Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6989–7014.
DOI: 10.3934/math.2026287
Received: 08 January 2026
Revised: 24 February 2026
Accepted: 04 March 2026
Published: 18 March 2026

Research article

Hopf bifurcation of predator-prey model with age structure

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Abstract: In this paper, a predator-prey model with a constant harvesting rate, fear effect, Holling Type II Function, and age structure is studied. Using algebraic methods, we derive all critical values for the two time delays at which the characteristic equation admits purely imaginary roots. This yields an explicit stability region in the parameter plane which corresponds to the positive equilibrium. By employing the integrated semigroup theory and the Hopf bifurcation theorem for abstract Cauchy problems with non-dense domains, we establish that the Hopf bifurcation occurs when the time delays cross these critical values. Notably, stability switches can also be observed as the delays vary. Finally, numerical simulations are performed to verify our analytical results.

Keywords: predator-prey model; Hopf bifurcation; age structure; delay; stability switches
Mathematics Subject Classification: 34K18, 92D25

1. Introduction

Predator-prey models [1–3] constitute a core research topic in population dynamics, thereby serving as a fundamental framework to reveal the evolutionary laws of ecological systems and the dynamic balance between different species. In the study of predator-prey models, the introduction of realistic ecological factors is crucial to improve the accuracy and practicality of the model analysis, among which the age structure is a key factor that cannot be ignored—predators at different age stages exhibit significant differences in foraging, reproduction, and other biological behaviors, and ignoring the age structure will lead to a large deviation between model results and actual ecological phenomena. For this reason, predator-prey models with an age structure [4–6] have attracted extensive attention from scholars, and a wealth of research results have been accumulated in the exploration of the model stability, bifurcation, and other dynamic properties.

Cushing [7] characterized the reproductive capacity of predators by incorporating a maturity function into the predator model, and further studied the predator-prey model with an age structure.

Additionally, he investigated the existence and stability of the positive equilibrium point of an age-structured model in [8]. Over the past 20 years, with the development of the integral semigroup theory, transforming age structured models into an abstract non-densely defined Cauchy problem and analyzing the dynamic properties of abstract non steady Cauchy problems has become one of the main methods to study age structured models. Thieme [9] studied the semilinear Cauchy problem with operator perturbations and extended the concept proposed in the theory of weakly continuous semigroups on dual Banach spaces to "integral semigroups" using the theory of integral semigroups [10, 11]. He established the relationship between homogeneous and non-homogeneous abstract Cauchy problems for integral decomposition, which allows the age structure model to be rewritten as an abstract non dense Cauchy problem; for more information, see [9, 12, 13]. Magal and Ruan [14] obtained some results on the integral semigroup theory when the generator is not a Hille Yosida operator and not dense. They obtained sufficient and necessary conditions for the existence of solutions to non-densely non-homogeneous Cauchy problems and applied the results to the study of age structure models. Magal and Ruan [15] established a center manifold theory for semilinear Cauchy problems with non steady states, and studied the existence and smoothness of the center manifold for semilinear non steady Cauchy problems when dealing with infinite dimensional systems; Liu et al. [16] established the Hopf bifurcation theorem for abstract Cauchy problems, and proved this theorem using the central manifold theory for non viscous Cauchy problems related to the integral semigroup theory. This theorem is mainly used to obtain Hopf bifurcation results for functional differential equations and general Hopf bifurcation theorems for age structured systems. Liu et al. [17] studied the normal form of age structured models, provided detailed calculations for the Taylor expansion of reduced systems on the central manifold, and derived explicit formulas to determine the direction of a Hopf bifurcation and the stability and amplitude of bifurcation periodic solutions. Magal and Ruan [18] elaborated on the theory and application of abstract semilinear Cauchy problems, including the central manifold theorem, the Hopf bifurcation theorem, the computation of normanl forms, and discussions on age structure models. These results have been successfully applied to many age-structured or size-structured models [19–21] and the references therein. For example, Qiao et al. [22] studied a double-time-delay Holling II predator model with weak Allee effect and age structure, and obtained the global stability of boundary equilibrium points, the local stability of positive equilibrium points, and the conditions of the Hopf bifurcation for the system. Wu et al. [23] studied Hopf bifurcation in an age-structured predator–prey system with Beddington–DeAngelis functional response. They have obtained the direction of Hopf bifurcation and the stability of bifurcating periodic solutions and constant harvesting. Cao et al. [24] studied the global stability of boundary equilibrium of an an age-structured predator–prey model incorporating time delay.

On the basis of the age structure, the Holling Type II functional response, fear effect, and constant harvesting rate are important ecological factors that affect the interaction between predators and prey. The Holling Type II functional response can accurately describe the change law of predator predation rate with prey density when there is a handling time for predation. When investigating predator-prey models that incorporate the Holling II functional response, particularly those that integrate complex factors such as age structure, time delays, and fear effects, the nonlinear characteristics inherent in this response function present fundamental challenges across multiple dimensions, including equation resolution, dynamical analyses, as well as qualitative and quantitative deductions. Moreover, when coupled with the age structure, dual time delays, and other elements, these challenges are further exacerbated, thus constituting the central difficulty in the study of such models. The fear effect reflects the indirect impact of predators on the prey population reproduction and survival by causing the prey stress response; moreover, the constant harvesting rate simulates the human interference of fixed-yield fishing or hunting on the prey population, which makes the model more in line with the actual ecological management scenario.

Zou [25] proposed the following predator-prey model:

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} = \dfrac{r_0 U(t)}{1+kV(t)} - \mathrm{d}U(t) - a(U(t))^2 - \dfrac{pU(t)V(t)}{1+qU(t)} - h_0, \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \dfrac{cpU(t)V(t)}{1+qU(t)} - mV(t), \end{cases} \tag{1.1}$$

where $U(t)$ is the density of the prey, $V(t)$ is the density of the predator and $\frac{r_0 U}{1+kV}$ is the birth rate of the prey. In the absence of predators ($V(t) = 0$), the intrinsic growth rate of the prey is r_0. When there are predators ($V(t) > 0$), the denominator $1 + kV(t)$ will increase, thus reducing the effective growth rate of the prey. The decline in this growth rate is a mathematical manifestation of the fear effect exerted by the presence of predators on prey, thus reflecting the indirect inhibitory effect of predation pressure on prey reproduction or survival. $\frac{pU(t)}{1+qU(t)}$ is a Holling Type II functional response function, a is the mortality from competition within the prey, d is the natural mortality rate of the prey, m is the natural mortality rate of the predators, c is the percent conversion, and h_0 is the constant yield for the prey. In [25], Zou studied the equilibrium points, limit cycles, and Hopf bifurcation of the model, and drew phase diagrams of stable and unstable limit cycles.

However, predators do not reproduce at all times in the ecosystem. To describe this phenomenon, a mature period is added to the predator to describe the reproductive capacity of the predator, that is, the predator cannot reproduce at an early age, and only the mature predator can reproduce. Therefore, in the paper, we shall consider the following predator-prey model with age structure in the predator as follows:

$$\begin{cases} \dfrac{\partial u(t,\varepsilon)}{\partial t} + \dfrac{\partial u(t,\varepsilon)}{\partial \varepsilon} = -\mu u(t,\varepsilon), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \dfrac{r_0 V(t)}{1+k\int_0^{+\infty} u(t,\varepsilon)\mathrm{d}\varepsilon} - \mathrm{d}V(t) - a(V(t))^2 - \dfrac{pV(t)\int_0^{+\infty} u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV(t)} - h_0, \\ u(t,0) = \dfrac{pcV(t)\int_0^{+\infty} \beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV(t)}, \\ u(0,\cdot) = u_0 \in L_1((0,+\infty),\mathbb{R}), \quad V(0) = V_0 \geq 0, \end{cases} \tag{1.2}$$

where t is time, ε is age, u is the density of the predator populations, V is the density of the prey, c is the percent conversion, h_0 is the constant yield for the prey, d is the natural mortality rate of the prey and $\beta(\varepsilon)$ is the maturity function, with the following assumption:

$$\beta(\varepsilon) =: \begin{cases} \beta^*, & \varepsilon \geq \tau_1, \\ 0, & \varepsilon \in (0,\tau_1), \end{cases}$$

where $\tau_1 > 0, \beta^* > 0$ and

$$\int_0^{+\infty} \beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon = M < +\infty.$$

In this equation, τ_1 is the delay in the predator's maturation, which means that predators under the age of τ_1 cannot reproduce. In addition to the predator maturation delay τ_1, the proposed model further incorporates a time delay τ_2, which specifically characterizes the gestation period that elapses from the conception of the predator's offspring to their actual birth. This second delay captures the inherent biological time lag in the predator's reproductive process, which is a key physiological characteristic that reflects the time required for embryonic development and offspring maturation inside the parent's body, thus making the model more biologically realistic in depicting the predator population's reproductive dynamics. Therefore, we will consider the model with two delays as follows:

$$\begin{cases} \dfrac{\partial u(t,\varepsilon)}{\partial t} + \dfrac{\partial u(t,\varepsilon)}{\partial \varepsilon} = -\mu u(t,\varepsilon), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \dfrac{r_0 V(t)}{1 + k\int_0^{+\infty} u(t,\varepsilon)\mathrm{d}\varepsilon} - \mathrm{d}V(t) - aV(t)^2 - \dfrac{pV(t-\tau_2)\int_0^{+\infty} u(t,\varepsilon)\mathrm{d}\varepsilon}{1 + qV(t-\tau_2)} - h_0, \\ u(t,0) = \dfrac{pcV(t-\tau_2)\int_0^{+\infty} \beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1 + qV(t-\tau_2)}, \\ u(0,\cdot) = u_0 \in L_1((0,+\infty),\mathbb{R}^+), \quad V(0) = V_0 = \phi \in C([-\tau_2,0],\mathbb{R}). \end{cases} \tag{1.3}$$

The mathematical goal of this paper focuses on the Hopf bifurcation problem of (1.3) by the Hopf theory developed in [16, 26]. We tackle this model to directly consider the equation in an abstract space directly. Applying the theory for an abstract non-densely defined Cauchy problem, which is developed under the framework of integrated semigroups [9]. Recall that the delay τ may also destablize the positive equilibrium of (1.3), thus generating periodic solutions. Therefore, we are interested in investigating the interactive impact of τ_1 and τ_2 on the dynamics of (1.3). By analyzing the characteristic equation of (1.3) at the positive equilibrium, we determine all the values of (τ_1, τ_2) such that the characteristic equation has roots with zero real parts. This will give the sharp region on the (τ_1, τ_2)-plane, where (1.3) has a locally stable positive equilibrium. Furthermore, as (τ_1, τ_2) passes through the boundary of this region, We can prove the existence of periodic solutions with different periods by the Hopf bifurcation theorem, where the periods of these periodic solutions are closely related to the time delay τ-; specifically, different critical values of the time delay τ correspond to periodic solutions with different periods.

