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Research article

On Hadamard fractional operator and three-point fractional boundary value problem in integral-form Hölder spaces

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Abstract: The paper aims to put forward and discuss the proper assumptions for the existence, in addition to the uniqueness, of the solutions to the non-local fractional boundary value problem containing a Hadamard-type fractional operator in integral-form Hölder Banach space $J_{\alpha,\beta}$, which has much better properties than both classical Hölder spaces C^α and the space of continuous solutions C. Based on these results, we investigate and prove some essential properties of the Hadamard fractional operators, such as the boundedness, acting, and continuity in the studied spaces. Our analysis is based on Darbo's fixed point principle and the measure of noncompactness with fractional calculus. The results are confirmed with a numerical example.

Keywords: integral-form Hölder space; Darbo's fixed point theorem; Hadamard-type fractional operator; measure of noncompactness
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Abbreviations

RL: Riemann-Liouville
$\mathbb{BVP}$: boundary value problem
FPT: Fixed point theorem
MNC: measure of noncompactness

1. Introduction

Fractional-order models offer greater flexibility compared to integer-order models, resulting in higher accuracy. Hadamard-type integrals are used in formulating various problems in mechanics, such as fracture analysis [1–3]. For more applications of these operators, see [4–6]. Non-local conditions contribute to both intermediate processes within a domain and to boundary processes. Such conditions arise in wave propagation, elasticity, and thermodynamics, where a controller at the endpoints of a domain can dissipate or supply energy in response to signals from sensors within the domain [7–9].

Three-point $\mathbb{BVP}$s (boundary value problems) have been widely studied in the classical (integer-order) framework. A standard prototype is the second-order nonlinear problem

$$\begin{cases} x''(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = 0, \quad x(1) = bx(\eta), & 0 < \eta < 1, \ b > 0, \end{cases}$$

which incorporates an interior-point (nonlocal) boundary condition.

Gupta [10] investigated the solvability of such nonlinear three-point problems and established existence results by applying topological degree methods, thereby highlighting the role of nonlocal conditions in nonlinear analysis. In a related direction, Il'in and Moiseev [11] analyzed nonlocal boundary-value problems of the second kind for Sturm–Liouville operators and established well-posedness results under interior constraints. Furthermore, Anderson [12] proved the existence of multiple positive solutions for nonlinear three-point $\mathbb{BVP}$s by employing fixed-point techniques in cones.

In contrast to the widely studied Riemann–Liouville (RL) fractional integral [13–15], Hadamard fractional operators possess a logarithmic kernel and distinct structural properties [16–18]. Consequently, the analytical techniques developed for RL-type problems cannot be applied directly, and Hadamard-type fractional three-point $\mathbb{BVP}$s require a separate and careful investigation.

Motivated by these results, we check and investigate the existence results as well as the uniqueness of the solution for non-local $\mathbb{BVP}$ involving the Hadamard fractional derivative of order $1 < \nu \leq 2$, of the form

$$\begin{cases} {}_{\mathrm{H}}D^{\nu} x(t) = f(t, x(t)), & t \in \mathsf{I}_e = (1, e), \\ x(1) = 0, \quad x(e) = bx(\eta), & 1 < \eta < e, \ b > 0, \end{cases} \tag{1.1}$$

in the Hölder spaces having integral of moduli of continuity $J_{\alpha,\beta}$, for $0 < \alpha < 1$, and $\beta > \nu > \alpha$.

To this aim, we will investigate and prove some essential properties of the Hadamard fractional operators, such as the boundedness, acting, and continuity in the space $J_{\alpha,\beta}$.

The three-point boundary condition in problem (1.1) introduces a nonlocal constraint that relates the solution at the terminal point to its value at an intermediate point $\eta \in (1, e)$. Such conditions naturally arise in systems with memory or feedback effects and are consistent with the intrinsic nonlocal character of the Hadamard fractional derivative. From an analytical viewpoint, three-point fractional $\mathbb{BVP}$s are less restrictive than classical two-point problems and often allow the existence of solutions under weaker assumptions on the nonlinearity. Therefore, problem (1.1) provides a more general and flexible framework than standard two-point fractional $\mathbb{BVP}$s.

An essential technique for examining the existence results of differential or integral issues is the fixed point approach (FPT). Due to restricted assumptions on superposition operators F_f on the investigated function spaces $J_{\alpha,\beta}$, we cannot anticipate operators in the examined $\mathbb{BVP}$ (1.1) to be either norm-contractions or compact. This indicates that Banach FPT or Schauder FPT might be difficult to apply. For the analyzed $\mathbb{BVP}$ (1.1), the choice $J_{\alpha,\beta}$ enables us to apply methods associated with Darbo FPT with a measure of noncompactness (MNC) to obtain solutions that are more regular than simply continuous [19]. It should be noted that when we demand not only continuity of solutions [20] but also anticipate more regularity, the Hölder space C^{α} is occasionally utilized as the solution space [21–23].

However, the space C^{α} poses unanticipated problems, as the acting conditions do not guarantee the continuity of operators ([24, Theorems 3.8, 3.9 and 3.13]), and this may also be true for discontinuous functions f [19]. To highlight natural and less restrictive assumptions, we skip these limitations by focusing on the space $J_{\alpha,\beta}$ as the space of solutions. The space $J_{\alpha,\beta}$ is indicated to contain C^{α}-spaces and to be a suitable choice for solving the $\mathbb{BVP}$ of the form

$$\begin{cases} {}_{\mathrm{RL}}D^{\nu}x(t) = f(t, x(t)), \quad t \in (a, b), \\ x(a) = 0, \quad x(b) = B, \quad B \in \mathbb{R}, \end{cases}$$

where $\nu > 0$ [19, 25]. In [26], the authors examined the characteristics of the Banach algebra $J_{\alpha,\beta}$ and used their results to prove the existence results of a quadratic equation of RL-type fractional order,

$$x(t) = \left[\int_1^t \frac{f_1(\theta, x(\theta))}{\Gamma(\nu_1)(t-\theta)^{1-\nu_1}} \, \mathrm{d}\theta \right] \left[g(t) + \int_1^t \frac{f_2(\theta, x(\theta))}{\Gamma(\nu)(t-\theta)^{1-\nu}} \, \mathrm{d}\theta \right],$$

in the Banach algebra $J_{\alpha,\beta}$, where $\beta \geq \max\{\nu_1, \nu\}$ and $0 < \alpha < \nu_i < 1$, $i = 1, 2$. Caballero et al., in [27] proved the existence of results in addition to numerical solutions to the equation

$$x(t) = \int_0^t K(t, \theta) f(\theta, x(\theta)) \, \mathrm{d}\theta,$$

in the space $J_{\alpha,\beta}$, and apply their results to the case of the capillary rise equation, which describes the motion of a liquid in a thin tube

$$x(t) = \int_0^1 \left[1 - e^{-(t-\theta)} \left(1 - \sqrt{2x(\theta)} \right) \right] \mathrm{d}\theta.$$

For more existence results on Hadamard fractional problems in various function spaces, we refer the interested reader to [28–30].

The following concepts are presented in this work:

- Describe and demonstrate some characteristics of the Hadamard-type fractional operator in the space $J_{\alpha,\beta}$.
- Prove the existence of the results together with the unique solution to the nonlocal $\mathbb{BVP}$, which contains contributions at both the domain's boundary and intermediate processes.
- By examining our problem in the space $J_{\alpha,\beta}$, we propose to eliminate the weaknesses of previously proposed spaces and combine their merits.
- We will employ the techniques related to Darbo FPT with an MNC.
- Because we studied our problem in the space $J_{\alpha,\beta}$, our results are more regular than the prior ones, which is a suitable choice for solving $\mathbb{BVP}$ (1.1).
- We conclude with some examples to support our theorems.

2. Preliminaries

Denote by $C = C(\mathrm{I}_e)$ the space of continuous functions on the interval $\mathrm{I}_e = [1, e]$, where $e = 2.718$ is the base of the natural logarithm, equipped with the supremum norm $\|\cdot\|_C$. For $0 < \alpha \le 1$, $\alpha < \beta < \infty$, we represent the integral-form Hölder space using the norm by $J_{\alpha,\beta}(\mathrm{I}_e)$ (cf. [19]),

$$\left\|\|x\|\right\|_{\alpha,\beta} = \|x\|_C + j_{\alpha,\beta}(x, \mathrm{I}_e)^{\alpha/\beta},$$

where

$$j_{\alpha,\beta}(x, \mathrm{I}_e) = \int_{\mathrm{I}_e} \sigma^{-(\beta+1)} \omega(x, \sigma)^{\beta/\alpha}\, \mathrm{d}\sigma,$$

with the modulus of continuity $\omega(x, \sigma)$ of a function $x \in C$, which is described by

$$\omega(x, \sigma) = \sup_{s,t \in \mathrm{I}_e} \left\{ |x(t) - x(s)| \;:\; |t - s| \le \sigma \right\}.$$

Further, for a function of two variables $f(t, x) : \mathrm{I}_e \times \mathbb{R} \to \mathbb{R}$, denoting the modulus of continuity by (cf. [19]),

$$\omega(f, \sigma, \mu) = \sup_{s,t \in \mathrm{I}_e} \left\{ |f(s, u) - f(t, v)| \;:\; |s - t| \le \sigma,\ u, v \in \mathbb{R}, |u - v| \le \mu \right\}.$$

If $\beta \to \infty$, we get the classical Hölder space $C^\alpha(\mathrm{I}_e)$ i.e., $\lim_{\beta\to\infty} J_{\alpha,\beta}(\mathrm{I}_e) = C^\alpha(\mathrm{I}_e)$.

