



Research article

Solitary waves and bifurcation analysis in a double-chain DNA model

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Abstract: This study investigated the double-chain deoxyribonucleic acid (DNA) model that is at the heart of conservation and transmission of genetic information in biological systems. The model consists of a pair of strands that simulates DNA's intertwined chains of polynucleotides. These chains are bonded together by an elastic membrane that simulates hydrogen bonds between base pairs. To analyze the nonlinear dynamics of this system, the modified extended direct algebraic method was employed to derive exact analytical solutions, including bright solitons, dark solitons, Jacobi elliptic (JE) solutions, and periodic structures. The 3D, 2D, and polar visualizations obtained show longitudinal and transverse dynamics of the DNA helix. Visual examination of the outcomes confirmed the presence of solitary waves on DNA strands. In addition to presenting some dynamical analyses, such as bifurcation and stability analyses. The current study is a significant landmark in the area of DNA dynamics, with emphasis on prominent aspects of genetic information synthesis and transmission. The novelty of this work lies in the derivation of new exact soliton solutions to the governing DNA model that, to the best of our knowledge, have not been previously reported. The findings are expected to contribute to a deeper understanding of DNA dynamics and may also have broader implications in soliton theory, nonlinear science, plasma physics, fiber optics, and physical engineering.

Keywords: soliton solutions; bifurcation analysis; traveling wave solutions; Jacobi elliptic method;

bright and dark solitons; DNA dynamics

Mathematics Subject Classification: 35Q51, 35Q92, 37G15, 92C40

1. Introduction

Genetic information in biological systems is primarily encoded and stored within each cell as DNA. This essential bio-molecule is composed of nucleotide sequences that contain the genetic blueprint unique to every species. The structural organization and functional mechanisms of DNA have long been a central focus in modern biophysics, owing to their fundamental role in both biological processes and the origins of life. Structurally, DNA exists as a double-stranded helix in which two polynucleotide chains run in opposite directions, forming what are known as antiparallel strands. A landmark achievement in molecular biology came in 1953 when Watson and Crick elucidated the double-stranded DNA (dsDNA) architecture [1], a discovery that provided not only insight into DNA's spatial configuration [2] but also its replication mechanisms.

Building upon this structural foundation, recent investigations have increasingly turned to nonlinear analytical techniques to achieve a deeper characterization of dsDNA properties [3]. DNA exhibits complex dynamical behavior, including torsional, longitudinal, and transverse displacements, which present significant challenges for mathematical modeling. Recognizing that these diverse motions operate on distinct temporal scales, researchers can refine computational approaches by focusing on dominant movements within specific parameter ranges, thereby improving both accuracy and efficiency in simulations.

Central to the biological function of DNA are two fundamental processes: replication and transcription [4]. DNA replication ensures the faithful duplication of genetic material, producing two genetically identical daughter strands. This process begins when helicase enzymes unwind the double helix, generating a replication fork at which DNA polymerase synthesizes new complementary strands. The result is a semi-conservative mechanism in which each daughter molecule contains one original parental strand and one newly synthesized strand. Transcription, in contrast, selectively produces RNA molecules from specific DNA segments. RNA polymerase binds to promoter regions, locally unwinds the DNA, and generates RNA transcripts using the template strand. Through this process, cells regulate protein synthesis and gene expression, thereby maintaining essential biological functions [5].

The theoretical study of DNA dynamics has its origins in the pioneering work of Davydov and was significantly advanced by Englander and colleagues in 1980, with particular attention to nitrogen-related properties [6]. Later, Muto et al. [7] classified DNA denaturation dynamics into transverse motions across hydrogen bonds and longitudinal displacements along the molecular backbone. Djine et al. presented an investigation of waves in the Peyrard–Bishop–Dauxois model of DNA; while extending the polynomial approximation of the Morse potential up to the sixth order [8]. Another recent contribution analyzes soliton collision dynamics in a double-chain DNA helix, emphasizing how nonlinear interactions can lead to complex energy exchange between solitary waves, shedding light on multi-soliton behaviors in DNA models [9]. These foundational studies laid the groundwork for a series of refinements through diverse analytical approaches aimed at capturing different dynamical modes and deriving exact solutions. Of particular significance are nonlinear wave phenomena, which

can transport energy without dissipation, thereby conserving energy within biological processes. This integration of nonlinear scientific principles with biological mechanisms has proven to be a powerful paradigm for elucidating DNA's physical behavior.

Solitons are localized wave packets that preserve their shape and velocity during propagation as a result of the balance between nonlinearity and dispersion [10]. They have been widely studied due to their significant role in diverse physical systems such as fluid dynamics, plasma physics, and nonlinear optics [11]. Understanding soliton dynamics not only provides deep insight into stable wave phenomena but also offers practical applications in areas like optical communication and signal transmission.

The motivation behind this research is to grasp the nonlinear energy transfer and genetic information transfer in DNA molecules using exact analytical solutions. Various studies have been devoted to numerical solutions or certain types of solutions for this equation. Few studies have utilized the β -fractional derivative of a function and more advanced analytical techniques to derive various solutions of this equation and investigate their stability and bifurcation properties. This research intends to fill these gaps and yield new solutions such as soliton solutions, periodic solutions, singular solutions, and rational solutions of this equation along with a thorough graphical and phase plane analysis to develop a better understanding of DNA dynamics and its applications.

The search for exact solutions to nonlinear evolution equations has attracted sustained interest across disciplines such as mathematical physics, fluid dynamics, optical communications, biological systems, and plasma physics. A variety of analytical methods have been developed for this purpose, including Kudryashov's method [12], the Hirota bilinear approach [13], the generalized $\frac{G'}{G}$ -expansion method [14], symbolic bilinear methods [15], the modified extended mapping [16], extended generalized Riccati equation mapping [17], and the generalized exponential rational function (GERF) technique [18]. For the dsDNA model, prior investigations have reported Riccati-based solitary wave solutions, singular and kink solutions via the ϕ^6 method [19], graphical solution representations through the GERF approach [20], and Lie symmetry analysis of solitary waves [21].

In the study of nonlinear differential equations arising in physical and biological models, a variety of analytical techniques have been developed. Symbolic calculation methods, including the modified extended direct algebraic method used in this work, rely on symbolic manipulation and structured ansätze to derive exact analytical solutions in closed form. These methods are powerful in constructing a wide range of solution types, such as solitons, periodic waves, and rational structures, but can involve extensive algebraic computation. In contrast, non-symbolic methods such as the direct mapping method [22] and the variational method [23] offer simpler and more direct procedures that can avoid some of the symbolic complexity by translating the problem into integral or mapping relations, as demonstrated in recent studies. Regardless of their efficiency, non-symbolic methods may be limited in the diversity of solution structures they can systematically generate or may not readily capture multi-parameter families of solutions. The modified extended direct algebraic method, while mathematically involved, provides a systematic framework for deriving multiple classes of exact solutions and for analyzing their parameter dependence, which is particularly appropriate for the fractional double-chain DNA model considered here. By combining analytical rigor with flexibility in ansatz selection, symbolic methods like modified extended direct algebraic method (MEDAM) complement non-symbolic techniques and allow a more complete characterization of nonlinear excitations within the model.

In recent years, fractional calculus has been widely adopted in diverse scientific and engineering disciplines to model memory effects and nonlocal interactions in complex dynamical systems. Numerous definitions of fractional derivatives, such as Riemann–Liouville, Caputo–Fabrizio, and conformable fractional derivatives. Wang derived a new fractal active low-pass filter (LPF) within the local fractional derivative (LFD) calculus on the Cantor set [24]. Also, he developed another new fractional low-pass transmission line model using local fractional derivatives [25]. It introduces two special functions from the Mittag-Leffler function on a Cantor set to find exact, non-differentiable solutions through an auxiliary function and Yang’s transformation. These methods are suitable for studying other problems in the fractal electrical systems.

In the proposed research, the β -fractional derivative is used to extend the time and space evolution of the double-chain DNA model due to the ability of this derivative to maintain integral physical attributes of the model, such as energy conservation and wave transmission, and at the same time make possible the derivation of exact analytical solutions for these models. In contrast to other fractional derivatives, the β -fractional derivative allows for memory-dependent and nonlocal interactions between the DNA molecules.

Recent advances in the qualitative analysis of nonlinear partial differential equations (NLPDEs) have emphasized the importance of bifurcation and phase-plane techniques for understanding complex wave structures and transitions between solution types. For example, a very recent study by Wang et al. [26] investigated the qualitative behavior of the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation using a combination of variational principles, Hamiltonian formalism, phase-plane portraits, and bifurcation analysis to reveal diverse wave solutions and their stability properties. In that work, the detailed bifurcation diagrams illuminate how variations in system parameters lead to changes in wave characteristics such as solitary, periodic, and singular solutions, and also connect to chaotic and sensitivity phenomena under perturbations. Such studies underscore the value of bifurcation analysis in nonlinear wave dynamics and provide a broader context for the present investigation, where we similarly employ bifurcation and phase-plane analyses to systematically characterize solution regimes of the fractional double-chain DNA model.

Expanding on this body of work, the present investigation employs the MEDAM to derive soliton solutions for the dsDNA model in the presence of a β -fractional derivative. This method yields a wide range of solution types, including bright, dark, and singular solitons, and many other traveling wave patterns. The novelty of this study lies in obtaining solutions in a compact form distinct from those previously reported in Refs. [20, 21, 27], thereby enriching the theoretical understanding of the governing equations and providing a foundation for biologically inspired applications and future research [28–30].

The research work in [27] discusses the study of the double-chain DNA model and highlights its function in genetic information transmission; nevertheless, the focus, structure, and depth of contribution of the current research are different. With a focus on visualization and possible applications in many scientific domains, they provided a more comprehensive descriptive overview, emphasizing the physical interpretation of the model, the biological motivation, and the variety of soliton and wave solutions. The current paper, on the other hand, takes a more formal and method-oriented approach, clearly describing the mathematical framework, including the analytical derivation of exact solutions and the modified extended direct algebraic method. By explicitly explaining the derivation of new accurate soliton solutions and their lack in earlier investigations, it emphasizes

novelty more. Additionally, our study presents issues relating to stability and bifurcation, and presents the findings as a major development in DNA dynamics modeling.

The remainder of the manuscript is organized as follows: Section 2 outlines the proposed model and method scheme. Section 3 introduces the wave transformation that reduces the (3+1)-dimensional dsDNA model to a nonlinear ordinary differential equation, in addition to presenting the closed-form solutions obtained using MEDAM and their physical interpretation. Section 4 offers comparative analysis and contextual interpretation through bifurcation and stability analysis and provides 3D, 2D, and polar graphical representations of some selected solutions, accompanied by discussions of their dynamical behavior and physical significance. Section 5 includes limitations and future work, while Section 6 concludes with a summary of the principal findings.

2. Mathematical preliminaries

In this section, we provide basic mathematical preliminaries for the fractional derivative used in this paper, the governing equations for our model, and also the main steps of the algorithm used.

2.1. β -fractional derivative

Let $\Psi : [0, \infty) \rightarrow \mathbb{R}$ be a function. Then its beta fractional derivative of order $0 < \beta \leq 1$ is defined as [31]:

$$D_t^\beta \Psi(t) = \lim_{\epsilon \rightarrow 0} \frac{\Psi\left(t + \epsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - \Psi(t)}{\epsilon}, \quad \forall t \geq 0.$$

The fractional derivative satisfies several important properties:

- Linearity:

$$D^\beta (a\Psi(t) + bS(t)) = aD^\beta \Psi(t) + bD^\beta S(t), \quad \forall a, b \in \mathbb{R}.$$

- Product rule:

$$D^\beta (\Psi(t)S(t)) = \Psi(t)D^\beta S(t) + S(t)D^\beta \Psi(t).$$

- Quotient rule:

$$D^\beta \left(\frac{\Psi(t)}{S(t)} \right) = \frac{S(t)D^\beta \Psi(t) - \Psi(t)D^\beta S(t)}{(S(t))^2}.$$

- Fundamental definition:

$$D^\beta \Psi(t) = \left(t + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} \frac{d\Psi(t)}{dt}.$$

- Fractional chain rule:

$$D^\beta (\Psi(S(t))) = S'(t)^\beta D^\beta \Psi(S(t)).$$

2.2. The governing equations

The present study investigates a (3+1)-dimensional dsDNA model governed by the following NLPDE [27, 32, 33]:

$$\Phi_{tt} - \sigma_1^2 \Phi_{xx} - \sigma_1^2 \Phi_{yy} - \sigma_1^2 \Phi_{zz} - \mathcal{A}_1 \Phi - \mathcal{J}_1 \Phi \Psi - \mathcal{U}_1 \Phi^3 - \mathcal{U}_3 \Phi \Psi^2 = 0, \quad (2.1)$$

$$\Psi_{tt} - \sigma_2^2 \Psi_{xx} - \sigma_2^2 \Psi_{yy} - \sigma_2^2 \Psi_{zz} - \mathcal{A}_2 \Psi - \mathcal{J}_2 \Phi^2 - \mathcal{U}_2 \Phi^2 \Psi - \mathcal{U}_4 \Psi^3 - \mathbb{C}_0 = 0, \quad (2.2)$$

where $\Phi(x, y, z, t)$ characterizes longitudinal displacement variations between the two DNA strands, while $\Psi(x, y, z, t)$ represents transverse motion differences.