The paper is organized as follows: In Subsections 2.1 and 2.2, we transform the age structure model (1.3) into a non-densely defined Cauchy problem; in Subsection 2.3, we derive the equilibrium and characteristic equations; in Section 3, the Hopf bifurcation at the equilibrium point is obtained; and in Section 4, we conduct some numerical simulations to verify these results and summarize the results of this paper.

2. Transform

2.1. The transition of age and time

Let $\hat{\varepsilon}=\frac{\varepsilon}{\tau_1}$, $\hat{t}=\frac{t}{\tau_1}$ thus $\hat{V}(\hat{t})=V(\tau_1\hat{t})$, $\hat{u}(\hat{t},\hat{\varepsilon})=\tau_1 u(\tau_1\hat{t},\tau_1\hat{\varepsilon})$. By dropping the "^", Model (1.3) becomes the following:

$$\begin{cases} \dfrac{\partial u(t,\varepsilon)}{\partial t}+\dfrac{\partial u(t,\varepsilon)}{\partial \varepsilon}=-\mu\tau_1 u(t,\varepsilon), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t}=\tau_1\left(\dfrac{r_0V(t)}{1+k\int_0^{+\infty}u(t,\varepsilon)\mathrm{d}\varepsilon}-\mathrm{d}V(t)-aV(t)^2-\dfrac{pV\left(t-\frac{\tau_2}{\tau_1}\right)\int_0^{+\infty}u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV\left(t-\frac{\tau_2}{\tau_1}\right)}-h_0\right), \\ u(t,0)=\dfrac{\tau_1 pcV\left(t-\frac{\tau_2}{\tau_1}\right)\int_0^{+\infty}\beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV\left(t-\frac{\tau_2}{\tau_1}\right)}, \\ u(0,\cdot)=u_0\in L^1((0,+\infty),\mathbb{R}^+),\quad V(0)=V_0=\phi\in C([-\tau_2,0],\mathbb{R}). \end{cases} \tag{2.1}$$

Suppose that $\beta(\varepsilon)$ can be expressed as follows:

$$\beta(\varepsilon):=\begin{cases}\beta^*, & \varepsilon\geq 1,\\ 0, & \varepsilon\in(0,1),\end{cases} \tag{2.2}$$

where $\beta^*=M\mu e^{\mu\tau_1}$.

2.2. Convert to non-densely defined Cauchy problem

We can define $V(t)$ as follows:

$$V(t):=\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon,$$

where $\rho(t,\varepsilon)$ is expressed as the density of the prey with an age of ε. Then, the second equation of system (2.1) becomes the following:

$$\begin{cases} \dfrac{\partial \rho(t,\varepsilon)}{\partial t}+\dfrac{\partial \rho(t,\varepsilon)}{\partial \varepsilon}=-\tau_1 d\rho(t,\varepsilon), \\ \rho(t,0)=G(u(t,\cdot),\rho(t,\cdot)), \\ \rho(0,\cdot)=\rho_0\in L^1((0,+\infty),\mathbb{R}), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} G(u(t,\cdot),\rho(t,\cdot))=\tau_1\Bigg[&\frac{r_0\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon}{1+k\int_0^{+\infty}u(t,\varepsilon)\mathrm{d}\varepsilon}-d\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon-a\left(\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon\right)^2 \\ &-\frac{p\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon\int_0^{+\infty}u(t,\varepsilon)\mathrm{d}\varepsilon}{1+q\int_0^{+\infty}\rho(t,\varepsilon)\mathrm{d}\varepsilon}-h_0\Bigg]. \end{aligned} \tag{2.4}$$

Let

$$\omega(t,\varepsilon)=\begin{pmatrix}u(t,\varepsilon)\\ \rho(t,\varepsilon)\end{pmatrix},\quad D=\begin{pmatrix}\tau_1\mu & 0\\ 0 & \tau_1 d\end{pmatrix};$$

we can rewrite (2.1) as follows:

$$\begin{cases} \dfrac{\partial \omega(t,\varepsilon)}{\partial t} + \dfrac{\partial \omega(t,\varepsilon)}{\partial \varepsilon} = -D\omega(t,\varepsilon), \\ \omega(t,0) = \begin{pmatrix} \frac{\tau_1 pcV\left(t-\frac{\tau_2}{\tau_1}\right)\int_0^{+\infty}\beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV\left(t-\frac{\tau_2}{\tau_1}\right)} \\ G(u(t,\cdot),\rho(t,\cdot)) \end{pmatrix}, \\ \omega(0,\cdot) = \rho_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in L^1((0,+\infty),\mathbb{R}^2). \end{cases} \tag{2.5}$$

Consider the Banach space $Y =: \mathbb{R}^2 \times L^1((0,+\infty),\mathbb{R}^2)$. The norm is as follows:

$$\left\|\begin{pmatrix} \zeta \\ \varphi \end{pmatrix}\right\|_Y = |\zeta|_{\mathbb{R}^2} + \|\varphi\|_{L^1}, \quad \begin{pmatrix} \zeta \\ \varphi \end{pmatrix} \in Y.$$

Define the linear operator L: $D(L) \subset Y \longrightarrow Y$ by the following:

$$L\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ -\varphi' - D\varphi \end{pmatrix} \tag{2.6}$$

with $D(L) = 0 \times W^{1,1}((0,+\infty),\mathbb{R}^2)$. Let

$$C_A = \left\{\begin{pmatrix} \zeta(\cdot) \\ \phi(\cdot) \end{pmatrix} \in C\left(\left[-\frac{\tau_2}{\tau_1}, 0\right], Y\right) : \zeta(0) = 0\right\}.$$

Define the operator F: $C_A \to Y$ by the following:

$$F\left(\begin{pmatrix} \zeta(\cdot) \\ \phi(\cdot) \end{pmatrix}\right) = \begin{pmatrix} B(\phi(\cdot)) \\ 0_{L_1} \end{pmatrix},$$

where

$$B(\phi(\cdot)) = \begin{pmatrix} \frac{\tau_1 pcV\left(t-\frac{\tau_2}{\tau_1}\right)\int_0^{+\infty}\beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV\left(t-\frac{\tau_2}{\tau_1}\right)} \\ G(\phi_1(0)(\cdot),\phi_2(0)(\cdot)) \end{pmatrix}.$$

Let

$$y(t) = \begin{pmatrix} 0 \\ \omega(t) \end{pmatrix};$$

rewrite system (2.5) as the following Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = Ly(t) + F(y_t), t > 0, \\ y_0 = \varphi = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \omega_0 \end{pmatrix} \in C_A. \end{cases} \tag{2.7}$$

Observe that L is non-densely defined, since

$$Y_0 =: \overline{D(L)} = 0 \times L^1((0,+\infty),\mathbb{R}^2).$$

Define $h \in C\left([0,+\infty] \times \left[-\frac{\tau_2}{\tau_1}, 0\right]; Y \right)$ when $t \geq 0$ and $\theta \in \left(-\frac{\tau_2}{\tau_1}, 0\right)$; then, we get $h(t,\theta) = y(t+\theta)$. Therefore, we have the following:

$$\frac{\partial h(t,\theta)}{\partial t} = y'(t+\theta) = \frac{\partial h(t,\theta)}{\partial \theta};$$

then,

$$\frac{\partial h(t,\theta)}{\partial t} - \frac{\partial h(t,\theta)}{\partial \theta} = 0.$$

When $\theta = 0$, $\frac{\partial h(t,\theta)}{\partial \theta} = y'(t) = Ly(t) + F(y_t) = Lh(t,0) + F(h(t,\cdot)), t \geq 0$. We obtain the following the partial differential equation:

$$\begin{cases} \dfrac{\partial h(t,\theta)}{\partial t} - \dfrac{\partial h(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial h(t,\theta)}{\partial \theta} = Lh(t,0) + F(h(t,\cdot)), t \geq 0, \\ x(0,\cdot) = y_0 \in C_A. \end{cases} \tag{2.8}$$

Define the space $Z := Y \times C$, $C := C\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y \right)$ with

$$\begin{pmatrix} f \\ \phi \end{pmatrix} = \|f\|_Y + \|\phi\|_C.$$

Let

$$z := \begin{pmatrix} 0 \\ h(t) \end{pmatrix}.$$

The partial differential Eq (2.8) can be written as a non-densely defined Cauchy problem as follows:

$$\begin{cases} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = Az(t) + H(z_t), \quad t > 0, \\ z(0) = \begin{pmatrix} 0_Y \\ y_0 \end{pmatrix} \in Z_0, \end{cases} \tag{2.9}$$

where A: $D(A) \subset Z \to Z$, and H: $Z_0 \to Z$ with $Z_0 = \{0_Y\} \times C_A$ are given by the following:

$$A\begin{pmatrix} 0_Y \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi'(0) + L\phi(0) \\ \phi' \end{pmatrix}, \quad H\begin{pmatrix} 0_Y \\ \phi \end{pmatrix} = \begin{pmatrix} F(\phi) \\ 0_{C_A} \end{pmatrix}.$$

It is straightforward that (2.9) is an abstract non-densely defined Cauchy problem, due to the fact that

$$D(A) = \{0_Y\} \times \left\{ \phi \in C^1\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y \right), \phi(0) \in D(L) \right\}$$

and $\overline{D(A)} = Z_0 \neq Z$. The global existence, uniqueness, and positive solutions for system (2.9) directly follow from the results in [27].

