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Research article

On anisotropic parabolic equations: from regular solutions to finite-time blow-up

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Abstract: This work established fundamental distinctions in the dynamical evolution between anisotropic and isotropic non-Newtonian fluid systems, where the harmonic mean $\overline{p}$ of $\{p_i\}_{i=1}^N$ emerged as a critical bifurcation parameter governing solution behaviors. Using the parabolic regularization method, we established the local existence of a weak solution. By applying the Poincar inequality in the single-variable sense and imposing certain restrictions on the nonlinear term $f(x,t,u,u_{x_i})$, we proved the existence of a global solution. Moreover, if $f(x,t,u,u_{x_i}) = f(u)$ and $f(u)/u^{\overline{p}}$ was nondecreasing on $\mathbb{R}^+$, then the local solution blowed up in finite time. The proposed methodology revealed how directional diffusivity creates distinct evolutionary patterns in solution behavior.

Keywords: non-Newtonian fluid equation; Poincare inequality; global solution; blow-up
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1. Introduction

According to the relationship between the shear stress, and the rate of strain and its derivatives, the fluids can be characterized as one of the two categories: 1) Newtonian fluids: where stress is directly proportional to the strain, and 2) non-Newtonian fluids: where stress is not proportional to the rate of strain, its higher powers and derivatives. Newtonian fluids obey Newton's law of viscosity and are the simplest mathematical models of fluids. Most common liquids and gases, such as water and air, can be assumed to be Newtonian for practical conditions.

Non-Newtonian fluids do not follow Newton's law of viscosity, which exhibit behavior that deviates from that of classical Newtonian fluids. These fluids are characterized by a non-linear relationship between shear stress and shear rate. Examples include polymers, slurries, blood, and many other industrial and biological fluids. Key categories include:

1) Shear-thinning (pseudoplastic): Viscosity decreases with shear rate (e.g., ketchup, polymer melts).

2) Shear-thickening (dilatant): Viscosity increases with shear rate (e.g., cornstarch-water mixtures).
3) Viscoelastic fluids: Exhibit both viscous and elastic properties (e.g., blood, synovial fluid).

In some details, the non-Newtonian fluid model describes shear-dependent viscosity:

$$\tau = \eta(\dot{\gamma})\dot{\gamma},$$

where τ is shear stress, $\dot{\gamma}$ is shear rate, and $\eta(\dot{\gamma})$ is the apparent viscosity, e.g.,

$$\eta = K\dot{\gamma}^{n-1},$$

where the parameter n is indeed a critical factor in non-Newtonian fluid models, known as the power-law index, which describes how a fluid's viscosity changes with shear rate. Specifically, $n = 1$ represents a Newtonian fluid, $n < 1$ represents a shear-thinning fluid (pseudoplastic fluid) and $n > 1$ is a shear-thickening fluid (dilatant fluid). Integrating the heat conduction equation and reaction-diffusion equations, in the isotropic case, mathematicians have derived the following general form of non-Newtonian fluid equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(x,t,u,\nabla u), \tag{1.1}$$

where f will be explained later.

Though Eq (1.1) model effectively describes isotropic non-Newtonian behavior (where viscosity varies uniformly with shear rate), many complex fluids exhibit directional dependence in their transport properties. This anisotropy arises from microstructural alignment (e.g., polymer chains, fibrous tissues, or stratified sediments), requiring more sophisticated models. The governing equation generalizes to

$$\tau = \eta(\dot{\gamma}, \vec{n})\dot{\gamma},$$

where $\vec{n}$ is the preferred direction (e.g., fiber orientation) and the corresponding fluids are called anisotropic non-Newtonian fluids.

Additionally, anisotropic diffusion models are widely used in various fields where diffusion processes are directionally dependent or influenced by underlying structures, earlier works can be found in [1–3], and more recent ones in [4–6]. Basing on these models, through systematic induction and synthesis, mathematicians subsequently derived the following representative anisotropic nonlinear evolution equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right),\ (x,t) \in Q_T, \tag{1.2}$$

where $p_i > 1$ is a constant. For the more details, one can refer to [7–9] for the isotropic case and [10,11] for the anisotropic case. We may call Eq (1.2) (or Eq (1.3) below) as anisotropic non-Newtonian fluids equation.

Instead of Eqs (1.1) and (1.2), we consider

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,u,\nabla u), \quad (x,t) \in Q_T, \tag{1.3}$$

with the usual initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \tag{1.5}$$

where $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^N$ is a bound domain with smooth boundary $\partial\Omega$ and the nonlinear term $f(x,t,u,\nabla u)$ originates from numerous concrete application problems in physics, mechanics, biology, among others, and exerts a significant impact on the properties of solutions to the equation. When

$$f(x,t,u,\nabla u) = f(x,t,u) \geq 0,$$

it is called a source term; or when

$$f(x,t,u,\nabla u) = f(x,t,u) \leq 0,$$

it is called an absorption term. When $f(x,t,u,\nabla u)$ depends on the gradient ∇u but does not depend on u, it is called a convective term or a damping term. It is well-known that, even for the heat conduction equation (i.e., $p = 2$ in Eq (1.1)), if f is a source term, the solutions may blow up; and if f is a damping term, the uniqueness of solution is not true. The precise conditions of f and u_0 will be given later. We denote

$$p^+ = \max\{p_1, p_2, \cdots p_N\}, \ p^- = \min\{p_1, p_2, \cdots p_N\},$$

and assume that $p^- > 1$.

For the usual evolutionary p-Laplacian Eq (1.1), it is well-known that, by the comparison theorem, the local existence of weak solutions to Eq (1.1) can be proved, but even for $p = 2$, various sufficient conditions for the nonexistence of global time solutions of (1.1) have been found in [12, 13]. Such conditions ordinarily involve the order of growth of f. In fact, if f has at most linear growth in u, then there is a global solution of the initial-boundary problem (1.1), (1.4), and (1.5) in [14]. If $f(x,t,u,\nabla u) \in C^1(\bar{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$ satisfy

$$f(x,t,u,\nabla u)\operatorname{sgn} u \leq C_0\left(\phi + |u|^\beta + |\nabla u|^\gamma\right),$$

where $\phi \in L^r(Q_T)$, $r > (N+p)/p$, and $\beta, \gamma < p-1$, then there is a global solution of the initial-boundary problems (1.1), (1.4), and (1.5) in [15]. Here and the after, $\operatorname{sgn}(u)$ denotes the sign function. Certainly, from 1980s, numerous studies have investigated properties of solutions to Eq (1.1), including existence and non-existence, long-time behavior, finite-time blow-up, and extinction phenomena. We briefly cite several examples that are comparatively related to the research topic of this work. For the evolutionary p-Laplacian equation with a source term

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q, \tag{1.6}$$

it is well known (see, e.g., [16]) that when $p > 2$ and $1 < q \leq p - 1 + \frac{p}{N}$, the solution of (1.6) always blows up in finite time; while for $q > p - 1 + \frac{p}{N}$, the blow-up occurs if the initial data $u_0(x)$ is large enough. In the latter case, there also exists a global solution with small initial data. If the source term u^q is replaced by $|u|^\alpha u$, some new results on global existence of solutions were established in [17] by introducing a family of potential wells. Also, asymptotic behavior and finite time blow-up of solutions were obtained in the case of subcritical initial energy and critical initial energy, respectively, in [17]. When

$$f(x,t,u,\nabla u) = f(u)$$

in Eq (1.1), and by defining the energy functional as

$$E(0) := \frac{1}{p}\int_\Omega |\nabla u_0|^p dx - \int_\Omega F(u_0)dx,$$

where

$$F(u) = \int_0^u f(s)dx,$$

the blow-up phenomena of weak solutions was investigated in [15, 18] with additional restrictions on f and $E(0)$. Moreover, by imposing a new condition

$$\alpha\int_0^u f(s)ds \le uf(u) + \beta u^p + \gamma,\ u > 0 \tag{1.7}$$

for some α, β, γ with

$$0 < \beta \le \frac{(\alpha - p)\lambda_{1,p}}{p},$$

the blow-up phenomena of the solutions to Eq (1.1) was studied by the concavity method in [19], where $\lambda_{1,p}$ is the first eigenvalue of p-Laplacian Δ_p.

