Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(3): 6935–6954.
DOI: 10.3934/math.2026285
Received: 23 January 2026
Revised: 05 March 2026
Accepted: 09 March 2026
Published: 17 March 2026

Research article

Optimal controllability of stochastic multi-term Hilfer fractional systems with delayed controls

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Abstract: This paper investigates the optimal controllability of stochastic multi-term Hilfer fractional differential systems with finite delays in the control function. The considered model combines multi-order fractional dynamics, stochastic perturbations, and delayed control actions, which provide a unified framework to describe complex dynamical processes with memory and uncertainty. By employing the theory of fractional calculus and the concept of a (δ, β_κ)-resolvent family, we establish the existence and uniqueness of mild solutions for the proposed system. Balder's theorem and variational techniques demonstrate the existence of an optimal state-control pair minimising a quadratic cost functional. A diffusion-type fractional stochastic model with delayed control is presented to illustrate the applicability of the theoretical findings. The obtained results not only extend several existing works on fractional stochastic control systems, but also provide new insights into the analysis of Hilfer-type operators with multiple fractional orders and control delays, which are relevant in viscoelastic materials, heat transfer processes, and neural network models with memory effects.

Keywords: Hilfer fractional derivative; optimal control; stochastic system; delay equation; fixed-point theorem
Mathematics Subject Classification: 26A33, 34K50, 47H10, 60H15, 93E20

1. Introduction

Classical differential equations often lag in modeling complex real-world phenomena with memory effects, hereditary characteristics, or spatial non-locality. With ongoing advancements in science and technology, it has become well established that many physical, biological, and engineering systems cannot be adequately described using traditional integer-order models. This limitation has led to the development of fractional calculus, which is an extension of classical calculus that deals with derivatives and integrals of arbitrary (non-integer) order.

The conceptual roots of fractional calculus trace back to 1695, when Leibniz first proposed the idea of a fractional derivative. Subsequent foundational contributions from Euler, Liouville, Riemann, and Caputo have formalized and expanded the theory [1–3]. Unlike classical (local) derivatives, fractional derivatives are nonlocal operators, meaning that their evaluation at a point depends on the function's behavior over an interval. This nonlocal property makes them particularly well-suited for modeling systems with memory, long-range dependence, and spatial heterogeneity. Consequently, fractional differential equations (FDEs) have gained substantial attention and found applications in diverse disciplines, including viscoelasticity, anomalous diffusion, fluid dynamics, electromagnetism, control theory, and biological modeling [4, 5].

Among the various formulations, the Hilfer fractional derivative provides a unified framework that interpolates between the Riemann–Liouville and Caputo derivatives, offering greater flexibility in capturing different memory effects [6, 7]. This semi-local operator has been extensively applied in processes with intermediate levels of memory, making it especially relevant in fractional control systems.

At the same time, stochastic differential equations (SDEs) have become indispensable to model systems influenced by uncertainty and noise. When combined with fractional dynamics, stochastic perturbations give rise to stochastic fractional differential equations (SFDEs), a powerful class of models that more accurately capture the behavior of complex, uncertain systems [8, 9]. Furthermore, in practical applications, control delays frequently arise due to information-processing lags, actuator response times, or transport delays. Such delayed control systems are standard in population dynamics, neuroscience, industrial processes, and aerospace engineering. The interplay of stochasticity, memory effects, and delays, particularly within nonlinear Hilfer fractional stochastic differential equations with delayed control, presents significant mathematical challenges and constitutes a vibrant area of contemporary research [10, 11].

The study of optimal control has become increasingly important in addressing the regulation and performance optimization of complex dynamical systems [12]. Particular emphasis has been placed on infinite-dimensional nonlinear systems, where the existence of optimal controls has been rigorously investigated [13, 14]. The development of fractional optimal control frameworks, particularly those based on the Riemann–Liouville derivative, has introduced generalized strategies to formulate and solve control problems [15].

In recent years, multi-term time-fractional differential systems have attracted considerable interest for modeling complex processes in engineering and applied sciences, particularly due to their capability to describe memory-dependent dynamics. Several contributions have addressed both theoretical and applied aspects of such systems. For instance, [16, 17] studied two-term time-fractional systems in an abstract framework and demonstrated their applications to diffusion–wave problems. Singh and Pandey [18] established new controllability results for multi-term fractional systems. Subsequently, Chaudhary et al. [19] extended the analysis to systems incorporating state-dependent delays, while Singh et al. [20] examined multi-term time-fractional stochastic systems and analyzed their qualitative properties.

More recently, Afreen et al. [21] investigated the qualitative behavior of multi-term Caputo fractional stochastic systems and studied their controllability by analyzing systems of the form:

$$\begin{cases} {}^{c}D_t^{1+\alpha} z(t) + \sum_{\iota=1}^{m} \beta_\iota \, {}^{c}D_t^{\gamma_\iota} z(t) = Az(t) + \sum_{i=0}^{r} E_i \, u(t-\eta_i) + g_1(t, z(t), u(t)) + g_2(t, z(t), u(t)) \, \dfrac{d\nu(t)}{dt}, \\ \hfill t \in [0, \ell], \\ z(0) = z_0, \; z'(0) = z_1, \\ u(t) = 0, \quad t \in [-\eta_r, 0]. \end{cases}$$

Moreover, Bouguetof et al. [22] studied the existence and uniqueness of solutions for multi-fractional stochastic delay differential equations driven by Riemann–Liouville multi-fractional Brownian motion and standard Brownian motion. Advanced control protocols for high-order nonlinear systems, such as fast finite-time adaptive controllers for multi-agent systems, have also been developed in [23].

Despite these significant contributions, most existing studies focus on either on single-term fractional stochastic systems, deterministic multi-term models, or controllability problems without considering optimal control in the presence of finite control delays. For instance, Aydin [24] investigated the controllability of nonlinear fractional systems with multiple state and control delays, however, the corresponding optimal control problem was not addressed. The simultaneous consideration of multiple fractional orders, the Hilfer derivative, stochastic perturbations, and delayed control actions introduces substantial analytical challenges.

Motivated by this research gap, the present work establishes the existence and uniqueness of mild solutions for the proposed stochastic multi-term Hilfer fractional system. Furthermore, we prove the existence of an optimal state–control pair that minimizes a prescribed cost functional. These results extend and strengthen several existing contributions in the field of the fractional stochastic control theory.

In this work, we examine the following system:

$$\begin{cases} {}^{H}D_{0^+}^{1+\delta,\gamma}(y(\rho)) + \sum_{k=1}^{m} C_\kappa \, {}^{H}D_{0^+}^{\beta_\kappa,\gamma}(y(\rho)) = Ay(\rho) + \sum_{i=0}^{m} E_i \mu(\rho - \eta_i) \\ \qquad + f\left(\rho, y(\rho), \int_0^\rho y(s)\,ds\right) + g\left(\rho, y(\rho), \int_0^\rho y(s)\,ds\right) \dfrac{dw(\rho)}{d\rho}, \quad \rho \in \mathcal{J}' = (0, \ell], \\ \left(I_{0^+}^{2-\nu} y\right)(0) = y_0, \quad \left(I_{0^+}^{2-\nu} y\right)'(0) = y_1, \\ \mu(\rho) = 0, \quad \rho \in [-\eta_m, 0], \end{cases} \tag{1.1}$$

where $0 < \delta \leq \beta_m \leq \cdots \leq \beta_1 < 1$. The operators ${}^{H}D_{0^+}^{1+\delta,\gamma}$ and ${}^{H}D_{0^+}^{\beta_\kappa,\gamma}$ denote the Hilfer fractional derivatives of order $1 + \delta$ and β_κ, respectively, with type γ. The operator $I_{0^+}^{2-\nu}(\cdot)$ represents the Riemann–Liouville fractional integral of order $2 - \nu$, where $\nu = \delta + \gamma - \gamma\delta + 1$.

Let $A : D(A) \subseteq Y \to Y$ be a closed linear operator on the separable Hilbert space Y, endowed with the norm $\|\cdot\|$. Let $\mathcal{J} = [0, \ell]$ be the time interval.