2.3. Equilibrium and characteristic equation

2.3.1. Equilibrium

Suppose that the equilibrium solution of Eq (2.9) is as follows:

$$\bar{z} = \begin{pmatrix} 0_Y \\ \overline{\psi} \end{pmatrix};$$

thus, the right-hand side of Eq (2.9) is zero, $A\overline{z(t)} + H(\overline{z(t)}) = 0$. Then, from the second equation of Eq (2.8), solve Eq (2.9), and the following equation:

$$\begin{cases} \overline{\psi}'(0) - L\overline{\psi}(0) - H(\overline{\psi}) = 0, \\ \overline{\psi}' = 0. \end{cases} \tag{2.10}$$

Theorem 2.1. *System* (2.9) *always has the boundary equilibria*

$$\bar{z}_1 = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \overline{\zeta}_1(\cdot) \\ \begin{pmatrix} \overline{\phi}_{11}(\cdot) \\ \overline{\phi}_{12}(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix} \quad \textit{and} \quad \bar{z}_2 = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \overline{\zeta}_2(\cdot) \\ \begin{pmatrix} \overline{\phi}_{21}(\cdot) \\ \overline{\phi}_{22}(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

with

$$\overline{\zeta}_1(\theta) = \overline{\zeta}_2(\theta) = 0_{\mathbb{R}^2}, \quad \begin{pmatrix} \overline{\phi}_{11}(\theta)(\varepsilon) \\ \overline{\phi}_{12}(\theta)(\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \overline{\phi}_{21}(\theta)(\varepsilon) \\ \overline{\phi}_{22}(\theta)(\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ \tau_1 d\overline{V} e^{-\tau_1 da} \end{pmatrix}.$$

Furthermore, if

$$(H1) \quad cM\kappa^2 + \frac{4kr_0}{pcM - q} - (4 + cM)\kappa + \frac{4}{cM} > 0,$$

where

$$\kappa = \frac{1}{cM} + \frac{ka + k(h_0 + d)(pcM - q)^2}{(pcM - q)^2}$$

holds, then there exists a unique positive equilibrium of system (2.9) *given by*

$$\bar{z} = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \overline{\zeta}(\cdot) \\ \begin{pmatrix} \overline{\phi}_1(\cdot) \\ \overline{\phi}_2(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix}, \quad \overline{\zeta}(\theta) = 0_{\mathbb{R}^2}, \quad \begin{pmatrix} \overline{\phi}_1(\theta)(\varepsilon) \\ \overline{\phi}_2(\theta)(\varepsilon) \end{pmatrix} = \begin{pmatrix} C_1 e^{-\tau_1 \mu \varepsilon} \\ C_2 e^{-\tau_1 d\varepsilon} \end{pmatrix},$$

where

$$\begin{cases} C_1 = \tau_1 \mu \dfrac{\left(-\frac{p\overline{V}}{1+q\overline{V}} - k(a\overline{V}^2 + d + h_0) + \left(\frac{p\overline{V}}{1+q\overline{V}} + k(a\overline{V}^2 + d + h_0)\right)^2 + \frac{4kp\overline{V}}{1+q\overline{V}}(\overline{V}r_0 - a\overline{V}^2 - d - h_0)\right)}{\frac{2kp\overline{V}}{1+q\overline{V}}}, \\ C_2 = \tau_1 d\overline{V} \end{cases}$$

with

$$\overline{V} = \frac{1}{pcM - q}.$$

Proof. By solving Eq (2.10), we can get

$$\begin{pmatrix} 0_Y \\ B(\phi(\cdot)) \\ 0_{L_1} \end{pmatrix} = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \frac{\tau_1 pcV\left(t-\frac{\tau_2}{\tau_1}\right)\int_0^{+\infty}\beta(\varepsilon)u(t,\varepsilon)\mathrm{d}\varepsilon}{1+qV\left(t-\frac{\tau_2}{\tau_1}\right)} \\ G(\phi_1(0)(\cdot),\phi_2(0)(\cdot)) \end{pmatrix} \\ 0_{L_1} \end{pmatrix} = -L\begin{pmatrix} 0 \\ \phi_1(0) \\ \phi_2(0) \\ 0 \end{pmatrix},$$

and

$$L\begin{pmatrix} 0 \\ \omega(\varepsilon) \end{pmatrix} + F\left(\begin{pmatrix} 0 \\ \omega(\varepsilon) \end{pmatrix}\right) = 0,$$

where

$$\omega(\varepsilon) = \begin{pmatrix} \overline{u}(\varepsilon) \\ \overline{\rho}(\varepsilon) \end{pmatrix} = \begin{pmatrix} e^{-\tau_1\mu\varepsilon}\tau_1(\frac{cp\overline{V}}{1+q\overline{V}})\int_0^{+\infty}\beta(\varepsilon)\overline{u}(\varepsilon)\mathrm{d}\varepsilon \\ \tau_1(\frac{r_0\overline{V}}{1+k\int_0^{+\infty}\overline{u}(\varepsilon)d\varepsilon} - d\overline{V} - a\overline{V}^2 - \frac{p\overline{V}\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon}{1+q\overline{V}} - h_0)e^{-\tau_1 d\varepsilon} \end{pmatrix},$$

so

$$\begin{aligned} 1 &= \tau_1\left(\frac{cp\overline{V}}{1+q\overline{V}}\right)\int_0^{+\infty}\beta(\varepsilon)e^{-\tau_1\mu\varepsilon}\mathrm{d}\varepsilon, \\ \overline{V} &= \tau_1\left(\frac{r_0\overline{V}}{1+k\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon} - d\overline{V} - a\overline{V}^2 - \frac{p\overline{V}\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon}{1+q\overline{V}} - h_0\right)\int_0^{+\infty}e^{-\tau_1 d\varepsilon}\mathrm{d}\varepsilon, \end{aligned}$$

or

$$\begin{aligned} \overline{V}\left(\frac{r_0}{1+k\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon} - a\overline{V} - d\right) &= \frac{p\overline{V}\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon}{1+q\overline{V}} + h_0, \\ \frac{\overline{V}\left(\frac{r_0}{1+k\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon} - a\overline{V} - d\right) - h_0}{\tau_1\left(\frac{cp\overline{V}}{1+q\overline{V}}\right)} &= \frac{p\overline{V}\int_0^{+\infty}\overline{u}(\varepsilon)\beta(\varepsilon)\mathrm{d}\varepsilon}{1+q\overline{V}}\int_0^{+\infty}e^{-\tau_1\mu\varepsilon}\mathrm{d}\varepsilon. \end{aligned}$$

Therefore,

$$\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon = \frac{\left(-\frac{p\overline{V}}{1+q\overline{V}} - k(a\overline{V}^2 + d + h_0) + \left(\frac{p\overline{V}}{1+q\overline{V}} + k(a\overline{V}^2 + d + h_0)\right)^2 + \frac{4kp\overline{V}}{1+q\overline{V}}(\overline{V}r_0 - a\overline{V}^2 - d - h_0)\right)}{\frac{2kp\overline{V}}{1+q\overline{V}}}.$$

In conclusion, we have the following:

$$\begin{aligned} \overline{V} &= \frac{1}{\tau_1 cp\int_0^{+\infty}\beta(\varepsilon)e^{-\tau_1\mu\varepsilon}\mathrm{d}\varepsilon - q}, \\ \overline{u}(\varepsilon) &= e^{-\tau_1\mu\varepsilon}\tau_1\mu\frac{\left(-\frac{p\overline{V}}{1+q\overline{V}} - k(a\overline{V}^2 + d + h_0) + \left(\frac{p\overline{V}}{1+q\overline{V}} + k(a\overline{V}^2 + d + h_0)\right)^2 + \frac{4kp\overline{V}}{1+q\overline{V}}(\overline{V}r_0 - a\overline{V}^2 - d - h_0)\right)}{\frac{2kp\overline{V}}{1+q\overline{V}}}, \\ \overline{\rho}(\varepsilon) &= \tau_1 d\overline{V}e^{-\tau_1 d\varepsilon}. \end{aligned} \tag{2.11}$$

When

$$\int_0^{+\infty} \beta(\varepsilon)\overline{u}(\varepsilon)\mathrm{d}\varepsilon = 0,$$

we have

$$\begin{cases}\overline{u}(\varepsilon) \equiv 0,\\ \overline{\rho}(\varepsilon) \equiv 0,\end{cases} \quad \text{and} \quad \begin{cases}\overline{u}(\varepsilon) = 0,\\ \overline{\rho}(\varepsilon) = \tau_1 d\overline{V}e^{-\tau_1 d\varepsilon},\end{cases}$$

which are two equilibriums of the equation. The following is a unique positive equilibrium point in system (2.9):

$$\overline{u}(\varepsilon) = e^{-\tau_1\mu\varepsilon}\tau_1\mu\frac{\left(-\frac{p\overline{V}}{1+q\overline{V}} - k(a\overline{V}^2 + d + h_0) + \left(\frac{p\overline{V}}{1+q\overline{V}} + k(a\overline{V}^2 + d + h_0)\right)^2 + \frac{4kp\overline{V}}{1+q\overline{V}}(\overline{V}r_0 - a\overline{V}^2 - d - h_0)\right)}{\frac{2kp\overline{V}}{1+q\overline{V}}},$$

$$\overline{V} = \frac{1}{pcM - q}.$$

Let

$$\begin{pmatrix}\phi_1(\theta)(\varepsilon)\\ \phi_2(\theta)(\varepsilon)\end{pmatrix} = \begin{pmatrix}\overline{u}(\varepsilon)\\ \overline{\rho}(\varepsilon)\end{pmatrix};$$

then, we have the following:

$$\begin{pmatrix}\phi_1(\theta)(\varepsilon)\\ \phi_2(\theta)(\varepsilon)\end{pmatrix} = \begin{pmatrix} e^{-\tau_1\mu\varepsilon}\tau_1\mu\frac{\left(-\frac{p\overline{V}}{1+q\overline{V}} - k(a\overline{V}^2 + d + h_0) + \left(\frac{p\overline{V}}{1+q\overline{V}} + k(a\overline{V}^2 + d + h_0)\right)^2 + \frac{4kp\overline{V}}{1+q\overline{V}}(\overline{V}r_0 - a\overline{V}^2 - d - h_0)\right)}{\frac{2kp\overline{V}}{1+q\overline{V}}} \\ \tau_1 d\overline{V}e^{-\tau_1 d\varepsilon}\end{pmatrix}.$$

This completes the proof. □

Set $\tilde{z}(t) =: z(t) - \overline{z}$; system (2.9) can be written as follows:

$$\begin{cases}\dfrac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t} = A\tilde{z}(t) + H(\tilde{z}(t) + \overline{z}) - H(\overline{z}), \quad t \geq 0,\\ \tilde{z}(0) = \begin{pmatrix}0\\ \omega(0) - \overline{\psi}(\varepsilon)\end{pmatrix} \triangleq \tilde{z}(0) \in \overline{D(A)}.\end{cases} \tag{2.12}$$

The linearization equation at the positive equilibrium point $\bar{z}$ is

$$\frac{\mathrm{d}\tilde{z}(t)}{\mathrm{d}t} = A\tilde{z}(t) + DH(\overline{z})\tilde{z}, \quad t \geq 0,$$

where

$$DH(\overline{z})\begin{pmatrix}0_Y\\ \psi\end{pmatrix} = \begin{pmatrix}DF(\overline{\psi})(\psi)\\ 0_{C_A}\end{pmatrix}, \quad \forall\begin{pmatrix}0_Y\\ \psi\end{pmatrix} \in D(A), \quad \psi = \begin{pmatrix}\zeta(\cdot)\\ \phi(\cdot)\end{pmatrix}$$

and

$$DF(\overline{\psi})(\psi) = \begin{pmatrix}DB(\phi)\phi\\ 0_{L_1}\end{pmatrix}.$$

2.3.2. Characteristic equation

Let

$$\vartheta := min\{\tau_1 d, \tau_1\mu\} > 0, \quad \Omega := \{\lambda \in \mathbb{C} : Re(\lambda) > -\vartheta\}.$$

The following results are obtained from [21, Lemma 4.1].