For the sake of knowledge, the previous system was studied in [27], but with total derivative form, and produced some short solitons with complicated expressions. In this study, we will investigate the system in Eqs (2.1)-(2.2) but with the influence of the β -fractional derivative to produce novel and simple compact forms of the solutions, which take the following form:

$$D_t^{2\beta} \Phi - \sigma_1^2 \Phi_{xx} - \sigma_1^2 \Phi_{yy} - \sigma_1^2 \Phi_{zz} - \mathcal{A}_1 \Phi - \mathcal{J}_1 \Phi \Psi - \mathcal{U}_1 \Phi^3 - \mathcal{U}_3 \Phi \Psi^2 = 0, \quad (2.3)$$

$$D_t^{2\beta} \Psi - \sigma_2^2 \Psi_{xx} - \sigma_2^2 \Psi_{yy} - \sigma_2^2 \Psi_{zz} - \mathcal{A}_2 \Psi - \mathcal{J}_2 \Phi^2 - \mathcal{U}_2 \Phi^2 \Psi - \mathcal{U}_4 \Psi^3 - \mathbb{C}_0 = 0, \quad (2.4)$$

in which we define the physical parameters as follows:

$$\sigma_1 = \pm \frac{\epsilon}{\rho_1}, \quad \sigma_2 = \pm \frac{F}{\rho_1}, \quad \mathcal{A}_1 = \frac{-2\mu_1}{\rho_1 s_1 h} (c - L_0), \quad (2.5)$$

$$\mathcal{U}_3 = \mathcal{U}_4 = \frac{4\mu_1 L_0}{\rho_1 s_1 h^3}, \quad \mathbb{C}_0 = \frac{\sqrt{2}\mu_1 (h - L_0)}{\rho_1 s_1}, \quad (2.6)$$

$$\mathcal{A}_2 = \frac{-2\mu_1}{\rho_1 s_1}, \quad \mathcal{J}_1 = 2\mathcal{J}_2 = \frac{2\sqrt{2}\mu_1 L_0}{\rho_1 s_1 h^2}, \quad \mathcal{U}_1 = \mathcal{U}_2 = \frac{-2\mu_1 L_0}{\rho_1 s_1 h^3}, \quad (2.7)$$

where ρ_1 denotes density, s_1 is the transverse cross-sectional area, ϵ is Young's modulus, F is the strand tension density, μ_1 is the elastic membrane rigidity, h is the inter-strand distance, and L_0 is the membrane height at equilibrium.

2.3. Main steps of the MEDAM

This subsection provides a concise overview of the MEDAM (see [34, 35]). Any NLPDE may be represented as follows:

$$\mathcal{F}(\Phi, \Phi_x, \Phi_y, \Phi_z, D_t^\beta \Phi, \Phi_{xx}, \Phi_{yy}, \Phi_{zz}, D_t^{2\beta} \Phi, \dots) = 0. \quad (2.8)$$

The subsequent expression represents the solution corresponding to the last equation:

$$\Phi(x, y, z, t) = \mathcal{H}(\zeta), \quad \zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}. \quad (2.9)$$

Accordingly, Eq (2.8) may be reformulated as:

$$\mathcal{G}(\mathcal{H}, \mathcal{H}', \mathcal{H}'', \dots) = 0. \quad (2.10)$$

The upcoming formula is useful in the solution of Eq (2.10):

$$\mathcal{H}(\zeta) = \sum_{i=-N}^N \mathcal{A}_i \mathbb{Z}(\zeta)^i, \quad (2.11)$$

in which \mathbb{Z} fulfills the following relation:

$$\mathbb{Z}'(\zeta) = \sqrt{\omega_0 + \omega_1\mathbb{Z}(\zeta) + \omega_2\mathbb{Z}(\zeta)^2 + \omega_3\mathbb{Z}(\zeta)^3 + \omega_4\mathbb{Z}(\zeta)^4 + \omega_6\mathbb{Z}(\zeta)^6}. \quad (2.12)$$

The following scenarios exist for the solutions of Eq (2.12):

Case 1: When $\omega_0 = \omega_1 = \omega_3 = \omega_6 = 0$, then

$$\mathbb{Z}(\zeta) = \sqrt{-\frac{\omega_2}{\omega_4}} \operatorname{sech} \left[\zeta \sqrt{\omega_2} \right], \quad \omega_2 > 0, \omega_4 < 0,$$

$$\mathbb{Z}(\zeta) = \sqrt{\frac{\omega_2}{\omega_4}} \operatorname{csch} \left[\zeta \sqrt{\omega_2} \right], \quad \omega_2 > 0, \omega_4 > 0,$$

$$\mathbb{Z}(\zeta) = \sqrt{-\frac{\omega_2}{\omega_4}} \sec \left[\zeta \sqrt{-\omega_2} \right], \quad \omega_2 < 0, \omega_4 > 0,$$

$$\mathbb{Z}(\zeta) = \sqrt{-\frac{\omega_2}{\omega_4}} \operatorname{csch} \left[\rho \zeta \sqrt{\omega_2} \right], \quad \omega_2 = 1, \omega_4 = \frac{1}{2}.$$

Case 2: When $\omega_1 = \omega_3 = \omega_6 = 0, \omega_0 = \frac{\omega_2^2}{4\omega_4}$, then

$$\mathbb{Z}(\zeta) = \sqrt{\frac{\omega_2}{2\omega_4}} \tan \left[\zeta \sqrt{\frac{\omega_2}{2}} \right], \quad \omega_2 > 0, \omega_4 > 0,$$

$$\mathbb{Z}(\zeta) = \sqrt{\frac{-\omega_2}{2\omega_4}} \tanh \left[\zeta \sqrt{\frac{-\omega_2}{2}} \right], \quad \omega_2 < 0, \omega_4 > 0.$$

Case 3: When $\omega_3 = \omega_4 = \omega_6 = 0$, then

$$\mathbb{Z}(\zeta) = e^{\zeta \sqrt{\omega_2}} - \frac{\omega_1}{2\omega_2}, \quad \omega_2 > 0, \omega_0 = \frac{\omega_1^2}{4\omega_2}.$$

Case 4: When $\omega_0 = \omega_1 = \omega_6 = 0$, then

$$\mathbb{Z}(\zeta) = -\frac{\omega_2}{\omega_3} \left(\tanh \left[\frac{1}{2} \zeta \sqrt{\omega_2} \right] + 1 \right), \quad \omega_2 > 0, \omega_3^2 = 4\omega_2\omega_4,$$

$$\mathbb{Z}(\zeta) = -\frac{\omega_2}{\omega_3} \left(\coth \left[\frac{1}{2} \zeta \sqrt{\omega_2} \right] + 1 \right), \quad \omega_2 > 0, \omega_3^2 = 4\omega_2\omega_4,$$

$$\mathbb{Z}(\zeta) = \frac{\omega_2 \operatorname{sech}^2 \left[\frac{1}{2} \zeta \sqrt{\omega_2} \right]}{2 \sqrt{\omega_2\omega_4} \tanh \left[\frac{1}{2} \zeta \sqrt{\omega_2} \right] - \omega_3}, \quad \omega_2 > 0, \omega_4 > 0, \omega_3^2 = 4\omega_2\omega_4,$$

$$\mathbb{Z}(\zeta) = \frac{\omega_2 \sec^2 \left[\frac{1}{2} \zeta \sqrt{-\omega_2} \right]}{2 \sqrt{-\omega_2\omega_4} \tan \left[\frac{1}{2} \zeta \sqrt{-\omega_2} \right] + \omega_3}, \quad \omega_2 < 0, \omega_4 > 0, \omega_3^2 = 4\omega_2\omega_4.$$

Case 5: When $\omega_2 = \omega_4 = \omega_6 = 0$, then

$$\mathbb{Z}(\zeta) = \wp\left(\frac{\sqrt{\omega_3}}{2}\zeta; -\frac{4\omega_1}{\omega_3}, -\frac{4\omega_0}{\omega_3}\right), \quad \omega_3 > 0.$$

Case 6: When $\omega_0 = \omega_1 = \omega_2 = \omega_6 = 0$, then

$$\mathbb{Z}(\zeta) = \frac{4\omega_3}{\omega_3^2 \zeta^2 - 4\omega_4}.$$

Case 7: When $\omega_1 = \omega_3 = \omega_6 = 0$ and $0 \leq \tau \leq 1$, then we consider the following sub-cases:

Sub-case 7.1:

$$\omega_0 = 1, \quad \omega_2 = -1 - \tau^2, \quad \omega_4 = \tau^2, \quad \mathbb{Z}(\zeta) = \operatorname{sn}(\zeta, \tau) \text{ or } \operatorname{cd}(\zeta, \tau).$$

Sub-case 7.2:

$$\omega_0 = \tau^2 - 1, \quad \omega_2 = 2 - \tau^2, \quad \omega_4 = -1, \quad \mathbb{Z}(\zeta) = \operatorname{dn}(\zeta, \tau).$$

Sub-case 7.3:

$$\omega_0 = -\tau^2, \quad \omega_2 = 2\tau^2 - 1, \quad \omega_4 = 1 - \tau^2, \quad \mathbb{Z}(\zeta) = \operatorname{nc}(\zeta, \tau).$$

Sub-case 7.4:

$$\omega_0 = -1, \quad \omega_2 = 2 - \tau^2, \quad \omega_4 = \tau^2 - 1, \quad \mathbb{Z}(\zeta) = \operatorname{nd}(\zeta, \tau).$$

Sub-case 7.5:

$$\omega_0 = 1, \quad \omega_2 = 2 - 4\tau^2, \quad \omega_4 = 1, \quad \mathbb{Z}(\zeta) = \operatorname{dn}(\zeta, \tau) \operatorname{nc}(\zeta, \tau) \operatorname{sn}(\zeta, \tau).$$

Sub-case 7.6:

$$\omega_0 = \tau^4 - 2\tau^3 + \tau^2, \quad \omega_2 = -\frac{4}{\tau}, \quad \omega_4 = -\tau^2 + 6\tau - 1, \quad \mathbb{Z}(\zeta) = \frac{\tau \operatorname{cn}(\zeta, \tau) \operatorname{dn}(\zeta, \tau)}{\tau \operatorname{sn}^2(\zeta, \tau) + 1}.$$

Sub-case 7.7:

$$\omega_0 = \frac{1}{4}, \quad \omega_2 = \frac{1}{2}(\tau^2 - 2), \quad \omega_4 = \frac{\tau^4}{4}, \quad \mathbb{Z}(\zeta) = \frac{\operatorname{cn}(\zeta, \tau)}{\operatorname{dn}(\zeta, \tau) + \sqrt{1 - \tau^2}},$$

where τ is the modulus of the Jacobi elliptic functions $0 \leq \tau \leq 1$.

To determine the integer N , the balance principle is applied to Eq (2.10) by matching the dominant derivative term with the highest nonlinear term. Substituting Eqs (2.11)-(2.12) into Eq (2.10) reduces it to a polynomial in $\mathbb{Z}(\zeta)$. Equating coefficients of corresponding powers to zero produces a system of algebraic equations, which is solved with Wolfram Mathematica.

3. Exact analytical solutions and physical interpretations

3.1. Generation of exact solutions for the proposed system

By considering the next transformation,

$$\Psi = \delta_1 \Phi + \delta_2, \quad (3.1)$$

in Eqs (2.3)-(2.4), hence Eq (2.3) will be reduced to:

$$D_t^{2\beta} \Phi - \sigma_1^2 \Phi_{xx} - \sigma_1^2 \Phi_{yy} - \sigma_1^2 \Phi_{zz} = \Phi^3 (\bar{U}_1 + \delta_1^2 \bar{U}_3) + \Phi^2 (2\delta_2 \delta_1 \bar{U}_3 + \delta_1 \mathcal{J}_1) + \Phi (\mathcal{A}_1 + \delta_2^2 \bar{U}_3 + \delta_2 \mathcal{J}_1), \quad (3.2)$$

and Eq (2.4) will be:

$$D_t^{2\beta} \Psi - \sigma_2^2 \Psi_{xx} - \sigma_2^2 \Psi_{yy} - \sigma_2^2 \Psi_{zz} = \mathcal{A}_2 \Psi + \mathcal{J}_2 \Psi \Phi - \bar{U}_2 \Psi \Phi^2 + \bar{U}_4 \Psi^3 + \mathcal{C}_0. \quad (3.3)$$

Equations (3.2)-(3.3) are the same when:

$$\delta_2 = \frac{h}{\sqrt{2}}, \quad \sigma_1 = \sigma_2. \quad (3.4)$$

As a result, Eqs (3.2)-(3.3) are reduced to the following single partial differential equation:

$$D_t^{2\beta} \Phi - \sigma_1^2 \Phi_{xx} - \sigma_1^2 \Phi_{yy} - \sigma_1^2 \Phi_{zz} - \mathcal{P} \Phi^3 - \mathcal{Q} \Phi^2 - \mathcal{R} \Phi = 0, \quad (3.5)$$

where: $\mathcal{P} = \frac{\Omega(4\delta_1^2 - 2)}{h^3}$, $\mathcal{Q} = \frac{6\sqrt{2}\Omega\delta_1}{h^2}$, $\mathcal{R} = \frac{6\Omega}{h} - \frac{2\Omega}{L_0}$, $\Omega = \frac{\mu_1 L_0}{\rho_1 s_1}$, $\sigma_1 = \sigma_2$.