The state function $y(\cdot)$ takes values in Y, while the control function $\mu(\cdot)$ takes values in the admissible control set Θ_{ad}, and $E_1, E_2, \ldots, E_m \in \mathcal{L}(X, Y)$ be bounded linear operators, where X is another separable Hilbert space, and $0 = \eta_0 < \eta_1 < \cdots < \eta_m$ are constant point delays.

Moreover, let $\{w(\rho)\}_{\rho\geq 0}$ be an H-valued Wiener process with a covariance operator $Q \geq 0$, where H is also a separable Hilbert space. The admissible control set Θ_{ad}, as well as the functions $f : \mathcal{J} \times Y \times Y \to Y$, $g : \mathcal{J} \times Y \times Y \to L^2(H, Y)$ will be specified later.

The main contributions of this paper are summarized as follows:

- We formulate and analyze an optimal control problem for stochastic multi-term Hilfer fractional differential systems with finite control delays in an abstract Hilbert space framework.
- We establish the existence of mild solutions using the (δ, β_κ)-resolvent family.
- We prove the existence of an optimal control pair minimizing a quadratic cost functional via variational techniques and Balder's theorem.
- An illustrative fractional stochastic diffusion model is provided to demonstrate the applicability of the results.

The structure of this article is as follows: Section 2 presents the essential concepts and preliminary definitions that serve as the foundation for the analysis of the problem. Section 3 is devoted to investigating the existence of mild solutions for the optimal control of multi-term Hilfer stochastic fractional systems with finite delays in control. Section 4 focuses on the existence of an optimal control associated with the proposed system. Section 5 provides an example to illustrate and validate the theoretical results.

2. Preliminaries and assumptions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho\geq 0}, \mathbb{P})$ be a complete probability space, equipped with a standard filtration $\{\mathcal{F}_\rho\}$ that satisfies the usual conditions. Let $\{w(\rho)\}_{\rho\geq 0}$ be an H-valued Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho\geq 0}, \mathbb{P})$ with a covariance operator $Q \geq 0$ that satisfies $\mathrm{Tr}(Q) < \infty$. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis of H, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a bounded sequence of non-negative real numbers such that

$$Qe_n = \alpha_n e_n.$$

Moreover, for any $e \in H$ and $\rho \geq 0$,

$$\langle w(\rho), e\rangle = \sum_{n=1}^{\infty} \sqrt{\alpha_n}\langle e_n, e\rangle \lambda_n(\rho),$$

where $\{\lambda_n\}_{n\geq 1}$ are mutually independent one-dimensional standard Wiener processes. The space $L^2(Q^{\frac{1}{2}}H, Y)$ denotes the collection of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}}H$ to Y, endowed with the inner product $\langle ð, ð\rangle = \mathrm{Tr}(ðQð^*) < \infty$.

Let $L^2(\Omega, \mathcal{F}_\rho, \mathbb{P}; Y) \equiv L^2(\Omega, Y)$ denote the space of all Y-valued, strongly measurable, square-integrable random variables, equipped with the norm

$$\left\|y(\cdot)\right\|_{L^2(\Omega,Y)} = \left(\mathbb{E}\left\|y(\cdot, w)\right\|_Y^2\right)^{1/2}.$$

The expectation is represented by $\mathbb{E}(y) = \int_\Omega y(\omega)\, d\mathbb{P}(\omega)$. We consider the subspace of $L^2(\Omega, Y)$ given by:

$$L_0^2(\Omega, Y) = \left\{y \in L^2(\Omega, Y) : y \text{ is } \mathcal{F}_0 - \text{measurable}\right\}.$$

$\mathfrak{C}(\mathcal{J}, L^2(\Omega, Y))$ represents the Banach space of all continuous functions from $\mathcal{J}$ to $L^2(\Omega, Y)$ provided that $\sup_{\rho\in\mathcal{J}} \mathbb{E}\|y(\rho)\|^2 < \infty$. Let us define the Banach space $\mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y))$

$$\mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y)) = \left\{ y \in \mathfrak{C}(\mathcal{J}', L^2(\Omega, Y)) | \lim_{\rho\to 0} \rho^{2-\nu} y(\rho) \text{ exists and finite} \right\},$$

with the norm

$$\|y\|_{\mathfrak{C}_{2-\nu}} = \left(\sup_{\rho\in\mathcal{J}} \mathbb{E}\|\rho^{2-\nu} y(\rho)\|^2 \right)^{1/2}.$$

For simplicity, we denote this Banach space by $\mathfrak{C}_{2-\nu}$. Clearly, $\mathfrak{C}_{2-\nu}$ is a closed subspace of $\mathfrak{C}(\mathcal{J}', L^2(\Omega, Y))$.

Definition 2.1. *[25] The left-sided Hilfer fractional derivative of order* $1 < 1 + \delta < 2$ *and type* $0 \le \gamma \le 1$ *of function* $y(\cdot)$ *is defined as:*

$${}^{H}D_{0^+}^{\delta+1,\gamma} y(\rho) = \left(I_{0^+}^{\gamma[2-(\delta+1)]} \frac{d^2}{d\rho^2} I_{0^+}^{(1-\gamma)[2-(\delta+1)]} y \right)(\rho).$$

Definition 2.2. *[25] The left-sided Riemann-Liouville fractional integral of order* $1+\delta$ *of function* $y(\cdot)$ *is defined as:*

$$I_{0^+}^{\delta+1} y(\rho) = \frac{1}{\Gamma(\delta+1)} \int_0^{\rho} (\rho - s)^{\delta} y(s)\, ds.$$

Lemma 2.3. *[26] If* $\left(I_{0^+}^{2-\nu} y\right)(\rho)$ *is continuous and* $\left(I_{0^+}^{2-\nu} y\right)'(\rho)$ *is absolutely continuous, then*

$$I_{0^+}^{\nu} D_{0^+}^{\nu} y(\rho) = y(\rho) - \frac{1}{\Gamma(\nu)} \rho^{\nu-1} y_1 - \frac{1}{\Gamma(\nu-1)} \rho^{\nu-2} y_0.$$

Definition 2.4. *[27] Let* C_κ, δ, *and* β_κ *be positive real numbers for all* $\kappa = 1, 2, \ldots, m$. *The closed linear operator A generates a* (δ,β_κ)*-resolvent family if there exists a strongly continuous function* $S_{\delta,\beta_\kappa} : \mathbb{R}^+ \to \mathcal{L}(Y)$ *and a constant* $\omega > 0$ *such that* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$, *for every* $y \in Y$ *and* $\operatorname{Re}\lambda > \omega$, *we have*

$$\lambda^{\delta} \left(\lambda^{\delta+1} + \sum_{\kappa=1}^{m} C_\kappa \lambda^{\beta_\kappa} - A \right)^{-1} y = \int_0^{\infty} e^{-\lambda\rho} S_{\delta,\beta_\kappa}(\rho)\, y\, d\rho.$$

Theorem 2.5. *[27] Let* $0 < \delta \le \beta_m \le \cdots \le \beta_1 \le 1$, *and assume that* $C_\kappa \ge 0$ *for every* $\kappa = 1, 2, \ldots, m$. *If A is the infinitesimal generator of a bounded, strongly continuous cosine family* $\{C(\rho)\}_{\rho\in[0,\ell]}$, *then A admits a bounded* (δ,β_κ)*-resolvent family* $\{S_{\delta,\beta_\kappa}(\rho)\}_{\rho\ge 0}$.

