 \cap U_2 \neq \emptyset$ and, hence, $(Y, \sigma, \mathcal{J})$ is a ω-$\mathcal{HYC}$ space.

($\Longleftarrow$) Conversely, let Y be a ω-$\mathcal{HYC}$ space and $\emptyset \neq U_1, U_2 \in \sigma_\omega$. Then, $\emptyset \neq U_1 \cap U_2 \in \sigma_\omega$. Since $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, $U_1 \cap U_2 \notin \mathcal{J}$. Therefore, Y is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$. □

Lemma 10. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then, Y is a $\widetilde{\omega}$-$\mathcal{HYC}$ space if, and only if, Y is a ω-$\mathcal{HYC}$ space and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

Proof. ($\Longrightarrow$) It is clear that every $\widetilde{\omega}$-$\mathcal{HYC}$ space is ω-$\mathcal{HYC}$. Let W be a nonempty, ω-open set that belongs to the ideal. Then, $\widetilde{\omega}(W) = Y$. On the other hand, since $W \in \mathcal{J}$, $\widetilde{\omega}(W) = \emptyset$. Hence, $Y = \emptyset$. This is a contradiction. Therefore, $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

($\Longleftarrow$) Conversely, let $\emptyset \neq W \in \sigma_\omega$. Let $y \in Y$. Due to the ω-$\mathcal{HYC}$ space of Y, every ω-open set W_1 containing y meets W. Moreover, $W \cap W_1$ is a ω-open set and $W \cap W_1 \notin \mathcal{J}$ because $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Thus, $y \in \widetilde{\omega}(W)$. This demonstrates that W is a $\widetilde{\omega}$-dense. □

Remark 11. Let $\mathcal{J}_f$ be the ideal of every finite subset of $\mathbb{R}$. Let σ be left ray topological as Example 5. It is clear that $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, and since $(\mathbb{R}, \sigma)$ is a ω-$\mathcal{HYC}$ space by Example 5, hence, $(\mathbb{R}, \sigma, \mathcal{J})$ is a $\widetilde{\omega}$-$\mathcal{HYC}$ space.

Theorem 12. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**, where $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Then, a set H is $\widetilde{\omega}$-dense if, and only if, $(W \backslash J) \cap H \neq \emptyset$ whenever; $\emptyset \neq W \in \sigma_\omega$ and $J \in \mathcal{J}$.

Proof. Let H be $\widetilde{\omega}$-dense. Then, $W \cap H \notin \mathcal{J}$ for all nonempty ω-open sets W. Hence, for all $J \in \mathcal{J}$, $(W \backslash J) \cap H \neq \emptyset$, for otherwise $(W \backslash J) \cap H = \emptyset$ and, hence,

$$\emptyset = W \cap (Y \backslash J) \cap H = (W \cap H) \cap (Y \backslash J).$$

Therefore, $W \cap H \subseteq J$. Since $J \in \mathcal{J}$, $W \cap H \in \mathcal{J}$ which is contrary to $W \cap H \notin \mathcal{J}$. Therefore, $(W \backslash J) \cap H \neq \emptyset$.

Conversely, let $(W \backslash J) \cap H \neq \emptyset$ whenever $\emptyset \neq W \in \sigma_\omega$ and $J \in \mathcal{J}$. Then, we claim that H is $\widetilde{\omega}$-dense. Let H be non $\widetilde{\omega}$-dense. Then, there is some nonempty ω-open set W such that $W \cap H \in \mathcal{J}$. Let $W \cap H = J$. Then, since $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, $W \backslash J$ is nonempty but $(W \backslash J) \cap H = \emptyset$. This is contrary to our assumption. □

Theorem 13. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**, where $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Then, Y is ω-$\mathcal{HYC}$ space modulo $\mathcal{J}$ if, and only if, $(W - J) \cap H \neq \emptyset$ whenever $\emptyset \neq W, H \in \sigma_\omega$, and $J \in \mathcal{J}$.