Applying the transformation specified in Eq (2.9) to Eq (3.5), hence the partial derivatives associated with Eq (3.5) will be:

$$\begin{aligned} D_t^{2\beta} \Phi &= \mathbf{f}^2 \mathcal{H}'' \left(\frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} + \mathbf{a}x + \mathbf{b}y + \mathbf{d}z \right), \\ \Phi_{xx} &= \mathbf{a}^2 \mathcal{H}'' \left(\frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} + \mathbf{a}x + \mathbf{b}y + \mathbf{d}z \right), \\ \Phi_{yy} &= \mathbf{b}^2 \mathcal{H}'' \left(\frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} + \mathbf{a}x + \mathbf{b}y + \mathbf{d}z \right), \\ \Phi_{zz} &= \mathbf{d}^2 \mathcal{H}'' \left(\frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} + \mathbf{a}x + \mathbf{b}y + \mathbf{d}z \right). \end{aligned}$$

Therefore, Eq (3.5) will convert to the next nonlinear ordinary differential equation:

$$\left(\mathbf{f}^2 - \sigma_1^2 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) \right) \mathcal{H}'' - \mathcal{P} \mathcal{H}^3 - \mathcal{Q} \mathcal{H}^2 - \mathcal{R} \mathcal{H} = 0. \quad (3.6)$$

Thus, using the balancing principle between \mathcal{H}'' , and \mathcal{H}^3 , $N = 1$, and the solution of Eq (3.6), we arrive at the following:

$$\mathcal{H}(\zeta) = \lambda_0 + \lambda_1 \mathbb{Z}(\zeta) + \frac{\lambda_{-1}}{\mathbb{Z}(\zeta)}. \quad (3.7)$$

The following outcomes can be achieved by using Mathematica to solve a sequence of non-linear algebraic equations that are created when the coefficients of the same powers are grouped and set to zero: Eq (3.7) with the help of the auxiliary (2.12) and inserting them in Eq (3.6):

Case 1: $\omega_0 = \omega_1 = \omega_3 = \omega_6 = 0$

$$(1.1) \lambda_{-1} = 0, \lambda_0 = 0, \lambda_1 = \sqrt{\frac{2\omega_4\mathcal{R}}{\mathcal{P}\omega_2}}, \mathbf{f} = \sqrt{\frac{\sigma_1^2\omega_2(\mathbf{a}^2+\mathbf{b}^2+\mathbf{d}^2)+\mathcal{R}}{\omega_2}}, \mathcal{Q} = 0.$$

$$(1.2) \lambda_{-1} = 0, \lambda_0 = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}}, \lambda_1 = \sqrt{-\frac{\omega_4\mathcal{R}}{\mathcal{P}\omega_2}}, \mathbf{f} = \sqrt{\frac{2\sigma_1^2\omega_2(\mathbf{a}^2+\mathbf{b}^2+\mathbf{d}^2)-\mathcal{R}}{2\omega_2}}, \mathcal{Q} = -\sqrt{\frac{9\mathcal{P}\mathcal{R}}{2}}.$$

The solution of Eqs (2.1)-(2.2) will be raised for result (1.1) as follows:

(1.1.1) If $\omega_2 > 0$ and $\omega_4 < 0$, a bright soliton solution emerges in the following form:

$$\Phi_{1.1.1} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech} \left[\sqrt{\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right], \quad (3.8)$$

and

$$\Psi_{1.1.1} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech} \left[\sqrt{\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right] + \delta_2, \quad \mathcal{R}\mathcal{P} < 0. \quad (3.9)$$

(1.1.2) If $\omega_2 > 0$ and $\omega_4 > 0$, a singular soliton solution emerges in the following form:

$$\Phi_{1.1.2} = \sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{csch} \left[\sqrt{\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right], \quad (3.10)$$

and

$$\Psi_{1.1.2} = \delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{csch} \left[\sqrt{\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right] + \delta_2, \quad \mathcal{R}\mathcal{P} > 0. \quad (3.11)$$

(1.1.3) If $\omega_2 < 0$ and $\omega_4 > 0$, a singular periodic solution emerges in the following form:

$$\Phi_{1.1.3} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sec} \left[\sqrt{-\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right], \quad (3.12)$$

and

$$\Psi_{1.1.3} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sec} \left[\sqrt{-\omega_2} \left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} \right) \right] + \delta_2, \quad \mathcal{R}\mathcal{P} < 0. \quad (3.13)$$

(1.1.4) If $\omega_2 = 1$ and $\omega_4 = \frac{1}{2}$, a singular soliton solution emerges in the following form:

$$\Phi_{1.1.4} = -\sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{csch} \left[\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} - \varrho \right], \quad (3.14)$$

and

$$\Psi_{1.1.4} = -\delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{csch} \left[\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \frac{\mathbf{f} \left(\frac{1}{\Gamma(\beta)} + t \right)^\beta}{\beta} - \varrho \right] + \delta_2, \quad \mathcal{R}\mathcal{P} < 0, \quad (3.15)$$

where ϱ is a constant.

The solution of Eqs (2.1)-(2.2) will be raised for result (1.2) as follows:

Case 2: $\omega_1 = \omega_3 = \omega_6 = 0$, $\omega_0 = \frac{\omega_2^2}{4\omega_4}$

$$(2.1) \quad \lambda_{-1} = \frac{3\mathcal{R}}{4\mathcal{Q}} \sqrt{\frac{\omega_4}{\omega_2}}, \quad \lambda_0 = -\frac{3\mathcal{R}}{2\mathcal{Q}}, \quad \lambda_1 = \frac{3\mathcal{R}}{2\mathcal{Q}} \sqrt{\frac{\omega_4}{\omega_2}}, \quad \mathcal{P} = \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \quad \mathbf{f} = -\frac{1}{2} \sqrt{\mathcal{R} + 4\sigma_1^2 \omega_2 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2)}.$$

$$(2.2) \quad \lambda_{-1} = \frac{3\mathcal{R}}{4\mathcal{Q}} \sqrt{-\frac{\omega_4}{2\omega_2}}, \quad \lambda_0 = -\frac{3\mathcal{R}}{2\mathcal{Q}}, \quad \lambda_1 = -\frac{3\mathcal{R}}{2\mathcal{Q}} \sqrt{-\frac{\omega_4}{2\omega_2}}, \quad \mathcal{P} = \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \quad \mathbf{f} = -\sqrt{\frac{8\sigma_1^2 \omega_2 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \mathcal{R}}{8\omega_2}}.$$

$$(2.3) \quad \lambda_{-1} = \frac{3\mathcal{R}}{4\mathcal{Q}} \sqrt{-\frac{\omega_4}{2\omega_2}}, \quad \lambda_0 = -\frac{3\mathcal{R}}{2\mathcal{Q}}, \quad \lambda_1 = 0, \quad \mathcal{P} = \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \quad \mathbf{f} = \sqrt{\frac{2\sigma_1^2 \omega_2 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \mathcal{R}}{2\omega_2}}.$$

$$(2.4) \quad \lambda_{-1} = 0, \quad \lambda_0 = \frac{3\mathcal{R}}{2\mathcal{Q}}, \quad \lambda_1 = \frac{3\mathcal{R}}{2\mathcal{Q}} \sqrt{-\frac{\omega_4}{2\omega_2}}, \quad \mathcal{P} = \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \quad \mathbf{f} = \sqrt{\frac{2\sigma_1^2 \omega_2 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \mathcal{R}}{2\omega_2}}.$$

Due to result (2.1), a singular periodic solution for Eqs (2.1)-(2.2) emerges in the following form when $\omega_2 > 0$ and $\omega_4 > 0$:

$$\Phi_{2.1} = -\frac{3\mathcal{R}}{2\sqrt{2}\mathcal{Q}} \left(\sqrt{2} - \tan \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] - \frac{\omega_4}{\omega_2} \cot \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] \right), \quad (3.16)$$

and

$$\Psi_{2.1} = -\delta_1 \frac{3\mathcal{R}}{2\sqrt{2}\mathcal{Q}} \left(\sqrt{2} - \tan \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] - \frac{\omega_4}{\omega_2} \cot \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] \right) + \delta_2, \quad (3.17)$$

where $\zeta = \mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

Due to result (2.2), a (dark-singular) soliton solution for Eqs (2.1)-(2.2) emerges in the following form when $\omega_2 < 0$ and $\omega_4 > 0$:

$$\Phi_{2.2} = -\frac{3\mathcal{R}}{4\mathcal{Q}} \left(2 + \tanh \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] - \frac{\omega_2}{\omega_4} \coth \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] \right), \quad (3.18)$$

and

$$\Psi_{2.2} = -\delta_1 \frac{3\mathcal{R}}{4\mathcal{Q}} \left(2 + \tanh \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] - \frac{\omega_2}{\omega_4} \coth \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] \right) + \delta_2, \quad (3.19)$$

where $\zeta = \mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

Due to result (2.3), a singular soliton solution for Eq (2.3) emerges in the following form when $\omega_2 < 0$ and $\omega_4 > 0$:

$$\Phi_{2.3} = -\frac{3\mathcal{R}}{4\mathcal{Q}} \left(2 - \frac{\omega_2}{\omega_4} \coth \left[\frac{\left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right) \sqrt{-\omega_2}}{\sqrt{2}} \right] \right), \quad (3.20)$$

and

$$\Psi_{2.3} = -\delta_1 \frac{3\mathcal{R}}{4\mathcal{Q}} \left(2 - \frac{\omega_2}{\omega_4} \coth \left[\frac{\left(\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y} + \mathbf{d}\mathbf{z} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right) \sqrt{-\omega_2}}{\sqrt{2}} \right] \right) + \delta_2. \quad (3.21)$$

Due to result (2.4), a dark soliton solution for Eq (2.3) emerges in the following form when $\omega_2 < 0$ and $\omega_4 > 0$:

$$\Phi_{2.4} = -\frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 - \tanh \left[\frac{\left(\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)+t} \right)^\beta}{\beta} \right) \sqrt{-\omega_2}}{\sqrt{2}} \right] \right), \quad (3.22)$$

and

$$\Psi_{2.4} = -\delta_1 \frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 - \tanh \left[\frac{\left(\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)+t} \right)^\beta}{\beta} \right) \sqrt{-\omega_2}}{\sqrt{2}} \right] \right) + \delta_2. \quad (3.23)$$

Case 3: $\omega_3 = \omega_4 = \omega_6 = 0$, $\lambda_{-1} = \sqrt{\frac{2\omega_0\mathcal{R}}{\mathcal{P}\omega_2}}$, $\lambda_0 = 0$, $\lambda_1 = 0$, $\mathcal{Q} = \frac{3}{2} \sqrt{\frac{\mathcal{P}\omega_1\mathcal{R}}{2\omega_0\omega_2}}$, $\mathbf{b} = \frac{\sqrt{\omega_2(\mathbf{f}^2 - \sigma_1^2(\mathbf{a}^2 + \mathbf{d}^2)) - \mathcal{R}}}{\sigma_1\sqrt{\omega_2}}$.

Due to result (3), an exponential solution for Eqs (2.1)-(2.2) emerges in the following form when $\omega_2 > 0$ and $\omega_0 = \frac{\omega_1^2}{4\omega_2}$:

$$\Phi_3 = \sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \frac{\omega_1}{2\omega_2 e^{\left(\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)+t} \right)^\beta}{\beta} \right) \sqrt{\omega_2}} - \omega_1}, \quad (3.24)$$

$$\Psi_3 = \delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}}} \frac{\omega_1}{2\omega_2 e^{\left(\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)+t} \right)^\beta}{\beta} \right) \sqrt{\omega_2}} - \omega_1} + \delta_2, \quad (3.25)$$

where $\mathcal{R}\mathcal{P} > 0$, and $\left(2\omega_2 e^{\left(\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)+t} \right)^\beta}{\beta} \right) \sqrt{\omega_2}} - \omega_1 \right) \neq 0$.

Case 4: If $\omega_0 = \omega_1 = \omega_6 = 0$, the following set of outcomes can be obtained:

$$\lambda_{-1} = 0, \lambda_0 = 0, \lambda_1 = \frac{3\omega_3\mathcal{R}}{2\mathcal{Q}\omega_2}, \mathcal{P} = \frac{8\mathcal{Q}^2\omega_2\omega_4}{9\omega_3^2\mathcal{R}}, \mathbf{f} = -\sqrt{\frac{\sigma_1^2\omega_2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) + \mathcal{R}}{\omega_2}}.$$

The solution of Eqs (2.1)-(2.2) will be raised for result (4) as follows:

(4.1) If $\omega_2 > 0$ and $\omega_3^2 = 4\omega_2\omega_4$, a dark soliton or singular soliton solution emerges in the following form:

$$\Phi_{4.1.a} = -\frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 + \tanh \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] \right), \quad (3.26)$$

and

$$\Psi_{4.1.a} = -\delta_1 \frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 + \tanh \left[\frac{\zeta \sqrt{\omega_2}}{\sqrt{2}} \right] \right) + \delta_2, \quad (3.27)$$

or

$$\Phi_{4.1.b} = -\frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 + \coth \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] \right), \quad (3.28)$$

and

$$\Psi_{4.1.b} = -\delta_1 \frac{3\mathcal{R}}{2\mathcal{Q}} \left(1 + \coth \left[\frac{\zeta \sqrt{-\omega_2}}{\sqrt{2}} \right] \right) + \delta_2, \quad (3.29)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

(4.2) If $\omega_2 > 0$, $\omega_4 > 0$, and $\omega_3^2 \neq 4\omega_2\omega_4$, a (bright-dark) soliton solution emerges in the following form:

$$\Phi_{4.2} = -\frac{3\omega_3\mathcal{R}}{2\mathcal{Q}} \frac{\operatorname{sech}^2 \left(\frac{\zeta \sqrt{\omega_2}}{2} \right)}{\omega_3 - 2\sqrt{\omega_2\omega_4} \tanh \left(\frac{\zeta \sqrt{\omega_2}}{2} \right)}, \quad (3.30)$$

and

$$\Psi_{4.2} = -\delta_1 \frac{3\omega_3\mathcal{R}}{2\mathcal{Q}} \frac{\operatorname{sech}^2 \left(\frac{\zeta \sqrt{\omega_2}}{2} \right)}{\omega_3 - 2\sqrt{\omega_2\omega_4} \tanh \left(\frac{\zeta \sqrt{\omega_2}}{2} \right)} + \delta_2, \quad (3.31)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