Definition 2.1. [19] For a function $f(t, u) : \mathrm{I}_e \times \mathbb{R} \to \mathbb{R}$, we denote the superposition (Nemytskii) operator F_f by $F_f(u)(t) = f(t, u(t))$ for $t \in \mathrm{I}_e$.

Theorem 2.2. *[19] Let $f : \mathrm{I}_e \times \mathbb{R} \to \mathbb{R}$ verify the following conditions:*
a) *For any $\rho > 0$, there is $c_\rho > 0$ such that*

$$|f(s, u) - f(t, u)| \le c_\rho |s - t|, \qquad \forall s, t \in \mathrm{I}_e,\ u \in [-\rho, \rho];$$

b) *The function $f(t, \cdot)$ is differentiable for each $t \in \mathrm{I}_e$, and there is $B_\rho > 0$ such that*

$$\left|\partial_2 f(\cdot, u) - \partial_2 f(\cdot, v)\right| \le B_\rho \cdot |u - v|, \qquad u, v \in \mathbb{R}.$$

Then, the operator $F_f : J_{\alpha,\beta}(\mathrm{I}_e) \to J_{\alpha,\beta}(\mathrm{I}_e)$ is continuous.

Moreover, we have the following results for the acting conditions for the operator F_f in the space $J_{\alpha,\beta}$.

Theorem 2.3. *[19] Let $\rho > 0$, $b_\rho > 0$, $a_\rho \in L_1(\mathrm{I}_e)$, $0 < \alpha < 1$, and $\beta > \alpha$. Suppose that the function $f : \mathrm{I}_e \times \mathbb{R} \to \mathbb{R}$ satisfies the estimate*

$$\omega(f, \sigma, \omega(u, \sigma))^{\beta/\alpha} \le a_\rho(\sigma)\sigma^{\beta+1} + b_\rho \omega(u, \sigma)^{\beta/\alpha},$$

for $\sigma \ge 0$. Then, the superposition operator F_f generated by f maps the ball $B_\rho(J_{\alpha,\beta})$ into the ball $B_R(J_{\alpha,\beta})$, where

$$R = \max\left\{ |f(t, u)| \;:\; t \in \mathrm{I}_e, |u| \le \rho \right\} + \left(\|a_\rho\|_{L_1} + b_\rho \rho^{\beta/\alpha} \right)^{\alpha/\beta}.$$

Definition 2.4. [31] The Hausdorff MNC $\chi(X)$ of a bounded set $X \in J_{\alpha,\beta}(\mathrm{I}_e)$ is known as

$$\chi(X) = \inf\Big\{\varepsilon > 0 \,:\, X \text{ admits a finite } \varepsilon\text{-net in } X\Big\}.$$

The MNC in the space $J_{\alpha,\beta}$ is given as follows.

Proposition 2.5. *[19] The Hausdorff MNC in the space $J_{\alpha,\beta}$ is defined by*

$$c(X) = \limsup_{s\to 1} \sup_{x\in X} j_{\alpha,\beta}(x, [1, s]),$$

and the following estimation holds true $2^{-\beta/\alpha}\chi(X) \le c(X) \le 2^{\beta/\alpha}\chi(X)$.

Theorem 2.6. *[32] (Darbo-FPT) Let $\emptyset \neq \Omega \subset J_{\alpha,\beta}$ be a convex, bounded, and closed set, and $T : \Omega \to \Omega$ be a continuous mapping, and verify $\chi(T(X)) \le l \cdot \chi(X)$, $0 \le l < 1$ (Contraction condition) for any $\varnothing \neq X \subset \Omega$. Then, the map T has at least one fixed point in Ω.*

3. The Hadamard fractional operators in $J_{\alpha,\beta}$

We will introduce certain ideas and demonstrate several fundamental characteristics, including acting conditions, boundedness, and continuity conditions of Hadamard fractional integral operators in $J_{\alpha,\beta}$.

Definition 3.1. [3] The Hadamard fractional integral ${}_{\mathrm{H}}I^{\nu}$ of a given integrable function x of order $\nu > 0$ is given by

$${}_{\mathrm{H}}I^{\nu} x(t) = \int_1^t \left(\ln \tfrac{t}{\theta}\right)^{\nu-1} \tfrac{x(\theta)}{\theta\Gamma(\nu)} \mathrm{d}\theta, \qquad t > 0,$$

with public Gamma function Γ.

Definition 3.2. [3] The Hadamard derivative ${}_{\mathrm{H}}D^{\nu}$ of fractional order $\nu > 0$ for a function $x : [1, \infty) \to \mathbb{R}$ is given by

$$\left({}_{\mathrm{H}}D^{\nu}\right) x(t) = \left(t\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\ln \tfrac{t}{\theta}\right)^{n-\nu-1} \tfrac{x(\theta)}{\Gamma(n-\nu)\theta}\, \mathrm{d}\theta,$$

where $n - 1 < \nu < n,\ n = [\nu] + 1$, and $[\nu]$ refers to the integer part of the number ν.

Theorem 3.3. *[3] Letting $n - 1 < \nu < n, \nu > 0$, then:*

i) *The Hadamard fractional differential equation ${}_{\mathrm{H}}D^{\nu} x(t) = 0$ is verified if and only if*

$$x(t) = \sum_{i=1}^{n} c_i(\ln t)^{\nu-i}, \qquad c_i \in \mathbb{R}.$$

Particularly, if $1 < \nu < 2$, the relation ${}_{\mathrm{H}}D^{\nu} x(t) = 0$ is verified if and only if

$$x(t) = c_1(\ln t)^{\nu-1} + c_2(\ln t)^{\nu-2}, \qquad c_1, c_2 \in \mathbb{R}.$$

ii) ${}_{\mathrm{H}}D^{\nu}\ {}_{\mathrm{H}}I^{\nu} x(t) = x(t)$ *is verified for every $x(t) \in L^p(\mathrm{I}_e)$.*

iii) *Let $x \in C([1, \infty]) \cap L^p[1, \infty]$. The following formula is verified:*

$${}_{\mathrm{H}}I^{\nu}\ {}_{\mathrm{H}}D^{\nu} x(t) = x(t) - \sum_{i=1}^{n} c_i(\ln t)^{\nu-i}.$$

iv) ${}_{\mathrm{H}}I^{\nu_1}\ {}_{\mathrm{H}}I^{\nu_2} x(t) = {}_{\mathrm{H}}I^{\nu_1+\nu_2} x(t), \qquad \nu_1, \nu_2 > 0.$

First, we show that the operator ${}_{\mathrm{H}}I^{\nu} : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is continuous.