Proof. The proof is derived from Lemmas 10 and 12. □

3. On $\widetilde{\omega}$-resolvable spaces via ideal

Reference [15] states that if Y is the union of two disjoint dense subsets, then a topological space (Y, σ) is a resolvable. The existence of two disjoint $\mathcal{J}$-dense subsets of Y indicates that the ideal topological space $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-resolvable [16]. When two disjoint $\widetilde{\omega}$-dense sets exist in a topological ideal space $(Y, \sigma, \mathcal{J})$, it is said to be $\widetilde{\omega}$-resolvable; if not, it is said to be $\widetilde{\omega}$-irresolvable.

Theorem 14. [21] Let $(Y, \sigma, \mathcal{J})$ be an **ITS**, then the following characteristics are equivalent:

(1) $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$;
(2) If $J \in \mathcal{J}$, then $Int_\omega(J) = \{\emptyset\}$;
(3) For each ω-open set U, $U \subseteq \widetilde{\omega}(U)$;
(4) $Y = \widetilde{\omega}(Y)$.

Lemma 15. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then,
(1) If Y is $\widetilde{\omega}$-resolvable, then $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$;
(2) Y is $\widetilde{\omega}$-resolvable if, and only if, Y is the union of two disjoint $\widetilde{\omega}$-dense sets.

Proof. (1) There are two disjoint $\widetilde{\omega}$-dense sets A and B since Y is $\widetilde{\omega}$-resolvable space. Then, by [20, Lemma 2.4 (1)], we have

$$Y = \widetilde{\omega}(A) \subseteq \widetilde{\omega}(Y).$$

Therefore, Y is $\widetilde{\omega}$-dense. Thus, by Theorem 14, $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

(2) Assuming Y is $\widetilde{\omega}$-resolvable, A and B are two disjoint $\widetilde{\omega}$-dense sets. Thus,

$$\widetilde{\omega}(A) = Y \quad \text{and} \quad Y = \widetilde{\omega}(B) \subseteq \widetilde{\omega}(Y \setminus A)$$

and, hence, $Y = \widetilde{\omega}(Y \setminus A)$. Therefore, Y is the union of $\widetilde{\omega}$-dense sets A and $Y \setminus A$. The converse is clear. □

Example 16. Let $\mathbb{R}$ be the real numbers with topology $\sigma = \{\emptyset, \mathbb{R}, Q, Q^c\}$. Let $\mathcal{J}_f$ be the ideal of all finite subsets of $\mathbb{R}$. Then,

$$\sigma_\omega = \{A \subseteq \mathbb{R} : A = (U \setminus C)$$

for every open set $U \in \sigma$ and C is a countable set}. It is clear that $\sigma_\omega \cap \mathcal{J} \neq \{\emptyset\}$ and it is a non $\widetilde{\omega}$-resolvable space by item (1) of Lemma 15.

Example 17. Let $\mathbb{R}$ be the real numbers with co-finite topology $\tau = \{A \subseteq \mathbb{R} : A^c \text{ is finite}\} \cup \{\emptyset\}$. Let $\mathcal{J}_f$ be the ideal of all finite subset of $\mathbb{R}$. Then, $\tau_\omega = \{A \subseteq \mathbb{R} : A^c \text{ is countable}\} \cup \{\emptyset\}$. Let $H = (-\infty, 0)$ and $K = [0, \infty)$ be subsets of $\mathbb{R}$, then for any ω-open set U in τ_ω containing $x \in \mathbb{R}$ haveing infinite negative real numbers, we get that $\widetilde{\omega}(H) = \{x \in \mathbb{R} : (-\infty, 0) \cap U \notin \mathcal{J}_f$ for every $U \in \tau_\omega(x)\} = \mathbb{R}$ and $\mathbb{R} = \widetilde{\omega}(K) = \{x \in \mathbb{R} : [0, \infty) \cap U \notin \mathcal{J}_f$ for every $U \in \tau_\omega(x)\}$, where $\tau_\omega(x)$ is the set of all ω-open sets in $\mathbb{R}$ containing x. So, H and K are two disjoint $\widetilde{\omega}$-dense and $\mathbb{R} = H \cup k$. Hence, $(\mathbb{R}, \tau, \mathcal{J}_f)$ is a $\widetilde{\omega}$-resolvable space by item (2) of Lemma 15.