(4.3) If $\omega_2 < 0$, $\omega_4 > 0$, and $\omega_3^2 \neq 4\omega_2\omega_4$, a singular periodic solution emerges in the following form:

$$\Phi_{4.3} = -\frac{3\omega_3\mathcal{R}}{2\mathcal{Q}} \frac{\sec^2 \left(\frac{1}{2}\zeta \sqrt{-\omega_2} \right)}{2\sqrt{-\omega_2\omega_4} \tan \left(\frac{1}{2}\zeta \sqrt{-\omega_2} \right) + \omega_3}, \quad (3.32)$$

and

$$\Psi_{4.3} = -\delta_1 \frac{3\omega_3\mathcal{R}}{2\mathcal{Q}} \frac{\sec^2 \left(\frac{1}{2}\zeta \sqrt{-\omega_2} \right)}{2\sqrt{-\omega_2\omega_4} \tan \left(\frac{1}{2}\zeta \sqrt{-\omega_2} \right) + \omega_3} + \delta_2, \quad (3.33)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

Case 5: If $\omega_2 = \omega_4 = \omega_6 = 0$, the following set of outcomes can be obtained:

$$\lambda_{-1} = 0, \lambda_0 = -\frac{\mathcal{R}}{2\mathcal{Q}}, \lambda_1 = \frac{\sqrt{-\frac{3\omega_3}{\omega_1}}\mathcal{R}}{2\mathcal{Q}}, \mathcal{P} = 0, \mathbf{f} = \frac{\sqrt{3\sigma_1^2\omega_3(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2)} + \sqrt{-\frac{3\omega_3}{\omega_1}}\mathcal{R}}{\sqrt{3\omega_3}}.$$

If $\omega_3 > 0$, a Weierstrass elliptic doubly periodic solution is gained as demonstrated:

$$\Phi_5 = -\frac{\mathcal{R}}{2\mathcal{Q}} \left(1 - \sqrt{-\frac{3\omega_3}{4\omega_1}} \wp \left(\frac{\zeta \sqrt{\omega_3}}{2}, -\frac{4\omega_1}{\omega_3}, -\frac{4\omega_0}{\omega_3} \right) \right), \quad (3.34)$$

and

$$\Psi_5 = -\delta_1 \frac{\mathcal{R}}{2\mathcal{Q}} \left(1 - \sqrt{-\frac{3\omega_3}{4\omega_1}} \wp \left(\frac{\zeta \sqrt{\omega_3}}{2}, -\frac{4\omega_1}{\omega_3}, -\frac{4\omega_0}{\omega_3} \right) \right) + \delta_2, \quad (3.35)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$.

Case 6: If $\omega_0 = \omega_1 = \omega_2 = \omega_6 = 0$, one can acquire the next set of results:

$$\lambda_{-1} = 0, \lambda_0 = \sqrt{\frac{\mathcal{R}}{\mathcal{P}}}, \lambda_1 = \frac{4\omega_4}{3\omega_3} \sqrt{\frac{\mathcal{R}}{\mathcal{P}}}, \mathbf{f} = \frac{\sqrt{9\sigma_1^2\omega_3^2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) + 8\omega_4\mathcal{R}}}{3\omega_3}, \mathcal{Q} = -2\sqrt{\mathcal{P}\mathcal{R}}.$$

If $\omega_3 > 0$, a rational solution is gained as demonstrated:

$$\Phi_6 = \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{16\omega_4}{3(\zeta^2\omega_3^2 - 4\omega_4)} \right), \quad (3.36)$$

and

$$\Psi_6 = -\delta_1 \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{16\omega_4}{3(\zeta^2\omega_3^2 - 4\omega_4)} \right) + \delta_2, \quad (3.37)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, and $\mathcal{R}\mathcal{P} > 0$.

Case 7: $\omega_1 = \omega_3 = \omega_6 = 0$

$$(7.1) \lambda_{-1} = \sqrt{\frac{2\omega_0\mathcal{R}}{\mathcal{P}\omega_2}}, \lambda_0 = 0, \lambda_1 = 0, \mathbf{f} = \frac{\sqrt{\sigma_1^2\omega_2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) + \mathcal{R}}}{\sqrt{\omega_2}}, \mathcal{Q} = 0.$$

$$(7.2) \lambda_{-1} = 0, \lambda_0 = 0, \lambda_1 = \sqrt{\frac{2\omega_4\mathcal{R}}{\mathcal{P}\omega_2}}, \mathbf{f} = \frac{\sqrt{\sigma_1^2\omega_2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) + \mathcal{R}}}{\sqrt{\omega_2}}, \mathcal{Q} = 0.$$

$$(7.3) \lambda_{-1} = \sqrt{-\frac{\omega_0\mathcal{R}}{\mathcal{P}\omega_2}}, \lambda_0 = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}}, \lambda_1 = 0, \mathbf{f} = \frac{\sqrt{2\sigma_1^2\omega_2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \mathcal{R}}}{\sqrt{2\omega_2}}, \mathcal{Q} = -\sqrt{\frac{9\mathcal{P}\mathcal{R}}{2}}.$$

$$(7.4) \lambda_{-1} = 0, \lambda_0 = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}}, \lambda_1 = \sqrt{-\frac{\omega_4\mathcal{R}}{\mathcal{P}\omega_2}}, \mathbf{f} = \frac{\sqrt{2\sigma_1^2\omega_2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \mathcal{R}}}{\sqrt{2\omega_2}}, \mathcal{Q} = -\sqrt{\frac{9\mathcal{P}\mathcal{R}}{2}}.$$

$$(7.5) \lambda_{-1} = \sqrt{-\frac{\omega_0\mathcal{R}}{\mathcal{P}(\omega_2 - 6\sqrt{\omega_0\omega_4})}}, \lambda_0 = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}}, \lambda_1 = \sqrt{-\frac{\omega_4\mathcal{R}}{\mathcal{P}(\omega_2 - 6\sqrt{\omega_0\omega_4})}}, \mathcal{Q} = -\sqrt{\frac{9\mathcal{P}\mathcal{R}}{2}},$$

$$\mathbf{f} = \sqrt{\frac{\sigma_1^2(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{d}^2) - \frac{\mathcal{R}}{2(\omega_2 - 6\sqrt{\omega_0\omega_4})}}{2(\omega_2 - 6\sqrt{\omega_0\omega_4})}}.$$

The solution of Eqs (2.1)-(2.2) will be raised for result (7.1) as follows:

(7.1.1) If $\omega_0 = 1$, $\omega_2 = -\tau^2 - 1$, and $\omega_4 = \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.1.1.a} = \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \text{ns} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right], \quad (3.38)$$

and

$$\Psi_{7.1.1.a} = \delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \text{ns} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right] + \delta_2, \quad (3.39)$$

or

$$\Phi_{7.1.1.b} = \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \text{dc} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right], \quad (3.40)$$

and

$$\Psi_{7.1.1.b} = \delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \text{dc} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right] + \delta_2, \quad (3.41)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.38)-(3.39), a singular soliton solution emerges in the following form:

$$\Phi_{7.1.1.a.1} = \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \coth \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.42)$$

and

$$\Psi_{7.1.1.a.1} = \delta_1 \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \coth \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2. \quad (3.43)$$

When putting $\tau = 0$ in Eqs (3.38)-(3.39) or Eqs (3.39)-(3.40), a singular periodic solution emerges in the following form:

$$\Phi_{7.1.1.a.2} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \csc \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.44)$$

and

$$\Psi_{7.1.1.a.2} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \csc \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2, \quad (3.45)$$

or

$$\Phi_{7.1.1.b.1} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \sec \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.46)$$

and

$$\Psi_{7.1.1.b.1} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \sec \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2. \quad (3.47)$$

(7.1.2) If $\omega_0 = \tau^2 - 1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = -1$, a JE solution emerges in the following form:

$$\Phi_{7.1.2} = \sqrt{\frac{2\mathcal{R}(\tau^2 - 1)}{\mathcal{P}(2 - \tau^2)}} \operatorname{nd} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.48)$$

and

$$\Psi_{7.1.2} = \delta_1 \sqrt{\frac{2\mathcal{R}(\tau^2 - 1)}{\mathcal{P}(2 - \tau^2)}} \operatorname{nd} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2, \quad (3.49)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau < 1$.

(7.1.3) If $\omega_0 = -\tau^2$, $\omega_2 = 2\tau^2 - 1$, and $\omega_4 = 1 - \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.1.3} = \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(1 - 2\tau^2)}} \operatorname{cn} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.50)$$

and

$$\Psi_{7.1.3} = \delta_1 \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(1 - 2\tau^2)}} \operatorname{cn} \left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2. \quad (3.51)$$

When putting $\tau = 1$ in Eqs (3.50)-(3.51), a bright soliton solution emerges in the following form:

$$\Phi_{7.1.3.a} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech}\left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}\right], \quad (3.52)$$

and

$$\Psi_{7.1.3.a} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech}\left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}\right] + \delta_2. \quad (3.53)$$

(7.1.4) If $\omega_0 = -1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = \tau^2 - 1$, a JE solution emerges in the following form:

$$\Phi_{7.1.4} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn}\left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}\right], \quad (3.54)$$

and

$$\Psi_{7.1.4} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn}\left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}\right] + \delta_2, \quad (3.55)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau \leq 1$.

(7.1.5) If $\omega_0 = 1$, $\omega_2 = 2 - 4\tau^2$, and $\omega_4 = 1$, a JE solution emerges in the following form:

$$\Phi_{7.1.5} = \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(2 - 4\tau^2)}} \operatorname{nd}[\zeta] \operatorname{cn}[\zeta] \operatorname{ns}[\zeta], \quad (3.56)$$

and

$$\Psi_{7.1.5} = \delta_1 \sqrt{\frac{2\mathcal{R}}{\mathcal{P}(2 - 4\tau^2)}} \operatorname{nd}[\zeta] \operatorname{cn}[\zeta] \operatorname{ns}[\zeta] + \delta_2, \quad (3.57)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} < 0$, and $\frac{1}{\sqrt{2}} \leq \tau \leq 1$.

(7.1.6) If $\omega_0 = \tau^4 - 2\tau^3 + \tau^2$, $\omega_2 = -\frac{4}{\tau}$, and $\omega_4 = -\tau^2 + 6\tau - 1$, a JE solution emerges in the following form:

$$\Phi_{7.1.6} = \sqrt{-\frac{\tau(\tau^4 - 2\tau^3 + \tau^2)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{sn}[\zeta]^2 + 1}{\sqrt{2} \tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}, \quad (3.58)$$

and

$$\Psi_{7.1.6} = \delta_1 \sqrt{-\frac{\tau(\tau^4 - 2\tau^3 + \tau^2)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{sn}[\zeta]^2 + 1}{\sqrt{2} \tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]} + \delta_2, \quad (3.59)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} < 0$, and $0 < \tau < 1$.

(7.1.7) If $\omega_0 = \frac{1}{4}$, $\omega_2 = \frac{1}{2}(\tau^2 - 2)$, and $\omega_4 = \frac{\tau^4}{4}$, a JE solution emerges in the following form:

$$\Phi_{7.1.7} = \sqrt{\frac{\mathcal{R}}{\mathcal{P}(\tau^2 - 2)}} \frac{\sqrt{1 - \tau^2} + \operatorname{dn}[\zeta]}{\operatorname{cn}[\zeta]}, \quad (3.60)$$

and

$$\Psi_{7.1.7} = \delta_1 \sqrt{\frac{\mathcal{R}}{\mathcal{P}(\tau^2 - 2)}} \frac{\sqrt{1 - \tau^2 + \operatorname{dn}[\zeta]}}{\operatorname{cn}[\zeta]} + \delta_2, \quad (3.61)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau \leq 1$.

The solution of Eqs (2.1)-(2.2) will be raised for result (7.2) as follows:

(7.2.1) If $\omega_0 = 1$, $\omega_2 = -\tau^2 - 1$, and $\omega_4 = \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.2.1.a} = \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{sn}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right], \quad (3.62)$$

and

$$\Psi_{7.2.1.a} = \delta_1 \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{sn}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right] + \delta_2, \quad (3.63)$$

or

$$\Phi_{7.2.1.b} = \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{cd}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right], \quad (3.64)$$

and

$$\Psi_{7.2.1.b} = \delta_1 \sqrt{\frac{2\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{cd}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right] + \delta_2, \quad (3.65)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 < \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.62)-(3.63), a dark soliton solution emerges in the following form:

$$\Phi_{7.2.1.a.1} = \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \operatorname{tanh}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right], \quad (3.66)$$

and

$$\Psi_{7.2.1.a.1} = \delta_1 \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \operatorname{tanh}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right] + \delta_2. \quad (3.67)$$

(7.2.2) If $\omega_0 = \tau^2 - 1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = -1$, a JE solution emerges in the following form:

$$\Phi_{7.2.2} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right], \quad (3.68)$$

and

$$\Psi_{7.2.2} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right] + \delta_2, \quad (3.69)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau \leq 1$.

(7.2.3) If $\omega_0 = -\tau^2$, $\omega_2 = 2\tau^2 - 1$, and $\omega_4 = 1 - \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.2.3} = \sqrt{\frac{2(1 - \tau^2)\mathcal{R}}{\mathcal{P}(2\tau^2 - 1)}} \operatorname{nc}\left[\mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}\right], \quad (3.70)$$

and

$$\Psi_{7.2.3} = \delta_1 \sqrt{\frac{2(1-\tau^2)\mathcal{R}}{\mathcal{P}(2\tau^2-1)}} \operatorname{nc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2, \quad (3.71)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau < 1$.