From a dynamical systems perspective, the anisotropic nature of Eq (1.3) introduces multiple timescales in the evolution process, with different spatial directions exhibiting distinct rates of diffusion and energy dissipation. The multi-scale dynamics poses significant challenges in characterizing long-term behavior and stability properties. It means that the aforementioned methods can hardly be directly applied to the study of the existence and blow-up phenomena of solutions for anisotropic evolution equations. Here, we present only very elect research progress in this field (including the authors' prior work) that has come to our attention. Even for a simple case $f \equiv 0$, the fundamental solution was constructed in [20, 21] in recent years. When $f(x,t,u,\nabla u)$ is replaced by a convection term, the existence of a weak solution to Eq (1.3) was proved in our previous works [22–24]. For some other important studies in anisotropic parabolic equations with variable nonlinearity, one can refer to [11] and the references cited therein.

In this paper, by the Poincare inequality and the embedding theorem in the anisotropic Sobolev space, and imposing some restrictions on the nonlinear term $f(x,t,u,\nabla u)$, we will study the existence or no existence of the global solution of the initial-boundary value problems (1.3)–(1.5). Moreover, when

$$f(x,t,u,\nabla u) = f(u) \ge 0$$

and assuming that there is a r with

$$r \ge \overline{p}, \tag{1.8}$$

where $\overline{p}$ is the harmonic mean of $\vec{p} = \{p_i\}$, defined as

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} = \frac{1}{\overline{p}},$$

$f(u)$ is an odd function, $f \geq 0$ on $\mathbb{R}^+$, and $F(u)/u^r$ is nondecreasing on $\mathbb{R}^+$, where

$$F(u) = \int_0^u f(s)ds,$$

then we can show that the local solution may blow up in finite time. This conclusion elucidates the distinctive characteristics unique to anisotropic non-Newtonian fluid equations.

The paper is arranged as follows: after this introduction, in Section 2, the basic spaces and the main results are given; in Section 3, we prove the local existence of solutions of (1.3)–(1.5) under fairly general condition of f; in Section 4, we give sufficient conditions such that (1.3)–(1.5) has a global solution; in Section 5, we give some sufficient conditions to assure solutions of (1.3)–(1.5) must blow-up, in a pointwise sense, in finite time,and at last, a simple conclusion is given.

2. The basic space and the main results

Let Ω be a bounded, smooth domain in $\mathbb{R}^N$. Then the anisotropic Sobolev spaces are defined as follows:

$$W_0^{1,\vec{p}}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) : \frac{\partial v}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, 2, \cdots, N \right\}.$$

$W_0^{1,\vec{p}}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)}.$$

Lemma 1. *For any $u \in W_0^{1,\vec{p}}(\Omega)$, there is*

$$\|u\|_r \leq c(|\Omega|) r \left\| \frac{\partial u}{\partial x_i} \right\|_r, \ \forall r \geq 1,\ \cdots, \quad i = 1, 2, 3, \cdots N. \tag{2.1}$$

where c is a constant independent of u.

This lemma can be found in [25, 26], and inequality (2.1) can be called as the Poincare inequality in the single-variable sense.

Lemma 2. *Sobolev embedding theorem: let $1 \leq r \leq \frac{N\bar{p}}{N-\bar{p}}$ and $N - \bar{p} > 0$. Then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ and*

$$\|u\|_r \leq c \prod_{i=1}^{N} \|u_{x_i}\|_{p_i}^{\frac{1}{N}},\ r \in [1, \bar{p}] \tag{2.2}$$

for all $u \in W_0^{1,\vec{p}}(\Omega)$. Moreover, if $r < \frac{N\bar{p}}{N-\bar{p}}$, then the embedding is compact.

This lemma can be found in [27–29], and from (2.2), we have

$$\|u\|_r \leq c \sum_{i=1}^{N} \|u_{x_i}\|_{p_i},\ r \in [1, \bar{p}]. \tag{2.3}$$

By this lemma, we easily obtain the generalized Galiardo-Nirenberg inequality.

Lemma 3. *If* $1 \le r \le \frac{N\bar{p}}{N-\bar{p}}$ *and* $N - \bar{p} > 0$, *then for any* $q \in \left[r, \frac{N\bar{p}}{N-\bar{p}}\right]$, *there is*

$$\|u\|_q \le c\|u\|^{\alpha_1}_{W_0^{1,\vec{p}}(\Omega)}\|u\|^{1-\alpha_1}_{\bar{p}},$$

for all $u \in W_0^{1,\vec{p}}(\Omega)$, *where*

$$\frac{1}{q} = \alpha_1\left(\frac{1}{\bar{p}} - \frac{1}{N}\right) + \frac{1-\alpha_1}{r}.$$

Since, the anisotropic is the main characteristic of Eq (1.3), to obtain the needed estimate used in the proof of the existence of the global solutions, we additionally prove the Galiardo-Nirenberg inequality in a single direction.

Lemma 4. *It holds that*

$$\|u\|_q \le c(|\Omega|, p_i)\|u_{x_i}\|^{\alpha}_{p_i}\|u\|^{1-\alpha}_{r_i}, \tag{2.4}$$

where

$$r_i = \frac{(1-\alpha)qp_i}{p_i - \alpha q}$$

and

$$\frac{1}{q} = \frac{1-\alpha}{r_i} + \frac{\alpha}{r_i p_i}.$$

Proof. By the Hölder inequality, we have

$$\begin{aligned}\|u\|_q &= \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} = \left(\int_\Omega |u^{\alpha+1(1-\alpha)}|^q dx\right)^{\frac{1}{q}}\\ &\le \left(\int_\Omega |u|^{\alpha q\frac{p_i}{\alpha q}} dx\right)^{\frac{\alpha q}{p_i}\frac{1}{q}}\left(\int_\Omega |u|^{(1-\alpha)q\frac{p_i}{p_i-\alpha q}} dx\right)^{\frac{p_i-\alpha q}{p_i(1-\alpha)}\frac{1-\alpha}{q}}\\ &= \|u\|^{\alpha}_{p_i}\|u\|^{1-\alpha}_{r_i}.\end{aligned}$$

Then, Lemma 1 yields the relationship (2.4). □

Definition 5. *By a solution of the initial boundary value problems (1.3)–(1.5), we mean a function* $u \in L^\infty(Q_T) \cap L^1(0,T; W_0^{1,\vec{p}}(\Omega))$, $u_t \in L^2(Q_T)$, *satisfying*

$$\iint_{Q_T}\left[u\psi_t - \sum_{i=1}^N |u_{x_i}|^{p_i-2}u_{x_i}\psi_{x_i} + f\psi\right]dxdt + \int_\Omega u_0\psi(x,0)dx = 0 \tag{2.5}$$

for all $\psi \in C^1(\overline{Q}_T)$ *such that* $\psi(x,T) = 0$, *and* $\psi = 0$ *on* $\partial\Omega \times (0,T)$. *Equation (2.5) implies*

$$\begin{aligned}&\int_0^t\int_\Omega\left[u\psi_t - \sum_{i=1}^N |u_{x_i}|^{p_i-2}u_{x_i}\psi_{x_i} + f\psi\right]dxds\\ &= \int_\Omega u(x,t)\psi(x,t)dx - \int_\Omega u_0\psi(x,0)dx, \quad \textit{for a.e. } t \in (0,T).\end{aligned} \tag{2.6}$$

In this paper, by the so-called "concavity" method (see [15, 30]), we are able to obtain some theorems in the existence and nonexistence of the global weak solutions to the initial boundary value conditions (1.3)–(1.5).