Lemma 2.2. *For the operators L and A defined above, the following statements are valid.*

(i) If $\lambda \in \Omega$, then $\lambda \in \rho(L)$ and

$$(\lambda I - L)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix} \Leftrightarrow \varphi(a) = e^{-\int_0^a(\lambda I + D)\mathrm{d}l}\delta + \int_0^a e^{-\int_s^a(\lambda I+D)\mathrm{d}l}\psi(s)\mathrm{d}s$$

with $\begin{pmatrix}\delta\\ \psi\end{pmatrix} \in X$ and $\begin{pmatrix}0\\ \varphi\end{pmatrix} \in D(L)$.

(ii) $\rho(L) = \rho(A)$. Moreover, for each $\lambda \in \rho(A)$, we also have the following explicit formula for the resolvent of A:

$$(\lambda I - A)^{-1}\begin{pmatrix}f\\ \phi\end{pmatrix} = \begin{pmatrix}0_X\\ \phi\end{pmatrix} \Leftrightarrow \phi(\theta) = e^{\lambda\theta}(\lambda I - L)^{-1}[\psi(0) + f] + \int_\theta^0 e^{\lambda(\theta - s)}\psi(s)\,\mathrm{d}s.$$

Note that $(\lambda I - L)$ is reversible, and $\lambda I - (L + DF(\bar{y}))$ is reversible if and only if $I - DF(\bar{y})(\lambda I - L)^{-1}$ is reversible; thus, we can obtain the following:

$$(\lambda I - (L + DF(\bar{y})))^{-1} = (\lambda I - L)^{-1}(I - DF(\bar{y})(\lambda I - L)^{-1})^{-1},$$

where

$$DF(\bar{y})\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}DB(\bar{\omega})(\varphi)\\ 0\end{pmatrix}$$

and

$$\begin{aligned}DB(\bar{\omega})(\varphi) = &\begin{pmatrix}0 & \frac{\tau_1 pcu}{(1+qV)^2}\\ \tau_1\left(\frac{-r_0kV}{(1+ku)^2} - \frac{pV}{1+qV}\right) & \tau_1\left(\frac{r_0}{1+ku} - d - 2aV - \frac{pu}{(1+qV)^2}\right)\end{pmatrix}\int_0^{+\infty}\varphi(\varepsilon)\mathrm{d}\varepsilon\\ &+\begin{pmatrix}\frac{\tau_1 pcV}{1+qV} & 0\\ 0 & 0\end{pmatrix}\int_0^{+\infty}\beta(\varepsilon)\varphi(\varepsilon)\mathrm{d}\varepsilon.\end{aligned}$$

Let

$$(I - DF(\bar{y})(\lambda I - L)^{-1})\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}\gamma\\ \varpi\end{pmatrix};$$

then,

$$\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} - DF(\bar{y})\begin{pmatrix}0\\ e^{-\int_0^\varepsilon(\lambda I + D(l))\mathrm{d}l}\alpha + \int_0^\varepsilon e^{\int_0^\varepsilon(\lambda I + D(l))\mathrm{d}l}\psi(s)\mathrm{d}s\end{pmatrix} = \begin{pmatrix}\gamma\\ \varpi\end{pmatrix}.$$

The new system can be acquired as follows:

$$\begin{cases}\alpha - DB(\bar{\omega})\left(e^{-\int_0^\varepsilon(\lambda I + D(l))\mathrm{d}l}\alpha + \displaystyle\int_0^\varepsilon e^{\int_0^\varepsilon(\lambda I + D(l))\mathrm{d}l}\psi(s)\mathrm{d}s\right) = \gamma,\\ \varphi = \varpi,\end{cases}$$

or

$$\begin{cases}\alpha - DB(\bar{\omega})\left(e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\alpha\right) = \gamma + DB(\bar{\omega})\left(\int_0^{\varepsilon} e^{\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\psi(s)\mathrm{d}s\right),\\ \varphi = \varpi.\end{cases}$$

Denote

$$\Delta(\lambda) = I - DB(\bar{\phi})\left(e^{\lambda\theta}e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\right),$$

and

$$K(\lambda,\varpi) = DB(\bar{\omega})\left(\int_0^{\varepsilon} e^{\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\varpi(s)\mathrm{d}s\right).$$

Then,

$$\Delta(\lambda)\alpha = \gamma + K(\lambda,\varpi).$$

When $\Delta(\lambda)$ is reversible, then

$$\alpha = (\Delta(\lambda))^{-1}(\gamma + K(\lambda,\varpi)).$$

When considering a single delay,

$$\begin{aligned}\Delta(\lambda) = I &- \begin{pmatrix} \frac{\tau_1 pc\overline{V}}{1+q\overline{V}} & 0\\ 0 & 0\end{pmatrix}\int_0^{+\infty}\beta(\varepsilon)e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\mathrm{d}\varepsilon\\ &- \begin{pmatrix} 0 & \frac{\left(\tau_1 pc\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon\right)}{(1+q\overline{V})^2}\\ \tau_1\begin{pmatrix}\frac{-r_0k\overline{V}}{(1+k\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon)^2}\\ -\frac{p\overline{V}}{1+q\overline{V}}\end{pmatrix} & \tau_1\begin{pmatrix}\frac{r_0}{1+k\int_0^{+\infty}\overline{u}(\varepsilon)\mathrm{d}\varepsilon} - d-\\ 2a\overline{V} - \frac{p\int_0^{+\infty}\beta(\varepsilon)\overline{u}(\varepsilon)\mathrm{d}\varepsilon}{(1+q\overline{V})^2}\end{pmatrix}\end{pmatrix}\int_0^{+\infty} e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\mathrm{d}\varepsilon.\end{aligned} \tag{2.13}$$

When considering two delays, $\Delta(\lambda)$ becomes then following:

$$\begin{aligned}\Delta(\lambda) =I &- \begin{pmatrix} 0 & 0\\ -\tau_1\begin{pmatrix}\frac{-r_0k\int_0^{+\infty}\phi_2(0)\mathrm{d}\varepsilon}{\left(1+k\int_0^{+\infty}\phi_1(0)\mathrm{d}\varepsilon\right)^2}\\ -\frac{p\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon}{1+q\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon}\end{pmatrix} & \tau_1\begin{pmatrix}\frac{r_0}{1+k\int_0^{+\infty}\phi_1(0)\mathrm{d}\varepsilon} - d-\\ 2a\int_0^{+\infty}\phi_2(0)\mathrm{d}\varepsilon\end{pmatrix}\end{pmatrix}\int_0^{+\infty} e^{-\int_0^{\varepsilon}(\lambda I+D(l))dl}\mathrm{d}\varepsilon\\ &- \begin{pmatrix}\frac{\tau_1 pc\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon}{1+q\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon} & 0\\ 0 & 0\end{pmatrix}\int_0^{+\infty}\beta(\varepsilon)e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\mathrm{d}\varepsilon\\ &- \begin{pmatrix} 0 & \frac{\left(\tau_1 pc\int_0^{+\infty}\beta(\varepsilon)\phi_1(0)\mathrm{d}\varepsilon\right)}{(1+q\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon)^2}\\ 0 & -\frac{\left(\tau_1 p\int_0^{+\infty}\int_0^{+\infty}\phi_1(0)\mathrm{d}\varepsilon\right)}{(1+q\int_0^{+\infty}\phi_2\left(-\frac{\tau_2}{\tau_1}\right)\mathrm{d}\varepsilon)^2}\end{pmatrix}\int_0^{+\infty} e^{\lambda\left(-\frac{\tau_2}{\tau_1}\right)}e^{-\int_0^{\varepsilon}(\lambda I+D(l))\mathrm{d}l}\mathrm{d}\varepsilon.\end{aligned} \tag{2.14}$$

By [21, Lemma 3.3], we know

$$\sigma(A + DH(\bar{z})) \cap \Omega = \sigma_p(A + DH(\bar{z})) \cap \Omega = \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\}.$$

Under Assumption $(H1)$, we obtain the following characteristic equation:

$$\det(\Delta(\lambda)) = \frac{\lambda^2 + \tau_1 A_1 \lambda + \tau_1^2 B_1 + (\tau_1^2 C_1 + \tau_1 D_1 \lambda)e^{-\lambda} + (\tau_1 Q_1 \lambda + \tau_1^2 L)e^{\lambda\left(-\frac{\tau_2}{\tau_1}\right)} + \tau_1^2 H_1 e^{\lambda\left(-\frac{\tau_2}{\tau_1}\right)} e^{-\lambda}}{(\lambda + \tau_1 \mu)(\lambda + \tau_1 d)}$$
$$:= \frac{f(\lambda)}{h(\lambda)} = 0,$$

where

$$\overline{V} = \frac{1}{pcM - q}, \quad U_1 = \int_0^{+\infty} \overline{u}(\varepsilon)\mathrm{d}\varepsilon, \quad U_2 = \int_0^{+\infty} \beta(\varepsilon)\overline{u}(\varepsilon)\mathrm{d}\varepsilon, \quad A_1 = d + \mu - \frac{r_0}{1 + kU_1} + 2a\overline{V},$$
$$B_1 = d\mu - \frac{\mu r_0}{1 + kU_1} + 2\mu a\overline{V}, \quad C_1 = pc\overline{V}M\mu\frac{r_0}{1 + kU_1} - dpc\overline{V}M\mu - 2pca\overline{V}^2 M\mu,$$
$$D_1 = -pc\overline{V}M\mu, \quad H_1 = -\frac{p^2 c\overline{V}U_1 M\mu}{(1 + q\overline{V})^2}, \quad Q_1 = \frac{pU_1}{(1 + q\overline{V})^2},$$
$$L_1 = \frac{\mu pU_1}{(1 + q\overline{V})^2} + \frac{p^2 U_1 cU_2}{(1 + q\overline{V})^3} + \frac{r_0 k\overline{V}pcU_2}{(1 + kU_1)^2(1 + q\overline{V})^2}.$$