Theorem 3.4. *Let $0 < \alpha < 1$, $1 < \nu < 2$, and $\beta > \nu > \alpha$. Then, the operator ${}_{\mathrm{H}}I^{\nu} : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is continuous with the modulus of continuity*

$$\omega\left({}_{\mathrm{H}}I^{\nu}x, \sigma\right) \leq \frac{\omega(x,\sigma)+\nu e^{\nu-1}\sigma\|x\|_C}{\Gamma(\nu+1)}, \qquad 0 \leq \sigma \leq 1,$$

where $\left\|\left|{}_{\mathrm{H}}I^{\nu}x\right|\right\|_{\alpha,\beta} \leq c_{11} \cdot \left\|\left|x\right|\right\|_{\alpha,\beta}$, where

$$c_{11} = \max\left\{\frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)},\ \frac{1}{\Gamma(\nu+1)} + \left[\frac{\nu e^{\nu-1}2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\right]\left[\frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right\}.$$

Proof. For $s, t, \theta \in \mathrm{I}_e$, set $u = \frac{t}{\theta}$ (resp. $u = \frac{s}{\theta}$), which yields

$$\begin{aligned}
\Gamma(\nu)\left|{}_{\mathrm{H}}I^{\nu}x(t) - {}_{\mathrm{H}}I^{\nu}x(\theta)\right| &\leq \left|\int_1^t \left(\ln\frac{t}{\theta}\right)^{\nu-1} x(\theta)\frac{d\theta}{\theta} - \int_1^s \left(\ln\frac{s}{\theta}\right)^{\nu-1} x(\theta)\frac{d\theta}{\theta}\right| \\
&\leq \left|\int_1^t (\ln u)^{\nu-1} x\left(\frac{t}{u}\right)\frac{du}{u} - \int_1^s (\ln u)^{\nu-1} x\left(\frac{s}{u}\right)\frac{du}{u}\right| \\
&\leq \left|\int_1^t (\ln u)^{\nu-1} x\left(\frac{t}{u}\right)\frac{du}{u} - \int_1^t (\ln u)^{\nu-1} x\left(\frac{s}{u}\right)\frac{du}{u}\right| \\
&\quad + \left|\int_1^t (\ln u)^{\nu-1} x\left(\frac{s}{u}\right)\frac{du}{u} - \int_1^s (\ln u)^{\nu-1} x\left(\frac{s}{u}\right)\frac{du}{u}\right| \\
&\leq \int_1^t (\ln u)^{\nu-1}\left|x\left(\frac{t}{u}\right) - x\left(\frac{s}{u}\right)\right|\frac{du}{u} + \left|\int_s^t (\ln u)^{\nu-1} x\left(\frac{s}{u}\right)\frac{du}{u}\right| \\
&\leq \frac{\omega(x,\sigma)}{\nu}(\ln u)^{\nu}\Big|_1^t + \frac{\|x\|_C}{\nu}(\ln u)^{\nu}\Big|_s^t \\
&\leq \frac{\omega(x,\sigma)}{\nu} + \frac{\|x\|_C}{\nu}\left|t^{\nu} - s^{\nu}\right|.
\end{aligned}$$

Now, for $t, s \in \mathrm{I}_e$, $|t - s| \leq \sigma \leq 1$, then by using the mean value theorem, we get

$$|t^{\nu} - s^{\nu}| \leq \nu\, e^{\nu-1}|t - s|, \quad 1 < \nu < 2.$$

Consequently,

$$\omega\left({}_{\mathrm{H}}I^{\nu}x, \sigma\right) \leq \frac{\omega(x,\sigma)+\nu\, e^{\nu-1}\sigma\|x\|_C}{\Gamma(\nu+1)}.$$

Therefore,

$$\begin{aligned}
j_{\alpha,\beta}\left({}_{\mathrm{H}}I^{\nu}x, \mathrm{I}_e\right)^{\alpha/\beta} &= \left[\int_1^e \sigma^{-(\beta+1)}\omega\left({}_{\mathrm{H}}I^{\nu}x,\sigma\right)^{\beta/\alpha}\, d\sigma\right]^{\alpha/\beta} \\
&= \left[\int_1^e \sigma^{-(\beta+1)}\left(\frac{\omega(x,\sigma)+\nu\, e^{\nu-1}\sigma\|x\|_C}{\Gamma(\nu+1)}\right)^{\beta/\alpha} d\sigma\right]^{\alpha/\beta} \\
&\leq \left[\frac{2^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\int_1^e \sigma^{-(\beta+1)}\left(\omega(x,\sigma)^{\beta/\alpha} + (\nu\, e^{\nu-1}\sigma)^{\beta/\alpha}\,\|x\|_C^{\beta/\alpha}\right) d\sigma\right]^{\alpha/\beta} \\
&\leq \frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[\int_1^e \sigma^{-(\beta+1)}\omega(x,\sigma)^{\beta/\alpha} d\sigma + (\nu\, e^{\nu-1})^{\beta/\alpha}\|x\|_C^{\beta/\alpha}\int_1^e \sigma^{-\left(\beta+1-\frac{\beta}{\alpha}\right)} d\sigma\right]^{\alpha/\beta}
\end{aligned}$$

$$\leq \frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[j_{\alpha,\beta}\left(x,\mathsf{I}_e\right)^{\alpha/\beta}+\nu e^{\nu-1}\left[\frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\|x\|_C\right]. \tag{3.1}$$

Moreover,

$$\left\|{}_{\mathrm{H}}I^{\nu}x(t)\right\|_C\leq\frac{\|x\|_C}{\Gamma(\nu)}\int_1^t\left(\ln\frac{t}{\theta}\right)^{\nu-1}\frac{1}{\theta}\mathrm{d}\theta\leq\frac{\|x\|_C}{\Gamma(\nu+1)}. \tag{3.2}$$

Combining estimations (3.1) and (3.2), we get

$$\begin{aligned}\left\|\left\|{}_{\mathrm{H}}I^{\nu}x\right\|\right\|_{\alpha,\beta}&\leq\left\|{}_{\mathrm{H}}I^{\nu}x\right\|_C+j_{\alpha,\beta}\left({}_{\mathrm{H}}I^{\nu}x,\mathsf{I}_e\right)^{\alpha/\beta}\\&\leq\frac{\|x\|_C}{\Gamma(\nu+1)}+\frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left(j_{\alpha,\beta}\left(x,\mathsf{I}_e\right)^{\alpha/\beta}+\nu\,e^{\nu-1}\left[\frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\|x\|_C\right)\\&=\left[\frac{1}{\Gamma(\nu+1)}+\left[\frac{\nu e^{\nu-1}2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\right]\left[\frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right]\|x\|_C+\frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}j_{\alpha,\beta}\left(x,\mathsf{I}_e\right)^{\alpha/\beta}\\&\leq c_{11}\left\|\|x\|\right\|_{\alpha,\beta},\end{aligned}$$

where

$$c_{11}=\max\left\{\frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)},\ \frac{1}{\Gamma(\nu+1)}+\left[\frac{\nu e^{\nu-1}2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\right]\left[\frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right\},$$

which completes our claim. □

4. Three-point Hadamard-type fractional $\mathbb{BVP}$

Next, we study the "non-local" three-point $\mathbb{BVP}$ of Hadamard-type $\mathbb{FDE}$ (1.1):

Lemma 4.1. *For* $1<\nu\leq 2,\ b\,(\ln\eta)^{\nu-1}\neq 1$, *and* $\zeta\in J_{\alpha,\beta}(\mathsf{I}_e)$, *the* $\mathbb{BVP}$

$$\begin{cases}{}_{\mathrm{H}}D^{\nu}x(t)=\zeta(t), & t\in(1,e),\\ x(1)=0,\ x(e)=bx(\eta), & 1<\eta<e,\end{cases} \tag{4.1}$$

is equivalent to the integral equation

$$\begin{aligned}x(t)=&\int_1^t\left(\ln\frac{t}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\Gamma(\nu)\theta}\,\mathrm{d}\theta+\frac{(\ln t)^{\nu-1}}{\Gamma(\nu)\left(1-b(\ln\eta)^{\nu-1}\right)}\left[b\cdot\int_1^{\eta}\left(\ln\frac{\eta}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\theta}\,\mathrm{d}\theta\right.\\&\left.-\int_1^{e}\left(\ln\frac{e}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\theta}\,\mathrm{d}\theta\right],\qquad t\in\mathsf{I}_e,\ b\,(\ln\eta)^{\nu-1}\neq 1.\end{aligned} \tag{4.2}$$

Proof. The solution of the Hadamard $\mathbb{FDE}$ in (4.1) can be given by (cf. [3])

$$x(t)=\int_1^t\left(\ln\frac{t}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\Gamma(\nu)\theta}\,\mathrm{d}\theta+c_1\,(\ln t)^{\nu-1}+c_2\,(\ln t)^{\nu-2}. \tag{4.3}$$

Using our boundary conditions, we get that $c_2=0$, and

$$c_1=\frac{1}{1-b(\ln\eta)^{\nu-1}}\left[\frac{b}{\Gamma(\nu)}\int_1^{\eta}\left(\ln\frac{\eta}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\theta}\,\mathrm{d}\theta-\int_1^{e}\left(\ln\frac{e}{\theta}\right)^{\nu-1}\frac{\zeta(\theta)}{\Gamma(\nu)s}\,\mathrm{d}\theta\right].$$

Substituting the values of the constants c_1,c_2 in (4.3), we get (4.2). Conversely, by direct calculations, it can be shown that (4.2) verifies the $\mathbb{BVP}$ (4.1). This completes the proof. □

4.1. Existence of solution in $J_{\alpha,\beta}$

Regarding Lemma 4.1, the solution to $\mathbb{BVP}$ (1.1) can be written as

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\nu)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\nu-1} \frac{f(\theta,x(\theta))}{\theta} \,\mathrm{d}\theta \\ & + \frac{(\ln t)^{\nu-1}}{\Gamma(\nu)\big(1-b(\ln\eta)^{\nu-1}\big)}\Big[b \cdot \int_1^\eta \left(\ln \frac{\eta}{\theta}\right)^{\nu-1} \frac{f(\theta,x(\theta))}{\theta} \,\mathrm{d}\theta - \int_1^e \left(\ln \frac{e}{\theta}\right)^{\nu-1} \frac{f(\theta,x(\theta))}{\theta} \,\mathrm{d}\theta\Big], \qquad t \in \mathsf{I}_e. \end{aligned} \tag{4.4}$$

Let us write Eq (4.4) in operator form:

$$x(t) = \mathcal{H}(x)(t) = {}_{\mathrm{H}}I^{\nu} F_f(x)(t) + \frac{(\ln t)^{\nu-1}}{\big(1-b(\ln\eta)^{\nu-1}\big)}\Big(b \cdot {}_{\mathrm{H}}I^{\nu} F_f(x)(\eta) - {}_{\mathrm{H}}I^{\nu} F_f(x)(e)\Big),$$

where F_f is the superposition operator as in Def. 2.1, and ${}_{\mathrm{H}}I^{\nu}$ is the fractional integral of Hadamard type as in Def. 3.1.