We will prove the following theorem.

Theorem 6. *Let* $p^+ \geq p^- > 2$ *and* $u_0 \in L^\infty(\Omega) \cap W_0^{1,p^+}(\Omega)$*, and* $f(x,t,u,\nabla u) \in C^1(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$ *satisfy*

$$f(x,t,u,\nabla u)\text{sgn}\, u \leq C_0 \left(\phi + |u|^\beta + |\nabla u|^\gamma\right). \tag{2.7}$$

If $\phi \in L^r(Q_T), r \geq 2,$ *and* $\beta, \gamma < p^- - 1$*, then (1.3)–(1.5) has a solution* $u \in L^\infty(Q_T)$ *for any* $T > 0$*.*

Remark 7. *In the isotropic case, i.e., for Eq (1.1), when* $\phi \in L^r(Q_T), r \geq 1 + \frac{p}{N}$*, a result similar to Theorem 6 was obtained in [15]. One may conjecture that for the anisotropic non-Newtonian Eq (1.3),* $r \geq 1 + \frac{\overline{p}}{N}$ *is enough to ensure Theorem 6, but it is hard to verify.*

Instead of Eq (1.3), sometimes, mathematicians use

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p_i-2}u_{x_i}\right) + f(x,t,u,\nabla u),\ (x,t) \in Q_T \tag{2.8}$$

to represent the anisotropic non-Newtonian fluids equation. In such a case, by Lemmas 2–4, we believe that the conjecture in Remark 7, i.e.,

$$\phi \in L^r(Q_T),\ r \geq 1 + \frac{\overline{p}}{N},$$

is a sufficient condition to show the existence of the global solution to Eq (2.8).

Another goal of our paper is to investigate the blow-up solution demonstrating anisotropic characteristics. By assumption (H3) below, we are able to prove the following theorem.

(H1): Suppose $f(x,t,u,\nabla u) = f(u) \in C^1(\mathbb{R})$, $f(u)$ is an odd function, $f \geq 0$ on $\mathbb{R}^+$, and $F(u)/u^r$ is nondecreasing on $\mathbb{R}^+$, $r \geq \overline{p}$,

Theorem 8. *Assume that* f *satisfies assumption (H1),* $p^- > 2$*, and the positive constant* r *satisfies Eq (1.8). If* $u_0 \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ *satisfies*

$$\int_\Omega F(u_0)dx > \sum_{i=1}^{N} \int_\Omega \frac{r}{p_i}|u_{0x_i}|^{p_i}dx + \frac{2(r-1)}{r}\frac{2}{T(r-2)^2}\int_\Omega u_0^2 dx. \tag{2.9}$$

Then, there exists $t_1 \in (0,T]$ *such that*

$$\lim_{t \to t_1} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \tag{2.10}$$

where u *is the local solution of (1.3)–(1.5).*

Remark 9. *This theorem demonstrates that whenever the capability grows too rapidly compared to* u^r, $r \geq \overline{p}$*, blow-up phenomena are prone to occur. This precisely characterizes the distinctive property of solutions to anisotropic reaction-diffusion equations (including anisotropic non-Newtonian fluid equations as a particular case)!*

Last but not least, if we denote that

$$E(\Omega,T)=\left\{u_0\in L^\infty(\Omega)\cap W_0^{1,\vec{p}}(\Omega):\int_\Omega F(u_0)dx>\sum_{i=1}^N\frac{1}{p_i}\int_\Omega|u_{0x_i}|^{p_i}dx+\frac{2(r-1)}{r}\frac{2}{T(r-2)^2}\int_\Omega u_0^2dx\right\},$$

then we are able to show that $E(\Omega,T)$ is not empty.

Proposition 10. *Let $f(s)$ be an odd function, $f(s)\geq 0$, $s\in\mathbb{R}^+$,*

$$\lim_{u\to\infty}\left(F(u)\ /|u|^r\right)=+\infty,$$

and $u_0\in L^\infty(\Omega)\cap W_0^{1,\vec{p}}(\Omega)$, $u_0\not\equiv 0$. Then $\tau u_0\in E(\Omega,T)$ for sufficiently large τ.

In addition, interested readers may refer to the method in [19] and replace condition (H1) in this paper with a condition similar to inequality (1.7), thereby obtaining more general blow-up properties for solutions to anisotropic reaction-diffusion equations. Also, one can consider whether the limitation $p^->2$ can be weakened to $p^->1$ or not.

3. Local existence

In this section, only $f(x,t,u,\nabla u)$ independent of ∇u is considered. We make the following assumption.

Theorem 11. *If $u_0\in L^\infty(\Omega)\cap W_0^{1,p^+}(\Omega)$ and the following assumption holds:*
(H2) $f(x,t,u)\in C(\bar{\Omega}\times[0,T]\times\mathbb{R})$, and there exists a function $g(u)\in C^1(\mathbb{R})$ such that

$$|f(x,t,u)|\leq g(u).$$

Then, the initial-boundary value problems (1.3)–(1.5) has a solution $u\in Q_{T_1}$, where $T_1\in[0,T]$. Moreover, if $f(x,t,u)$ is Lipschitz continuous on $\overline{\Omega}\times[0,T]\times(-M,M)$ for any $M<\infty$, then u is unique.

Proof. Consider the approximate problem:

$$u_t=\sum_{i=1}^N\left(\left(|u_{x_i}|^2+\frac{1}{n}\right)^{(p_i-2)/2}u_{x_i}\right)+f_n(x,t,u),\quad (x,t)\in Q_T,\tag{3.1}$$

$$u(x,0)=u_{0n}(x),\quad x\in\Omega,\tag{3.2}$$

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T).\tag{3.3}$$

Here $f_n\in C^1(\bar{\Omega}\times[0,T]\times\mathbb{R})$, $|f_n|\leq n$, $f_n\to f$ uniformly on bounded subset of $\bar{\Omega}\times[0,T]\times\mathbb{R}$ and

$$|f_n|\leq g(u);$$

$u_{0n}\in C_0^\infty(\Omega)$, such that $\|u_{0n}\|_{L^\infty(\Omega)}\leq\|u_0\|_{L^\infty(\Omega)}$, $\|u_{0nx_i}\|_{L^{p^+}(\Omega)}\leq c\|u_{0x_i}\|_{L^{p^+}(\Omega)}$, and

$$u_{0n}\to u_0 \text{ in } W_0^{1,p^+}(\Omega).$$

According to the classical parabolic theory [8, 14, 31], we know that the approximate problems (3.1)–(3.3) has a classical solution u_n and there exists $T_1 \in (0, T]$ and a constant M such that

$$\|u_n\|_{L^\infty(Q_{T_1})} \le M \quad \text{for all}\, n. \tag{3.4}$$

To see this, let $v^\pm(t)$ be the solutions of the ordinary differential equations

$$\frac{dv^\pm}{dt} = \pm g(v),\ v^\pm(0) = \pm\|u_0\|_{L^\infty(\Omega)}.$$

From [32, Chapter 1], we know there is $T^* \in (0, T)$ such that $v^\pm$ exists on $[0, T^*]$, and T^* depends only on $|u_0|_{L^\infty(\Omega)}$. By the comparison theorem,

$$|u_n(x,t)| \le \max\{v^+(t), -v^-(t)\}.$$

Setting $T_1 = T^*/2$ and $M = \max\{v^+(T_1), -v^-(T_1)\}$, we obtain (3.4).

Multiplying (3.1) by u_n and integrating over Q_{T_1} yields

$$\|u_{nx_i}\|_{L^{p_i}(Q_{T_1})} \le c,\ i = 1, 2, \cdots, N, \tag{3.5}$$

where c denotes the constants independent of n.