Lemma 2.6. *Equation* (1.1) *is equivalent to the following integral equation:*

$$\begin{aligned}
&y(\rho) + \sum_{k=1}^{m} \frac{C_\kappa}{\Gamma(\delta+1-\beta_\kappa)} \int_0^{\rho} (\rho - s)^{\delta-\beta_\kappa} y(s)\, ds \\
=& \frac{1}{\Gamma(\nu)} \rho^{\nu-1} y_1 + \frac{1}{\Gamma(\nu-1)} \rho^{\nu-2} y_0 \\
&+ \frac{1}{\Gamma(\delta+1)} \int_0^{\rho} (\rho - s)^{\delta} \left(Ay(s) + \sum_{i=0}^{m} E_i \mu(s - \eta_i) + f\left(s, y(s), \int_0^{s} y(\xi)\, d\xi\right)\right) ds \\
&+ \frac{1}{\Gamma(\delta+1)} \int_0^{\rho} (\rho - s)^{\delta} \left(g\left(s, y(s), \int_0^{s} y(\xi)\, d\xi\right)\right) dw(s).
\end{aligned} \tag{2.1}$$

Proof. From Definition 2.1,we have

$$ {}^{H}D_{0^+}^{\delta+1,\gamma}y(\rho) = \left(I_{0^+}^{(-1-\delta)}I_{0^+}^{\nu}D_{0^+}^{\nu}y\right)(\rho). $$

From Lemma 2.3, Definition 2.2, applying the operator $I_{0^+}^{\delta+1}$ both side of (1.1), we obtain

$$ \begin{aligned} y(\rho) + \sum_{k=1}^{m}\frac{C_\kappa}{\Gamma(\delta+1-\beta_\kappa)}\int_0^\rho(\rho-s)^{\delta-\beta_\kappa}y(s)\,ds &= \frac{1}{\Gamma(\nu)}\rho^{\nu-1}y_1 + \frac{1}{\Gamma(\nu-1)}\rho^{\nu-2}y_0 \\ &+ \frac{1}{\Gamma(\delta+1)}\int_0^\rho(\rho-s)^\delta\left(Ay(s) + \sum_{i=0}^{m}E_i\mu(s-\eta_i) + f\left(s, y(s), \int_0^s y(\xi)\,d\xi\right)\right)ds \\ &+ \frac{1}{\Gamma(\delta+1)}\int_0^\rho(\rho-s)^\delta\left(g\left(s, y(s), \int_0^s y(\xi)\,d\xi\right)\right)dw(s). \end{aligned} $$

The proof is completed. □

Lemma 2.7. *A stochastic process $y \in \mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y))$ is said to be a* mild solution *of (1.1) if y satisfies*

$$ \begin{aligned} y(\rho) =& \frac{y_0}{\Gamma(\nu-1)}\int_0^\rho(\rho-s)^{\nu-2}S_{\delta,\beta_\kappa}(s)\,ds + \frac{y_1}{\Gamma(\nu)}\int_0^\rho(\rho-s)^{\nu-1}S_{\delta,\beta_\kappa}(s)\,ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(\sum_{i=0}^{m}E_i\mu(s-\eta_i) + f\left(s, y(s), \int_0^s y(\xi)\,d\xi\right)\right)ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)g\left(s, y(s), \int_0^s y(\xi)d\xi\right)dw(s) \qquad \rho\in\mathcal{J}', \end{aligned} \tag{2.2} $$

where

$$ \mathcal{I}_{\delta,\beta_\kappa}(\rho) = \frac{1}{\Gamma(\delta)}\int_0^\rho(\rho-\tau)^{\delta-1}S_{\delta,\beta_\kappa}(\tau)\,d\tau. $$

Proof. Applying the Laplace transform to both side of (2.1), we obtain

$$ \begin{aligned} Y(\lambda) =& \lambda^{1-\nu}\lambda^\delta\left(\lambda^{\delta+1} + \sum_{k=1}^{m}C_\kappa\lambda^{\beta_\kappa} - A\right)^{-1}y_0 + \lambda^{-\nu}\lambda^\delta\left(\lambda^{\delta+1} + \sum_{k=1}^{m}C_\kappa\lambda^{\beta_\kappa} - A\right)^{-1}y_1 \\ &+ \lambda^{-\delta}\lambda^\delta\left(\lambda^{\delta+1} + \sum_{k=1}^{m}C_\kappa\lambda^{\beta_\kappa} - A\right)^{-1}(U(\lambda) + F(\lambda) + G(\lambda)), \end{aligned} \tag{2.3} $$

where

$$ \begin{aligned} Y(\lambda) &= \int_0^\infty e^{-\lambda s}y(s)\,ds, \\ F(\lambda) &= \int_0^\infty e^{-\lambda s}f\left(s, y(s), \int_0^s y(\xi)\,d\xi\right)ds, \\ U(\lambda) &= \int_0^\infty e^{-\lambda s}\sum_{i=0}^{m}E_i\mu(s-\eta_i)\,ds, \\ G(\lambda) &= \int_0^\infty e^{-\lambda s}\left(g\left(s, y(s), \int_0^s y(\xi)\,d\xi\right)\frac{dw(s)}{d(s)}\right)ds. \end{aligned} $$

Taking the inverse Laplace transform on both side of (1.1) and using Definition 2.4, we get

$$\begin{aligned} y(\rho) =& \frac{y_0}{\Gamma(\nu-1)} \int_0^\rho (\rho-s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\, ds + \frac{y_1}{\Gamma(\nu)} \int_0^\rho (\rho-s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\, ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) \left(\sum_{i=0}^{m} E_i \mu(s-\eta_i) + f\left(s, y(s), \int_0^s y(\xi)\, d\xi\right)\right) ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) g\left(s, y(s), \int_0^s y(\xi) d\xi\right) dw(s). \end{aligned}$$

The proof is completed. □

Lemma 2.8. *Suppose that* $M = \sup_{\rho\in\mathcal{J}} \|S_{\delta,\beta_\kappa}(\rho)\|$. *Then, for any* $y \in \mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y))$, *we have*

$$\|\mathcal{I}_{\delta,\beta_\kappa}(\rho) y\| \leq \frac{M}{\Gamma(\delta+1)} \rho^\delta \|y\|.$$

Proof. Taking the norm, we have

$$\begin{aligned} \|\mathcal{I}_{\delta,\beta_\kappa}(\rho) y\| &= \left\| \frac{1}{\Gamma(\delta)} \int_0^\rho (\rho-\tau)^{\delta-1} S_{\delta,\beta_\kappa}(\tau) y\, d\tau \right\| \\ &\leq \frac{1}{\Gamma(\delta)} \int_0^\rho (\rho-\tau)^{\delta-1} \|S_{\delta,\beta_\kappa}(\tau) y\|\, d\tau \\ &\leq \frac{M}{\Gamma(\delta)} \int_0^\rho (\rho-\tau)^{\delta-1} \|y\|\, d\tau \\ &\leq \frac{M}{\Gamma(\delta+1)} \rho^\delta \|y\|. \end{aligned}$$

The proof is completed. □

Lemma 2.9. *[28] Let* $\varphi : [0, l] \times \Omega \to L_0^2$ *be a strongly measurable mapping such that* $\int\limits_0^\rho E\|\varphi(s)\|^2_{L_0^2} ds < \infty$. *Then we have*

$$\mathbb{E} \left\| \int\limits_0^\rho \varphi(s) dw(s) \right\|^2 \leq \omega_\varphi \int\limits_0^\rho \mathbb{E} \|\varphi(s)\|^2_{L_0^2} ds,$$

where $\omega_\varphi > 0$ *is a constant.*

3. Existence of mild solution

We require the following assumptions:

(H1) The operator $\mathcal{I}_{\delta,\beta_\kappa}(\rho)$ is compact for every $\rho > 0$.

(H2) For each $\rho \in \mathcal{J}$, the mapping $f(\rho, \cdot, \cdot) : Y \times Y \to Y$ is continuous. Moreover, there exists a constant $M_1 > 0$ such that for all $y, \tilde{y} \in Y$, and $\rho \in \mathcal{J}$, the function f satisfies

$$\mathbb{E}\left\| f\left(\rho, y(\rho), \int_0^\rho y(\xi)\, d\xi\right) - f\left(\rho, \tilde{y}(\rho), \int_0^\rho \tilde{y}(\xi)\, d\xi\right)\right\|^2 \leq M_1\, \mathbb{E}\, \|y(\rho) - \tilde{y}(\rho)\|^2 (1 + \rho).$$

(H3) For each $\rho \in \mathcal{J}$, the mapping $g(\rho, \cdot, \cdot) : Y \times Y \to L^2(H, Y)$ is continuous. Moreover, there exists a constant $M_3 > 0$ such that for all $y, \tilde{y} \in Y$, and $\rho \in \mathcal{J}$, the function g satisfies:

$$\mathbb{E}\left\| g\left(\rho, y(\rho), \int_0^\rho y(\xi)\, d\xi\right) - g\left(\rho, \tilde{y}(\rho), \int_0^\rho \tilde{y}(\xi)\, d\xi\right)\right\|^2 \leq M_3\, \mathbb{E}\, \|y(\rho) - \tilde{y}(\rho)\|^2 (1 + \rho).$$

(H4) The functions E_i are bounded, i.e., $\max\limits_{0 \leq i \leq m} \|E_i\| \leq M_4$, for some $M_4 > 0$.