Theorem 4.2. *Let* $0 < \alpha < 1$, $1 < \nu < 2$, *and* $\beta > \nu > \alpha$ *in addition to the following set of assumptions:*
i) Let the functions $f : \mathsf{I}_e \times \mathbb{R} \to \mathbb{R}$ *verify for all* $s, t \in \mathsf{I}_e$, $u \in C(\mathsf{I}_e)$, *and* $x, y \in \mathbb{R}$,

1) $\exists \rho > 0, b_\rho > 0$ *and functions* $a_\rho \in L_1(\mathsf{I}_e)$ *such that*

$$\omega(f, \sigma, \omega(u, \sigma))^{\beta/\alpha} \leq a_\rho(\sigma)\sigma^{\beta+1} + b_\rho \omega(u, \sigma)^{\beta/\alpha}, \quad \sigma \geq 0. \tag{4.5}$$

2) For any $\rho > 0$, *there exists* $c_\rho > 0$ *such that* $|f(s, x) - f(t, x)| \leq c_\rho |s - t|$.
3) For any $\rho > 0$, *there exists* $B_\rho > 0$ *such that* $|f(s, x) - f(s, y)| \leq B_\rho |x - y|$.
4) The function $f(t, \cdot)$ *is differentiable, and for any* $\rho > 0$, *there exists a constant* $B_{\rho_1} > 0$ *such that*

$$|\partial_2 f(t, x) - \partial_2 f(t, y)| \leq B_{\rho_1} |x - y|.$$

ii) Denote $N = \|f(t, 0)\|_C$, *and assume that*

$$L := \frac{b_\rho 8^{\frac{2\beta}{\alpha}-1}}{\Gamma^{\frac{\beta}{\alpha}}(\nu + 1)} < 1.$$

Then, $\mathbb{B}$*VP (1.1) has at least one solution* $x \in J_{\alpha\beta}$ *defined on* I_e.

Proof. We present the proof due to Darbo-FPT 2.6.

`Step I.` First, in view of Theorem 3.4, the Hadamard operator ${}_{\mathrm{H}}I^{\nu} : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is continuous. Assumption (i) and Theorem 2.2 indicate that $F_f : J_{\alpha,\beta} \to J_{\alpha,\beta}$, is continuous. Consequently, the operator $\mathcal{H} : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is continuous.

`Step II.` We shall demonstrate that $\mathcal{H}$ is bounded on the ball

$$B_r(J_{\alpha,\beta}) = \{x \in J_{\alpha,\beta} : \ \big\||x|\big\|_{\alpha,\beta} \leq r\},$$

where r is given by

$$r = \frac{1}{1-c_{22}} \left[\frac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)} \left\|a_\rho\right\|_{L_1}^{\alpha/\beta} + c_{22} \cdot N\right],$$

where

$$c_{22} = \max\left\{\frac{4^{1-\frac{\alpha}{\beta}} b_\rho^{\alpha/\beta}}{\Gamma(\nu+1)}, \frac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta) + 4^{1-\frac{\beta}{\alpha}}\left[\frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta} + \delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\right\},$$

with $\delta = \frac{b(\ln\eta)^\nu+1}{M}$, $M = \left(1 - b(\ln\eta)^{\nu-1}\right) \neq 0$, and $N = \|f(t,0)\|_C$. For $x \in J_{\alpha,\beta}$, we have

$$\begin{aligned}
\left\||\mathcal{H}(x)|\right\|_{\alpha,\beta} &= \left\|\left|{}_{\mathrm{H}}I^\nu F_f(x)(t) + \tfrac{b}{M}(\ln t)^{\nu-1}\cdot{}_{\mathrm{H}}I^\nu F_f(x)(\eta) + \tfrac{1}{M}(\ln t)^{\nu-1}\cdot{}_{\mathrm{H}}I^\nu F_f(x)(e)\right|\right\|_{\alpha,\beta}\\
&\leq \left\|\left|{}_{\mathrm{H}}I^\nu F_f(x)\right|\right\|_{\alpha,\beta} + \tfrac{\left|b\cdot{}_{\mathrm{H}}I^\nu F_f(x)(\eta)\right|}{M}\left\|\left|(\ln t)^{\nu-1}\right|\right\|_{\alpha,\beta} + \tfrac{\left|{}_{\mathrm{H}}I^\nu F_f(x)(e)\right|}{M}\left\|\left|(\ln t)^{\nu-1}\right|\right\|_{\alpha,\beta}\\
&\leq \left\|{}_{\mathrm{H}}I^\nu F_f(x)\right\|_C + \left[\tfrac{b\cdot\left|{}_{\mathrm{H}}I^\nu F_f(x)(\eta)\right|}{M} + \tfrac{\left|{}_{\mathrm{H}}I^\nu F_f(x)(e)\right|}{M}\right]\left\|(\ln t)^{\nu-1}\right\|_C\\
&\quad + j_{\alpha,\beta}\left({}_{\mathrm{H}}I^\nu F_f(x), \mathrm{I}_e\right)^{\alpha/\beta} + \left[\tfrac{b\cdot\left|{}_{\mathrm{H}}I^\nu F_f(x)(\eta)\right|}{M} + \tfrac{\left|{}_{\mathrm{H}}I^\nu F_f(x)(e)\right|}{M}\right]\cdot j_{\alpha,\beta}\left((\ln t)^{\nu-1}, \mathrm{I}_e\right)^{\alpha/\beta}.
\end{aligned} \tag{4.6}$$

Note that $\left\|(\ln t)^{\nu-1}\right\|_C = \sup\{(\ln t)^{\nu-1} : t \in \mathrm{I}_e\} \leq 1$. Then, we get the estimation

$$\begin{aligned}
\|\mathcal{H}(x)\|_C &= \left\|{}_{\mathrm{H}}I^\nu F_f(x)\right\|_C + \tfrac{b}{M}\left\|{}_{\mathrm{H}}I^\nu F_f(x)(\eta)\right\|_C + \tfrac{1}{M}\left\|{}_{\mathrm{H}}I^\nu_{t=e} F_f(x)(e)\right\|_C\\
&\leq \tfrac{\|F_f(x)\|_C}{\Gamma(\nu)}\left[\int_1^t \left(\ln\tfrac{t}{\theta}\right)^{\nu-1}\tfrac{d\theta}{\theta} + \tfrac{b}{M}\int_1^\eta \left(\ln\tfrac{\eta}{\theta}\right)^{\nu-1}\tfrac{d\theta}{\theta} + \tfrac{1}{M}\int_1^e \left(\ln\tfrac{e}{\theta}\right)^{\nu-1}\tfrac{d\theta}{\theta}\right]\\
&= \tfrac{\|f(t,x)-f(t,0)+f(t,0)\|_C}{\nu\Gamma(\nu)}\left[-\left(\ln\tfrac{t}{\theta}\right)^\nu\Big|_1^t + \tfrac{-b}{M}\left(\ln\tfrac{\eta}{\theta}\right)^\nu\Big|_1^\eta + \tfrac{-1}{M}\left(\ln\tfrac{e}{\theta}\right)^\nu\Big|_1^e\right]\\
&\leq \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)}\left[1 + \tfrac{b(\ln\eta)^\nu+1}{M}\right] = \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)}\cdot(1+\delta).
\end{aligned} \tag{4.7}$$