Multiplying (3.1) by u_{nt} and integrating, we may derive

$$\iint_{Q_{T_1}} u_{nt}^2 dxdt = -\sum_{i=1}^{N} \iint_{Q_{T_1}} \left(|u_{nx_i}|^2 + \frac{1}{n}\right)^{(p_i-2)/2} u_{nx_i} \cdot u_{ntx_i} dxdt + \iint_{Q_{T_1}} f_n(x,t,u_n) u_{nt} dxdt.$$

By the Hölder inequality and integrating by parts, we have

$$\iint_{Q_{T_1}} u_{nt}^2 dxdt \le c\left[\int_\Omega \left(|u_{0nx_i}|^{p_i} + |u_{0nx_i}|^2\right) dx\right] dx + c\iint_{Q_{T_1}} f_n^2 dxdt. \tag{3.6}$$

From inequalities (3.5) and (3.6), we know that there is a subsequence $n_k \to \infty$ and function $u \in L^\infty(Q_{T_1})$ such that

$$\begin{aligned}
&u_{n_k} \to u, \quad f_{n_k}(u_{n_k}, x, t) \to f(x,t,u) \quad \text{a.e. on } Q_{T_1},\\
&u_{n_k x_i} \rightharpoonup u_{x_i} \quad \text{in } L^{p_i}(Q_{T_1}),\\
&u_{n_k t} \rightharpoonup u_t \quad \text{in } L^2(Q_{T_1}),\\
&\sum_{i=1}^{N} \left(|u_{n_k x_i}|^{p_i-2} u_{n_k x_i}\right) \rightharpoonup \sum_{i=1}^{N} w_i \quad \text{in } L^1(Q_{T_1}),
\end{aligned}$$

as $n_k \to \infty$.

As in [15], we can show that

$$\sum_{i=1}^{N} w_i = \sum_{i=1}^{N} |u_{x_i}|^{p_i-2} u_{x_i}.$$

In details, multiplying (3.1) by $(u_n - u)\phi$, we have

$$\iint_{Q_{T_1}} \phi u_{nt}(u_n - u)dxdt + \sum_{i=1}^{N} \iint_{Q_{T_l}} \phi \left(|u_{nx_i}|^2 + \frac{1}{n}\right)^{(p_i-2)/2} u_{nx_i}(u_{nx_i} - u_{x_i})dxdt$$
$$= \iint_{Q_{T_1}} f_n(x,t,u_n)(u_n - u)dxdt$$

and

$$\lim_{n\to\infty} \iint_{Q_{T_1}} \phi \sum_{i=1}^{N} |u_{nx_i}|^{p_i-2} u_{nx_i}(u_{nx_i} - u)dxdt = 0, \tag{3.7}$$

where $\phi \in C_0^1(Q_{T_1}), \phi \geq 0$.

On the other hand, from $u_{x_i} \in L^{p_i}(Q_{T_1})$, we have

$$\lim_{n\to\infty} \iint_{Q_{T_1}} \sum_{i=1}^{N} |u_{x_i}|^{p_i-2} u_{x_i}(u_{nx_i} - u)\phi dxdt = 0. \tag{3.8}$$

Noticing that

$$\left(|u_{nx_i}|^{p_i-2}u_{nx_i} - |u_{x_i}|^{p_i-2}u_{x_i}\right)(u_{nx_i} - u_{x_i}) \geq \int_0^1 |\nabla(su_n + (1-s)u)|^{p_i-2} ds |\nabla(u_n - u)|^2; \tag{3.9}$$

we may derive from (3.7) and (3.8) that

$$\lim_{n\to\infty} \iint_{Q_{T_1}} \phi \int_0^1 |\nabla(su_n + (1-s)u)|^{p_i-2} ds \nabla(u_n - u)|^2 dxdt = 0.$$

Hence by

$$\iint_{Q_{T_1}} \int_0^1 |\nabla(su_n + (1-s)u)|^{p_i-2} dsdxdt \leq c,$$

we have

$$\begin{aligned}\left||u_{nx_i}|^{p_i-2}u_{nx_i} - |u_{x_i}|^{p_i-2}u_{x_i}\right| &= \left|\int_0^1 \frac{d}{ds}\left\{|su_{nx_i} + (1-s)u_{x_i}|^{p_i-2}\left(su_{nx_i} + (1-s)u_{x_i}\right)\right\} ds\right| \\ &\leq \left|\int_0^1 |su_{nx_i} + (1-s)u_{x_i}|^{p_i-2}(u_{nx_i} - u_{x_i})ds\right| + \left|\int_0^1 (p_i - 2)|su_{nx_i} + (1-s)u_{x_i}|^{p_i-4}\right. \\ &\quad \left.\times (su_{nx_i} + (1-s)u_{x_i})\left(su_{nx_j} + (1-s)u_j\right)(u_{nx_j} - u_{x_j})ds\right| \\ &\leq c\int_0^1 |su_{nx_i} + (1-s)u_{x_i}|^{p_i-2} ds |\nabla(u_n - u)|.\end{aligned} \tag{3.10}$$

Then, as $n \to \infty$, we obtain

$$\begin{aligned}\left|\iint_{Q_{T_1}} \phi\left(|u_{nx_i}|^{p_i-2}u_{nx_i} - |u_{x_i}|^{p_i-2}u_{x_i}\right)dxdt\right| &\leq c\left(\iint_{Q_{T_1}} \phi \int_0^1 |\nabla(su_n + (1-s)u)|^{p_i-2} ds |u_{nx_i} - u_{x_i}|^2 dxdt\right)^{1/2} \\ &\quad \times \left(\iint_{Q_{T_1}} \phi \int_0^1 |\nabla(su_n + (1-s)u)|^{p_i-2} dsdxdt\right)^{1/2} \to 0.\end{aligned}$$

Accordingly, it holds that

$$\iint_{Q_{T_1}} \sum_{i=1}^{N} \left[w_i - |u_{x_i}|^{p_i-2} u_{x_i} \right] \phi dxdt = 0.$$

Thus

$$\sum_{i=1}^{N} w_i = \sum_{i=1}^{N} |u_{x_i}|^{p_i-2} u_{x_i}.$$

Theorem 11 can be followed by a standard limiting process. □

The uniqueness of the solution is clear. Let u, v be two solutions of (1.3)–(1.5). Then (2.6) yields

$$\begin{aligned}
\frac{1}{2}\int_{\Omega} (u(x,t) - v(x,t))^2 \, dx = & - \iint_{Q_t} \left(|u_{x_i}|^{p_i-2} u_{x_i} - |\nabla v|^{p_i-2} \nabla v \right) (u_{x_i} - \nabla v) \, dxdt \\
& - \iint_{Q_t} \left(|u_{x_i}|^{p_i-2} u_{x_i} - |\nabla v|^{q-2} \nabla v \right) (u_{x_i} - \nabla v) \, dxdt \\
& + \iint_{Q_t} (f(x,t,u) - f(v,x,t)) (u - v) dxdt \\
\leq & \, c \iint_{Q_t} (u - v)^2 dxdt.
\end{aligned}$$

This implies $u = v$.

4. The global solution

In this section, we will use anisotropic embedding to obtain Moser-type bounds and prove Theorem 6.