3. Hopf bifurcation

3.1. *Case of one delay*

Consider the case of a single delay. Let $\tau_1 = \tau_2 = \tau$,

$$\det(\Delta(\lambda)) = \frac{\lambda^2 + \tau A_2 \lambda + \tau^2 B_2 + (\tau^2 C_2 + \tau D_2 \lambda)e^{-\lambda}}{(\lambda + \tau\mu)(\lambda + \tau d)} := \frac{f_1(\lambda)}{f_2(\lambda)} = 0,$$

where

$$A_2 = d + \mu - \frac{r_0}{1 + kU_1} + 2a\overline{V} + \frac{pU_1}{(1 + q\overline{V})^2},$$
$$B_2 = d\mu - \frac{\mu r_0}{1 + kU_1} + 2\mu a\overline{V} + \frac{\mu pU_1}{(1 + q\overline{V})^2} + \frac{p^2 U_1 cU_2}{(1 + q\overline{V})^3} + \frac{r_0 k\overline{V}pcU_2}{(1 + kU_1)^2(1 + q\overline{V})^2},$$
$$C_2 = pc\overline{V}M\mu\frac{r_0}{1 + kU_1} - dpc\overline{V}M\mu - 2pca\overline{V}^2 M\mu - \frac{p^2 c\overline{V}U_1 M\mu}{(1 + q\overline{V})^2},$$
$$D_2 = -pc\overline{V}M\mu.$$

Let $\lambda = \tau\zeta$. Then, we have the following:

$$f_1(\lambda) = f_1(\tau\zeta) = \tau^2 g(\zeta),$$

where

$$g(\zeta) = \zeta^2 + A_2\zeta + B_2 + (C_2 + D_2\zeta)e^{-\tau\zeta}. \tag{3.1}$$

In the following, we will analyze the distribution of purely imaginary roots of $g(\zeta) = 0$. Suppose that $i\omega(\omega > 0)$ is a root of $g(\zeta) = 0$, that is,

$$-\omega^2 + iA_2\omega + B_2 + C_2 e^{-i\omega\tau} + iD_2\omega e^{-i\omega\tau} = 0. \tag{3.2}$$

With Euler's formula,

$$C_2 e^{-i\omega\tau} = C_2\cos(\omega\tau) - C_2 i\sin(\omega\tau),$$
$$D_2 i\omega e^{-i\omega\tau} = D_2 i\omega\cos(\omega\tau) + D_2\omega\sin(\omega\tau).$$

Separating real and imaginary parts, we get

$$\begin{cases} -\omega^2 + B_2 = -D_2\omega\sin(\omega\tau) - C_2\cos(\omega\tau), \\ A_2\omega = C_2\sin(\omega\tau) - D_2\omega\cos(\omega\tau), \end{cases} \tag{3.3}$$

which follows that

$$\omega^4 + (A_2^2 - 2B_2 - D_2^2)\omega^2 + B_2^2 - C_2^2 = 0. \tag{3.4}$$

According to Veda's theorem, the two roots of Eq (3.3) are as follows:

$$\omega_k = \left[\frac{1}{2}\left(-(A_2^2 - 2B_2 - D_2^2) \pm \sqrt{(A_2^2 - 2B_2 - D_2^2)^2 - 4(B_2^2 - C_2^2)} \right)\right]^{1/2}, \quad k = 1, 2,$$

provided that

$$(H2)\ \ A_2^2 - 2B_2 - D_2^2 < 0, (A_2^2 - 2B_2 - D_2^2)^2 > 4(B_2^2 - C_2^2) \text{ and } B_2 - C_2 > 0.$$

Moreover, Eq (3.1) has two pairs of purely imaginary roots $\pm i\omega_k, k = 1, 2$ at τ_k^j, $j \in \mathbb{N}_0$, with

$$\tau_k^j = \begin{cases} \frac{1}{\omega_k}\left[\arccos\frac{(C_2 - A_2D_2)(\omega_k)^2 - B_2C_2}{C_2^2 + D_2^2(\omega_k)^2} + 2j\pi\right], & \frac{(C_2 - A_2D_2)(\omega_k)^2 - B_2C_2}{C_2^2 + D_2^2(\omega_k)^2} \geq 0, \\ \frac{1}{\omega_k}\left[2\pi - \arccos\frac{(C_2 - A_2D_2)(\omega_k)^2 - B_2C_2}{C^2 + D^2(\omega_k)^2} + 2j\pi\right], & \frac{(C_2 - A_2D_2)(\omega_k)^2 - B_2C_2}{C^2 + D^2(\omega_k)^2} < 0. \end{cases} \tag{3.5}$$

In order to check the transversal condition, one needs to compute the following:

$$\left(\frac{\mathrm{d}\zeta(\tau)}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_k^j} = -\frac{\tau}{\zeta} + \frac{D_2}{\zeta(C_2 + D_2\zeta)} - \frac{2\zeta + A_2}{\zeta(\zeta^2 + A_2\zeta + B_2)}.$$

Since

$$\begin{aligned} Sign\left\{\frac{\mathrm{d}(\mathrm{Re}\zeta(\tau))}{\mathrm{d}\tau}\right\}^{-1}_{\tau=\tau_k^j} &= Sign\left\{\mathrm{Re}\left[\frac{D_2}{\zeta(C_2 + D_2\zeta)}\right]_{\tau=\tau_k^j} + \mathrm{Re}\left[\frac{2\zeta + A_2}{\zeta(\zeta^2 + A_2\zeta + B_2)}\right]_{\tau=\tau_k^j}\right\} \\ &= Sign\left[\frac{-(D_2)^2 + 2((\omega_k)^2 - B_2) + (A_2)^2}{(C_2)^2 + (D_2)^2(\omega_k)^2}\right]. \end{aligned}$$

Hence, if we further assume

$$(H3)\quad -(D_2)^2 + 2((\omega_k)^2 - B_2) + (A_2)^2 \neq 0,$$

then

$$\left.\frac{\mathrm{d}(\mathrm{Re}\zeta(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_k^j} \neq 0.$$

Let $\tau^0 := min\{\tau_1^0, \tau_2^0\}$. Summarizing the above analysis, we have the following theorem.

Theorem 3.1. *Assume that* $(H1)$–$(H3)$ *hold. Then, the positive equilibrium* $\bar{z}$ *is unstable for* $\tau \in [0, \tau^0)$ *and Eq* (1.3) *undergoes a Hopf bifurcation at the positive equilibrium* $\bar{z}$ *when* $\tau = \tau_k^j$*, for* $j \in \mathbb{N}_0$*, k=1,2.*

3.2. Case of two delays

We further extend our analysis to the case that incorporates two delays $\tau_1 \neq \tau_2$. Let $\lambda = \tau_1\zeta$; then, we have the following:

$$f(\lambda) = \tau_1^2 g_1(\zeta) = \tau_1^2 \left(\zeta^2 + A_1\zeta + B_1 + (C_1 + D_1\zeta)e^{-\zeta\tau_1} + (L_1 + Q_1\zeta)e^{-\zeta\tau_2} + H_1 e^{-\zeta(\tau_2+\tau_1)}\right).$$

In the following, we will use the method in [28] to study $g_1(\zeta) = 0$. Substituting $\zeta = i\omega$, $\omega > 0$ into $g_1(\zeta) = 0$, we have

$$-\omega^2 + A_1 i\omega + B_1 + (C_1 + D_1 i\omega)e^{-\tau_1 i\omega} + (L_1 + Q_1 i\omega)e^{-\tau_2 i\omega} + H_1 e^{-(\tau_2+\tau_1)i\omega} = 0,$$

or

$$-\omega^2 + A_1 i\omega + B_1 + (C_1 + D_1 i\omega)e^{-\tau_1 i\omega} + ((L_1 + Q_1 i\omega) + H_1 e^{-\tau_1 i\omega})e^{-\tau_2 i\omega} = 0.$$

It follows from Euler's formula that $\left|e^{-\tau_2 i\omega}\right| = 1$; therefore,

$$\left|-\omega^2 + A_1 i\omega + B_1 + (C_1 + D_1 i\omega)e^{-\tau_1 i\omega}\right| = \left|(L_1 + Q_1 i\omega) + H_1 e^{-\tau_1 i\omega}\right|,$$

which implies

$$\omega^4 - (2B_1 - A_1^2 - D_1^2 + Q_1^2)\omega^2 + (B_1^2 + C_1^2 - L_1^2 - H_1^2) = 2U(\omega)\cos(\omega\tau_1) - 2V(\omega)\sin(\omega\tau_1), \tag{3.6}$$

where

$$U_1(\omega) = (C_1 - A_1 D_1)\omega^2 + (L_1 H_1 - B_1 C_1),$$
$$V_1(\omega) = (Q_1 H_1 - A_1 C_1 + B_1 D_1)\omega - D_1\omega^3.$$

Let

$$\phi_1(\omega) = arg\left[(C_1 - A_1 D_1)\omega^2 + L_1 H_1 - B_1 C_1 + (-D_1\omega^2 + Q_1 H_1 - A_1 C_1 + B_1 D_1)i\omega\right].$$

Then, $U_1(\omega)$ and $V_1(\omega)$ can be written as follows:

$$\begin{aligned} U_1(\omega) &= \sqrt{((C_1 - A_1 D_1)\omega^2 + (L_1 H_1 - B_1 C_1))^2 + ((Q_1 H_1 - A_1 C + B_1 D_1)\omega - D_1\omega^3)^2}\cos(\phi_1(\omega)), \\ V_1(\omega) &= \sqrt{((C_1 - A_1 D_1)\omega^2 + (L_1 H_1 - B_1 C_1))^2 + ((Q_1 H_1 - A_1 C_1 + B_1 D_1)\omega - D_1\omega^3)^2}\sin(\phi_1(\omega)). \end{aligned} \tag{3.7}$$

By substituting (3.7) into (3.6), we obtain the following:

$$\begin{aligned} &\omega^4 - (2B_1 - A_1^2 - D_1^2 + Q_1^2)\omega^2 + (B_1^2 + C_1^2 - L_1^2 - H_1^2) \\ &= 2\sqrt{((C_1 - A_1 D_1)\omega^2 + (L_1 H_1 - B_1 C_1))^2 + ((Q_1 H_1 - A_1 C_1 + B_1 D_1)\omega - D_1\omega^3)^2}\cos(\phi_1(\omega) + \omega\tau_1). \end{aligned} \tag{3.8}$$

For (3.8) to have positive roots for ω, it is necessary that

$$\begin{aligned} F(\omega) \triangleq &\left[\omega^4 - (2B_1 - A_1^2 - D_1^2 + Q_1^2)\omega^2 + (B_1^2 + C_1^2 - L_1^2 - H_1^2)\right]^2 \\ &- 4\left[((C_1 - A_1 D_1)\omega^2 + (L_1 H_1 - B_1 C_1))^2 + ((Q_1 H_1 - A_1 C_1 + B_1 D_1)\omega - D_1\omega^3)^2\right] \leq 0. \end{aligned} \tag{3.9}$$

On the other hand, if $\omega > 0$ satisfies (3.9), then we can always find τ_1 such that (ω, τ_1) is the root of Eq (3.8). The set of all possible values of $\omega > 0$ that satisfy (3.9) is denoted by Ω.