(7.2.4) If $\omega_0 = -1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = \tau^2 - 1$, a JE solution emerges in the following form:

$$\Phi_{7.2.4} = \sqrt{\frac{2(\tau^2-1)\mathcal{R}}{\mathcal{P}(2-\tau^2)}} \operatorname{nd} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.72)$$

and

$$\Psi_{7.2.4} = \delta_1 \sqrt{\frac{2(\tau^2-1)\mathcal{R}}{\mathcal{P}(2-\tau^2)}} \operatorname{nd} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2, \quad (3.73)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau < 1$.

(7.2.5) If $\omega_0 = 1$, $\omega_2 = 2 - 4\tau^2$, and $\omega_4 = 1$, a JE solution emerges in the following form:

$$\Phi_{7.2.5} = \sqrt{\frac{2(\tau^2-1)\mathcal{R}}{\mathcal{P}(2-\tau^2)}} \operatorname{dn}[\zeta] \operatorname{nc}[\zeta] \operatorname{sn}[\zeta], \quad (3.74)$$

and

$$\Psi_{7.2.5} = \delta_1 \sqrt{\frac{2(\tau^2-1)\mathcal{R}}{\mathcal{P}(2-\tau^2)}} \operatorname{dn}[\zeta] \operatorname{nc}[\zeta] \operatorname{sn}[\zeta] + \delta_2, \quad (3.75)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} < 0$, and $0 \leq \tau < 1$.

When putting $\tau = 0$ in Eqs (3.74)-(3.75), a singular periodic solution emerges in the following form:

$$\Phi_{7.2.5.1} = \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \tan \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.76)$$

and

$$\Psi_{7.2.5.1} = \delta_1 \sqrt{-\frac{\mathcal{R}}{\mathcal{P}}} \tan \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] + \delta_2. \quad (3.77)$$

(7.2.6) If $\omega_0 = \tau^4 - 2\tau^3 + \tau^2$, $\omega_2 = -\frac{4}{\tau}$, and $\omega_4 = -\tau^2 + 6\tau - 1$, a JE solution emerges in the following form:

$$\Phi_{7.2.6} = \sqrt{-\frac{\tau(-\tau^2+6\tau-1)\mathcal{R}}{2\mathcal{P}}} \frac{\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}{\tau \operatorname{sn}[\zeta]^2 + 1}, \quad (3.78)$$

and

$$\Psi_{7.2.6} = \delta_1 \sqrt{-\frac{\tau(-\tau^2+6\tau-1)\mathcal{R}}{2\mathcal{P}}} \frac{\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}{\tau \operatorname{sn}[\zeta]^2 + 1} + \delta_2, \quad (3.79)$$

where $\mathcal{R}\mathcal{P} < 0$, and $0 < \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.78)-(3.79), a bright soliton solution emerges in the following form:

$$\Phi_{7.2.6.1} = \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech} \left[2 \left(\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right) \right], \quad (3.80)$$

and

$$\Psi_{7.2.6.1} = \delta_1 \sqrt{-\frac{2\mathcal{R}}{\mathcal{P}}} \operatorname{sech} \left[2 \left(\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right) \right] + \delta_2. \quad (3.81)$$

(7.2.7) If $\omega_0 = \frac{1}{4}$, $\omega_2 = \frac{1}{2}(\tau^2 - 2)$, and $\omega_4 = \frac{\tau^4}{4}$, a JE solution emerges in the following form:

$$\Phi_{7.2.7} = \sqrt{\frac{\tau^4 \mathcal{R}}{\mathcal{P}(\tau^2 - 2)}} \frac{\operatorname{cn}[\zeta]}{\sqrt{1 - \tau^2 + \operatorname{dn}[\zeta]}}, \quad (3.82)$$

and

$$\Psi_{7.2.7} = \delta_1 \sqrt{\frac{\tau^4 \mathcal{R}}{\mathcal{P}(\tau^2 - 2)}} \frac{\operatorname{cn}[\zeta]}{\sqrt{1 - \tau^2 + \operatorname{dn}[\zeta]}} + \delta_2, \quad (3.83)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} < 0$, and $0 < \tau \leq 1$.

The solution of Eqs (2.1)-(2.2) will be raised for result (7.3) as follows:

(7.3.1) If $\omega_0 = 1$, $\omega_2 = -\tau^2 - 1$, and $\omega_4 = \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.3.1.a} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{ns} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.84)$$

and

$$\Psi_{7.3.1.a} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{ns} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.85)$$

or

$$\Phi_{7.3.1.b} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{dc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.86)$$

and

$$\Psi_{7.3.1.b} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{dc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.87)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.84)-(3.85), a singular soliton solution emerges in the following form:

$$\Phi_{7.3.1.a.1} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} \left(1 + \operatorname{coth} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right), \quad (3.88)$$

and

$$\Psi_{7.3.1.a.1} = \delta_1 \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} \left(1 + \operatorname{coth} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2. \quad (3.89)$$

When putting $\tau = 0$ in Eqs (3.84)-(3.85), a singular periodic solution emerges in the following form:

$$\Phi_{7.3.1.a.2} = \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{csc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right), \quad (3.90)$$

and

$$\Psi_{7.3.1.a.2} = \delta_1 \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{csc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.91)$$

or

$$\Phi_{7.3.1.b.1} = \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{sec} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right), \quad (3.92)$$

and

$$\Psi_{7.3.1.b.1} = \delta_1 \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{sec} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2. \quad (3.93)$$

(7.3.2) If $\omega_0 = -\tau^2$, $\omega_2 = 2\tau^2 - 1$, and $\omega_4 = 1 - \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.3.2} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\tau^2 \mathcal{R}}{\mathcal{P}(2\tau^2 - 1)}} \operatorname{cn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.94)$$

and

$$\Psi_{7.3.2} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\tau^2 \mathcal{R}}{\mathcal{P}(2\tau^2 - 1)}} \operatorname{cn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.95)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.94)-(3.95), a bright soliton solution emerges in the following form:

$$\Phi_{7.3.2.1} = \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{sech} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right), \quad (3.96)$$

and

$$\Psi_{7.3.2.1} = \delta_1 \sqrt{\frac{\mathcal{R}}{\mathcal{P}}} \left(\frac{1}{\sqrt{2}} + \operatorname{sech} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2. \quad (3.97)$$

(7.3.3) If $\omega_0 = -1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = \tau^2 - 1$, a JE solution emerges in the following form:

$$\Phi_{7.3.3} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.98)$$

and

$$\Psi_{7.3.3} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.99)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

(7.3.4) If $\omega_0 = 1$, $\omega_2 = 2 - 4\tau^2$, and $\omega_4 = 1$, a JE solution emerges in the following form:

$$\Phi_{7.3.4} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(2-4\tau^2)}} \operatorname{nd}[\zeta] \operatorname{cn}[\zeta] \operatorname{ns}[\zeta], \quad (3.100)$$

and

$$\Psi_{7.3.4} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(2-4\tau^2)}} \operatorname{nd}[\zeta] \operatorname{cn}[\zeta] \operatorname{ns}[\zeta] \right) + \delta_2, \quad (3.101)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.3.5) If $\omega_0 = \tau^4 - 2\tau^3 + \tau^2$, $\omega_2 = -\frac{4}{\tau}$, and $\omega_4 = -\tau^2 + 6\tau - 1$, a JE solution emerges in the following form:

$$\Phi_{7.3.5} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau(\tau^4 - 2\tau^3 + \tau^2)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{sn}[\zeta]^2 + 1}{2\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}, \quad (3.102)$$

and

$$\Psi_{7.3.5} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau(\tau^4 - 2\tau^3 + \tau^2)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{sn}[\zeta]^2 + 1}{2\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]} \right) + \delta_2, \quad (3.103)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau < 1$.

(7.3.6) If $\omega_0 = \frac{1}{4}$, $\omega_2 = \frac{1}{2}(\tau^2 - 2)$, and $\omega_4 = \frac{\tau^4}{4}$, a JE solution emerges in the following form:

$$\Phi_{7.3.6} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{2\mathcal{P}(\tau^2 - 2)}} \frac{\sqrt{1 - \tau^2} + \operatorname{dn}[\zeta]}{\operatorname{cn}[\zeta]}, \quad (3.104)$$

and

$$\Psi_{7.3.6} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{2\mathcal{P}(\tau^2 - 2)}} \frac{\sqrt{1 - \tau^2} + \operatorname{dn}[\zeta]}{\operatorname{cn}[\zeta]} \right) + \delta_2, \quad (3.105)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

The solution of Eqs (2.1)-(2.2) will be raised for result (7.4) as follows:

(7.4.1) If $\omega_0 = 1$, $\omega_2 = -\tau^2 - 1$, and $\omega_4 = \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.4.1.a} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{sn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right], \quad (3.106)$$

and

$$\Psi_{7.4.1.a} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{sn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.107)$$

or

$$\Phi_{7.4.1.b} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^2\mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{cd} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta} \right], \quad (3.108)$$

and

$$\Psi_{7.4.1.b} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^2 \mathcal{R}}{\mathcal{P}(-\tau^2 - 1)}} \operatorname{sn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.109)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

When putting $\tau = 1$ in Eqs (3.106)-(3.107), a singular soliton solution emerges in the following form:

$$\Phi_{7.4.1.a.1} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} \left(1 + \tanh \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right), \quad (3.110)$$

and

$$\Psi_{7.4.1.a.1} = \delta_1 \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} \left(1 + \tanh \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2. \quad (3.111)$$

(7.4.2) If $\omega_0 = \tau^2 - 1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = -1$, a JE solution emerges in the following form:

$$\Phi_{7.4.2} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.112)$$

and

$$\Psi_{7.4.2} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{dn} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.113)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

(7.4.3) If $\omega_0 = -\tau^2$, $\omega_2 = 2\tau^2 - 1$, and $\omega_4 = 1 - \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.4.3} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{(1 - \tau^2)\mathcal{R}}{\mathcal{P}(2\tau^2 - 1)}} \operatorname{nc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.114)$$

and

$$\Psi_{7.4.3} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{(1 - \tau^2)\mathcal{R}}{\mathcal{P}(2\tau^2 - 1)}} \operatorname{nc} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.115)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau < 1$.

(7.4.4) If $\omega_0 = -1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = \tau^2 - 1$, a JE solution emerges in the following form:

$$\Phi_{7.4.4} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{(\tau^2 - 1)\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{nd} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right], \quad (3.116)$$

and

$$\Psi_{7.4.4} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{(\tau^2 - 1)\mathcal{R}}{\mathcal{P}(2 - \tau^2)}} \operatorname{nd} \left[\mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{\left(\frac{1}{\Gamma(\beta)} + t\right)^\beta}{\beta} \right] \right) + \delta_2, \quad (3.117)$$

where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau < 1$.

(7.4.5) If $\omega_0 = 1$, $\omega_2 = 2 - 4\tau^2$, and $\omega_4 = 1$, a JE solution emerges in the following form:

$$\Phi_{7.4.5} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(2-4\tau^2)}} \operatorname{dn}[\zeta] \operatorname{nc}[\zeta] \operatorname{sn}[\zeta], \quad (3.118)$$

and

$$\Psi_{7.4.5} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(2-4\tau^2)}} \operatorname{dn}[\zeta] \operatorname{nc}[\zeta] \operatorname{sn}[\zeta] \right) + \delta_2, \quad (3.119)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.4.6) If $\omega_0 = \tau^4 - 2\tau^3 + \tau^2$, $\omega_2 = -\frac{4}{\tau}$, and $\omega_4 = -\tau^2 + 6\tau - 1$, a JE solution emerges in the following form:

$$\Phi_{7.4.6} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\tau(-\tau^2 + 6\tau - 1)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}{2(\tau \operatorname{sn}[\zeta]^2 + 1)}, \quad (3.120)$$

and

$$\Psi_{7.4.6} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\tau(-\tau^2 + 6\tau - 1)\mathcal{R}}{\mathcal{P}}} \frac{\tau \operatorname{cn}[\zeta] \operatorname{dn}[\zeta]}{2(\tau \operatorname{sn}[\zeta]^2 + 1)} \right) + \delta_2, \quad (3.121)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.4.7) If $\omega_0 = \frac{1}{4}$, $\omega_2 = \frac{1}{2}(\tau^2 - 2)$, and $\omega_4 = \frac{\tau^4}{4}$, a JE solution emerges in the following form:

$$\Phi_{7.4.7} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^4 \mathcal{R}}{2\mathcal{P}(\tau^2 - 2)}} \frac{\operatorname{cn}[\zeta]}{\sqrt{1 - \tau^2 + \operatorname{dn}[\zeta]}}, \quad (3.122)$$

and

$$\Psi_{7.4.7} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\tau^4 \mathcal{R}}{2\mathcal{P}(\tau^2 - 2)}} \frac{\operatorname{cn}[\zeta]}{\sqrt{1 - \tau^2 + \operatorname{dn}[\zeta]}} \right) + \delta_2, \quad (3.123)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

The solution of Eqs (2.1)-(2.2) will be raised for result (7.5) as follows:

(7.5.1) If $\omega_0 = 1$, $\omega_2 = -\tau^2 - 1$, and $\omega_4 = \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.5.1.a} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\tau - 1)}} (\operatorname{sn}[\zeta] + \operatorname{ns}[\zeta]), \quad (3.124)$$

and

$$\Psi_{7.5.1.a} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{\tau^2} - 1)}} (\operatorname{sn}[\zeta] + \operatorname{ns}[\zeta]) \right) + \delta_2, \quad (3.125)$$

or

$$\Phi_{7.5.1.b} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{-\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{\tau^2} - 1)}} (\operatorname{dc}[\zeta] + \operatorname{cd}[\zeta]), \quad (3.126)$$

and

$$\Psi_{7.5.1.b} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{\tau^2 - 1})}} (\operatorname{dc}[\zeta] + \operatorname{cd}[\zeta]) \right) + \delta_2, \quad (3.127)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