(H5) The admissible set Θ_{ad} is defined as

$$\Theta_{ad} = \left\{ \mu(\cdot) : \mathcal{J} \times \Omega \to X \,\middle|\, \mu \text{ is an } \mathcal{F}_\rho\text{-adapted stochastic process and } \int_0^\rho \|\mu(s)\|^2\, ds < \infty \right\}.$$

Clearly, $\Theta_{ad} \neq \emptyset$ and $\Theta_{ad} \subset L^2(\mathcal{J}, X)$ is bounded, closed, and convex. Moreover, $E\mu \in L^2(\mathcal{J}, Y)$ for every $\mu \in \Theta_{ad}$.

Next, we assume the zero initial control on the interval $[-\eta_m, 0)$, and after interchanging the order of integration, the mild solution for $\eta_r < \rho \leq \eta_{r+1}$, with $r = 0, 1, 2, \ldots, m-1$, for $\rho \in [0, \eta_m]$, is given by (see [29])

$$\begin{aligned} y(\rho) = {} & \frac{y_0}{\Gamma(\nu - 1)} \int_0^\rho (\rho - s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\, ds + \frac{y_1}{\Gamma(\nu)} \int_0^\rho (\rho - s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\, ds \\ & + \sum_{i=0}^{r-1} \int_{\rho-\eta_{i+1}}^{\rho-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds \\ & + \int_0^{\rho-\eta_r} \left(\sum_{j=0}^{r} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds \\ & + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho - s) f\left(s, y(s), \int_0^s y(\xi) d\xi\right) ds \\ & + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho - s) g\left(s, y(s), \int_0^s y(\xi) d\xi\right) dw(s). \end{aligned} \tag{3.1}$$

Similarly for $\rho > \eta_m$, we have

$$\begin{aligned} y(\rho) = {} & \frac{y_0}{\Gamma(\nu - 1)} \int_0^\rho (\rho - s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\, ds + \frac{y_1}{\Gamma(\nu)} \int_0^\rho (\rho - s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\, ds \\ & + \sum_{i=0}^{m-1} \int_{\rho-\eta_{i+1}}^{\rho-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_m}(\rho - s - \eta_j) E_j \right) \mu(s) ds \\ & + \int_0^{\rho-\eta_m} \left(\sum_{j=0}^{m} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho - s) \\ & + f\left(s, y(s), \int_0^s y(\xi) d\xi\right) ds + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho - s) g\left(s, y(s), \int_0^s y(\xi) d\xi\right) dw(s). \end{aligned} \tag{3.2}$$

Let $\ell > 0$ be a fixed final time. Using the integral form of the mild solution, we introduce the associated operators. If $\eta_r < \ell < \eta_{r+1}$ for some $r \in 0, 1, 2, \dots, m-1$, we define the linear and bounded control operator $\mathcal{L}_\ell$: $L^2(\mathcal{J}, X) \to L^2(\mathcal{F}_\rho, Y)$ by

$$\mathcal{L}_\ell \mu = \sum_{i=0}^{r-1} \int_{\rho-\eta_{i+1}}^{\rho-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds + \int_0^{\rho-\eta_r} \left(\sum_{j=0}^{r} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds.$$

Similarly, for $\rho > \eta_m$, we have:

$$\mathcal{L}_\ell \mu = \sum_{i=0}^{m-1} \int_{\rho-\eta_{i+1}}^{\rho-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds + \int_0^{\rho-\eta_m} \left(\sum_{j=0}^{m} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \mu(s) ds.$$

The adjoint $\mathcal{L}_\ell^*$:$L^2(\mathcal{F}_\rho, Y) \to L^2(\mathcal{J}, X)$ is defined by :

$$\mathcal{L}_\ell^* y \;\; = \;\; \begin{cases} E_0^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - \rho) \mathbb{E}\{y \mid \mathcal{F}_\rho\}, & \rho \in [0, \ \ell - \eta_m], \\ \displaystyle\sum_{j=1}^{i} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - \rho - \eta_j) \mathbb{E}\{y \mid \mathcal{F}_\rho\}, & \rho \in (\ell - \eta_{i+1}, \ \ell - \eta_i], \ i = 0, 1, 2, \dots m-1. \end{cases}$$

We introduce the linear controllability operator [30] $\zeta_0^\ell \in \mathcal{L}(L^2(\mathcal{F}_\rho, Y), L^2(\mathcal{F}_\rho, Y))$, which is rigorously constructed in correspondence with the control operator $\mathcal{L}_\ell$, for $\ell \in (\eta_i, \ \eta_{i+1}), \ i = 0, 1, 2, \dots m-1$,

$$\begin{aligned} \zeta_0^\ell \;\; &= \;\; \mathcal{L}_\ell \mathcal{L}_\ell^* \{\cdot\} \\ &= \;\; \sum_{i=0}^{r-1} \int_{\ell-\eta_{i+1}}^{\ell-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \left(\sum_{j=0}^{i} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - s - \eta_j) \right) \mathbb{E}\{\cdot \mid \mathcal{F}_\rho\} d\rho \\ &\quad + \int_0^{\ell-\eta_r} \left(\sum_{j=0}^{r} \mathcal{I}_{\delta,\beta_\kappa}(\ell - s - \eta_j) E_j \right) \left(\sum_{j=0}^{r} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - s - \eta_j) \right) \mathbb{E}\{\cdot \mid \mathcal{F}_\rho\} d\rho. \end{aligned}$$

Similarly, for $\rho > \eta_m$, we have

$$\begin{aligned} \zeta_0^\ell \;\; &= \;\; \mathcal{L}_\ell \mathcal{L}_\ell^* \{\cdot\} \\ &= \;\; \sum_{i=0}^{m-1} \int_{\ell-\eta_{i+1}}^{\ell-\eta_i} \left(\sum_{j=0}^{i} \mathcal{I}_{\delta,\beta_\kappa}(\rho - s - \eta_j) E_j \right) \left(\sum_{j=0}^{i} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - s - \eta_j) \right) \mathbb{E}\{\cdot \mid \mathcal{F}_\rho\} d\rho \\ &\quad + \int_0^{\ell-\eta_m} \left(\sum_{j=0}^{m} \mathcal{I}_{\delta,\beta_\kappa}(\ell - s - \eta_j) E_j \right) \left(\sum_{j=0}^{m} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - s - \eta_j) \right) \mathbb{E}\{\cdot \mid \mathcal{F}_\rho\} d\rho. \end{aligned}$$

Clearly, ζ_ℓ^0 is a bounded linear operator, and its resolvent is given by $R(\lambda, \zeta_\ell^0) = (\lambda I + \zeta_\ell^0)^{-1}$.