Moreover, recalling Theorem 3.4, we obtain

$$\begin{aligned}
j_{\alpha,\beta}\left({}_{\mathrm{H}}I^\nu F_f(x), \mathrm{I}_e\right) &= \int_1^e \sigma^{-(\beta+1)}\omega\left({}_{\mathrm{H}}I^\nu F_f(x), \sigma\right)^{\beta/\alpha}\,d\sigma\\
&\leq 2^{\frac{\beta}{\alpha}-1}\int_1^e \sigma^{-(\beta+1)}\frac{\omega\left(F_f(x),\sigma\right)^{\beta/\alpha} + (\nu e^{\nu-1}\sigma)^{\beta/\alpha}\|F_f(x)\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\nu+1)}\,d\sigma\\
&\leq \frac{2^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\int_1^e \sigma^{-(\beta+1)}\omega\left(F_f(x),\sigma\right)^{\beta/\alpha}\,d\sigma\right.\\
&\quad\left. + \int_1^e \sigma^{-(\beta+1)}(\nu e^{\nu-1}\sigma)^{\beta/\alpha}\|F_f(x)\|_C^{\beta/\alpha}\,d\sigma\right]\\
&\leq \frac{4^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\int_1^e \sigma^{-(\beta+1)}\left(a_\rho(\sigma)\sigma^{\beta+1} + b_\rho\omega(x,\sigma)^{\beta/\alpha}\right)d\sigma\right.\\
&\quad\left. + \frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\left\|f(t,x) - f(t,0) + f(t,0)\right\|_C^{\beta/\alpha}\right]\\
&\leq \frac{4^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\int_1^e a_\rho(\sigma)\,d\sigma + b_\rho\int_1^e \sigma^{-(\beta+1)}\omega(x,\sigma)^{\beta/\alpha}\,d\sigma\right.\\
&\quad\left. + \frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}(B_\rho\|x\|_C + N)^{\beta/\alpha}\right]\\
&\leq \frac{4^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\|a_\rho\|_{L_1} + b_\rho j_{\alpha,\beta}(x,\mathrm{I}_e) + \frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}(B_\rho\|x\|_C + N)^{\beta/\alpha}\right].
\end{aligned} \tag{4.8}$$

Moreover, for $0 < \nu - 1 < 1$ and $|t - s| \le \sigma$, the function $(\ln t)^{\nu-1}$ satisfies

$$\left|(\ln t)^{\nu-1} - (\ln s)^{\nu-1}\right| \le |t - s|^{\nu-1} \le \sigma^{\nu-1}.$$

Consequently,

$$\omega((\ln t)^{\nu-1}, \sigma) \le \sigma^{\nu-1}$$

and

$$j_{\alpha,\beta}\big((\ln t)^{\nu-1},\ \mathsf{I}_e\big)^{\alpha/\beta} = \left[\int_1^e \sigma^{-(\beta+1)} \cdot \sigma^{\beta(\nu-1)/\alpha}\, \mathrm{d}\sigma\right]^{\alpha/\beta} = \left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}. \tag{4.9}$$

By substituting (4.7)–(4.9) into (4.6), we have

$$\begin{aligned}
\left\|\left|\mathcal{H}(x)\right|\right\|_{\alpha,\beta} &\le \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)} \cdot (1+\delta) + \tfrac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[\|a_\rho\|_{L_1} + b_\rho j_{\alpha,\beta}(x,\ \mathsf{I}_e) + \frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\Big[B_\rho\|x\|_C + N\Big]^{\beta/\alpha}\right]^{\alpha/\beta} \\
&\quad + \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)} \cdot \delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta} \\
&= \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)} \cdot (1+\delta) + \tfrac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[\|a_\rho\|_{L_1}^{\frac{\alpha}{\beta}} + b_\rho^{\alpha/\beta} j_{\alpha,\beta}(x,\ \mathsf{I}_e)^{\alpha/\beta} + \left[\frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\Big[B_\rho\|x\|_C + N\Big]\right] \\
&\quad + \tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)}\delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta} \\
&= \tfrac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\|a_\rho\|_{L_1}^{\alpha/\beta} + \tfrac{1}{\Gamma(\nu+1)}\left[(1+\delta) + 4^{1-\frac{\beta}{\alpha}}\left[\frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta} + \delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right] \cdot N \\
&\quad + \tfrac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta) + 4^{1-\frac{\beta}{\alpha}}\left[\frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta} + \delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\|x\|_C \\
&\quad + \tfrac{4^{1-\frac{\alpha}{\beta}} b_\rho^{\alpha/\beta}}{\Gamma(\nu+1)} j_{\alpha,\beta}(x,\ \mathsf{I}_e)^{\alpha/\beta} \\
&\le \tfrac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\|a_\rho\|_{L_1}^{\alpha/\beta} + c_{22} \cdot N + c_{22}\left\|\left|x\right|\right\|_{\alpha,\beta},
\end{aligned}$$

where

$$c_{22} = \max\left\{\frac{4^{1-\frac{\alpha}{\beta}} b_\rho^{\alpha/\beta}}{\Gamma(\nu+1)},\ \frac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta) + 4^{1-\frac{\beta}{\alpha}}\left[\frac{\alpha(\nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta} + \delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\right\}.$$

Consequently, $\mathcal{H} : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is continuous. Now, for $x \in B_r(J_{\alpha,\beta})$, we obtain

$$\left\|\left|\mathcal{H}(x)\right|\right\|_{\alpha,\beta} \le \tfrac{4^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\|a_\rho\|_{L_1}^{\alpha/\beta} + c_{22} \cdot N + c_{22} \cdot r \le r.$$

Then, $\mathcal{H} : B_r(J_{\alpha,\beta}) \to B_r(J_{\alpha,\beta})$ is continuous.

`Step III.` It is obvious that the ball $B_r(J_{\alpha,\beta}) \neq \emptyset$ is bounded and closed; in addition, it is convex.

`Step IV.` We prove that $\mathcal{H}$ verifies the contraction condition. Assume that $\varnothing \neq X \subset B_r(J_{\alpha,\beta})$ and for $x \in X$, $1 < s < e$, we have

$$\begin{aligned}
j_{\alpha,\beta}(\mathcal{H}(x), [1, s]) &= \int_1^s \sigma^{-(\beta+1)}\omega(\mathcal{H}(x), \sigma)^{\beta/\alpha}\, \mathrm{d}\sigma \\
&\le 2^{\frac{\beta}{\alpha}-1}\Big[\int_1^s \sigma^{-(\beta+1)}\omega\big({}_{\mathrm{H}}I^\nu F_f(x), \sigma\big)^{\beta/\alpha}\, \mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&+\left[\tfrac{B_\rho\|x\|_C+N}{\Gamma(\nu+1)}\cdot\delta\right]^{\beta/\alpha}\int_1^s \sigma^{-(\beta+1)}\omega((\ln t)^{\nu-1},\sigma)^{\beta/\alpha}\,\mathrm{d}\sigma\Big]\\
\le{}& \tfrac{4^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\Big[\int_1^s \sigma^{-(\beta+1)}\omega(F_f(x),\sigma)^{\beta/\alpha}\,\mathrm{d}\sigma+\int_1^s \sigma^{-(\beta+1)}(\nu e^{\nu-1}\sigma)^{\beta/\alpha}\|F_f(x)\|_C^{\beta/\alpha}\,\mathrm{d}\sigma\\
&+\left[(B_\rho\|x\|_C+N)\cdot\delta\right]^{\beta/\alpha}\int_1^s \sigma^{-(\beta+1)}\sigma^{\beta(\nu-1)/\alpha}\,\mathrm{d}\sigma\Big]\\
\le{}& \tfrac{8^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\Big[\int_1^s a_\rho(\sigma)\,\mathrm{d}\sigma+b_\rho\int_1^s \sigma^{-(\beta+1)}\omega(x,\sigma)^{\beta/\alpha}\,\mathrm{d}\sigma\\
&+\|F_f(x)\|_C^{\beta/\alpha}(\nu e^{\nu-1})^{\beta/\alpha}\int_1^s \sigma^{-(\beta+1)+\frac{\beta}{\alpha}}\,\mathrm{d}\sigma\\
&+\left[(B_\rho\|x\|_C+N)\cdot\delta\right]^{\beta/\alpha}\int_1^s \sigma^{-(\beta+1)+\beta(\nu-1)/\alpha}\,\mathrm{d}\sigma\Big].
\end{aligned}
$$

Taking the limit as $s\to 1$, we get

$$c(\mathcal{H}(X))\le \tfrac{b_\rho 8^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}c(X).$$

Employing Proposition 2.5, we obtain

$$\chi(\mathcal{H}(X))\le 2^{\frac{\beta}{\alpha}}c(\mathcal{H}(X))\le 2^{\frac{\beta}{\alpha}}\tfrac{b_\rho 8^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}c(X)\le 2^{\frac{\beta}{\alpha}}\tfrac{b_\rho 8^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\cdot 2^{\frac{\beta}{\alpha}}\chi(X).$$

Therefore,

$$\chi(\mathcal{H}(X))\le \tfrac{b_\rho 8^{\frac{2\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\chi(X). \tag{4.10}$$

We conclude that the operator $\mathcal{H}$ verifies all requirements of Theorem 2.6 with $L<1$, which completes the proof. □

4.2. Uniqueness of the solution

Now we show that $\mathbb{BVP}$ (1.1) has exactly one solution.