Remark 18. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Reference [20] obtained that $Cl^*_\omega(W) = W \cup \widetilde{\omega}(W)$ is a Kuratowski closure operator. We will denote by σ^*_ω the topology generated by Cl^*_ω, that is,

$$\sigma^*_\omega = \{W \subseteq Y : Cl^*_\omega(Y \setminus W) = Y \setminus W\}.$$

Theorem 19. [20] Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then, $\beta(\sigma_\omega, \mathcal{J}) = \{W - J : W$ is a ω-open set of Y and $J \in \mathcal{J}\}$ is a basis for σ^*_ω.

Theorem 20. An **ITS** $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-resolvable if, and only if, (Y, σ^*_ω) is resolvable and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

Proof. ($\Longrightarrow$) Let $(Y, \sigma, \mathcal{J})$ be $\widetilde{\omega}$-resolvable. Then, by Lemma 15 (2), $Y = U_1 \cup U_2$, where U_1 and U_2 are disjoint $\widetilde{\omega}$-dense sets of Y. We know that

$$Cl^*_\omega(U_1) = U_1 \cup \widetilde{\omega}(U_1) = U_1 \cup Y = Y.$$

Hence, U_1 and U_2 are τ^*_ω-dense. Thus, (Y, σ^*_ω) is resolvable. By Lemma 15 (1), $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

($\Longleftarrow$) Conversely, let (Y, σ^*_ω) be resolvable and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Suppose that

$$Y = U_1 \cup U_2, \quad U_1 \cap U_2 = \emptyset$$

and both U_1 and U_2 are σ^*_ω-dense. Let $y \in Y$ and $y \notin \widetilde{\omega}(U_1)$, then there is a ω-open set V_1 containing y such that

$$V_2 = V_1 \cap U_1 \in \mathcal{J}.$$

Since U_2 is σ^*_ω-dense and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, V_2 is nonempty and, moreover, $V_1 \nsubseteq U_1$. It follows from Theorem 19 that

$$\emptyset \neq W = V_1 \setminus V_2 \in \sigma^*_\omega \quad \text{and} \quad W \cap U_1 = \emptyset.$$

This is in conflict with the fact that U_1 is τ^*_ω-dense. Thus, $y \in \widetilde{\omega}(U_1)$ and, hence, U_1 is $\widetilde{\omega}$-dense. The same reasoning demonstrates that U_2 is $\widetilde{\omega}$-dense. Thus, $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-resolvable. □

Lemma 21. Let $(Y, \sigma, \mathcal{J})$ be an **ITS**. Then, the nonempty σ^*_ω-open subspace of an $\widetilde{\omega}$-resolvable space is an $\widetilde{\omega}$-resolvable.

Proof. First, we are aware that an open set intersecting a dense set is dense, so the resolvability is an open hereditary. Furthermore, for all $A \in \sigma^*_\omega$ we have $\sigma^*_{\omega_A} = (\sigma_A)^*_\omega$. Thus, by Theorem 20, if $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-resolvable and A is σ^*_ω-open, then (Y, σ^*_ω) is resolvable; hence,

$$(A, \sigma^*_{\omega_A}) = (A, (\sigma_A)^*_\omega)$$

is resolvable and, thus, $(A, \tau_A, \mathcal{J}_A)$ is $\widetilde{\omega}$-resolvable. □

Theorem 22. An **ITS** $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-resolvable if, and only if, there exists $\widetilde{\omega}$-dense set W such that for all nonempty $H \in \sigma_\omega$ and all $J \in \mathcal{J}$, $H \setminus J \neq \emptyset$ implies $(H \setminus J) \nsubseteq W$.