We defined the control function μ as

$$\mu(\rho) \;\; = \;\; \begin{cases} E_0^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - \rho) \mathbb{E}\{R(\lambda, \zeta_\ell^0) \phi(y_0, y_\ell, y) \mid \mathcal{F}_\rho\}, & \rho \in [0, \ \ell - \eta_m], \\ \displaystyle\sum_{j=1}^{i} E_j^* \mathcal{I}_{\delta,\beta_\kappa}^*(\ell - \rho - \eta_j) \mathbb{E}\{R(\lambda, \zeta_\ell^0) \phi(y_0, y_l, y) \mid \mathcal{F}_\rho\}, & \rho \in (\ell - \eta_{i+1}, \ \ell - \eta_i], \end{cases}$$

where

$$\phi(y_0, y_\ell, y) = y_\ell - \frac{y_0}{\Gamma(\nu-1)}\int_0^\ell (\ell-s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\,ds - \frac{y_1}{\Gamma(\nu)}\int_0^\ell (\ell-s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\,ds$$
$$- \int_0^\ell \mathcal{I}_{\delta,\beta_\kappa}(\ell-s)\, f\!\left(s, y(s), \int_0^s y(\xi)d\xi\right) ds$$
$$- \int_0^\ell \mathcal{I}_{\delta,\beta_\kappa}(\ell-s)\, g\!\left(s, y(s), \int_0^s y(\xi)\,d\xi\right) dw(s). \tag{3.3}$$

For simplicity, we take
$\delta_1 = \left(\frac{M_4 M \ell^{(2-\nu+\delta)}}{\lambda\Gamma(1+\delta)}\right)^2$, $\delta_2 = \ell^{(2\nu-3)}(1+\ell)\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2$, $\delta_3 = \ell^{(2\nu-3)}\omega_g(1+\ell)\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2$,
$\delta_4 = \sum_{i=0}^{r-1}(i+1)\ell^{2\nu-3}\left(\frac{M_4 M\ell^\delta}{\Gamma(1+\delta)}\right)^2$, $\delta_5 = (r+1)\ell^{2\nu-3}\left(\frac{M_4 M\ell^\delta}{\Gamma(1+\delta)}\right)^2$,
$\delta_6 = \sum_{i=0}^{m-1}(i+1)\ell^{2\nu-3}\left(\frac{M_4 M\ell^\delta}{\Gamma(1+\delta)}\right)^2$, $\delta_7 = (m+1)\ell^{2\nu-3}\left(\frac{M_4 M\ell^\delta}{\Gamma(1+\delta)}\right)^2$,
$N_1 = [2i\delta_1\delta_2 M_1 + 2i\delta_1\delta_3 M_3]$, $N_2 = [\delta_1\delta_2 M_1 + \delta_1\delta_3 M_3]$, $N = \max(N_1, N_2)$,
$K_1 = 4\ell^{2(2-\nu)}[\delta_4 N + \delta_5 N + \delta_2 M_1 + \delta_3 M_3]$, $K_2 = 4\ell^{2(2-\nu)}[\delta_6 N + \delta_7 N + \delta_2 M_1 + \delta_3 M_3]$,
$K = \max(K_1, K_2)$.

Using Lemma 2.9 and assumptions (H1–H5), take any $y, \tilde{y} \in \mathfrak{C}_{2-\nu}$, and $\mu, \tilde{\mu} \in \Theta_{ad}$, then for $\rho \in (\ell - \eta_{i+1},\ \ell - \eta_i]$, we have

$$\begin{aligned}
&\|\mu - \tilde{\mu}\|^2_{\mathfrak{C}_{2-\nu}} \\
&= \sup_{\rho\in\mathcal{J}} \mathbb{E}\left\|\rho^{2-\nu}\left[\mu(\rho) - \tilde{\mu}(\rho)\right]\right\|^2 \\
&\leq \left(\frac{M_4 M\ell^{(2-\nu+\delta)}}{\lambda\Gamma(1+\delta)}\right)^2 2i\Bigg[\mathbb{E}\left\|\int_0^\ell \mathcal{I}_{\delta,\beta_\kappa}(\ell-s)\left(f\left(s, y(s), \int_0^s y(\xi)d\xi\right) - f\left(s, \tilde{y}(s), \int_0^s \tilde{y}(\xi)d\xi\right) ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\ell \mathcal{I}_{\delta,\beta_\kappa}(\ell-s)\left(g\left(s, y(s), \int_0^s y(\xi)\,d\xi\right) - g\left(s, \tilde{y}(s), \int_0^s \tilde{y}(\xi)\,d\xi\right)\right) dw(s)\right\|^2\Bigg] \\
&\leq \left(\frac{M_4 M\ell^{(2-\nu+\delta)}}{\lambda\Gamma(1+\delta)}\right)^2 2i\Bigg[\ell^{(2\nu-3)}\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2 (1+\ell)[M_1\ \|y-\tilde{y}\|^2\,] \\
&\quad + \ell^{(2\nu-3)}\omega_g(1+\ell)\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2 M_3\ \|y-\tilde{y}\|^2\Bigg] \\
&\leq [2i\delta_1\delta_2 M_1 + 2i\delta_1\delta_3 M_3]\,\|y-\tilde{y}\|^2 \\
&\leq N_1\,\|y-\tilde{y}\|^2 \\
&\leq N\,\|y-\tilde{y}\|^2.
\end{aligned}$$

Similarly, for $\rho \in [0,\ \ell - \eta_m]$

$$\|\mu - \tilde{\mu}\|^2_{\mathfrak{C}_{2-\nu}} \leq N_2\,\|y-\tilde{y}\|^2 \leq N\,\|y-\tilde{y}\|^2.$$

Theorem 3.1. *If* (H2)–(H5) *hold, then* (1.1) *has a mild solution on* $\mathcal{J}$ *for every* $\mu \in \Theta_{ad}$*, provided that* $K = \max(K_1, K_2) < 1$.

Proof. Define an operator $\mathcal{H} : \mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y)) \to \mathfrak{C}_{2-\nu}(\mathcal{J}, L^2(\Omega, Y))$ by

$$
\begin{aligned}
(\mathcal{H}y)(\rho) = {} & \frac{y_0}{\Gamma(\nu-1)}\int_0^\rho (\rho-s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\,ds + \frac{y_1}{\Gamma(\nu)}\int_0^\rho (\rho-s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\,ds \\
& + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(\sum_{i=0}^{m} E_i\mu(s-\eta_i) + f\left(s, y(s), \int_0^s y(\xi)d\xi\right)\right) ds \\
& + \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) g\left(s, y(s), \int_0^s y(\xi)d\xi\right) dw(s).
\end{aligned} \tag{3.4}
$$

Next we show that $\mathcal{H}$ maps the space $\mathfrak{C}_{2-\nu}$ into itself.

For $\rho \in (\ell-\eta_{i+1},\ \ell-\eta_i]$

$$
\begin{aligned}
\|(\mathcal{H}y)\|^2_{\mathfrak{C}_{2-\nu}} &= \sup_{\rho\in\mathcal{J}} \mathbb{E}\|\rho^{2-\nu}(\mathcal{H}y)(\rho)\|^2 \\
&\le \sup_{\rho\in\mathcal{J}}\Bigg\{\rho^{2(2-\nu)} 7\Bigg[\mathbb{E}\left\|\frac{y_0}{\Gamma(\nu-1)}\int_0^\rho (\rho-s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\,ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\frac{y_1}{\Gamma(\nu)}\int_0^\rho (\rho-s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\,ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\sum_{i=0}^{r-1}\int_{\rho-\eta_{i+1}}^{\rho-\eta_i}\left(\sum_{j=0}^{i}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s-\eta_j)E_j\right)\mu(s)ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^{\rho-\eta_r}\left(\sum_{j=0}^{r}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s-\eta_j)E_j\right)\mu(s)ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) f\left(s, y(s), \int_0^s y(\xi)d\xi\right) ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) g\left(s, y(s), \int_0^s y(\xi)d\xi\right) dw(s)\right\|^2\Bigg]\Bigg\} < \infty.
\end{aligned}
$$