(7.5.2) If $\omega_0 = \tau^2 - 1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = -1$, a JE solution emerges in the following form:

$$\Phi_{7.5.2} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{1 - \tau^2} + 2)}} (\sqrt{1 - \tau^2} \operatorname{nd}[\zeta] + \operatorname{dn}[\zeta]), \quad (3.128)$$

and

$$\Psi_{7.5.2} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{1 - \tau^2} + 2)}} (\sqrt{1 - \tau^2} \operatorname{nd}[\zeta] + \operatorname{dn}[\zeta]) \right) + \delta_2, \quad (3.129)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.5.3) If $\omega_0 = -\tau^2$, $\omega_2 = 2\tau^2 - 1$, and $\omega_4 = 1 - \tau^2$, a JE solution emerges in the following form:

$$\Phi_{7.5.3} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2\tau^2 - 6\sqrt{-\tau^2(1 - \tau^2)} - 1)}} (\tau \operatorname{nc}[\zeta] + \sqrt{1 - \tau^2} \operatorname{nc}[\zeta]), \quad (3.130)$$

and

$$\Psi_{7.5.3} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(2\tau^2 - 6\sqrt{-\tau^2(1 - \tau^2)} - 1)}} (\tau \operatorname{nc}[\zeta] + \sqrt{1 - \tau^2} \operatorname{nc}[\zeta]) \right) + \delta_2, \quad (3.131)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.5.4) If $\omega_0 = -1$, $\omega_2 = 2 - \tau^2$, and $\omega_4 = \tau^2 - 1$, a JE solution emerges in the following form:

$$\Phi_{7.5.4} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{1 - \tau^2} + 2)}} (\operatorname{dn}[\zeta] + \sqrt{1 - \tau^2} \operatorname{nd}[\zeta]), \quad (3.132)$$

and

$$\Psi_{7.5.4} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(-\tau^2 - 6\sqrt{1 - \tau^2} + 2)}} (\operatorname{dn}[\zeta] + \sqrt{1 - \tau^2} \operatorname{nd}[\zeta]) \right) + \delta_2, \quad (3.133)$$

where $\zeta = \mathbf{ax} + \mathbf{by} + \mathbf{dz} + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 < \tau \leq 1$.

(7.5.5) If $\omega_0 = 1$, $\omega_2 = 2 - 4\tau^2$, and $\omega_4 = 1$, a JE solution emerges in the following form:

$$\Phi_{7.5.5} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{4\mathcal{P}(\tau^2 + 1)}} (\text{nd}[\zeta]\text{cn}[\zeta]\text{ns}[\zeta] + \text{dn}[\zeta]\text{nc}[\zeta]\text{sn}[\zeta]), \quad (3.134)$$

and

$$\Psi_{7.5.5} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{4\mathcal{P}(\tau^2 + 1)}} (\text{nd}[\zeta]\text{cn}[\zeta]\text{ns}[\zeta] + \text{dn}[\zeta]\text{nc}[\zeta]\text{sn}[\zeta]) \right) + \delta_2, \quad (3.135)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, where $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

(7.5.6) If $\omega_0 = \frac{1}{4}$, $\omega_2 = \frac{1}{2}(\tau^2 - 2)$, and $\omega_4 = \frac{\tau^4}{4}$, a JE solution emerges in the following form:

$$\Phi_{7.5.6} = \sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(\tau^2 + 1)}} \left(\frac{\tau^2 \text{cn}[\zeta]}{\sqrt{1 - \tau^2} + \text{dn}[\zeta]} + \frac{\text{dn}[\zeta] + \sqrt{1 - \tau^2}}{2\text{cn}[\zeta]} \right), \quad (3.136)$$

and

$$\Psi_{7.5.6} = \delta_1 \left(\sqrt{\frac{\mathcal{R}}{2\mathcal{P}}} + \sqrt{\frac{\mathcal{R}}{\mathcal{P}(\tau^2 + 1)}} \left(\frac{\tau^2 \text{cn}[\zeta]}{\sqrt{1 - \tau^2} + \text{dn}[\zeta]} + \frac{\text{dn}[\zeta] + \sqrt{1 - \tau^2}}{2\text{cn}[\zeta]} \right) \right) + \delta_2, \quad (3.137)$$

where $\zeta = \mathbf{a}x + \mathbf{b}y + \mathbf{d}z + \mathbf{f} \frac{(\frac{1}{\Gamma(\beta)} + t)^\beta}{\beta}$, $\mathcal{R}\mathcal{P} > 0$, and $0 \leq \tau \leq 1$.

3.2. Physical interpretation of the obtained solutions

The analytical solutions derived in this study represent distinct nonlinear excitation modes in the double-chain DNA model, each with specific physical implications. The hyperbolic solutions, which consist of both bright and dark solitons, describe localized wave structures that do not change their shape while propagating due to the balance between nonlinearity and dispersion. Bright soliton solutions correspond to the concentration of energy in certain regions of DNA strands; such solutions can be related to the localization and transfer of energy along DNA strands in some biological processes. Dark soliton solutions, on the other hand, correspond to the concentration of densities in some local regions with respect to the continuous background; they indicate the position where the excitations are reduced and propagate along the helix. The periodic solutions describe the spatially and temporally repeating wave patterns and have relevance to the oscillatory modes of DNA dynamics with possible sustaining or boundary-driven excitations. They provide a clear insight into the long-range collective motions of the DNA molecule. The singular solution thus represents extreme nonlinear responses of the system, and such solutions may be understood as a mathematical precursor of instability thresholds or transition regimes in DNA dynamics. Idealized as they are, they are helpful for the determination of critical parameters whereby qualitative changes in wave behavior occur. Rational solutions, describing localized structures with algebraic decay, can similarly be associated with transient, high-amplitude excitations emerging from nonlinear interactions.

4. Phase plane analysis

4.1. Transformation into a planar dynamical system

We begin with a second-order ordinary differential equation that is nonlinear,

$$(f^2 - (a^2 + b^2 + d^2)\sigma_1^2)\mathcal{H}'' - \mathcal{P}\mathcal{H}^3 - Q\mathcal{H}^2 - \mathcal{R}\mathcal{H} = 0, \quad (4.1)$$

where the parameters $a, b, d, f, \sigma_1, \mathcal{P}, Q, \mathcal{R} \in \mathbb{R}$. For convenience, set

$$L := f^2 - (a^2 + b^2 + d^2)\sigma_1^2.$$

If $L \neq 0$, dividing Eq (4.1) by L gives a normalized form. Introducing

$$\kappa_3 = -\frac{\mathcal{P}}{L}, \quad \kappa_2 = -\frac{Q}{L}, \quad \kappa_1 = -\frac{\mathcal{R}}{L}, \quad (4.2)$$

we obtain

$$\mathcal{H}'' + \kappa_3\mathcal{H}^3 + \kappa_2\mathcal{H}^2 + \kappa_1\mathcal{H} = 0. \quad (4.3)$$

Equation (4.3) resembles a Duffing-type oscillator but includes a quadratic contribution. To recast it as a planar system, define

$$x = \mathcal{H}, \quad y = \mathcal{H}'.$$

This yields

$$x' = y, \quad y' = -\kappa_3x^3 - \kappa_2x^2 - \kappa_1x,$$

or equivalently,

$$\begin{cases} x' = y, \\ y' = -\kappa_3x^3 - \kappa_2x^2 - \kappa_1x. \end{cases} \quad (4.4)$$

4.2. Equilibrium points and linearization

The equilibrium solutions of (4.4) satisfy $y = 0$, and

$$-\kappa_3x^3 - \kappa_2x^2 - \kappa_1x = 0.$$

Factoring gives

$$x(\kappa_3x^2 + \kappa_2x + \kappa_1) = 0.$$

Thus, the origin $E_0 = (0, 0)$ is always an equilibrium. Additional equilibria,

$$E_{\pm} = (x_{\pm}, 0),$$

exist when x_{\pm} are real solutions of

$$\kappa_3x^2 + \kappa_2x + \kappa_1 = 0, \quad (4.5)$$

which requires a nonnegative discriminant

$$\Delta = \kappa_2^2 - 4\kappa_3\kappa_1 \geq 0. \quad (4.6)$$

The Jacobian matrix of (4.4) at a point (x, y) is

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ -3\kappa_3x^2 - 2\kappa_2x - \kappa_1 & 0 \end{bmatrix}.$$

At the origin E_0 .

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -\kappa_1 & 0 \end{bmatrix}, \quad \mu^2 + \kappa_1 = 0 \quad \Rightarrow \quad \mu = \pm \sqrt{-\kappa_1}.$$

Proposition 1. *At $E_0 = (0,0)$, the system (4.4) has:*

- (1) a center if $\kappa_1 > 0$,
- (2) a saddle if $\kappa_1 < 0$,
- (3) a non-hyperbolic equilibrium if $\kappa_1 = 0$. □

At a nonzero equilibrium $E_* = (x_*, 0)$. Linearization gives eigenvalues satisfying

$$\mu^2 - (3\kappa_3 x_*^2 + 2\kappa_2 x_* + \kappa_1) = 0.$$

Proposition 2. *If x_* is a real solution of (4.5), then $E_* = (x_*, 0)$ is:*

- (1) a center if $3\kappa_3 x_*^2 + 2\kappa_2 x_* + \kappa_1 < 0$,
- (2) a saddle if $3\kappa_3 x_*^2 + 2\kappa_2 x_* + \kappa_1 > 0$,
- (3) non-hyperbolic if equality holds. □

4.3. Bifurcation scenarios

Let $\kappa_3 \neq 0$, and vary (κ_1, κ_2) .

Saddle–node bifurcation. Equation (4.5) produces two distinct equilibria when $\Delta > 0$, one double root when $\Delta = 0$, and none for $\Delta < 0$. Thus $\Delta = 0$ corresponds to a saddle-node bifurcation.

Symmetric case $\kappa_2 = 0$. When the quadratic term is absent, the system is odd in x . In this case, as κ_1 passes through zero with fixed κ_3 , a pitchfork bifurcation occurs: for $\kappa_1/\kappa_3 < 0$, two symmetric equilibria exist in addition to the origin, while for $\kappa_1/\kappa_3 > 0$, only the origin remains.

Symmetry breaking $\kappa_2 \neq 0$. If κ_2 is nonzero, the symmetry is destroyed and the pitchfork is unfolded into a pair of saddle–node bifurcations determined by the curve $\Delta = 0$ in parameter space.

Potential energy viewpoint. Since the system is Hamiltonian with a potential function,

$$V(x) = \frac{\kappa_3}{4}x^4 + \frac{\kappa_2}{3}x^3 + \frac{\kappa_1}{2}x^2,$$

equilibria correspond to critical points of $V(x)$, and their type is determined by the sign of $V''(x_*) = 3\kappa_3 x_*^2 + 2\kappa_2 x_* + \kappa_1$.

Remark 1 (Link to original coefficients). *From (4.2), the bifurcation condition $\Delta \geq 0$ translates back to*

$$Q^2 - 4PR \geq 0,$$

with $L \neq 0$. Thus, the emergence of additional equilibria is controlled directly by this inequality in the original parameters.

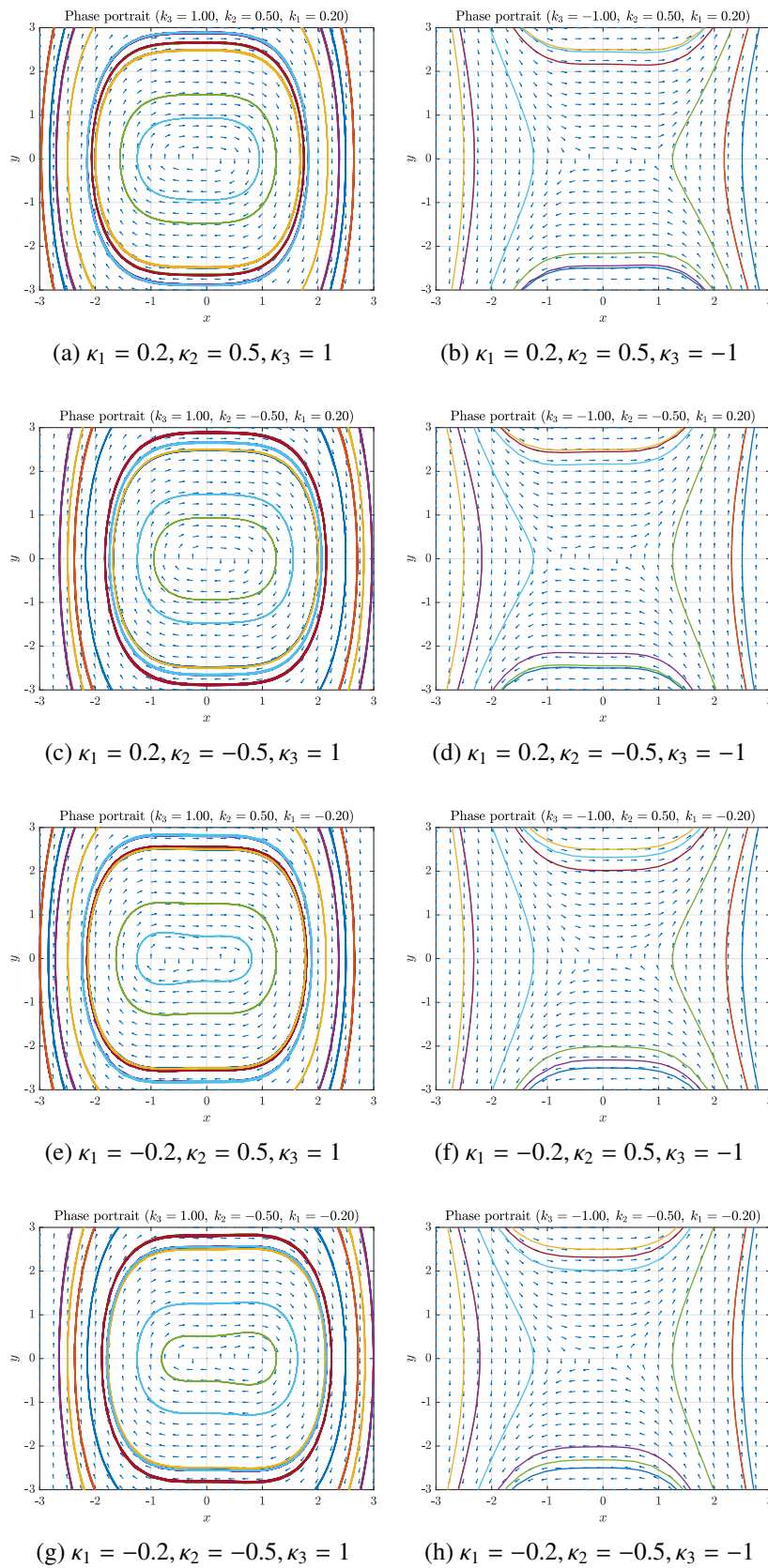


Figure 1. Phase portraits of the DNA model corresponding to various parameter settings.