From [28, Lemma 3.2], it concludes that Ω consists of a finite number of intervals of a finite length Ω_k, that is,

$$\Omega = \bigcup_{k=1}^{N} \Omega_k$$

with $|\Omega_k| < \infty$. Denote the following:

$$\psi_1(\omega) = \phi_1(\omega) + \omega\tau_1 \in [0, \pi];$$

then,

$$\cos(\psi_1(\omega)) = \frac{\omega^4 - (2B_1 - A_1^2 - D_1^2 + Q_1^2)\omega^2 + (B_1^2 + C_1^2 - L_1^2 - H_1^2)}{2\sqrt{((C_1 - A_1D_1)\omega^2 + (L_1H_1 - B_1C_1))^2 + ((Q_1H_1 - A_1C_1 + B_1D_1)\omega - D_1\omega^3)^2}}.$$

Therefore,

$$\tau^{\pm}_{1,n_1}(\omega) = \frac{\pm\psi_1(\omega) - \phi_1(\omega) + 2n_1\pi}{\omega}, \quad n_1 \in \mathbb{Z}, \tag{3.10}$$

for $\omega \in \Omega, n_1 \in \mathbb{Z}$, $\left(\omega, \tau^{\pm}_{1,n_1}(\omega)\right)$ satisfy (3.8). Similarly, we have $\left|e^{-\tau_1 i\omega}\right| = 1$ for τ_1; then,

$$\left|-\omega^2 + A_1 i\omega + B_1 + (L_1 + Q_1 i\omega)e^{-\tau_2 i\omega}\right| = \left|(C_1 + D_1 i\omega) + H_1 e^{-\tau_2 i\omega}\right|.$$

Hence,

$$\omega^4 + (A_1^2 - 2B_1 + D_1^2 - Q_1^2)\omega^2 + (B_1^2 + L_1^2 - C_1^2 - H_1^2) = 2U_2(\omega)\cos(\omega\tau_2) - 2V_2(\omega)\sin(\omega\tau_2), \tag{3.11}$$

where

$$U_2(\omega) = (L_1 - A_1Q_1)\omega^2 + (H_1C_1 - B_1L_1) = \sqrt{U_2(\omega)^2 + U_2(\omega)^2}\cos(\phi_2(\omega)),$$
$$V_2(\omega) = Q_1\omega^3 - (B_1Q_1 + A_1L_1)\omega - DH = \sqrt{U_2(\omega)^2 + U_2(\omega)^2}\sin(\phi_2(\omega)).$$

Let

$$\phi_2(\omega) = arg(U_2(\omega) + iU_2(\omega));$$

similarly, we have the following:

$$\begin{aligned} &\left[\omega^4 + (A_1^2 - 2B_1 + D_1^2 - Q_1^2)\omega^2 + (B_1^2 + L_1^2 - C_1^2 - H_1^2)\right]^2 \\ &- 4\left[((L_1 - A_1Q_1)\omega^2 + (H_1C_1 - B_1L_1))^2 + (Q_1\omega^3 - (B_1Q_1 + A_1L_1)\omega - D_1H_1)^2\right] \le 0. \end{aligned} \tag{3.12}$$

Denote

$$\psi_2(\omega) = \phi_2(\omega) + \omega\tau_2 \in [0, \pi];$$

then,

$$\cos(\psi_2(\omega)) = \frac{\omega^4 + (A_1^2 - 2B_1 + D_1^2 - Q_1^2)\omega^2 + (B_1^2 + L_1^2 - C_1^2 - H_1^2)}{2\sqrt{((L_1 - A_1Q_1)\omega^2 + (H_1C_1 - B_1L_1))^2 + (Q\omega^3 - (B_1Q_1 + A_1L_1)\omega - D_1H_1)^2}},$$

Therefore,

$$\tau^{\pm}_{2,n_2}(\omega) = \frac{\pm\psi_2(\omega) - \phi_2(\omega) + 2n_2\pi}{\omega}, \quad n_2 \in \mathbb{Z}. \tag{3.13}$$

According to [28, page 525], the crossing curves of $g_1(\zeta) = 0$ are given in the following theorem.

Theorem 3.2. *The crossing curves of* $g_1(\zeta) = 0$ *are given by the following:*

$$\Gamma = \left(\bigcup_{k=1,2,\ldots N} \Gamma^{\pm k}_{n_1,n_2} \right) \bigcap \mathbb{R}^2_+,$$

where

$$\Gamma^{\pm k}_{n_1,n_2} = \left\{ \left(\frac{\pm\psi_1(\omega) - \phi_1(\omega) + 2n_1\pi}{\omega}, \frac{\mp\psi_2(\omega) - \phi_2(\omega) + 2n_2\pi}{\omega} \right) : \omega \in \Omega_k \right\}.$$

Next, we will discuss the direction in which the roots of $g_1(\zeta) = 0$ cross the imaginary axis, as (τ_1, τ_2) deviates from the curve in Γ. As in [26], we call the direction of the curve that corresponds to increasing $\omega \in \Omega_k$ as the *positive direction*, and the region on the left-hand (right-hand) side as we head in the positive direction of the curve as *the region on the left (right)*. As a direct consequence of [26, Proposition 6.1], we have the following conclusions.

Theorem 3.3. *Let* $\omega \in \Omega_k$ *and* $(\tau_1, \tau_2) \in \Gamma^{\pm k}_{n_1,n_2}$ *such that* $i\omega$ *is a simple solution of* $g_1(\zeta) = 0$*. Then, as* (τ_1, τ_2) *moves from the region on the right to the region on the left of the crossing curve, a pair of complex roots of* $g_1(\zeta) = 0$ *cross the imaginary axis to the right if*

$$R_2 I_1 - R_1 I_2 > 0, \tag{3.14}$$

where

$$R_1 = \mathrm{Re}\left(\frac{\partial g_1(\zeta, \tau_1, \tau_2)}{\partial \tau_1} \right) = D_1\omega^2 \cos(\omega\tau_1) - \omega C_1 \sin(\omega\tau_1) - H_1\omega \sin(\omega(\tau_1 + \tau_2)),$$
$$I_1 = \mathrm{Im}\left(\frac{\partial g_1(\zeta, \tau_1, \tau_2)}{\partial \tau_1} \right) = -D_1\omega^2 \sin(\omega\tau_1) - \omega C_1 \cos(\omega\tau_1) - H_1\omega \cos(\omega(\tau_1 + \tau_2)),$$
$$R_2 = \mathrm{Re}\left(\frac{\partial g_1(\zeta, \tau_1, \tau_2)}{\partial \tau_2} \right) = Q_1\omega^2 \cos(\omega\tau_1) - \omega L_1 \sin(\omega\tau_1) - H_1\omega \sin(\omega(\tau_1 + \tau_2)),$$
$$I_2 = \mathrm{Im}\left(\frac{\partial g_1(\zeta, \tau_1, \tau_2)}{\partial \tau_2} \right) = -Q_1\omega^2 \sin(\omega\tau_1) - \omega L_1 \cos(0\omega\tau_1) - H_1\omega \cos(\omega(\tau_1 + \tau_2)).$$

The crossing is in the opposite direction if (3.14) *is reversed.*

4. Numerical simulation

We illustrate the occurrence of the Hopf bifurcation for system (1.3) by numerical simulations. This study employs the finite difference method to discretize system (1.3) with a time step of 0.01, a spatial step of 0.1, a time span of 50000, and a spatial span of 200. Choose the following set of parameters:

$$r_0 = 2.5,\ a = 0.04,\ k = 8,\ p = 1.2,\ q = 0.6,\ M = 2.5,\ \mu = 0.8,\ c = 1.8,\ d = 0.2,\ h_0 = 0.1. \tag{4.1}$$

When $\tau_1 = \tau_2$, we have the following:

$$A_2 = 0.9630; \quad B_2 = 1.8421; \quad C_2 = 0.0333; \quad \text{and} \quad D_2 = -0.9.$$

By direct calculation, we obtain the following:

$$\frac{\mathrm{d}(\mathrm{Re}\zeta(\tau))}{\mathrm{d}\tau}\Big|_{\tau=\tau_1^j} \approx -0.1847 < 0$$

and

$$\frac{\mathrm{d}(\mathrm{Re}\zeta(\tau))}{\mathrm{d}\tau}\Big|_{\tau=\tau_2^j} \approx 0.1882 > 0,$$

where $j \in \mathbb{N}_0$.

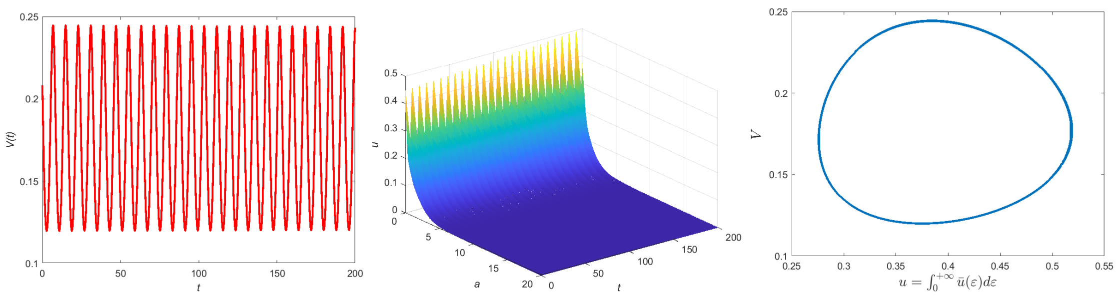
Furthermore, we can compute the values for

$$\omega_1 = 0.3216, \quad \omega_2 = 4.2176, \quad \tau_1^0 = 0.5547, \quad \text{and} \quad \tau_1^1 = 9.2034$$

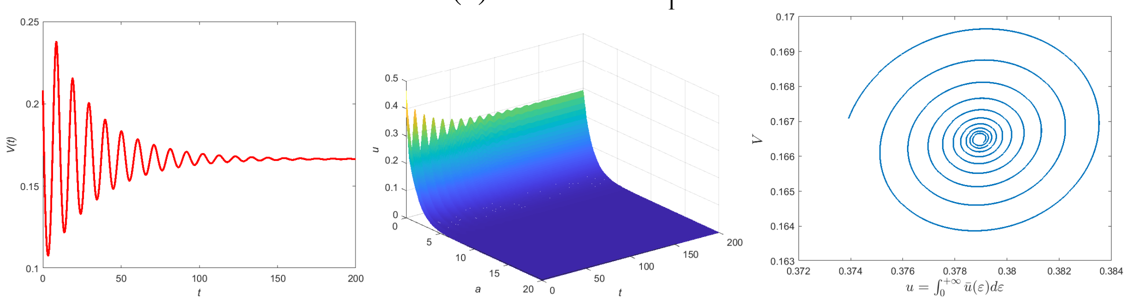
when $\omega_1 = 0.3216$ and compute

$$\tau_2^0 = 7.2986, \quad \tau_2^1 = 14.4225$$

when $\omega_2 = 4.2176$. It follows from Theorem 3.1 that (1.3) has periodic solutions for $\tau < \tau_1^0$, and the positive equilibrium of (1.3) is stable if $\tau > \tau_1^0$. Hence, increasing τ will stabilize the equilibrium (Figure 1). In addition, with an increasing τ, the stability switches are observed, that is, the positive equilibrium of system is unstable for $\tau \in [0, \tau_1^0), (\tau_2^0, \tau_1^1)$ and stable for $\tau \in (\tau_1^0, \tau_2^0), (\tau_1^1, \tau_2^1)$ (Figure 2).