We first try to derive an estimate for $\|(u-k)_+\|_{L^{\alpha+1}(Q_T)}$. Here, we denote $u = u_n$, $u_+ = \max\{0, u\}$, and $1 \leq \alpha < \infty$, and chose positive constant k such that $|u_{0n}| \leq k$ for all n. Let us multiply (3.1) by $(u-k)_+^{\alpha}$. Then, we have

$$\begin{aligned}
& \sup_{t\in(0,T)} \frac{1}{\alpha+1} \int_{\Omega} (u-k)_+^{(\alpha+1)} dx + \alpha \sum_{i=1}^{N} \iint_{Q_T} (u-k)_+^{\alpha-1} |(u-k)_{+x_i}|^{p_i} dxdt \\
= & \sup_{t\in(0,T)} \frac{1}{\alpha+1} \int_{\Omega} (u-k)_+^{(\alpha+1)} dx + \sum_{i=1}^{N} \frac{\alpha p_i^{p_i}}{(\alpha+p_i-1)^{p_i}} \iint_{Q_T} |(u-k)_{+x_i}^{(\alpha+p_i-1)/p_i}|^{p_i} dxdt \\
\leq & \, C_0 \iint_{Q_T} (\phi + |u|^{\beta} + |\nabla u|^{\gamma})(u-k)_+^{\alpha} dxdt.
\end{aligned} \tag{4.1}$$

Lemma 12. *There is a constant $c = c(\Omega, N, p^+, p^-, C_0)$ such that*

$$\begin{aligned}
& \iint_{Q_T} (u-k)_+^{\alpha} \phi dxdt \\
\leq & \, c(p^+, p^-, |\Omega|) \left\{ \sup_{t\in(0,T)} \int_{\Omega} (u-k)_+^{\alpha+1} dx + \left[\iint_{Q_T} \left| (u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}} \right|^{p_i} dxdt \right]^{1+\frac{(\alpha+p_i-1)(r_i-1)}{(p_i-1)(\alpha+1)}} \right\}.
\end{aligned} \tag{4.2}$$

Proof. Let

$$w = (u-k)_+^{\frac{\alpha+p_i-1}{p_i}}.$$

For any $r_i > 0$, we have

$$w^{r_i} = (u-k)_+^{\frac{\alpha+p_i-1}{p_i}(r_i-1)} w.$$

By the Hölder inequality and the embedding inequality in Lemma 3, we have

$$\begin{aligned}
\iint_{Q_T} w^{r_i} dxdt &= \iint_{Q_T} (u-k)_+^{\frac{(\alpha+p_i-1)}{p_i}(r_i-1)} w dxdt \\
&\le \left(\iint_{Q_T} w^{p_i} dxdt\right)^{\frac{1}{p_i}} \left(\iint_{Q_T} (u-k)_+^{\frac{\alpha+p_i-1}{p_i}\frac{p_i}{p_i-1}(r_i-1)} dxdt\right)^{\frac{p_i-1}{p_i}} \\
&\le c(r_i,|\Omega|)\left(\iint_{Q_T} \left|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right|^{p_i} dxdt\right)^{\frac{1}{p_i}} \left(\iint_{Q_T} (u-k)_+^{\frac{\alpha+p_i-1}{p_i}\frac{p_i}{p_i-1}(r_i-1)} dxdt\right)^{\frac{p_i-1}{p_i}} \\
&\le c(r_i,|\Omega|)\left(\iint_{Q_T} \left|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right|^{p_i} dxdt\right)^{\frac{1}{p_i}} \left(\iint_{Q_T} (u-k)_+^{\alpha+1} dxdt\right)^{\frac{p_i-1}{p_i}\frac{(\alpha+p_i-1)(r_i-1)}{(p_i-1)(\alpha+1)}} \\
&\le c(r_i,|\Omega|)\left[\iint_{Q_T} \left|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right|^{p_i} dxdt + \left(\iint_{Q_T} (u-k)_+^{\alpha+1} dxdt\right)^{\frac{(\alpha+p_i-1)(r_i-1)}{(p_i-1)(\alpha+1)}}\right] \\
&\le c(r_i,|\Omega|)\left[\iint_{Q_T} \left|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right|^{p_i} dxdt + \sup_{t\in(0,T)} \int_\Omega (u-k)_+^{\alpha+1} dxdt\right]^{1+\frac{(\alpha+p_i-1)(r_i-1)}{(p_i-1)(\alpha+1)}},
\end{aligned} \tag{4.3}$$

where r_i is chosen to satisfy

$$\alpha + 1 > (r_i - 1)\frac{\alpha + p_i - 1}{p_i - 1}$$

or

$$r_i < \frac{\alpha p_i + 2(p_i-1)}{\alpha + p_i - 1}.$$

If we choose

$$\delta_i = \frac{(\alpha+p_i-1)r_i}{(\alpha+p_i-1)r_i - \alpha p_i} < 2.$$

Since $\phi \in L^r(Q_T), r > 2$, we have

$$\iint_{Q_T} \phi^\delta dxdt < \infty. \tag{4.4}$$

Meanwhile, using the Hölder inequality and combing inequality (4.3) with (4.4), there is

$$\iint_{Q_T}(u-k)_+^\alpha \phi dxdt \le \left(\iint_{Q_T}(u-k)_+^{\alpha\frac{(\alpha+p_i-1)r_i}{\alpha p_i}}dxdt\right)^{\frac{\alpha p_i}{(\alpha+p_i-1)r_i}}\left(\iint_{Q_T}\phi^\delta dxdt\right)^{\frac{1}{\delta}}$$

$$\le \left(\iint_{Q_T}(u-k)_+^{\frac{(\alpha+p_i-1)r_i}{p_i}}dxdt\right)^{\frac{\alpha p_i}{(\alpha+p_i-1)r_i}}$$

$$\le c(r_i,|\Omega|)\left[\iint_{Q_T}\left|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right|^{p_i}dxdt+\sup_{t\in(0,T)}\int_\Omega (u-k)_+^{\alpha+1}dxdt\right]^{\left[1+\frac{(\alpha+p_i-1)(r_i-1)}{(p_i-1)(\alpha+1)}\right]\frac{\alpha p_i}{(\alpha+p_i-1)r_i}}.$$

Lemma 12 follows immediately. □

Second, if we choose $l=\frac{p_i(\alpha+\beta)}{\alpha+p_i-1}$, then $l\ge 1$, by (2.1), we have

$$\begin{aligned}\iint_{Q_T}(u-k)_+^{(\alpha+\beta)}dxdt &= \int_0^T\int_\Omega\left((u-k)_+^{\frac{\alpha+p_i-1}{p_i}}\right)^l dxdt\\ &\le c\int_0^T\left(\sum_{i=1}^N\left\|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}\right\|_{p_i}\right)^l dt\\ &\le c\int_0^T\sum_{i=1}^N\left(\int_\Omega|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dx\right)^{\frac{l}{p_i}}dt\\ &\le c\int_0^T\sum_{i=1}^N\left(\int_\Omega|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dx\right)^{\frac{p_i(\alpha+\beta)}{p_i(\alpha+p_i-1)}}dt.\end{aligned}\tag{4.5}$$

By the Jessen inequality, the Young inequality and Lemma 1, we have

$$\begin{aligned}\iint_{Q_T}|\nabla u|^\gamma(u-k)_+^\alpha dxdt &\le c\sum_{i=1}^N\iint_{Q_T}|(u-k)_{+x_i}|^\gamma(u-k)_+^\alpha dxdt\\ &\le \eta\sum_{i=1}^N\iint_{Q_T}|(u-k)_{+x_i}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dxdt+C(\eta)\iint_{Q_T}(u-k)_+^{\frac{p_i[\alpha+\gamma(\alpha-1)]}{p_i-\gamma}}dxdt\\ &\le \eta\sum_{i=1}^N\iint_{Q_T}|(u-k)_{x_i+}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dxdt\\ &\quad+c(\eta)\left(\iint_0^T(u-k)_+^{\frac{p_i[\alpha+\gamma(\alpha-1)]}{p_i-\gamma}\frac{(p_i-\gamma)\frac{\alpha+p_i-1}{p_i}}{p_i[\alpha+\gamma(\alpha-1)]}}dxdt\right)^{\frac{p_i[\alpha+\gamma(\alpha-1)]}{(p_i-\gamma)\frac{\alpha+p_i-1}{p_i}}}\\ &\le \eta\sum_{i=1}^N\iint_{Q_T}|(u-k)_{x_i+}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dxdt+c(\eta)\left(\iint_0^T(u-k)_+^{\frac{\alpha+p_i-1}{p_i}}dxdt\right)^{\frac{p_i[\alpha+\gamma(\alpha-1)]}{(p_i-\gamma)\frac{\alpha+p_i-1}{p_i}}}\\ &\le \eta\sum_{i=1}^N\iint_{Q_T}|(u-k)_{x_i+}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dxdt+c(\eta)\left(\sum_{i=1}^N\iint_{Q_T}|(u-k)_{x_i+}^{\frac{\alpha+p_i-1}{p_i}}|^{p_i}dxdt\right)^{\frac{p_i[\alpha+\gamma(\alpha-1)]}{(p_i-\gamma)\frac{\alpha+p_i-1}{p_i}}}.\end{aligned}\tag{4.6}$$