Moreover, Figure 1 presents phase portraits of the system for various values of the parameters κ_1, κ_2 , and κ_3 , highlighting the qualitative changes in trajectories as these parameters are varied. The simulations clearly capture the transitions between centers and saddles, in agreement with the predictions of the linearization and bifurcation analysis. In particular, the numerical results faithfully reproduce the pitchfork and saddle-node bifurcation scenarios identified in the theoretical study.

4.4. Graphical depictions of some obtained solutions

Using numerical simulations shown as 2D, 3D, and polar diagrams, the physical properties of many solutions are investigated in this section. Figure 2 displays a bright soliton solution related to Eqs (3.8) and (3.9) when the following assumption is made: $\delta_1 = 0.45$, $\delta_2 = 0.5$, $\Omega = 0.55$, $h = 0.6$, $L_0 = 0.65$, $\sigma_1 = 0.7$, $\mathbf{a} = 1$, $\mathbf{b} = 1.05$, $\mathbf{d} = 1.1$, $y = 0$, $z = 0$, $\omega_2 = 1.15$, and $-15 \leq x \leq 15$. Figure 3 displays a singular soliton solution related to Eqs (3.10) and (3.11) when the following assumption is made: $\delta_1 = 1.25$, $\delta_2 = 0.55$, $\Omega = 0.6$, $h = 0.65$, $L_0 = 0.7$, $\sigma_1 = 0.75$, $\mathbf{a} = 1.05$, $\mathbf{b} = 1.1$, $\mathbf{d} = 1.15$, $y = 0$, $z = 0$, $\omega_2 = 1.2$, and $-20 \leq x \leq 5$. Figure 4 shows a singular periodic solution related to Eqs (3.12) and (3.13) when the following assumption is made: $\delta_1 = 0.6$, $\delta_2 = 2$, $\Omega = 0.65$, $h = 0.7$, $L_0 = 0.75$, $\sigma_1 = 1.3$, $\mathbf{a} = 1.1$, $\mathbf{b} = 1.15$, $\mathbf{d} = 1.2$, $y = 0$, $z = 0$, $\omega_2 = -1.25$, and $-20 \leq x \leq 5$. Figure 5 depicts a dark soliton solution related to Eqs (3.66) and (3.67) when the following assumption is made: $\delta_1 = 0.55$, $\delta_2 = 0.75$, $\Omega = 0.75$, $h = 0.8$, $L_0 = 0.85$, $\sigma_1 = 0.9$, $\mathbf{a} = 0.95$, $\mathbf{b} = 1$, $\mathbf{d} = 1.05$, $y = 0$, $z = 0$, and $-20 \leq x \leq 5$.

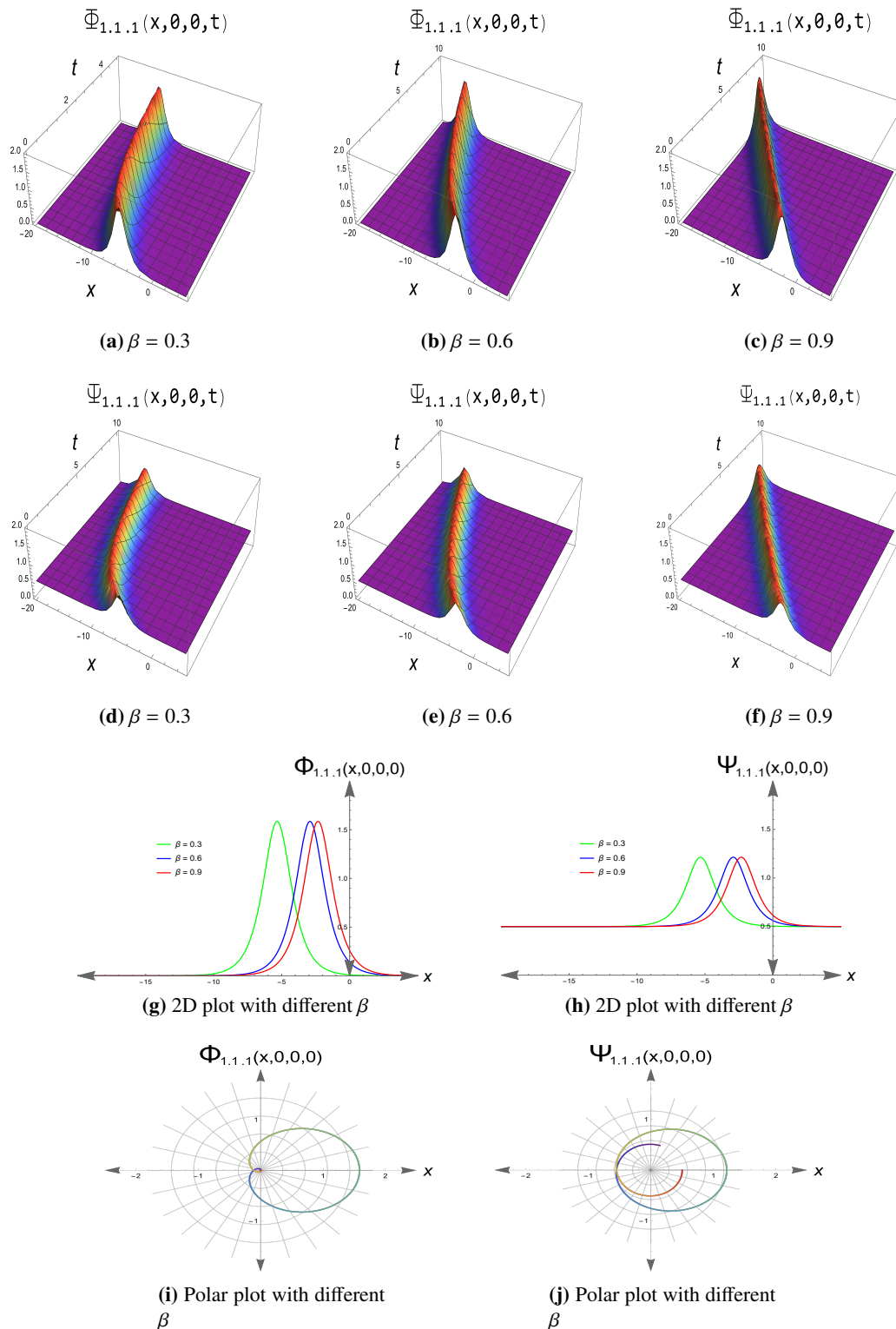


Figure 2. 3D, 2D, and polar plots for the bright soliton solution introduced by Eqs (3.8)-(3.9).

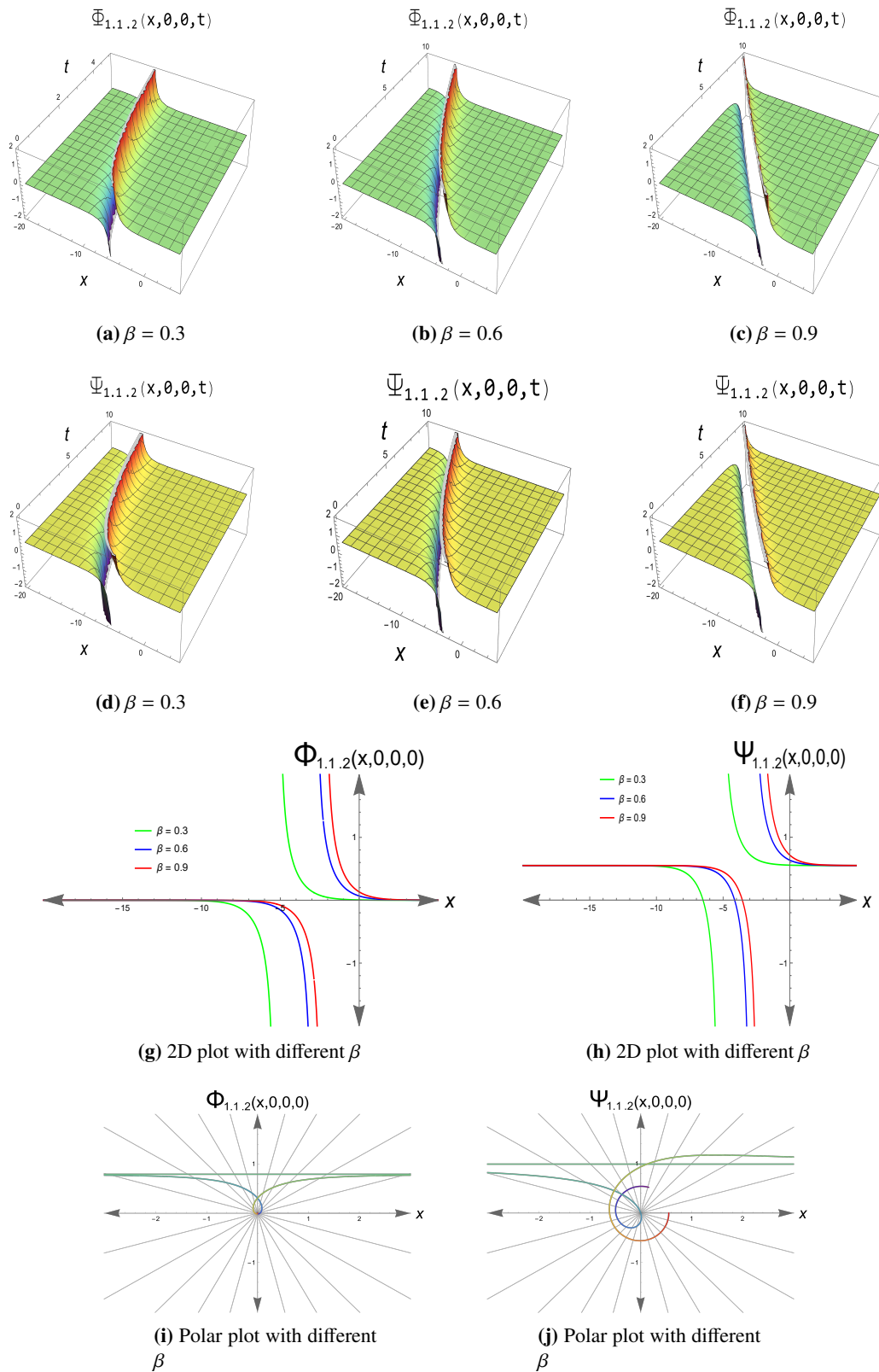


Figure 3. 3D, 2D, and polar plots for the singular soliton solution introduced by Eqs (3.10)-(3.11).

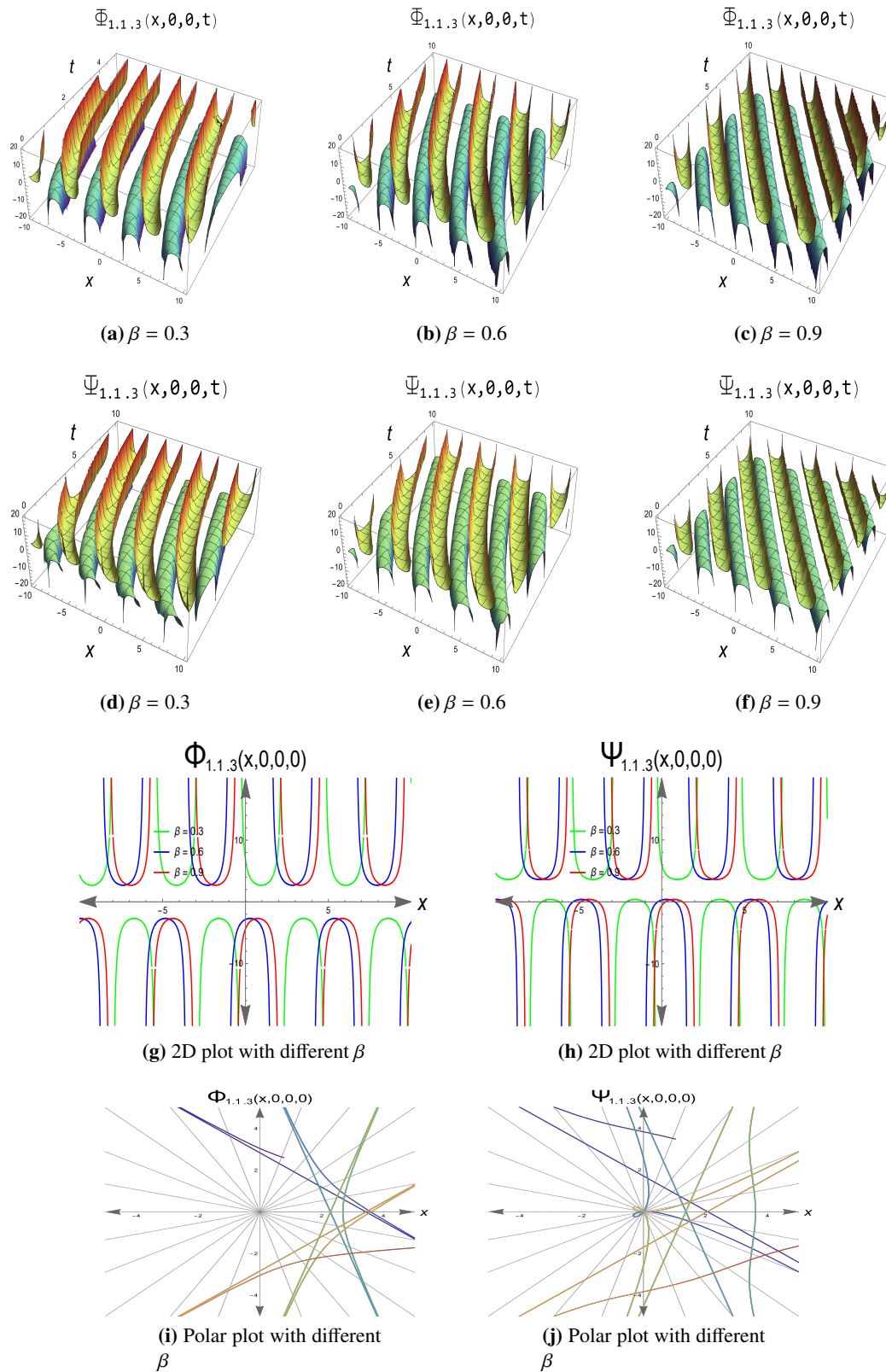


Figure 4. 3D, 2D, and polar plots for the singular periodic solution introduced by Eqs (3.12)-(3.13).