Theorem 4.3. *Let assumptions of Theorem 4.2 be verified by replacing inequality* (4.5) *by*

$$\omega(F_f(u)-F_f(v),\,\sigma)\le b_\rho^*\cdot\omega\,(u-v,\,\sigma)\,,\ \forall\, u,v\in C(\mathrm{I}_e),\ \sigma\ge 0 \tag{4.11}$$

and assume that

$$c_{33}=\max\left\{\tfrac{2^{1-\frac{\alpha}{\beta}}b_\rho^*}{\Gamma(\nu+1)},\ \tfrac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta)+\left[\tfrac{\alpha\left(\nu e^{\nu-1}\right)^{\frac{\beta}{\alpha}}2^{\frac{\beta}{\alpha}-1}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}+\delta\left[\tfrac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\right\}<1,$$

where δ is defined in Theorem 4.2. Then, problem (1.1) *has a unique solution $x\in B_r(J_{\alpha,\beta})$.*

Proof. By employing inequality (4.11), we have

$$\omega(F_f(u)-F_f(0),\,\sigma)\le b_\rho^*\cdot\omega\,(u,\,\sigma)\,. \tag{4.12}$$

Moreover,

$$\begin{aligned}\omega(F_f(u)-F_f(0),\,\sigma) &= \sup_{s,t\in \mathsf{I}_e}\left\{\left|f(s,u)-f(t,v)-f(s,0)+f(t,0)\right| \,:\, |s-t|\le\sigma, u,v\in\mathbb{R}, |u-v|\le\mu\right\}\\ &\ge \sup_{s,t\in \mathsf{I}_e}\left\{\left|f(s,u)-f(t,v)\right| \,:\, |s-t|\le\sigma,\, u,v\in\mathbb{R},\, |u-v|\le\mu\right\}\\ &\quad -\sup_{s,t\in \mathsf{I}_e}\left\{\left|f(s,0)-f(t,0)\right| \,:\, |s-t|\le\sigma\right\}\\ &\ge \omega\left(F_f(u),\,\sigma\right)-c_\rho\cdot\sup_{s,t\in \mathsf{I}_e}\left\{|s-t| \,:\, |s-t|\le\sigma\right\}\\ &= \omega\left(F_f(u),\,\sigma\right)-c_\rho\cdot\sigma. \end{aligned}\tag{4.13}$$

Combining (4.12) and (4.13), we get

$$\omega\left(F_f(u),\,\sigma\right)-c_\rho\cdot\sigma\le\omega(F_f(u)-F_f(0),\,\sigma)\le b_\rho^*\cdot\omega\left(u,\,\sigma\right),$$

and so $\omega(F_f(u),\,\sigma)\le c_\rho\cdot\sigma+b_\rho^*\cdot\omega(u,\sigma)$. Thus,

$$\omega(F_f(u),\,\sigma)^{\beta/\alpha}\le 2^{\frac{\beta}{\alpha}-1}\left(c_\rho^{\beta/\alpha}\cdot\sigma^{\beta/\alpha}+\left(b_\rho^*\right)^{\beta/\alpha}\cdot\omega(u,\sigma)^{\beta/\alpha}\right),$$

which implies that inequality (4.5) is verified with $a_\rho=2^{\beta/\alpha-1}c_\rho^{\beta/\alpha}\cdot\sigma^{\beta/\alpha-\beta-1}$ and $b_\rho=2^{\beta/\alpha-1}(b_\rho^*)^{\beta/\alpha}$. Then, Theorem 4.2 implies that $\mathbb{BVP}$ (1.1) has at least one solution $x\in B_r(J_{\alpha,\beta})$.

Now, letting x and z be any two different solutions of Eq (4.4), we get

$$\begin{aligned}\left\|\left|\mathcal{H}(x)-\mathcal{H}(z)\right|\right\|_{\alpha,\beta} &= \left\|\mathcal{H}(x)-\mathcal{H}(z)\right\|_C+j_{\alpha,\beta}\left(\mathcal{H}(x)-\mathcal{H}(z),\,\mathsf{I}_e\right)^{\alpha/\beta}\\ &\le \left\|\mathcal{H}(x)-\mathcal{H}(z)\right\|_C+j_{\alpha,\beta}\left({}_{\mathrm{H}}I^\nu\left(F_f(x)-F_f(z)\right),\,\mathsf{I}_e\right)^{\alpha/\beta}\\ &\quad+\frac{b\cdot\left|{}_{\mathrm{H}}I^\nu\left(F_f(x)-F_f(z)\right)(\eta)\right|+\left|{}_{\mathrm{H}}I^\nu\left(F_f(x)-F_f(z)\right)(e)\right|}{M}j_{\alpha,\beta}((\ln t)^{\nu-1},\,\mathsf{I}_e)^{\alpha/\beta}. \end{aligned}\tag{4.14}$$

Therefore,

$$\begin{aligned}\left\|\mathcal{H}(x)-\mathcal{H}(z)\right\|_C &\le \frac{\left\|F_f(x)-F_f(z)\right\|_C}{\Gamma(\nu)}\left[\int_1^t\left(\ln\frac{t}{\theta}\right)^{\nu-1}\frac{\mathrm{d}\theta}{\theta}+\frac{b}{M}\int_1^\eta\left(\ln\frac{\eta}{\theta}\right)^{\nu-1}\frac{\mathrm{d}\theta}{\theta}+\frac{1}{M}\int_1^e\left(\ln\frac{e}{\theta}\right)^{\nu-1}\frac{\mathrm{d}\theta}{\theta}\right]\\ &\le \frac{\left\|f(t,x)-f(t,z)\right\|_C}{\nu\Gamma(\nu)}\left[-\left(\ln\frac{t}{\theta}\right)^\nu\Big|_1^t+\frac{-b}{M}\left(\ln\frac{\eta}{\theta}\right)^\nu\Big|_1^\eta+\frac{-1}{M}\left(\ln\frac{e}{\theta}\right)^\nu\Big|_1^e\right]\\ &\le \frac{B_\rho\|x-z\|_C}{\Gamma(\nu+1)}\left[1+\frac{b(\ln\eta)^\nu+1}{M}\right]=\frac{B_\rho\|x-z\|_C}{\Gamma(\nu+1)}\cdot(1+\delta). \end{aligned}\tag{4.15}$$

Using assumption (iii), we have the following estimations:

$$\begin{aligned} j_{\alpha,\beta}\left({}_{\mathrm{H}}I^\nu(F_f(x)-F_f(z)),\,\mathsf{I}_e\right) &= \int_1^e\sigma^{-(\beta+1)}\omega\left({}_{\mathrm{H}}I^\nu(F_f(x)-F_f(z)),\,\sigma\right)^{\beta/\alpha}\,\mathrm{d}\sigma\\ \le \frac{2^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}&\int_1^e\sigma^{-(\beta+1)}\left[\omega(F_f(x)-F_f(z),\,\sigma)^{\beta/\alpha}+(\nu e^{\nu-1}\sigma)^{\beta/\alpha}\left\|F_f(x)-F_f(z)\right\|_C^{\beta/\alpha}\,\mathrm{d}\sigma\right]\\ \le \frac{2^{\frac{\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)}&\int_1^e\sigma^{-(\beta+1)}\left[\left(b_\rho^*\right)^{\beta/\alpha}\omega\left(x-z,\,\sigma\right)^{\beta/\alpha}+(\nu e^{\nu-1}\sigma)^{\beta/\alpha}\left\|B_\rho|x-z|\right\|_C^{\beta/\alpha}\right]\mathrm{d}\sigma \end{aligned}$$

$$\leq \frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\left(b_\rho^*\right)^{\beta/\alpha}\int_1^e \sigma^{-(\beta+1)}\omega\left(x-z,\sigma\right)^{\beta/\alpha}\,\mathrm{d}\sigma + (B_\rho \nu e^{\nu-1})^{\beta/\alpha}\|x-z\|_C^{\beta/\alpha}\int_1^e \sigma^{-(\beta+1)}\sigma^{\beta/\alpha}\,\mathrm{d}\sigma\right]$$
$$\leq \frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[b_\rho^{*\beta/\alpha}\int_1^e \sigma^{-(\beta+1)}\omega\left(x-z,\sigma\right)^{\beta/\alpha}\,\mathrm{d}\sigma + \frac{\alpha(B_\rho \nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\|x-z\|_C^{\beta/\alpha}\right]$$
$$\leq \frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\nu+1)}\left[\left(b_\rho^*\right)^{\beta/\alpha} j_{\alpha,\beta}\left(x-z,\,\mathsf{I}_e\right) + \frac{\alpha(B_\rho \nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\|x-z\|_C^{\beta/\alpha}\right]. \tag{4.16}$$