Proof. Let $(Y, \sigma, \mathcal{J})$ be $\widetilde{\omega}$-resolvable. Then, by Remark 6 and Theorem 14, $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Hence, there is two disjoint $\widetilde{\omega}$-dense sets, say W_1 and W_2. We show that $(H \setminus J) \nsubseteq W_1$ whenever $H \setminus J \neq \emptyset$ for all $\emptyset \neq H \in \sigma_\omega$ and $J \in \mathcal{J}$. If possible, let $(H \setminus J) \subseteq W_1$ for some $\emptyset \neq H \in \sigma_\omega$ and $J \in \mathcal{J}$. Then,

$$(H \setminus J) \cap W_2 = \emptyset.$$

Now, since $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, by Theorem 12, W_2 is not $\widetilde{\omega}$-dense. This is contrary that W_2 is $\widetilde{\omega}$-dense. Hence, $(H \setminus J) \nsubseteq W_1$ whenever, $H \setminus J \neq \emptyset$ for all $\emptyset \neq H \in \sigma_\omega$ and $J \in \mathcal{J}$.

Conversely, let the condition hold in $(Y, \sigma, \mathcal{J})$. Then, there exists a $\widetilde{\omega}$-dense set W such that $(H \setminus J) \nsubseteq W$ if $H \setminus J \neq \emptyset$ for all $\emptyset \neq H \in \sigma_\omega$ and $J \in \mathcal{J}$. We show that $Y \setminus W$ is $\widetilde{\omega}$-dense. Let $Y \setminus W$ be non $\widetilde{\omega}$-dense. Then, there exists $\emptyset \neq K \in \sigma_\omega$ such that $K \cap (Y \setminus W) \in \mathcal{J}$. Clearly, $K \cap (Y \setminus W) \neq \emptyset$, for otherwise $K \subseteq W$, which is contrary to our assumption. Let

$$J = K \cap (Y \setminus W).$$

Then, $K \setminus J \neq \emptyset$. For if $K \setminus J = \emptyset$, then $K \subseteq J$ and, hence, $K \in \mathcal{J}$ which implies $K \cap W \in \mathcal{J}$. This is contrary that W is $\widetilde{\omega}$-dense. Therefore, $K \setminus J \subseteq W$, which is again contrary to our assumption. Thus, $Y \setminus W$ is $\widetilde{\omega}$-dense and, hence, $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-resolvable. □

Lemma 23. [14] Let A be a subset of Y and $\mathcal{J}$ be an ideal in Y. Then,

$$\mathcal{J}_A = \{J \in \mathcal{J} : J \subseteq A\} = \{J \cap A : J \in \mathcal{J}\}$$

is an ideal in A.

Corollary 24. An **ITS** $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-irresolvable if, and only if, for each $\widetilde{\omega}$-dense set W, there exist $H \in \sigma_\omega$ and $J \in \mathcal{J}$ such that $\emptyset \neq (H \setminus J) \subseteq W$.

Theorem 25. Let $(Y, \sigma, \mathcal{J})$ be an **ITS** such that $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. If W is $\widetilde{\omega}$-dense in $(Y, \sigma, \mathcal{J})$, then for all $A = H \setminus J$, where $\emptyset \neq H \in \sigma_\omega$ and $J \in \mathcal{J}$, $A \cap W$ is $\widetilde{\omega}$-dense in $(A, \tau_A, \mathcal{J}_A)$.