Similarly, for $\rho \in [0,\ \ell-\eta_m]$,

$$
\|(\mathcal{H}y)\|^2_{\mathfrak{C}_{2-\nu}} < \infty.
$$

Hence, it follows that $\mathcal{H}$ maps the space $\mathfrak{C}_{2-\nu}$ into space $\mathfrak{C}_{2-\nu}$. Next we show that the operator $\mathcal{H}$ is a contraction. Take any $y, \tilde{y} \in \mathfrak{C}_{2-\nu}$, and $\mu, \tilde{\mu} \in \Theta_{ad}$. Then for $\rho \in (\ell-\eta_{i+1},\ \ell-\eta_i]$,

$$
\begin{aligned}
& \|(\mathcal{H}y) - (\mathcal{H}\tilde{y})\|^2_{\mathfrak{C}_{2-\nu}} \\
&= \sup_{\rho\in\mathcal{J}} \mathbb{E}\left\|\rho^{2-\nu}\left[(\mathcal{H}y)(\rho) - (\mathcal{H}\tilde{y})(\rho)\right]\right\|^2 \\
&\le \sup_{\rho\in\mathcal{J}}\Bigg\{\rho^{2(2-\nu)} 4\Bigg[\mathbb{E}\left\|\sum_{i=0}^{r-1}\int_{\rho-\eta_{i+1}}^{\rho-\eta_i}\left(\sum_{j=0}^{i}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s-\eta_j)E_j\right)(\mu(s)-\tilde{\mu}(s))ds\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^{\rho-\eta_r}\left(\sum_{j=0}^{r}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s-\eta_j)E_j\right)(\mu(s)-\tilde{\mu}(s))ds\right\|^2
\end{aligned}
$$

$$+\mathbb{E}\left\|\int_0^{\rho}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(f\left(s,y(s),\int_0^s y(\xi)d\xi\right)-f\left(s,\tilde{y}(s),\int_0^s\tilde{y}(\xi)d\xi\right)\right)ds\right\|^2$$

$$+\mathbb{E}\left\|\int_0^{\rho}\mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(g\left(s,y(s),\int_0^s y(\xi)d\xi\right)-g\left(s,\tilde{y}(s),\int_0^s\tilde{y}(\xi)d\xi\right)\right)dw(s)\right\|^2\Bigg]\Bigg\}$$

$$\leq 4\ell^{2(2-\nu)}\Bigg[\sum_{i=0}^{r-1}(i+1)\ell^{2\nu-3}\left(\frac{M_4M\ell^{\delta}}{\Gamma(1+\delta)}\right)^2\|\mu-\tilde{\mu}\|^2_{\Theta_{ad}}+(r+1)\ell^{2\nu-3}\left(\frac{M_4M\ell^{\delta}}{\Gamma(1+\delta)}\right)^2\|\mu-\tilde{\mu}\|^2_{\Theta_{ad}}$$

$$+\ell^{(2\nu-3)}\left(\frac{M\ell^{\delta}}{\Gamma(1+\delta)}\right)^2(1+\ell)[M_1\,\|y-\tilde{y}\|^2\,]$$

$$+\ell^{(2\nu-3)}\omega_g(1+\ell)\left(\frac{M\ell^{\delta}}{\Gamma(1+\delta)}\right)^2M_3\,\|y-\tilde{y}\|^2\Bigg],$$

$$\leq 4\ell^{2(2-\nu)}\Big[\delta_4N\,\|y-\tilde{y}\|^2+\delta_5N\,\|y-\tilde{y}\|^2+[\delta_2M_1+\delta_3M_3]\,\|y-\tilde{y}\|^2\Big]$$

$$\leq 4\ell^{2(2-\nu)}[\delta_4N+\delta_5N+\delta_2M_1+\delta_3M_3]\,\|y-\tilde{y}\|^2$$

$$\leq K_1\,\|y-\tilde{y}\|^2\,.$$

Similarly, for $\rho\in[0,\ \ell-\eta_m]$,

$$\|(\mathcal{H}y)-(\mathcal{H}\tilde{y})\|^2_{\mathfrak{C}_{2-\nu}}\leq K_2\,\|y-\tilde{y}\|^2\,.$$

Hence, for $\rho\in[0,\ \ell]$,

$$\|(\mathcal{H}y)-(\mathcal{H}\tilde{y})\|^2_{\mathfrak{C}_{2-\nu}}\leq K\,\|y-\tilde{y}\|^2\,,$$

where K <1, which implies that $\mathcal{H}$ is a contraction operator. Hence, by the Banach contraction principle, the operator $\mathcal{H}$ admits a unique fixed point $y\in\mathfrak{C}_{2-\nu}$, which corresponds to the mild solution of (1.1). □

4. Optimal controllability

In this section, we investigate the existence of an optimal control for (1.1). We formulate the Lagrange problem (LP) as follows: find a pair $(y^0,\mu^0)\in\mathfrak{C}_{2-\delta}\times\Theta_{ad}$ such that y^0 is the mild solution of (1.1) for the control μ^0, and the cost functional $\mathfrak{H}(y^0,\mu^0)$ achieves the minimum value over all admissible pairs, i.e.,

$$\mathfrak{H}(y^0,\mu^0)\leq\mathfrak{H}(y^{\mu},\mu),\quad for\ (y,\mu)\in\mathfrak{C}_{2-\delta}\times\Theta_{ad},$$

where

$$\mathfrak{H}(y^{\mu},\mu)=\mathbb{E}\left\{\int_0^{\ell}\mathfrak{L}\left(\rho,y^{\mu}(\rho),\mu(\rho)\right)d\rho\right\}.$$

We require the following assumptions:
(H6) The function $\mathfrak{L}:\mathcal{J}\times Y\times\Theta_{ad}\to\mathbb{R}\cup\{\infty\}$ satisfies the following conditions:

(i) $\mathfrak{L} : \mathcal{J} \times Y \times \Theta_{ad} \to \mathbb{R} \cup (\infty)$ is $\mathcal{F}_\rho$-measurable.
(ii) $\mathfrak{L}(\rho, \cdot, \cdot)$ is sequentially lower semicontinuous on $Y \times \Theta_{ad}$ for a.e. $\rho \in \mathcal{J}$.
(iii) $\mathfrak{L}(\rho, y, \cdot)$ is convex on X for every $y \in Y$ and for each $\rho \in \mathcal{J}$.
(iv) There exists a non negative function $\phi \in L^1([0, \ell], \mathbb{R})$ such that

$$\mathfrak{L}(\rho, y, \mu) \geq \phi(\rho) + M_5 \|y\|^2 + M_6 \|\mu\|_X^2,$$

where $M_5 \geq 0, M_6 > 0$ are constants.

Theorem 4.1. *If* (H1)–(H6) *hold. Moreover, if all the hypotheses of Theorem 3.1 are satisfied, then there exists at least one pair* $(y^0, \mu^0) \in \mathfrak{C}_{2-\delta} \times \Theta_{ad}$ *such that* (y^0, μ^0) *is an optimal pair for (LP), that is,*

$$\begin{aligned} \mathfrak{H}(y^0, \mu^0) &= \mathbb{E}\left\{ \int_0^\ell \mathfrak{L}\left(\rho, y^0(\rho), \mu^0(\rho)\right) d\rho \right\} \\ &\leq \mathfrak{H}(y^\mu, \mu), \quad \textit{for all } (y^\mu, \mu) \in \mathfrak{C}_{2-\delta} \times \Theta_{ad}, \end{aligned}$$

provided that

$$3 l^{2(2-\nu)} (1 + \ell) \left(\frac{M \ell^\delta}{\Gamma(1 + \delta)} \right)^2 \ell^{2\nu - 3} (M_1 + M_3 \omega_g) < 1.$$

Proof. If $\inf\{\mathfrak{H}(y^\mu, \mu) \mid (y^\mu, \mu) \in \mathfrak{C}_{2-\nu} \times \Theta_{ad}\} = \infty$, the result is trivially true. Therefore, without loss of generality, we may assume that

$$\inf\{\mathfrak{H}(y^\mu, \mu) \mid (y^\mu, \mu) \in \mathfrak{C}_{2-\nu} \times \Theta_{ad}\} = \zeta < \infty.$$

Moreover, by (H6), we know that $\zeta > -\infty$.