Therefore, by using (4.9) and (4.16), we have

$$j_{\alpha,\beta}(\mathcal{H}(x)-\mathcal{H}(z),\,\mathsf{I}_e)^{\alpha/\beta} \leq \frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[b_\rho^*\cdot j_{\alpha,\beta}\left(x-z,\,\mathsf{I}_e\right)^{\alpha/\beta} + \left[\frac{\alpha(B_\rho \nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\|x-z\|_C\right]$$
$$+\frac{B_\rho\cdot\delta}{\Gamma(\nu+1)}\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\|x-z\|_C. \tag{4.17}$$

By substituting (4.15) and (4.17) into (4.14), we have

$$\left\|\left|\mathcal{H}(x)-\mathcal{H}(z)\right|\right\|_{\alpha,\beta} \leq \frac{B_\rho\|x-z\|_C}{\Gamma(\nu+1)}(1+\delta) + \frac{2^{1-\frac{\alpha}{\beta}}}{\Gamma(\nu+1)}\left[b_\rho^* j_{\alpha,\beta}\left(x-z,\,\mathsf{I}_e\right)^{\alpha/\beta} + \left[\frac{\alpha(B_\rho \nu e^{\nu-1})^{\beta/\alpha}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}\|x-z\|_C\right]$$
$$+\frac{B_\rho\cdot\delta}{\Gamma(\nu+1)}\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\|x-z\|_C$$
$$=\frac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta)+\left[\frac{\alpha\left(\nu e^{\nu-1}\right)^{\frac{\beta}{\alpha}}2^{\frac{\beta}{\alpha}-1}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}+\delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\|x-z\|_C.$$
$$+\frac{2^{1-\frac{\alpha}{\beta}}b_\rho^*}{\Gamma(\nu+1)}j_{\alpha,\beta}\left(x-z,\,\mathsf{I}_e\right)^{\alpha/\beta} \leq c_{33}\left\|\left|x-z\right|\right\|_{\alpha,\beta},$$

where

$$c_{33}=\max\left\{\frac{2^{1-\frac{\alpha}{\beta}}b_\rho^*}{\Gamma(\nu+1)},\,\frac{B_\rho}{\Gamma(\nu+1)}\left[(1+\delta)+\left[\frac{\alpha\left(\nu e^{\nu-1}\right)^{\frac{\beta}{\alpha}}2^{\frac{\beta}{\alpha}-1}\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{\beta(1-\alpha)}\right]^{\alpha/\beta}+\delta\left[\frac{\alpha\left(e^{\beta(\nu-1-\alpha)/\alpha}-1\right)}{\beta(\nu-1-\alpha)}\right]^{\alpha/\beta}\right]\right\}<1.$$

This wraps up the proof. □

5. Application

Now, we conclude with an example that verifies our set of assumptions.

Example 5.1. For $t \in \mathsf{I}_e = [1, e]$, consider the non-local $\mathbb{FDE}$ of the form

$$\begin{cases} {}_{\mathrm{H}}D^{\nu}x(t) = \sin\left(\frac{t+x(t)}{100}\right), \\ x(1) = 0, \quad x(e) = \frac{3}{2}x(2), \end{cases} \tag{5.1}$$

in Hölder space with the module of continuity $J_{\alpha,\beta}$, for any $\beta > \nu > \alpha$, $0 < \alpha < 1$ in two cases as follows:

$$\texttt{Case 1:}\quad \alpha = 0.60 \xrightarrow{0.05} 0.95,\ \beta \in \left\{\frac{23}{20}, \frac{5}{4}, \frac{13}{10}\right\},\ \nu = \frac{11}{10},$$

$$\texttt{Case 2}: \quad \alpha = 0.60 \xrightarrow{0.05} 0.95, \beta = \frac{6}{5}, \nu \in \left\{\frac{21}{20}, \frac{5}{4}, \frac{23}{20}\right\}.$$

We can note that $f(t,x) = \sin\left(\frac{t+x}{100}\right)$ with $c_\rho = B_\rho = \frac{1}{100}$ and $B_{\rho_1} = \frac{1}{100}$, where

$$\begin{aligned}
\left|f(t,x) - f(s,x)\right| &= \left|\sin\left(\tfrac{t+x}{100}\right) - \sin\left(\tfrac{s+x}{100}\right)\right| \leq \left|\tfrac{t+x}{100} - \tfrac{s+x}{100}\right| = \tfrac{1}{100}|t-s|,\\
\left|f(t,x) - f(t,y)\right| &= \left|\sin\left(\tfrac{t+x}{100}\right) - \sin\left(\tfrac{t+y}{100}\right)\right| \leq \left|\tfrac{t+x}{100} - \tfrac{t+y}{100}\right| = \tfrac{1}{100}|x-y|,\\
\left|\partial_2 f(t,x) - \partial_2 f(t,y)\right| &\leq \tfrac{1}{10^4}|x-y|,
\end{aligned}$$

and

$$\begin{aligned}
\omega(f,\sigma,\omega(x,\sigma))^{\beta/\alpha} &= \left|\sin\left(\tfrac{t+x(t)}{100}\right) - \sin\left(\tfrac{s+x(\theta)}{100}\right)\right|^{\beta/\alpha}\\
&\leq \left|\tfrac{t+x(t)}{100} - \tfrac{s+x(\theta)}{100}\right|^{\beta/\alpha}\\
&\leq \tfrac{100^{\frac{\beta}{\alpha}-1}}{100^{\beta/\alpha}}|t-s|^{\beta/\alpha} + \tfrac{100^{\frac{\beta}{\alpha}-1}}{100^{\beta/\alpha}}|x(t)-x(\theta)|^{\beta/\alpha}\\
&\leq \tfrac{1}{100}\sigma^{\beta/\alpha} + \tfrac{1}{100}\omega(x,\sigma)^{\beta/\alpha},
\end{aligned}$$

then $a_\rho(\sigma) = \frac{1}{100}\sigma^{\frac{\beta}{\alpha}-\beta-1}$, $b_\rho = \frac{1}{100}$.

`Case 1:` By choosing the values of parameters in this case, we have

$$\|a_\rho\|_{L_1(\mathrm{I}_e)} = \frac{\alpha\left(e^{\beta(1-\alpha)/\alpha}-1\right)}{2\beta(1-\alpha)} \approx \begin{cases} 0.5154, & \beta = 23/20,\\ 0.5168, & \beta = 5/4,\\ 0.5175, & \beta = 13/10,\end{cases}$$

and

$$L = \frac{b_\rho 8^{\frac{2\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)} \approx \left\{\begin{array}{ll} 0.1817, & \beta = 23/20,\\ 0.2802, & \beta = 5/4,\\ 0.3480, & \beta = 13/10,\end{array}\right\} < 1.$$

These numerical estimations are shown in Table 1, and are plotted in Figures 1 and 2.

Table 1. The estimated results for non-local $\mathbb{FDE}$ (5.1) with $\alpha = 0.60 \xrightarrow{0.05} 0.95$ and three values of β in Example 5.1 for `Case 1`.

α	$\beta = 23/20$		$\beta = 5/4$		$\beta = 13/10$	
	$\|a_\rho\|_{L_1(\mathrm{I}_e)}$	$L<1$	$\|a_\rho\|_{L_1(\mathrm{I}_e)}$	$L<1$	$\|a_\rho\|_{L_1(\mathrm{I}_e)}$	$L<1$
0.50	0.9383	16.0597	0.9961	36.5617	1.0267	55.1660
0.55	0.8302	6.7953	0.8706	14.3554	0.8917	20.8651
0.60	0.7517	3.3184	0.7806	6.5868	0.7956	9.2799
0.65	0.6924	1.8094	0.7133	3.4071	0.7241	4.6752
0.70	0.6462	1.0759	0.6614	1.9364	0.6692	2.5977
0.75	0.6093	0.6857	0.6203	1.1866	0.6258	1.5611
0.80	0.5793	0.4623	0.5869	0.7731	0.5908	0.9997
0.85	0.5543	0.3265	0.5594	0.5297	0.5620	0.6747
0.90	0.5333	0.2397	0.5364	0.3785	0.5379	0.4757
0.95	0.5154	0.1817	0.5168	0.2802	0.5175	0.3480

Thus, all assumptions of Theorem 4.2 are verified, and then (5.1) has at least one solution $x \in J_{\alpha,\beta}$ in `Case 1`.