Proof. Clearly, we suppose $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Then, a ω-open set in A is of the form

$$A \cap K = (H \setminus J) \cap K = (H \cap K) \setminus J,$$

where K is a ω-open set in (Y, σ). Let $\emptyset \neq (H \cap K) \setminus J$. Consider, $\emptyset \neq ((H \cap K) \setminus J) \setminus J_A$, $J_A \in \mathcal{J}_A$. Then, since W is a $\widetilde{\omega}$-dense and $H \cap K$ is a ω-open set in (Y, σ), by Theorem 12, $((H \cap K) \setminus (J \cup J_A)) \cap W \neq \emptyset$. Hence,

$$(((H \cap K) \setminus J) \setminus J_A) \cap (A \cap W) = (((H \cap K) \setminus J) \setminus J_A) \cap W \neq \emptyset.$$

Therefore, again by Theorem 12, $A \cap W$ is $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$. □

Theorem 26. Let $(Y, \sigma, \mathcal{J})$ be an **ITS** such that

$$\sigma_\omega \cap \mathcal{J} = \{\emptyset\} \quad \text{and} \quad B \subseteq A = W_1 \setminus J,$$

where $\emptyset \neq W_1 \in \sigma_\omega$, $J \in \mathcal{J}$. Then, B is $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$ if, and only if, $B = A \cap D$, where D is $\widetilde{\omega}$-dense in $(Y, \sigma, \mathcal{J})$.

Proof. Let B be $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$. Consider the set $B \cup (Y \setminus A)$. Then,

$$(B \cup (Y \setminus A)) \cap W_2 = (B \cap W_2) \cup ((Y \setminus A) \cap W_2),$$

where $\emptyset \neq W_2 \in \sigma_\omega$. Now if $W_2 \subseteq Y \setminus A$, then $B \subseteq A$ and $B \cap W_2 = \emptyset$, and we have

$$(B \cup (Y \setminus A)) \cap W_2 = W_2,$$

which is not in $\mathcal{J}$ because $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Finally, if $W_2 \cap A \neq \emptyset$, then since B is $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$, $B \cap (W_2 \cap A) \notin \mathcal{J}_A$, and, hence, $B \cap W_2 \notin \mathcal{J}$. Therefore, $(B \cup (Y \setminus A)) \cap W_2 \notin \mathcal{J}$. Thus,

$$(B \cup (Y \setminus A)) = D,$$

which is $\widetilde{\omega}$-dense in $(Y, \sigma, \mathcal{J})$ and, hence, $B = A \cap D$. Conversely, let $B = A \cap D$, where D is $\widetilde{\omega}$-dense in $(Y, \sigma, \mathcal{J})$. Then, by Theorem 25, B is $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$. This completes the proof of the theorem. □

4. On ω-irresolvable spaces with ideal

We shall now define and discuss properties of a ω-irresolvable (briefly, ω-$\mathcal{IRR}$) space.

Definition 27. An **ITS** $(Y, \sigma, \mathcal{J})$ is said to be a ω-$\mathcal{IRR}$ space if for each $\widetilde{\omega}$-dense set D and each $\emptyset \neq W_1 \in \sigma_\omega$ and $J_1 \in \mathcal{J}$ such that $\emptyset \neq W_1 \setminus J_1$, there exist $\emptyset \neq W_2 \in \sigma_\omega$ and $J_2 \in \mathcal{J}$ such that $\emptyset \neq (W_2 \setminus J_2) \subseteq (W_1 \setminus J_1) \cap D$.

Theorem 28. An **ITS** $(Y, \sigma, \mathcal{J})$ is a ω-$\mathcal{IRR}$ space if, and only if, the intersection of any two $\widetilde{\omega}$-dense sets is a $\widetilde{\omega}$-dense set, when $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