By the definition of the infimum, there exists a sequence of admissible pairs $\{(y^{\tilde{n}}, \mu^{\tilde{n}})\} \subset Z_{ad}$ such that $\lim\limits_{\tilde{n} \to \infty} \mathfrak{H}(y^{\tilde{n}}, \mu^{\tilde{n}}) = \zeta$, where

$$Z_{ad} = \{(y, \mu) : y \text{ is a mild solution of (1.1) corresponding to } \mu \in \Theta_{ad}\}.$$

Since the sequence $\{\mu^{\tilde{n}}\} \subseteq \Theta_{ad},\ \tilde{n} = 1, 2 \ldots, \{\mu^{\tilde{n}}\}$ is a bounded sequence in the reflexive and separable Banach space $L^2(\mathcal{J}, X)$, there exists a weakly convergent subsequence still denoted by $\{\mu^{\tilde{n}}\}$ and $\{\mu^0\}$ $\in L^2(\mathcal{J}, X)$ such that

$$\mu^{\tilde{n}} \rightharpoonup \mu^0 \in L^2(\mathcal{J}, X).$$

Applying Mazur's lemma to the closed and convex set Θ_{ad} guarantees that the limit point μ^0 remains within Θ_{ad}. Let $\{y^{\tilde{n}}\}$ denote the sequence of mild solutions to (1.1), for the sequence of controls $\{\mu^{\tilde{n}}\}$. The solution$\{y^0\}$ corresponds to the control $\{\mu^0\}$, each $\{y^{\tilde{n}}\}$, and $\{y^0\}$ satisfy their respective integral representations as follows:

$$\begin{aligned} y^{\tilde{n}}(\rho) =& \frac{y_0}{\Gamma(\nu - 1)} \int_0^\rho (\rho - s)^{\nu - 2} S_{\delta, \beta_\kappa}(s)\, ds + \frac{y_1}{\Gamma(\nu)} \int_0^\rho (\rho - s)^{\nu - 1} S_{\delta, \beta_\kappa}(s)\, ds \\ &+ \int_0^\rho \mathcal{I}_{\delta, \beta_\kappa}(\rho - s) \left(\sum_{i=0}^m E_i \mu^{\tilde{n}}(s - \eta_i) + f\left(s, y^{\tilde{n}}(s), \int_0^s y^{\tilde{n}}(\xi) d\xi\right) \right) ds \\ &+ \int_0^\rho \mathcal{I}_{\delta, \beta_\kappa}(\rho - s) g\left(s, y^{\tilde{n}}(s), \int_0^s y^{\tilde{n}}(\xi) d\xi\right) dw(s), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} y^0(\rho) =& \frac{y_0}{\Gamma(\nu-1)}\int_0^\rho (\rho-s)^{\nu-2} S_{\delta,\beta_\kappa}(s)\,ds + \frac{y_1}{\Gamma(\nu)}\int_0^\rho (\rho-s)^{\nu-1} S_{\delta,\beta_\kappa}(s)\,ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(\sum_{i=0}^m E_i\mu^0(s-\eta_i) + f\left(s, y^0(s), \int_0^s y^0(\xi)d\xi\right)\right) ds \\ &+ \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) g\left(s, y^0(s), \int_0^s y^0(\xi)d\xi\right) dw(s). \end{aligned} \tag{4.2}$$

Using Lemma 2.9 and Holder's inequality, we have

$$\begin{aligned} \|y^{\tilde{n}} - y^0\|^2_{\mathfrak{C}_{2-\nu}} =& \sup_{\rho\in\mathcal{J}} \mathbb{E}\left\|\rho^{2-\nu}(y^{\tilde{n}}(\rho) - y^0(\rho))\right\|^2 \\ \le& \sup_{\rho\in\mathcal{J}}\Bigg\{\rho^{2(2-\nu)} 3\Bigg[\mathbb{E}\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\sum_{i=0}^m E_i\left(\mu^{\tilde{n}}(s-\eta_i) - \mu^0(s-\eta_i)\right) ds\right\|^2 \\ &+ \mathbb{E}\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\left(f\left(s, y^{\tilde{n}}(s), \int_0^s y^{\tilde{n}}(\xi)d\xi\right) - f\left(s, y^0(s), \int_0^s y^0(\xi)d\xi\right)\right) ds\right\|^2 \\ &+ \mathbb{E}\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s) g\left(s, y^{\tilde{n}}(s), \int_0^s y^{\tilde{n}}(\xi)d\xi\right) - g\left(s, y^0(s), \int_0^s y^0(\xi)d\xi\right) dw(s)\right\|^2\Bigg]\Bigg\} \\ \le& 3\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\sum_{i=0}^m E_i\left(\mu^{\tilde{n}}(s-\eta_i) - \mu^0(s-\eta_i)\right) ds\right\|^2 \\ &+ 3\ell^{2(2-\nu)}\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2 \ell^{2\nu-3}(1+\ell)(M_1 + M_3\omega_g)\left\|y^{\tilde{n}} - y^0\right\|^2. \end{aligned}$$

Since $\mu^{\tilde{n}} \rightharpoonup \mu^0$ in $L^2(\mathcal{J}, X)$, and by (H1),

$$\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\sum_{i=0}^m E_i\mu^{\tilde{n}}(s-\eta_i)\,ds \longrightarrow \int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\sum_{i=0}^m E_i\mu^0(s-\eta_i)\,ds$$

strongly in $\mathfrak{C}_{2-\nu}$. Therefore, $(1 - A2)\|y^{\tilde{n}} - y^0\|^2_{\mathfrak{C}_{2-\nu}} \le A^{\tilde{n}}$, where

$$A2 = 3\ell^{2(2-\nu)}\left(\frac{M\ell^\delta}{\Gamma(1+\delta)}\right)^2 \ell^{2\nu-3}(1+\ell)(M_1 + M_3\omega_g)$$

and

$$A^{\tilde{n}} = 3\left\|\int_0^\rho \mathcal{I}_{\delta,\beta_\kappa}(\rho-s)\sum_{i=0}^m E_i\left(\mu^{\tilde{n}}(s-\eta_i) - \mu^0(s-\eta_i)\right) ds\right\|^2.$$

Since $A2 < 1$ and $A^{\tilde{n}} \to 0$, we conclude that $y^{\tilde{n}} \to y^0$ strongly in $\mathfrak{C}_{2-\nu}$.

Since Θ_{ad} is nonempty, closed, convex, and bounded in the reflexive Banach space $L^2(\mathcal{J}, X)$, Moreover, by (H6)(i), the integrand $\mathfrak{L}$ is measurable. By (H6)(ii), $\mathfrak{L}$ is sequentially lower semicontinuous. By (H6)(iii), $\mathfrak{L}$is convex with respect to the control variable. Moreover, condition (H6)(iv) ensures coercivity, which guarantees the boundedness of the minimizing sequence in $L^2(\mathcal{J}, X)$.

Therefore, all hypotheses of Balder's theorem are satisfied. Hence, by Balder's theorem [31], we conclude that

$$(y, \mu) \mapsto \mathbb{E}\left\{\int_0^\ell \mathfrak{L}\,(\rho, y(\rho), \mu(\rho))\, d\rho\right\}$$

is sequentially lower semicontinuous with respect to the weak topology on $L^2[\mathcal{J}, X] \subset L^1[\mathcal{J}, X]$ and the strong topology on $L^1[\mathcal{J}, Y]$. As a result, $\mathfrak{H}$ is weakly lower semicontinuous on $L^2[\mathcal{J}, X]$. Therefore, the functional $\mathfrak{H}$ attains its infimum at $\mu^0 \in \Theta_{ad}$, that is,

$$\begin{aligned}\zeta &= \lim_{\tilde{n}\to\infty} \mathbb{E}\left\{\int_0^\ell \mathfrak{L}\left(\rho, y^{\tilde{n}}(\rho), \mu^{\tilde{n}}(\rho)\right) d\rho\right\} \\ &\geq \mathbb{E}\left\{\int_0^\ell \mathfrak{L}\left(\rho, y^0(\rho), \mu^0(\rho)\right) d\rho\right\} = \mathfrak{H}(y^0, \mu^0) \geq \zeta.\end{aligned}$$

The proof is complete. □

Remark 4.2. Influence of Delay on Mild Solutions: *The delay parameter η_i enters the mild solution representation through the shifted control term inside the integral operator. The presence of delay:*

- *Modifies the effective control action interval,*
- *Influences the regularity of the solution trajectory,*
- *May enlarge the norm of the solution due to accumulated memory effects,*
- *Alters stability margins depending on the magnitude of the delay.*

The boundedness and compactness framework established in Section 4 ensures that, for admissible delays within the prescribed interval, the existence and uniqueness of mild solutions remain valid. However, increasing the delay magnitude affects the growth estimates arising in the solution bounds.