Thus, if

$$\beta, \gamma < p^- - 1 = \min_{i=1}^{N}\{p_i\} - 1,$$

then by (4.1), (4.2), (4.5), and (4.6), we may obtain

$$\sup_{t\in(0,T)} \int_\Omega (u_n - k)_+^{(\alpha+1)} dx \le c, \tag{4.7}$$

where c depends on $\alpha, \beta, \gamma, C_0, |\Omega|$, and T.

Thus,

$$\|u_n\|_{L^{\alpha+1}(Q_T)} \le C(\alpha), \tag{4.8}$$

where $C(\alpha) \to \infty$ as $\alpha \to \infty$.

Thirdly, we show that the u_n are uniformly bounded for n. In (4.1) we pick $\alpha = 1$, and obtain

$$\begin{aligned} &\frac{1}{2} \sup_{t\in(0,T)} \int_\Omega (u-k)_+^2 dx + \sum_{i=1}^{N} \frac{p_i^{p_i}}{(p_i-1)^{p_i}} \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt \\ &\le C_0 \left\{ \iint_{Q_T} \left(\phi + |\nabla (u-k)_+|^\gamma + |u|^\beta\right)(u-k)_+ \, dxdt \right\}. \end{aligned} \tag{4.9}$$

Moreover, by Lemma 1 and (4.8), we have

$$\begin{aligned} \iint_{Q_T} \phi (u-k)_+ dxdt &\le \left(\iint_{Q_T} \phi^{s_1} dxdt\right)^{1/s_1} \left(\iint_{Q_T} (u-k)_+^{s_1/(s_1-1)} dxdt\right)^{(s_1-1)/s_1} \\ &\le c\left(\iint_{Q_T} (u-k)_+^{(p_i+2(p_i/N))} dxdt\right)^{\frac{N}{p_iN+2p_i}} \mu(k)^{\frac{s_1-1}{s_1}-\frac{N}{p_iN+2p_i}} \\ &\le c(|\Omega|, p_i)| \left\{ \sup_{t\in(0,T)} (u-k)_+^2 dx + \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt \right\}^{(N+p_i)/(p_iN+2p_i)} \\ &\quad \times \mu(k)^{(s_1-1)/s_1 - N/(p_iN+2p_i)}, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \iint_{Q_T} |u|^\beta (u-k)_+ dxdt &\le \left(\iint_{Q_T} |u|^{\beta s_1} dxdt\right)^{1/s_1} \left(\iint_{Q_T} (u-k)_+^{s_1/(s_1-1)} dxdt\right)^{(s_1-1)/s_1} \\ &\le c\left\{ \sup_{t\in(0,T)} \int_\Omega (u-k)_+^2 dx + \sum_{i=1}^{N} \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt \right\}^{(N+p_i)/(p_iN+2p_i)} \\ &\quad \times \mu(k)^{(s_1-1)/s_1 - N/(p_iN+2p_i)}, \end{aligned} \tag{4.11}$$

$$\iint_{Q_T} |\nabla u|^\gamma (u-k)_+ dxdt \le \eta \sum_{i=1}^N \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt + c(\eta)\left(\iint_{Q_T} (u-k)_+^{s_1/(s_1-1)} dxdt\right)^{(s_1-1)/s_1} \left(\iint_{Q_T} (u-k)_+^{\gamma s_1/(p_i-\gamma)} dxdt\right)^{1/s_1}$$
$$\le \eta \sum_{i=1}^N \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt + c(\eta)\left\{\sup_{t\in(0,T)} \int_\Omega (u-k)_+^2 dx + \sum_{i=1}^N \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt\right\}^{(N+p_i)/(p_iN+2p_i)} \mu(k)^{(s_1-1)/s_1 - N/(p_iN+2p_i)}, \tag{4.12}$$

where $\mu(k)$ denotes the measure of the set $\{(x,t)\in Q_T; u_n \ge k\}$.

Combining (4.9)–(4.12), we obtain

$$\sup_{t\in(0,T)} \int_\Omega (u-k)_+^2 dx + \sum_{i=1}^N \iint_{Q_T} |(u-k)_{+x_i}|^{p_i} dxdt \le c\mu(k)^{1+(s_1p_i - Np_i - 2p_i)/s_1(N(p_i-1)+p_i)}.$$

We take $k_h = M(2-2^{-h}), h = 0, 1, 2, \cdots$, where $M > 0$ as in (3.4). It is clear that

$$(k_{h+1}-k_h)\mu(k_{h+1})^{N/(p_iN+2p_i)} \le \left(\iint_{Q_T} (u-k_h)_+^{(p_i+2p_i/N)} dxdt\right)^{N/(p_iN+2p_i)}$$
$$\le c\left\{\sup_{t\in(0,T)} \int_\Omega (u-k_h)_+^2 dx + \sum_{i=1}^N \iint_{Q_T} |(u-k_h)_{+x_i}|^{p_i} dxdt\right\}^{(q+N)/(p_iN+2p_i)}$$
$$\le c\mu(k_h)^{((N+p_i)/(p_iN+2p_i))(1+(rp_i-Np_i-2p_i)/(s_1N(p_i-1)+s_1p_i))}.$$

Thus, by [33, Theorem 6.1], we are able to obtain

$$\|u_n\|_{L^\infty(Q_T)} \le c, \text{ for all } n. \tag{4.13}$$

Proof of Theorem 6. We define u_{0n}, f_n, u_n as in Section 3. If Eq (2.7) holds, then by Eq (4.13) and by the results of [34], we know $|u_{nx_i}|$ is uniformly Hölder continuous for n. Similarly, we may derive a lower bound, and this completes the proof of Theorem 6. □

We use some ideas from [15], and the details are given below.

5. Blow-up in finite time

In this section, we will prove Theorem 8. We need the following lemma.