Proof. $(\Longrightarrow)$ Let $(Y, \sigma, \mathcal{J})$ be a ω-$\mathcal{IRR}$ space and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Let D_1 and D_2 be two $\widetilde{\omega}$-dense sets in $(Y, \sigma, \mathcal{J})$. We show that $D_1 \cap D_2$ is $\widetilde{\omega}$-dense. Consider $W_1 \setminus J_1$, where $\emptyset \neq W_1 \in \sigma_\omega$ and $J_1 \in \mathcal{J}$. We show that $(W_1 \setminus J_1) \cap D_1 \cap D_2 \neq \emptyset$. Since D_1 is $\widetilde{\omega}$-dense, $\emptyset \neq W_1 \in \sigma_\omega$ and $J_1 \in \mathcal{J}$ with $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$; hence, by Theorem 12, $(W_1 \setminus J_1) \cap D_1 \neq \emptyset$. Since $(Y, \sigma, \mathcal{J})$ is a ω-$\mathcal{IRR}$ space, there exist $\emptyset \neq W_2 \in \sigma_\omega$ and $J_2 \in \mathcal{J}$ such that $\emptyset \neq (W_2 \setminus J_2) \subseteq (W_1 \setminus J_1) \cap D_1$. Again, since D_2 is $\widetilde{\omega}$-dense, there exist $\emptyset \neq W_3 \in \sigma_\omega$ and $J_3 \in \mathcal{J}$ such that $\emptyset \neq (W_3 \setminus J_3) \subseteq (W_2 \setminus J_2) \cap D_2$. Hence, $\emptyset \neq W_3 \setminus J_3 \subseteq (W_1 \setminus J_1) \cap D_1 \cap D_2$.

Therefore, $(W_1 \setminus J_1) \cap (D_1 \cap D_2) \neq \emptyset$, $\emptyset \neq W_1 \in \sigma_\omega$, and $J_1 \in \mathcal{J}$ with $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, hence, by Theorem 12, $D_1 \cap D_2$ is $\widetilde{\omega}$-dense.

($\Longleftarrow$) On the other hand, if the intersection of any two $\widetilde{\omega}$-dense sets is $\widetilde{\omega}$-dense, assume that $(Y, \sigma, \mathcal{J})$ is not ω-$\mathcal{IRR}$ space. Then, there exist a $\widetilde{\omega}$-dense set D_1, $\emptyset \neq W_1 \in \sigma_\omega$, and $J_1 \in \mathcal{J}$, where $\emptyset \neq W_1 \setminus J_1$, such that $(W_1 \setminus J_1) \cap D_1$ does not contain $W_4 \setminus J$, for any $\emptyset \neq W_4 \in \sigma_\omega$ and $J \in \mathcal{J}$. Consider the set

$$D_2 = (Y \setminus (W_1 \setminus J_1)) \cup ((W_1 \setminus J_1) \setminus (W_1 \setminus J_1) \cap D_1).$$

Since $(W_4 \setminus J) \cap D_2 \neq \emptyset$, $\emptyset \neq W_4 \in \sigma_\omega$, and $J \in \mathcal{J}$ with $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$, hence, by Theorem 12, D_2 is $\widetilde{\omega}$-dense. However,

$$(W_1 \setminus J_1) \cap D_1 \cap D_2 = \emptyset.$$

This is in opposition to the reality that the intersection of two $\widetilde{\omega}$-dense sets is an $\widetilde{\omega}$-dense set. Hence $(Y, \sigma, \mathcal{J})$ must be a ω-$\mathcal{IRR}$ space. The proof of the theorem is thus finished. □

Theorem 29. Let $(Y, \sigma, \mathcal{J})$ be an **ITS** and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. If $(Y, \sigma, \mathcal{J})$ is a ω-$\mathcal{IRR}$ space, then $(A, \sigma_A, \mathcal{J}_A)$ is a ω-$\mathcal{IRR}$ space whenever $A = W \setminus J$ for every $\emptyset \neq W \in \sigma_\omega$ and $J \in \mathcal{J}$.