Quantitative Impact on Optimal Control Performance: *The delay parameter influences:*

- *The structure of the reachable set,*
- *The control energy required to steer the system,*
- *The convergence behavior of minimizing sequences.*

Larger delays typically require stronger control amplitudes to compensate for the postponed system response, which may increase the value of the optimal cost functional. The delay enters the control operator through the shifted argument $\mu(s - \eta_i)$, thereby affecting the compactness-based convergence analysis and the associated optimality conditions.

5. Application

Let $Y = X = L^2[0, \pi]$ and $\delta, \beta_\kappa > 0$, $\kappa = 1, 2, 3, \cdots, m$ be given such that $0 < \delta \leq \beta_m \leq \cdots \leq \beta_1 < 1$. Consider the following Hilfer fractional differential equation:

$$\begin{cases} {}^H D_{0^+}^{1+\delta,\gamma}\,(y(\rho, x)) + \sum\limits_{k=1}^{m} C_\kappa\, {}^H D_{0^+}^{\beta_\kappa,\gamma}\,(y(\rho, x)) = \dfrac{\partial^2 y(\rho, x)}{\partial x^2} + E_0\mu(\rho, x) + E_1\mu(\rho - 2, x) \\ + g_1(\rho)\displaystyle\int_0^\rho \sin(\rho - s) y(s, x)\, ds + g_2(\rho)\int_0^\rho \sin(\rho - s) y(s, x) dw(s), \quad \rho \in (0, \ell], \\ \left(I_{0^+}^{2-\nu} y\right)(0, x) = y_0, \left(I_{0^+}^{2-\nu} y\right)'(0, x) = y_1, \quad \mu(\rho, x) = 0, \quad x \in [0, \pi], \\ y(\rho, 0) = y(\rho, \pi) = 0, \quad \rho \in [-2, \ell], \end{cases} \tag{5.1}$$

where ${}^{H}D_{0^+}^{1+\delta,\gamma}$ denotes the Hilfer fractional derivative. Consider a standard Brownian motion $w(\rho)$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho\geq 0}, \mathbb{P})$ with the filtration $\{\mathcal{F}_\rho\}_{\rho\geq 0}$, where the filtration satisfies the usual assumptions.

Define the operator A by $Ay = y''$ with domain as follows:

$$D(A) = \{y \in Y : y \text{ and } y' \text{ are absolutely continuous}, y'' \in Y, \text{ and } y(0) = y(\pi) = 0\}.$$

Then, the operator A is given by:

$$Ay = \sum_{n=1}^{\infty} -n^2 \langle y, e_n\rangle e_n, \quad y \in D(A),$$

where

$$e_n(\theta) = \sqrt{\frac{2}{\pi}} \sin(n\theta), \quad n \in \mathbb{N},$$

are the orthonormal basis of eigenfunctions of A. The operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(\rho) : \rho \geq 0\}$ on Y, defined by

$$C(\rho)y = \sum_{n=1}^{\infty} \cos(n\rho)\langle y, e_n\rangle e_n.$$

Since the operator A generates a bounded strongly continuous cosine family $\{C(\rho)\}_{\rho\geq 0}$ on Y, and the parameters satisfy $0 < \delta \leq \beta_m \leq \cdots \leq \beta_1 \leq 1$ with $C_\kappa \geq 0$ for all $\kappa = 1, 2, \ldots, m$, it follows from Theorem 2.5 that A admits a bounded (δ, β_κ)-resolvent family $\{S_{\delta,\beta_\kappa}(\rho)\}_{\rho\geq 0}$.

Define $y(\rho)(x) = y(\rho, x)$. Then, the nonlinear terms are given by as follows:

$$\begin{aligned}
f\left(\rho, y(\rho), \int_0^\rho y(s)ds\right) &= g_1(\rho) \int_0^\rho \sin(\rho - s)y(s, x)\, ds,\\
E_0\mu(\rho) &= \mu(\rho, x),\\
E_1\mu(\rho) &= E_1\mu(\rho - 2, x),\\
g\left(\rho, y(\rho), \int_0^\rho y(s)\, ds\right) &= g_2(\rho) \int_0^\rho \sin(\rho - s)y(s, x)\, ds.
\end{aligned}$$

Hence, system (5.1) can be reformulated as the abstract system given by (1.1). Since all the conditions of Theorem 3.1 are satisfied, we now consider the following cost function:

$$\mathfrak{H}(\mu) = \mathbb{E}\left\{\int_0^\ell \mathfrak{L}\left(\rho, y(\rho), \mu(\rho)\right) d\rho\right\},$$

where

$$\mathfrak{L}\left(\rho, y(\rho), \mu(\rho)\right) = \int_0^\pi \|y(\rho, x)\|^2\, dx + \int_0^\pi \|\mu(\rho, x)\|^2\, dx.$$

It follows that all the assumptions of Theorem 4.1 hold, which implies that system (5.1) has at least one optimal pair.

5.1. Physical interpretation

Model (5.1) can describe a variety of physical systems that exhibit memory, stochastic fluctuations, and delayed control actions. For instance, it represents the temperature evolution in a heat-conducting rod or the deformation of a viscoelastic beam, where the fractional Hilfer derivatives capture hereditary (memory) effects and an intermediate relaxation behavior between the Caputo and Riemann–Liouville types. The Laplacian term $\partial^2 y/\partial x^2$ models spatial diffusion or elastic strain, while the stochastic term $dw(\rho)$ accounts for random environmental disturbances such as fluctuating boundary conditions or measurement noise. The delayed control inputs $\mu(\rho - \tau, x)$ represent actuator or communication lags in feedback mechanisms, which are common in industrial, biological, and engineering systems. Thus, Eq (5.1) provides a unified framework framework for the analysis of the optimal regulation of complex fractional dynamical processes subject to uncertainty and finite control delays.

6. Conclusions and future work

In this paper, we studied the existence and optimal controllability of multi-term Hilfer stochastic fractional systems with finite delays in control. By applying fractional calculus, the (δ, β_κ)-resolvent family, and the Banach fixed-point theorem, we established the existence of a unique mild solution. Furthermore, the existence of an optimal control pair that minimizes a quadratic cost functional was demonstrated using Balder's theorem. An illustrative diffusion-type example confirmed the applicability of the theoretical results to real-world systems with memory, randomness, and control delays.

The results of this paper provide a solid theoretical basis for the optimal control of systems with memory, randomness, and delayed controls. The proposed framework can be applied to models in viscoelastic materials, heat transfer with memory effects, and neural network dynamics under uncertainty. Moreover, this work strengthens the theory of fractional stochastic control by incorporating multi-term Hilfer operators and delay effects in a unified setting, which may support further analytical and numerical developments.

The present study can be extended in several directions:

- Extension to impulsive and switching multi-term systems;
- Numerical approximation schemes for multi-term Hilfer stochastic systems;
- Investigation of multi-term Hilfer stochastic systems with distributed and state-dependent delays;
- Studying the uniqueness of the optimal control pair by imposing stronger convexity or structural assumptions;
- Analyzing the influence of key parameters, such as the fractional orders and control delay, on the structure and performance of the optimal control.

Moreover, the developed analytical framework may be connected with recent advances in the state feedback stabilization of stochastic high-order and low-order nonlinear systems with time-delay. Since the present model incorporates stochastic perturbations, multi-term Hilfer fractional dynamics, and control delays, the obtained resolvent-based approach can potentially be adapted to derive stabilization criteria under suitable state feedback laws. Such an extension would provide a bridge between fractional stochastic optimal control and the modern stabilization theory for nonlinear delay systems.

These extensions will further enhance the understanding and applicability of stochastic fractional control systems in complex environments.

Author contributions

All authors contributed equally to this work. All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the preparation of this article.

Acknowledgments

The second author acknowledges Science and Engineering Research Board (SERB), India for providing financial support through MATRICS research project with reference no. MTR/2023/000060.

Conflict of interest

The authors declare that they have no conflicts of interest.

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