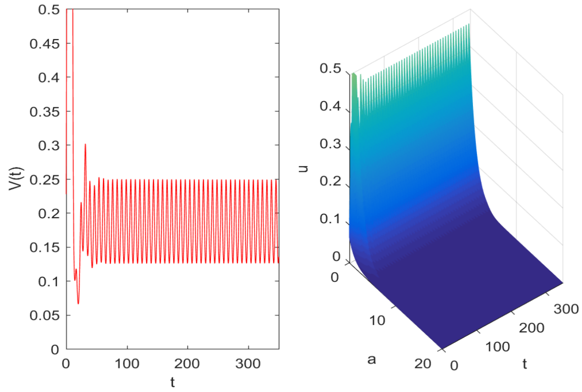


(*a*) $\tau = 0.5 < \tau_1^0$

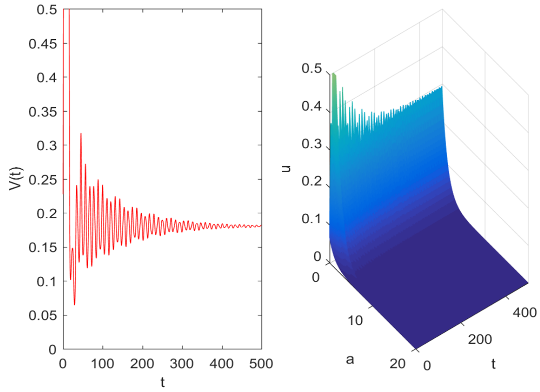


(*b*) $\tau_1^0 < \tau = 1.2 < \tau_2^0$

Figure 1. (*a*) The positive equilibrium of (1.3) has periodic solutions for $\tau = 0.5 < \tau_1^0$. (*b*) The positive equilibrium of (1.3) is stable $\tau_1^0 < \tau = 1.2 < \tau_2^0$.



$(a) \quad \tau_2^0 < \tau = 7.5 < \tau_1^1$



$(b) \quad \tau_1^1 < \tau = 12 < \tau_2^1$

Figure 2. Solutions of (1.3) with different choice of time delays.

When $\tau_1 \neq \tau_2$, we have

$$A_1 = 0.4006, \quad B_1 = -0.3195, \quad C_1 = 0.3595, \quad D_1 = -0.9, \quad L_1 = 2.1616, \quad H_1 = -0.3262,$$

and $Q_1 = 0.3624$. Here, the parameter values are given by (4.1), and the functions

$$V(0) = 0.2083 \quad \text{and} \quad u(0, a) = 0.3823e^{-0.8\varepsilon}$$

are assigned to the initial values. By plotting the graph of $F(\omega)$ (Figure 3), we have

$$\Omega = [1.015, 1.838];$$

thus, the crossing curves of

$$g_1(\zeta) = 0$$

can be obtained according to (3.10) and (3.13). Moreover, the quantity $R_2 I_1 - R_1 I_2$, which determines the crossing directions, can be also calculated by Theorem 3.3 (Figure 4). In Figure 4, Arrows represent the crossing directions, that is, the region on the end of an arrow has two more characteristic roots with positive real parts. These curve will determine the stable region of the positive steady state of (1.3) (i.e., the bottom-left region in (τ_1, τ_2)-plane, where the point B locates at). In addition, when the parameters (τ_1, τ_2) pass through the crossing curves, a periodic solution will be bifurcated from the positive steady state through the Hopf bifurcation. Additionally, from Figure 4, the phenomenon of stability switches takes place, as the parameters (τ_1, τ_2) move from point A (periodic solution) to B (stable steady state) and then to C (periodic solution again). The phase portraits of (1.3), with (τ_1, τ_2) determined by A, B and C, are shown in Figure 5.

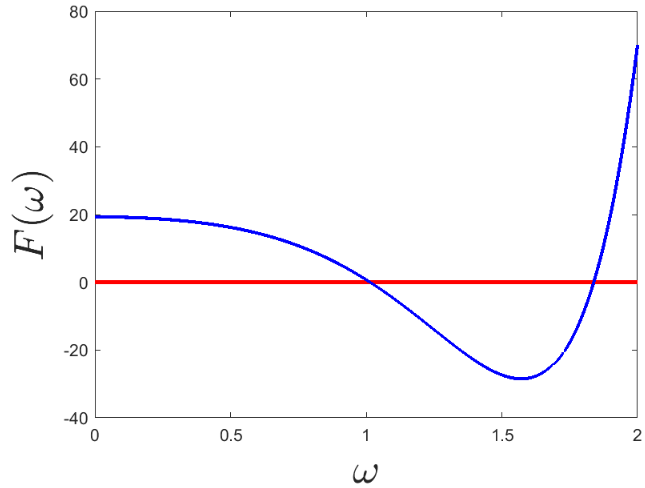


Figure 3. The graph of $F(\omega)$.

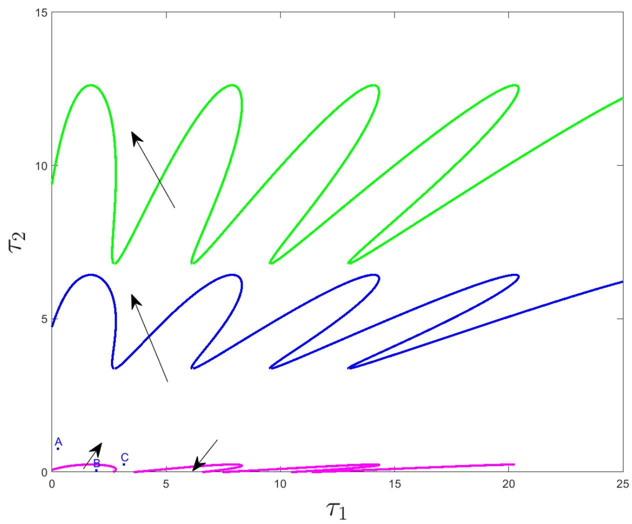
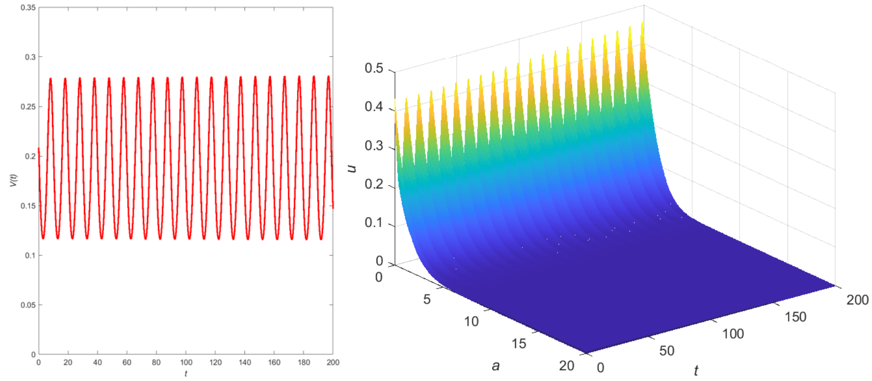
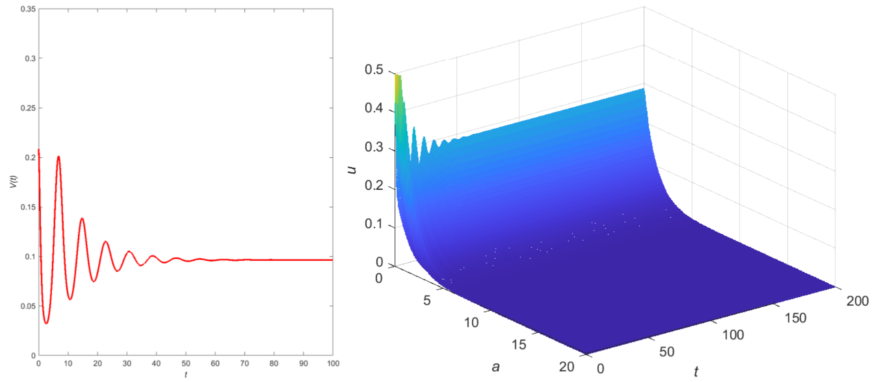


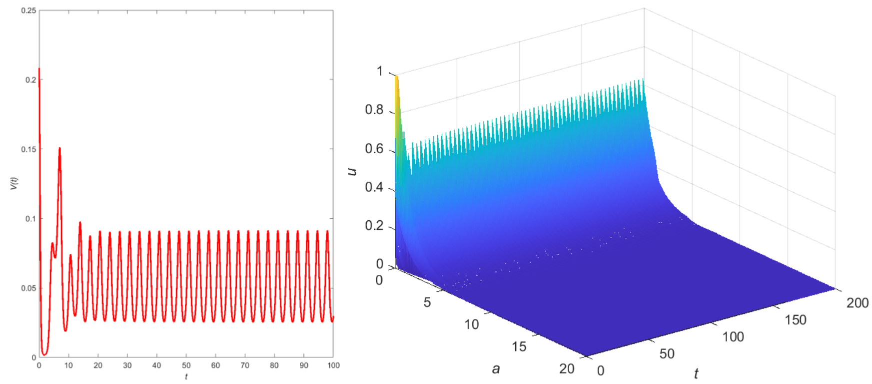
Figure 4. The crossing curves of $g(\zeta) = 0$.