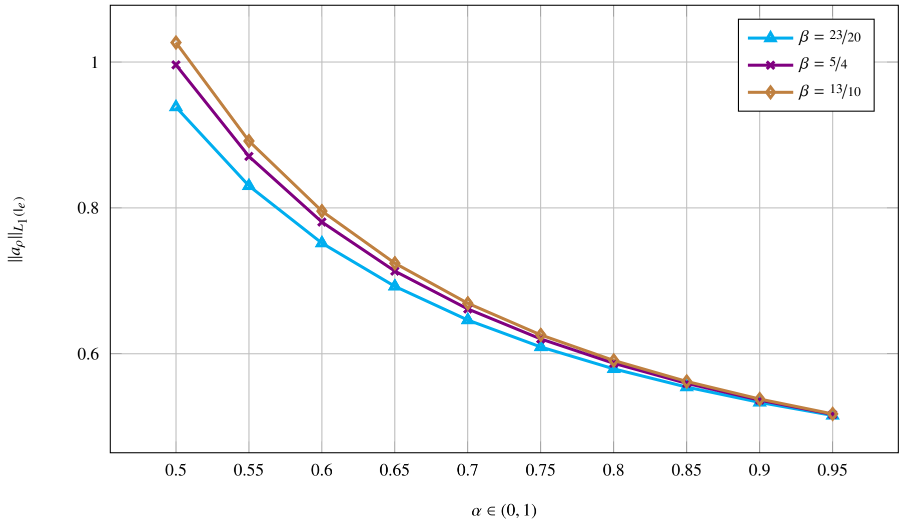

Figure 1. Representation of $\|a_\rho\|_{L_1(I_e)}$ for non-local $\mathbb{FDE}$ (5.1) with $\alpha = 0.60 \xrightarrow{0.05} 0.95$ and three values of β in Example 5.1 for `Case 1`.

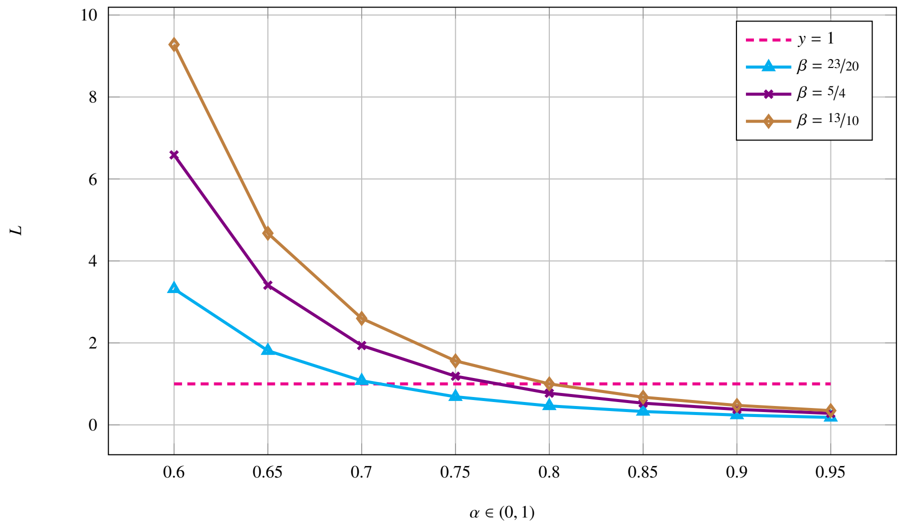

Figure 2. Representation of L for non-local $\mathbb{FDE}$ (5.1) with $\alpha = 0.60 \xrightarrow{0.05} 0.95$ and three values of β in Example 5.1 for `Case 1`.

`Case 2:` By choosing the values of parameters in this case, we have $\|a_\rho\|_{L_1(I_e)} \approx 0.5161$ and

$$L = \frac{b_\rho 8^{\frac{2\beta}{\alpha}-1}}{\Gamma^{\beta/\alpha}(\nu+1)} \approx \left\{ \begin{array}{ll} 0.2325, & \nu = 4/3, \\ 0.2257, & \nu = 3/2, \\ 0.2187, & \nu = 5/3, \end{array} \right\} < 1.$$

These numerical estimations are shown in Table 2 and are plotted in Figure 3.

Table 2. The estimated results for non-local $\mathbb{FDE}$ (5.1) with $\alpha = 0.60 \xrightarrow{0.05} 0.95$ and three derivative orders ν in Example 5.1 for `Case 2`.

α	$\nu = {}^{21}/_{20}$		$\nu = {}^{11}/_{10}$		$\nu = {}^{23}/_{20}$	
	$\|a_\rho\|_{L_1(I_e)}$	$L < 1$	$\|a_\rho\|_{L_1(I_e)}$	$L < 1$	$\|a_\rho\|_{L_1(I_e)}$	$L < 1$
0.50	0.9667	25.6376	0.9667	24.2316	0.9667	22.8195
0.55	0.8501	10.3963	0.8501	9.8767	0.8501	9.3520
0.60	0.7660	4.9002	0.7660	4.6752	0.7660	4.4471
0.65	0.7028	2.5930	0.7028	2.4829	0.7028	2.3709
0.70	0.6538	1.5027	0.6538	1.4434	0.6538	1.3828
0.75	0.6148	0.9366	0.6148	0.9020	0.6148	0.8666
0.80	0.5831	0.6193	0.5831	0.5978	0.5831	0.5758
0.85	0.5569	0.4299	0.5569	0.4159	0.5569	0.4014
0.90	0.5349	0.3108	0.5349	0.3012	0.5349	0.2913
0.95	0.5161	0.2325	0.5161	0.2257	0.5161	0.2187

Thus, all assumptions of Theorem 4.2 are verified, and then (5.1) has at least one solution $x \in J_{\alpha,\beta}$ in `Case 2`.

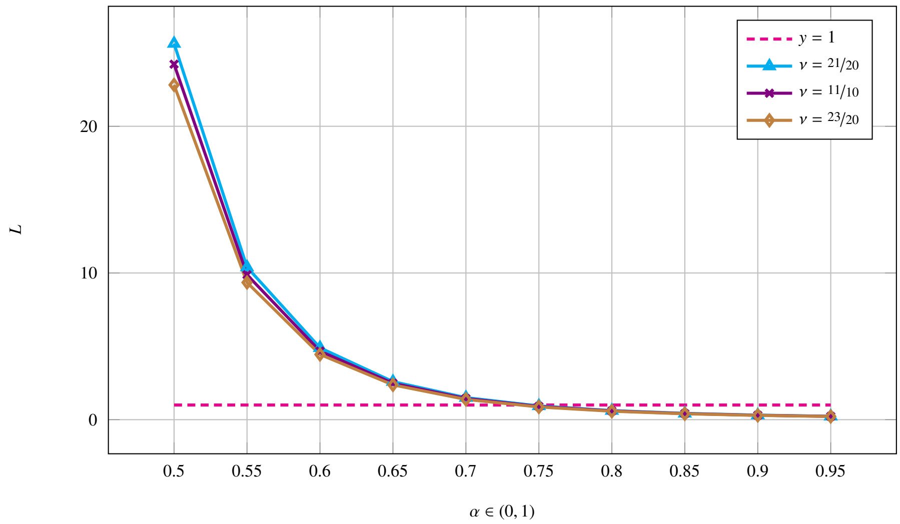

Figure 3. Representation of L for non-local $\mathbb{FDE}$ (5.1) with $\alpha = 0.60 \xrightarrow{0.05} 0.95$ and three values of derivative orders ν in Example 5.1 for `Case 2`.

6. Conclusions

In this article, some fundamental properties of Hadamard fractional operators are examined, including their boundedness, acting, and continuity in integral-form Hölder Banach space $J_{\alpha,\beta}$. It is better to consider the space $J_{\alpha,\beta}$ as a space of solutions to differential problems because it has much better properties than the space of continuous solutions C and the classical Hölder spaces C^α. The existence and uniqueness of the solutions to the non-local $\mathbb{FBVP}$ containing a Hadamard-type fractional operator $J_{\alpha,\beta}$ are established by utilizing such characteristics in conjunction with Darbo's FPT and

MNC techniques. The results are confirmed with a numerical example. Interested readers can solve a variety of problems and add some numerical results to the examined space, as well as extend and apply these conclusions to various fractional operators in the space $J_{\alpha,\beta}$.

Author contributions

MMAM: Methodology, validation, formal analysis, actualization, investigation, initial draft, and was a major contributor in writing the manuscript. **JA**: Methodology, validation, formal analysis, review and edit. **FMA**: Methodology, validation, formal analysis, review and edit **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft, and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the preparation of this article.

Availability of data and materials

The datasets used and/or analyzed during the current study available from the corresponding author on reasonable request.

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Conflict of interest

The authors declare that they have no competing interests.

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