Lemma 13. *Let the hypotheses of Theorem 8 hold with* $|u_{0x_i}| \in L^{p^+}(\Omega)$. *Then we have the following conclusion:*

(1) *There holds*

$$u_t \in L^2(Q_T), (u^2)_t \in L^2(Q_T).$$

(2) *There holds*

$$\frac{1}{2}\int_0^t\int_\Omega u_t^2dxds=\frac{1}{2}\int_\Omega\left(u^2(x,t)-u_0^2(x)\right)dx$$
$$=\int_0^t\int_\Omega\left[-\sum_{i=1}^N|u_{x_i}|^{p_i}+uf(u)\right]dxds.$$

(3) *There holds*

$$\int_0^t\int_\Omega u_t^2dxds\le-\int_\Omega\sum_{i=1}^N\frac{1}{p_i}\left(|u_{x_i}(x,t)|^{p_i}-|u_{0x_i}|^{p_i}\right)dx+\int_\Omega\left(F(u)-F(u_0)\right)dx.$$

Proof. Let u_n be the solutions of initial-boundary value problems (3.1)–(3.3). Then, by the uniqueness of weak solution to the initial-boundary value problems (1.3)–(1.5), we know

$$u=\lim_{n\to\infty}u_n.$$

Multiplying (3.1) by u_{nt} and integrating over Q_t, we have

$$\iint_{Q_t}u_{nt}^2dxdt+\frac{1}{2}\sum_{i=1}^N\iint_{Q_t}\frac{d}{ds}\int_0^{|u_{nx_i}|^2}\left(s+\frac{1}{n}\right)^{(p_i-2)/2}dsdxdt=\iint_{Q_t}f(u_n)u_{nt}dxdt. \tag{5.1}$$

By the Cauchy inequality, we have

$$\iint_{Q_t}u_{nt}^2dxdt+\sum_{i=1}^N\int_\Omega|u_{nx_i}(x,t)|^{p_i}dx\le c\left\{\int_\Omega\sum_{i=1}^N|u_{0x_i}|^{p_i}dx+\iint_{Q_t}f^2(u_n)dxdt\right\},\quad t\in(0,T), \tag{5.2}$$

and so

$$\iint_{Q_t}u_t^2dxdt\le c.$$

Moreover, if we choose $\psi=u$ in (2.6), then we have

$$\frac{1}{2}\int_0^t\int_\Omega u_t^2dxdt=\frac{1}{2}\int_\Omega\left(u^2(x,t)-u_0^2(x)\right)dx=\int_0^t\int_\Omega\left[-\sum_{i=1}^N|u_{x_i}|^{p_i}+uf(u)\right]dxdt.$$

At last, let us prove (3) of Lemma 13. From (5.1), we obtain

$$\int_0^t\int_\Omega u_{nt}^2dxds=-\frac{1}{2}\sum_{i=1}^N\int_\Omega\int_0^{|u_{nx_i}|^2}\left(s+\frac{1}{n}\right)^{(p_i-2)/2}dsdx\Big|_0^t+\int_\Omega F(u_n)dx\Big|_0^t. \tag{5.3}$$

By (5.2) and uniqueness, we obtain

$$\begin{aligned}u_n(x,t)&\to u(x,t), && \text{in } L^2(\Omega);\\ u_{nt}&\rightharpoonup u_t, && \text{in } L^2(Q_t);\\ \nabla u_n(x,t)&\rightharpoonup\nabla u(x,t), && \text{in } L^1(0,T;L^{p^-}(\Omega)).\end{aligned}$$

Thus, by letting $n\to\infty$ in (5.3), using the lower semicontinuity of the norm with respect to weak convergence, by that $p^-\ge2$, inequality (3) of Lemma 13 follows naturally. □

Lemma 14. *Let*

$$f(x,t,u,u_{x_i}) = f(u) \in C^1(\mathbb{R})$$

satisfy (H1). Then, for any solution $u \in L^\infty(Q_T)$ of Eqs (1.3)–(1.5), we have the estimate

$$\int_\Omega F(u_0)dx \le \int_\Omega \sum_{i=1}^N \frac{r}{p_i}|u_{0x_i}|^{p_i}dx + \frac{2(r-1)}{r}\frac{2}{T(r-2)^2}\int_\Omega u_0^2 dx. \tag{5.4}$$

Proof. Let

$$B = \frac{2}{T(r-2)^2}\int_\Omega u_0^2 dx, \quad t_0 = \frac{T}{2}(r-2)$$

and

$$H(t) = \int_0^t \int_\Omega \frac{1}{2}u^2 dxds + (T-t)\int_\Omega \frac{1}{2}u_0^2 dx + B(t+t_0)^2.$$

It is obvious that

$$\begin{aligned} H'(t) &= \frac{1}{2}\int_\Omega u^2 dx - \frac{1}{2}\int_\Omega u_0^2 dx + 2B(t+t_0) \\ &= \int_0^t \int_\Omega \left[-\sum_{i=1}^N |u_{x_i}|^{p_i} + uf(u)\right]dxds + 2B(t+t_0) \end{aligned}$$

and

$$\frac{d^2}{dt^2}H(t) = \int_\Omega \left[-\sum_{i=1}^N |u_{x_i}|^{p_i} + uf(u)\right]dx + 2B.$$

By the Hölder inequality, we have

$$\begin{aligned} \left|\frac{1}{2}\left(\int_\Omega u^2 dx - \int_\Omega u_0^2 dx\right)\right| &= \frac{1}{2}\left|\int_0^t \int_\Omega (u^2)_t dxds\right| \\ &\le \left(\int_0^t \int_\Omega u^2 dxds\right)^{1/2}\left(\int_0^t \int_\Omega u_t^2 dxds\right)^{1/2}. \end{aligned}$$

Then, by Schwarz's inequality and the definition of H, we have

$$\begin{aligned} (H'(t))^2 \le& \left(\int_0^t \int_\Omega u^2 dxds\right)\left(\int_0^t \int_\Omega u_t^2 dxds\right) + 4B^2(t+t_0)^2 \\ &+ 4\beta(t+t_0)\left(\int_0^t \int_\Omega u^2 dxds\right)^{1/2}\left(\int_0^t \int_\Omega u_t^2 dxds\right)^{1/2} \\ \le& 2\left(\int_0^t \int_\Omega u_t^2 dxds\right)\left(\int_0^t \int_\Omega \frac{1}{2}u^2 dxds\right) + 4B^2(t+t_0)^2 \\ &+ 2\int_0^t \int_\Omega u_t^2 dxds\left[(T-t)\int_\Omega \frac{1}{2}u_0^2 dx + B(t+t_0)^2\right] \\ &+ 4B^2(t+t_0)^2 \int_0^t \int_\Omega \frac{1}{2}u^2 dxds\left\{(T-t)\int_\Omega \frac{1}{2}u_0^2 dx + B(t+t_0)^2\right\}^{-1} \\ \le& H(t)\left(2\int_0^t \int_\Omega u_t^2 dxds + 4B\right). \end{aligned}$$

Then, by condition (1.8), Lemma 13 yields that

$$
\begin{aligned}
H(t)\frac{d^2}{dt^2}H(t)-\frac{r}{2}(H'(t))^2 &\geq H(t)\left[\int_\Omega\left(-\sum_{i=1}^{N}|u_{x_i}|^{p_i}+uf(u)\right)dx+2B-r\int_0^t\int_\Omega u_t^2dxds-2rB\right]\\
&\geq H(t)\left[r\int_\Omega F(u_0)dx-\int_\Omega\sum_{i=1}^{N}\frac{r}{p_i}|u_{0x_i}|^{p_i}dx-2B(r-1)\right.\\
&\qquad\left.+\int_\Omega(uf(u)-rF(u))\,dx\right].
\end{aligned}
\tag{5.5}
$$

Since $\frac{F(u)}{u^r}$ is nondecreasing on $\mathbb{R}^+$, we have

$$
\frac{d}{du}\left(\frac{F(u)}{u^r}\right)\geq 0,\quad \text{for } u\geq 0,
$$

and so

$$
uf(u)-rF(u)\geq 0. \tag{5.6}
$$

The same conclusion is also true for $u<0$, because that $f(u)$ is an odd function implies $F(u)$ is an even function. At the same time, if (5.4) is not true, then by the definition of B, we have

$$
r\int_\Omega F(u_0)dx>\int_\Omega\sum_{i=1}^{N}\frac{r}{p_i}|u_{0x_i}|^{p_i}dx+2B(r-1).
$$

Last but not least, by $H(t)\geq 0$ for $t\in(0,T)$, we derive from (5.5), (5.6) that

$$
H(t)\frac{d^2}{dt^2}H(t)-\frac{r}{2}(H'(t))^2\geq 0,\quad \text{for } t\in(0,T),
$$

which implies

$$
\frac{d^2}{dt^2}\left(H^{-\frac{r}{2}+1}(t)\right)\leq 0,\ \ t\in[0,T].
$$

At last, since

$$
H^{-\frac{r}{2}+1}(0)>0,\ \text{and}\ \left(H^{-\frac{r}{2}+1}\right)'(0)<0,
$$

we have

$$
H^{-\frac{r}{2}+1}(t_1)=0,\ \text{for some } t_1\in\left(0,\frac{-H^{-\frac{r}{2}+1}(0)}{\left(H^{-\frac{r}{2}+1}\right)'(0)}\right),
$$

where

$$
\frac{-H^{-\frac{r}{2}+1}(0)}{\left(H^{-\frac{r}{2}+1}\right)'(0)}=\frac{T\int_\Omega\frac{1}{2}u_0^2dx}{(r-2)Bt_0}\leq T.
$$

Then, $t_1\leq T$ and

$$
\lim_{t\to t_1^-}H(t)=\infty.
$$

This is impossible and the lemma follows naturally. □

By Lemma 14, we can prove Theorem 8.