(*a*) $\tau_1 = 0.13, \tau_2 = 1$



(*b*) $\tau_1 = 1.8, \tau_2 = 0.3$



(*c*) $\tau_1 = 3, \tau_2 = 0.5$

Figure 5. A solution of (1.3) with different choices of time delays.

5. Conclusions

This paper studied a predator-prey model with an age structure. The stability of the positive equilibrium and the existence of a Hopf Bifurcation in the positive equilibrium were studied by the integral semigroup theory and the Hopf Bifurcation theory for abstract non-densely defined Cauchy problems. The crossing curves in (τ_1, τ_2)-plane, on which the characteristic equation had purely imaginary roots, were obtained. From these curves, we showed that the model could exhibit rich dynamics, including a Hopf bifurcation and stability switches. The numerical simulation results further verified the theoretical conclusions, thus enabling ecologists to predict population dynamic changes and take targeted measures (e.g., adjusting habitat conditions to modify the maturation period of predators) to maintain ecological balance.

The model's integration of the predator age structure and dual time delays (maturation and gestation delays) accurately captures the biological characteristics of predator populations, the fact that juvenile predators cannot reproduce, and the reproductive process has an inherent gestation period, thus making the model a realistic reflection of the population development rules of predators such as mammals, birds, and fish in natural ecosystems. This overcomes the limitations of traditional models that treat predators as a homogeneous population, and provides a more accurate theoretical framework to analyze the dynamic changes of predator-prey communities with an obvious age differentiation. In summary, this research not only enriches the theoretical system of age-structured predator-prey dynamics, but also provides an actionable scientific basis for the conservation of biological diversity, the management of fishery and wildlife resources, and the restoration of damaged ecological systems, thus making it an important link between theoretical ecological mathematics and practical ecological governance.

Author contributions

Wenjie Li: completed the establishment and solution of the model, completed the numerical simulation, finished the writing of the manuscript; Dan Liu: finished the writing of the manuscript; Yuting Cai: completed the establishment and solution of the model, completed the numerical simulation, finished the writing of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The author's research is partially supported by Youth Fund of Shandong Provincial Natural Science Foundation of China (No. ZR2023QA068).

Conflict of interest

The authors declare that they have no conflicts of interest.

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Supplementary

Numerical simulation programs for Figures 1 and 3

```
clear;
r=2.5; a=0.04; K=8; p=1.2; q=0.6; M=2.5; mu=0.8; c=1.8;
d=0.2; h0=0; tau=1.2; beta=mu*M*exp(mu*tau);
dt=0.01; n=35000; da=0.1; m=200;k=floor(tau/da); V=zeros(1,n+1);
V(1:1)=feval(@(x)(0.2273));u=zeros(m+1,n+1);
u(:,1)=feval(@(x)(0.0674*exp(-mu*x)),0:da:m*da);
for i=1:n
V(i+1)=V(i)+(dt*(r*V(i)(1+K*da*sum(u(:,i)))-d*V(i)-a*V(i)^2)
-dt*p*V(i)*da*sum(u(:,i))(q*V(i)+1));
for j=2:m
u(j,i+1)=u(j,i)-dt*(u(j+1,i)-u(j-1,i))(2*da)-dt*mu*u(j,i);
end
u(m+1,i+1)=u(m+1,i)-dt*(u(m+1,i)-u(m,i))/da-dt*mu*u(m+1,i);
u(1,i+1)=c*p*V(i+1)*beta*da*sum(u(k:m+1,i+1))/(q*V(i)+1);
end
figure(10)
subplot(1,2,1)
t1=0:dt:dt*n;
plot(t1,V,'r','LineWidth',0.5) set(gca,'XLim',[0 350]) set(gca,'YLim',[0 0.5])
```

```
subplot(1,2,2);
t=0:dt:dt*n; t_size = size(t); t_n = t_size(2); m1=0:da:1*m*da; m1 = m1';
a_size = size(m1); a_n = a_size(1); T = ones(a_n,1)*t; A = m1*ones(1,t_n);
mesh(T,A,u); set(gca,'YDir','reverse')
set(gca,'XLim',[0 350]) set(gca,'ZLim',[0 0.5])
disp(V); disp(da*sum(u(:,i))); disp(u(1,1));
```

Numerical simulation programs for Figure 2

```
clear; clc;
A=0.7432;B= -0.0455;C= 0.0558;D= -0.9818;L= 1.5061; H=-0.0647;Q= 0.0537;
omega=0:0.001:2.5; y1=0*ones(1,length(omega));
F=(omega.4̂-(2*B-A2̂-D2̂+Q2̂).*omega.2̂+(B2̂+C2̂-L2̂-H2̂)).2̂-4*(((C-A*D).*omega.2̂+(L*H-
B*C)).2̂+((Q*H-A*C+B*D).*omega-D*omega.3̂).2̂);
plot(omega,y1,'r'); hold on; plot(omega,F,'b');
set(h,'Interpreter','latex','FontSize',25)
set(h,'Interpreter','latex','FontSize',25)
omegasl=fsolve(@(omega)(omega.4̂-(2*B-A2̂-D2̂+Q2̂).*omega.2̂+(B2̂+C2̂-L2̂-H2̂)).2̂
-4*(((C-A*D).*omega.2̂+(L*H-B*C)).2̂+((Q*H-A*C+B*D).*omega-D*omega.3̂).2̂),0.01)
omegasr=fsolve(@(omega)(omega.4̂-(2*B-A2̂-D2̂+Q2̂).*omega.2̂+(B2̂+C2̂-L2̂-H2̂)).2̂
-4*(((C-A*D).*omega.2̂+(L*H-B*C)).2̂+((Q*H-A*C+B*D).*omega-D*omega.3̂).2̂),6)
omega=omegasl:0.001:omegasr;
a1=(L*H-B*C+(C-A*D).*omega.2̂)+(Q*H-A*C+B*D-D.*omega.2̂).*omega.*i;
a2=(H*C-B*L+(L-A*Q).*omega.2̂)+(Q.*omega.3̂-(B*Q+A*L).*omega-D*H).*i;
module1=(omega.4̂-(2*B-A2̂-D2̂+Q2̂).*omega.2̂+(B2̂+C2̂-L2̂-H2̂));
module2=(omega.4̂-(2*B-A2̂-D2̂+Q2̂).*omega.2̂+(B2̂-C2̂+L2̂-H2̂));
PF1=(L*H-B*C+(C-A*D).*omega.2̂).2̂+((Q*H-A*C+B*D-D.*omega.2̂).*omega).2̂;
PF2=(H*C-B*L+(L-A*Q).*omega.2̂).2̂+(-Q.*omega.3̂+(B*Q+A*L).*omega+D*H).2̂;
temp1=module1./(2*sqrt(PF1))
temp2=module2./(2*sqrt(PF2))
psi1=acos(temp1); psi2=acos(temp2);
tau0=(psi1-angle(a1)+2*(0)*pi)./omega;sigma0=(-psi2-angle(a2)+2*(0)*pi)./omega;
ttau0=(-psi1-angle(a1)+2*(0)*pi)./omega;ssigma0=(psi2-angle(a2)+2*(0)*pi)./omega;
tau1=(psi1-angle(a1)+2*(1)*pi)./omega;sigma1=(-psi2-angle(a2)+2*(1)*pi)./omega;
tau2=(psi1-angle(a1)+2*(2)*pi)./omega;sigma2=(-psi2-angle(a2)+2*(2)*pi)./omega;
tau3=(psi1-angle(a1)+2*(3)*pi)./omega;sigma3=(-psi2-angle(a2)+2*(3)*pi)./omega;
tau4=(psi1-angle(a1)+2*(4)*pi)./omega;sigma4=(-psi2-angle(a2)+2*(4)*pi)./omega;
ttau1=(-psi1-angle(a1)+2*(1)*pi)./omega;ssigma1=(psi2-angle(a2)+2*(1)*pi)./omega;
ttau2=(-psi1-angle(a1)+2*(2)*pi)./omega;ssigma2=(psi2-angle(a2)+2*(2)*pi)./omega;
ttau3=(-psi1-angle(a1)+2*(3)*pi)./omega;ssigma3=(psi2-angle(a2)+2*(3)*pi)./omega;
ttau4=(-psi1-angle(a1)+2*(4)*pi)./omega;ssigma4=(psi2-angle(a2)+2*(4)*pi)./omega; figure
plot(tau0,sigma0,'m') hold on; plot(ttau0,ssigma0,'m'); plot(tau0,sigma1,'b');plot(ttau0,ssigma1,'b');
plot(tau0,sigma2,'g');plot(ttau0,ssigma2,'g'); plot(tau1,sigma0,'m');plot(ttau1,ssigma0,'m');
plot(tau1,sigma1,'b');plot(ttau1,ssigma1,'b'); plot(tau1,sigma2,'g');plot(ttau1,ssigma2,'g');
plot(tau2,sigma0,'m');plot(ttau2,ssigma0,'m'); plot(tau3,sigma0,'m');plot(ttau3,ssigma0,'m');
```

plot(tau1,sigma2,'g');plot(ttau1,ssigma2,'g'); plot(tau2,sigma1,'b');plot(ttau2,ssigma1,'b'); plot(tau3,sigma1,'b');plot(ttau3,ssigma1,'b'); plot(tau4,sigma1,'b');plot(ttau4,ssigma1,'b'); plot(tau2,sigma2,'g');plot(ttau2,ssigma2,'g'); plot(tau3,sigma2,'g');plot(ttau3,ssigma2,'g'); plot(tau4,sigma2,'g');plot(ttau4,ssigma2,'g'); h=xlabel('τ_1'); set(h,'Interpreter','latex','FontSize',25) h=ylabel('τ_2');
set(h,'Interpreter','latex','FontSize',25) axis([0,25,0,20]); text(2,1,'.','color','b','FontSize',20); text(2,1,'B','color','b','FontSize',10); text(0.5,1.5,'.','color','b','FontSize',20); text(0.5,1.5,'A','color','b','FontSize',10); text(3.8,1,'.','color','b','FontSize',20); text(3.8,1,'C','color','b','FontSize',10); omega1=0.7134; module1=(omega.$\hat{4}$-(2*B-A$\hat{2}$-D$\hat{2}$+Q$\hat{2}$).*omega.$\hat{2}$+(B$\hat{2}$+C$\hat{2}$-L$\hat{2}$-H$\hat{2}$)); PF1=(L*H-B*C+(C-A*D).*omega.$\hat{2}$).$\hat{2}$+((-Q*H-A*C+B*D-D.*omega.$\hat{2}$).*omega).$\hat{2}$; temp1=module1./(2*sqrt(PF1)); psi1=acos(temp1);