Proof. Let $D^{'}$ and $D^{''}$ be $\widetilde{\omega}$-dense sets in $(A, \sigma_A, \mathcal{J}_A)$. Then, by Theorem 26,

$$D^{'} = (W \setminus J) \cap D_1 \quad \text{and} \quad D^{''} = (W \setminus J) \cap D_2,$$

where D_1 and D_2 are $\widetilde{\omega}$-dense sets in $(Y, \sigma, \mathcal{J})$. Hence,

$$D^{'} \cap D^{''} = (W \setminus J) \cap D_1 \cap D_2$$

and, since $D_1 \cap D_2$ is a $\widetilde{\omega}$-dense set in $(Y, \sigma, \mathcal{J})$, again by Theorem 26, $D^{'} \cap D^{''}$ is $\widetilde{\omega}$-dense in $(A, \sigma_A, \mathcal{J}_A)$. Hence, by Theorem 28, $(A, \sigma_A, \mathcal{J}_A)$ is a ω-$\mathcal{IRR}$ space. □

Definition 30. An **ITS** $(Y, \sigma, \mathcal{J})$ is said to be a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space if each $W \setminus J \neq \emptyset$ is $\widetilde{\omega}$-dense, where $W \in \sigma_\omega$ and $J \in \mathcal{J}$.

Example 31. Let $\mathbb{R}$ be the real numbers with topology $\sigma = \{\emptyset, \mathbb{R}, Q, Q^c\}$. Let $\mathcal{J}_f$ be the ideal of all finite subsets of $\mathbb{R}$. Then, $\sigma_\omega = \{A \subseteq \mathbb{R} : A = U \setminus C$ for every open set $U \in \sigma$ and C is a countable set$\}$.

(1) Let $W = \{1, 2, 3\} \in \sigma_\omega$ and let $J = \{3\} \in \mathcal{J}$, then $W \setminus J = \{1, 2\}$ and $\widetilde{\omega}(W \setminus J) = \emptyset$, thus $W \setminus J$ is not $\widetilde{\omega}$-dense. Hence, $(\mathbb{R}, \sigma, \mathcal{J})$ is not a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space.

(2) It is clear that $\sigma_\omega \cap \mathcal{J} \neq \{\emptyset\}$ and by item (2) of Remark 6, $(\mathbb{R}, \sigma, \mathcal{J})$ is not a $\widetilde{\omega}$-$\mathcal{HYC}$ space. Hence, $(\mathbb{R}, \sigma, \mathcal{J})$ is not a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space which is illustrative the following theorem.

Theorem 32. An **ITS** $(Y, \sigma, \mathcal{J})$ is a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space if, and only if, $(Y, \sigma, \mathcal{J})$ is a $\widetilde{\omega}$-$\mathcal{HYC}$ space and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

Proof. ($\Longrightarrow$) Let $(Y, \sigma, \mathcal{J})$ be a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space. Clearly, $(Y, \sigma, \mathcal{J})$ is a $\widetilde{\omega}$-$\mathcal{HYC}$ space. Let $W_1 \neq \emptyset$ be an ω-open set and a member of the ideal. Then, $\widetilde{\omega}(W_1) = Y$ since $(Y, \sigma, \mathcal{J})$ is a $\widetilde{\omega}\mathcal{HYC}$ space. On the other hand, since $W_1 \in \mathcal{J}$, $\widetilde{\omega}(W_1) = \emptyset$, which is a contradiction. Consequently, $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$.

($\Longleftarrow$) Conversely, let $(Y, \sigma, \mathcal{J})$ be a $\widetilde{\omega}$-$\mathcal{HYC}$ space and $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Consider $W_1 \setminus J$, where $\emptyset \neq U \in \sigma_\omega$ and $J \in \mathcal{J}$. Then, $W_1 \setminus J \neq \emptyset$ because $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. We demonstrate that $W_1 \setminus J$ is

$\widetilde{\omega}$-dense. Let $y \in Y$ and W_2 be a ω-open set containing y. By Lemma 10, (Y, σ) is a ω-$\mathcal{HYC}$ space and $W_2 \cap (W_1 \setminus J) \neq \emptyset$ because

$$W_2 \cap (W_1 \setminus J) = W_2 \cap W_1 \setminus J \neq \emptyset \quad \text{and} \quad \sigma_\omega \cap \mathcal{J} = \{\emptyset\}.$$

Thus, $(Y, \sigma, \mathcal{J})$ is a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space. □

Theorem 33. If an **ITS** $(Y, \sigma, \mathcal{J})$ is a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space and $\widetilde{\omega}$-irresolvable space, then it is a ω-$\mathcal{IRR}$ space.