Proof of Theorem 8. By Theorem 11 and [34], u is a unique continuous solution of (1.3)–(1.5). Let $t_1 = \sup\{\tau \in (0, T] : u \in L^\infty(Q_\tau)\}$. If $t_1 < T$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} < M < \infty, \text{ for } t \le t_1,$$

then we let $f_M(u) = f(u)$ if $|u| \le M + 1$, $f_M = \pm f(M + 1)$ if $\pm u > M + 1$, and consider the following renormalized problem:

$$w_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|w_{x_i}|^{p_i-2} w_{x_i}\right) + f_M(w), \tag{5.7}$$

$$w(x, 0) = u_0(x), \quad \text{on } \Omega, \tag{5.8}$$

$$w(x, t) = 0, \quad x \in \partial\Omega. \tag{5.9}$$

According to Theorem 6, we know that since f is local Lipschitz continuous, by Theorem 6, we know that there is a unique solution $w \in L^\infty(Q_T)$ of the initial boundary value problems (5.7)–(5.9). Then $w = u$ on $(0, t_1]$. Since $|u| \le M$ on $(0, t_1]$, by the same comparison argument as in the proof of Theorem 11, we know that $|w| \le M + 1$ for $t \le t_1 + \varepsilon$ for some $\varepsilon > 0$. It follows that w is a solution of (1.3)–(1.5) on $Q_{t_1+\varepsilon}$, and this contradicts the definition of t_1. If $t_1 = T$, $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le M$ for $t \le T$, then we know the initial boundary value problems (1.3)–(1.5) has a bounded solution on Q_T, and Lemma 14 yields the estimate (5.4), which contradicts the assumption (2.9). Accordingly, we have

$$\overline{\lim_{t \to t_1}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Now, if there exists $t_n \to t_1$ and $M < \infty$ such that

$$\|u(\cdot, t_n)\|_{L^\infty(\Omega)} \le M,$$

then by comparing with the ordinary differential equation $V' = f(V)$, there is $h > 0$ such that

$$\|u(\cdot, s)\|_{L^\infty(\Omega)} \le 2M, \quad \text{for } s \in [t_n, t_n + h],$$

which means that

$$\overline{\lim_{t \to t_1}} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty,$$

and obtain the contradiction. Then, we have (2.10). □

At the end of this paper, we give the proof of Proposition 10.

Proof. If $\tau u_0 \notin E(\Omega, T)$ for all $\tau \ge 1$. Then, we have

$$\frac{1}{\tau^r} \int_\Omega F(\tau u_0) dx \le \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |u_{0x_i}|^{p_i} \tau^{p_i - r} dx + \frac{2(r-1)}{r} \frac{2}{T(r-2)^2} \int_\Omega u_0^2 \tau^{2-r} dx \tag{5.10}$$

for all $\tau \ge 1$. Then

$$\frac{1}{\tau^r} \int_\Omega F(\tau u_0) dx \le c, \text{ for all } \quad \tau \ge 1. \tag{5.11}$$

On the other hand, by the assumption, we have

$$\int_\Omega \frac{1}{\tau^r} F(\tau u_0)dx = \sum_{i=1}^{N} \int_\Omega |u_0|^{p_i} \frac{F(\tau u_0)}{|\tau u_0|^{p_i}} dx \to \infty.$$

This contradicts (5.11) and Proposition 10 is proved. □

6. Conclusions

This paper presents a systematic investigation of anisotropic parabolic equations associated with the p_i-Laplacian operator, with some methodological innovations in handling the inherent anisotropic nature of the problem. The key theoretical challenges stemmed from the inapplicability of the traditional Sobolev embedding theorems and the Poincare inequalities in anisotropic settings, requiring substantial modifications to our analytical framework.

The main theoretical contributions are:

1) Global existence: A sufficient condition for the integrability of the given quantity ϕ in inequality (2.7) is established, ensuring that the existence of global solutions can be proven using our extended anisotropic Poincare inequality and embedding theorems.

2) Finite-time blow-up: Theorem 8 demonstrates that whenever the capability grows faster than the harmonic mean growth order $u^{\overline{p}}$, blow-up phenomena are prone to occur. This precisely characterizes the distinctive property of solutions to anisotropic reaction-diffusion equations (including anisotropic non-Newtonian fluid equations as a particular case).

3) Applied significance: Results provide mathematical foundations for anisotropic diffusion processes in medical imaging, geophysics, and biological systems. Moreover, by replacing the Poincare inequality in the single-variable sense with the classical Poincare inequality, one may extend the results from references [17–19] to anisotropic equations. Furthermore, when considering the variable-exponent form of the Poincaré inequality in the single-variable sense, similar improvements could potentially be achieved for the results in [11] and related works. Last but not least, using some techniques in [35, subsection 5.4.2], the methods in our paper may be generalized to the case of $1 < p^- \leq p^+ < 2$.

Our principal methodological breakthroughs include:

1) Development of anisotropic-adapted functional analysis tools to replace classical isotropic approaches. One of the main difficulties arises because there is a damping part $|\nabla u|$ in the nonlinear term f, it is difficult to use the generalized Sobolev embedding theorem and the generalized Galiardo-Nirenberg inequality in the anisotropic Sobolev space to obtain the needed estimates. Thanks to the Poincare inequality in the single-variable sense, i.e., Lemma 1, the impact of nonlinear terms on bounded estimates was eliminated through repeated and meticulous application of this lemma. This demonstrates that the single-variable embedding theorem serves as a powerful tool for investigating various properties of solutions to anisotropic parabolic equations. However, it is slightly regrettable that the lemma only provides continuous embedding rather than compact embedding, leaving the study of equation regularity until the development of new tools.

2) By the Jessen inequality, the Poincare inequality in the single-variable sense can effectively control the growth order of $|\nabla u|$, the classical "concavity method" is successfully applied to obtain the L^∞-estimate for the anisotropic non-Newtonian fluids equation.

3) Because of the anisotropic characteristic of Eq (1.3), there is an essential difficulty in the treatment of nonlinear term growth rates and integrability conditions. This challenge is resolved through meticulous redesign of the exponent selection criteria in energy estimates, coupled with the construction of tailored test functions that account for anisotropic diffusion effects.

In summary, although formally following the classical framework for p-Laplacian equations to investigate solution properties, this work requires meticulous handling of essential technical difficulties arising from anisotropy. Theorem 8 has demonstrated the watershed role played by the harmonic mean in determining both the existence and blow-up of solutions for anisotropic non-Newtonian fluid equations. Our results reveal that the anisotropic dynamics governing Eq (1.3) produce solution behaviors markedly different from isotropic cases. The harmonic mean $\overline{p}$ emerges as a critical threshold separating stable dynamics from finite-time blow-up, highlighting how directional diffusivity collectively determine the system's dynamical regime.

Use of Generative-AI tools declaration

The author declares he has not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no competing interests.

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