Proof. Let $(Y, \sigma, \mathcal{J})$ be a strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space and $\widetilde{\omega}$-irresolvable space, then $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$ by Theorem 32. Let D_1 and D_2 be two $\widetilde{\omega}$-dense sets in $(Y, \sigma, \mathcal{J})$. Now, we show that $D_1 \cap D_2$ is $\widetilde{\omega}$-dense. By Theorem 12, it is sufficient to prove that $(D_1 \cap D_2) \cap (W_1 \setminus J_1) \neq \emptyset$ for all $\emptyset \neq W_1 \in \sigma_\omega$ and $J_1 \in \mathcal{J}$ with $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$. Since $(Y, \sigma, \mathcal{J})$ is $\widetilde{\omega}$-irresolvable, by Corollary 24, there exist $\emptyset \neq W_2 \in \sigma_\omega$ and $J_2 \in \mathcal{J}$ such that $\emptyset \neq W_2 \setminus J_2 \subseteq D_1$. Similarly, there exist $\emptyset \neq W_3 \in \sigma_\omega$ and $J_3 \in \mathcal{J}$ such that $\emptyset \neq W_3 \setminus J_3 \subseteq D_2$. Now, (Y, σ) is a ω-$\mathcal{HYC}$ space by Theorem 32, and $W_2 \cap W_3 \neq \emptyset$ since

$$\sigma_\omega \cap \mathcal{J} = \{\emptyset\}, \quad (W_2 \setminus J_2) \cap (W_3 \setminus J_3) = (W_2 \cap W_3) \setminus (J_2 \cup J_3) \neq \emptyset$$

and, hence, $(W_2 \cap W_3) \setminus (J_2 \cup J_3) \subseteq D_1 \cap D_2$. Therefore, by strongly $\widetilde{\omega}$-$\mathcal{HYC}$ space of $(Y, \sigma, \mathcal{J})$, $(W_2 \cap W_3) \setminus (J_2 \cup J_3)$ is $\widetilde{\omega}$-dense and we have $\emptyset \neq (W_1 \setminus J_1) \cap [(W_2 \cap W_3) \setminus (J_2 \cup J_3)]$ and, hence, $(W_1 \setminus J_1) \cap (D_1 \cap D_2) \neq \emptyset$ with $\emptyset \neq W_1 \in \sigma_\omega$ and $J_1 \in \mathcal{J}$. Also by Theorem 32, $\sigma_\omega \cap \mathcal{J} = \{\emptyset\}$; thus, by Theorem 12, $D_1 \cap D_2$ is $\widetilde{\omega}$-dense. Thus, by Theorem 28, $(Y, \sigma, \mathcal{J})$ is a ω-$\mathcal{IRR}$ space. □

5. Conclusions

In this paper, we introduced the notion of ω-hyperconnected spaces in the setting of ideal topological spaces and examined their fundamental properties. A key contribution is the characterization of strongly $\widetilde{\omega}$-hyperconnected spaces, where Theorem 32 establishes that this concept is equivalent to $\widetilde{\omega}$-hyperconnectedness under the condition $\sigma_\omega \cap \mathcal{J} = \emptyset$. Moreover, Lemma 10 clarifies the relationship between ω-hyperconnected and $\widetilde{\omega}$-hyperconnected spaces, showing that they coincide precisely when the family of ω-open sets intersects the ideal only empty set. These results highlight the essential role of the ideal in determining the hyperconnected structure of the space. Future research may further distinguish the proposed theoretical framework from rough-set-based approximation spaces induced by neighborhoods and ideals [22], while exploring potential applications in decision-making, medical modeling, data analysis, network theory, and information systems, as well as extensions to fuzzy [23] and intuitionistic environments and deeper studies of topological approximation methods for complex relational systems.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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