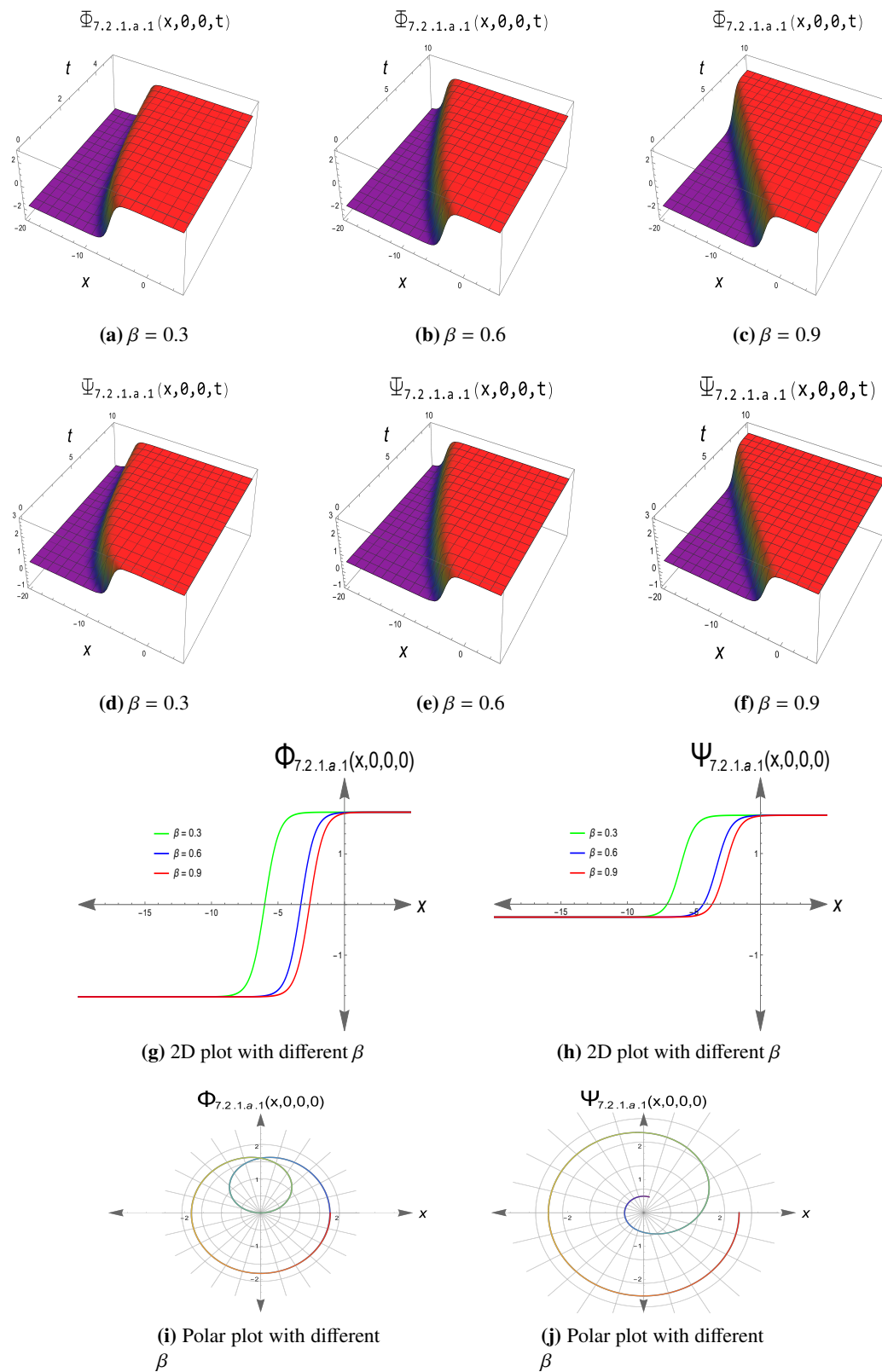


Figure 5. 3D, 2D, and polar plots for the dark soliton solution introduced by Eqs (3.66)-(3.67).

5. Limitations and future work

Despite the significant analytical insights offered by the results of this study, there remain some limitations that need to be pointed out. First, as far as the proposed double-chain DNA model is concerned, there remain some idealized considerations, such as homogeneity and elastic interaction, which might not serve as adequate models for biological systems in their entirety. There remain some considerations concerning thermal fluctuations, damping, and external forces, which might have been significant in DNA dynamics at the atomic levels, but have been overlooked in this proposed model. Results of the analytical solution via the modified extended direct algebraic method are centered around finding the exact soliton and traveling wave solutions based on certain parametric values. Although the analytical solutions are important for understanding the nonlinear wave motion in DNA from a theoretical point of view, numerical solutions can add to these findings in order to check the universality of the analytical solutions. Future studies might further extend the current model by considering thermal influences, inhomogeneity, viscous damping, or external perturbations, which might more realistically probe biological conditions. Furthermore, the study of stochastic influences and the linking of this model using experimental findings could lead to a gain in understanding of the role of nonlinear excitations in the transportation of genetic information. The use of different methods could also extend the set of admissible solutions, making this model more generally applicable to other nonlinear problems in physics.

6. Conclusions

In summary, the discovery of double-stranded DNA (dsDNA) not only unveiled its molecular composition but also clarified the mechanism by which genetic information is faithfully replicated. In this study, the modified extended direct algebraic method (MEDAM) was systematically employed to investigate the double-helix structure of DNA, a subject of profound significance in biological physics due to its central role in sustaining life processes. The application of this analytical technique provided deeper insight into the governing nonlinear equations, enabling the construction of exact solutions and thereby enriching our understanding of their mathematical properties.

The resulting solutions have clearly shown the existence of stable solitary wave propagation within the DNA strands and have described both the longitudinal and transverse dynamic behaviors of the DNA helix. The physical viability of these solutions has also been established using thorough two-dimensional, three-dimensional, and polar graphical plots. Furthermore, bifurcation and stability analyses have been employed to discuss the effect of the parameters of the systems on the qualitative solution behavior, and this has given a deeper understanding of the resulting model behavior.

Beyond their immediate application to DNA dynamics, the MEDAM framework possesses broad utility in addressing diverse challenges within nonlinear science. When coupled with computational modeling, these mathematical approaches provide a robust framework for advancing our comprehension of complex biological and physical systems. Moreover, the methodologies introduced here open new avenues for exploring interaction behaviors under extreme conditions, potentially leading to significant advancements in nonlinear dynamics, mathematical physics, and related interdisciplinary fields.

On a theoretical level, the importance of the paper is that it provides better insights on soliton

solutions and may contribute toward understanding biological mechanisms that deal with the localization and transport of energy within biological systems. On a general theoretical and applied level, the value of the paper is that it is expected to contribute toward understanding other nonlinear phenomena that may deal with soliton theory, plasmas, optical fibers, and other applications within the field of nonlinear engineering and dynamics.

Ultimately, this study not only deepens the mathematical understanding of DNA's structural dynamics but also highlights the potential of analytical and computational techniques to inspire future research directions and foster discoveries at the interface of biology, physics, and applied mathematics.

Author contributions

B.E.: Methodology, Writing—original draft; M.S.G.: Conceptualization, Methodology, Writing—review, and editing, Supervision; M.Y.H.: Formal analysis, Validation; H.E.S.: Resources, Investigation; H.M.A.: Software, Visualization; S.A.: Writing—review, and editing; K.K.A.: Formal analysis, Methodology, Writing—review, and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding statement

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2601).

Conflict of interest

The authors declare that they have no competing interests.

References

1. J. D. Watson, F. H. C. Crick, Molecular structure of nucleic acids: a structure for deoxyribose nucleic acid, *Nature*, **171** (1953), 737–738. <https://doi.org/10.1038/171737a0>
2. S. Li, K. Paloja, M. M. C. Bastings, Pattern and precision: DNA-based mapping of spatial rules for T cell activation, *Nanoscale Horizons*, 2025. <https://doi.org/10.1039/d5nh00412h>
3. C. Howard-Varona, N. E. Solonenko, M. Burris, M. Urvoy, C. M. Sanderson, B. Bolduc, et al., Infection and genomic properties of single- and double-stranded DNA cellulophaga phages, *Viruses*, **17** (2025), 365. <https://doi.org/10.3390/v17030365>
4. X. Meng, G. Hu, X. Li, C. Gao, W. Song, W. Wei, et al., A synthetic methylotroph achieves accelerated cell growth by alleviating transcription-replication conflicts, *Nature Commun.*, **16** (2025), 31. <https://doi.org/10.1038/s41467-024-55502-5>

5. Y. Zhang, T. Wang, Z. Wang, X. E. Shi, J. Jin, Functions and therapeutic potentials of long noncoding RNA in skeletal muscle atrophy and dystrophy, *J. Cachexia, Sarcopenia Muscle*, **16** (2025), e13747. <https://doi.org/10.1002/jcsm.13747>
6. S. W. Englander, N. R. Kallenbach, A. J. Heeger, J. A. Krumhansl, S. Litwin, Nature of the open state in long polynucleotide double helices: possibility of soliton excitations, *Proc. Natl. Acad. Sci.*, **77** (1980), 7222–7226. <https://doi.org/10.1073/pnas.77.12.7222> .
7. V. Muto, A. C. Scott, P. L. Christiansen, Thermally generated solitons in a toda lattice model of DNA, *Phys. Lett. A*, **136** (1989), 33–36. [https://doi.org/10.1016/0375-9601\(89\)90671-3](https://doi.org/10.1016/0375-9601(89)90671-3)
8. A. Djine, N. O. Nfor, G. R. Deffo, S. B. Yamgoué, Higher order investigation on modulated waves in the Peyrard–Bishop–Dauxois DNA model, *Chaos, Soliton. Fract.*, **181** (2024), 114706. <https://doi.org/10.1016/j.chaos.2024.114706>
9. A. Djine, S. B. Yamgoué, N. O. Nfor, Fifth order approximation, solvent interaction in helicoidal Peyrard-Bishop-Dauxois model of DNA, *Braz. J. Phys.*, **56** (2026), 13. <https://doi.org/10.1007/s13538-025-01933-4>
10. G. Arora, R. Rani, H. Emadifar, Soliton: a dispersion-less solution with existence and its types, *Heliyon*, **8** (2022), e12122. <https://doi.org/10.1016/j.heliyon.2022.e12122>
11. M. A. B. Iqbal, M. Z. Raza, A. Khan, D. K. Almutairi, T. Abdeljawad, An innovative multivariate method for exploring soliton dynamics in nonlinear models, *Results Eng.*, **26** (2025), 104892. <https://doi.org/10.1016/j.rineng.2025.104892>
12. N. A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **17** (2012), 2248–2253. <https://doi.org/10.1016/j.cnsns.2011.10.016>
13. R. Hirota, *The direct method in soliton theory*, Cambridge University Press, 2004. <https://doi.org/10.1017/CBO9780511543043>
14. J. Zhang, X. Wei, Y. Lu, A generalized $\frac{G'}{G}$ -expansion method and its applications, *Phys. Lett. A*, **372** (2008), 3653–3658. <https://doi.org/10.1016/j.physleta.2008.02.027>
15. Y. Liu, Z. Li, K. Wang, Symbolic computation of exact solutions for a nonlinear evolution equation, *Chaos, Soliton. Fract.*, **31** (2007), 1173–1180. <https://doi.org/10.1016/j.chaos.2005.09.055>
16. M. S. Ghayad, M. Y. Hamada, H. M. Ahmed, H. Emadifar, K. K. Ahmed, Dynamical behavior of analytical solutions and bifurcation analysis for a novel structured (2+1)-dimensional Kadomtsev–Petviashvili equation via analytic approach, *Sci. Rep.*, **15** (2025), 29832. <https://doi.org/10.1038/s41598-025-15823-x>
17. C. L. Zheng, Comments on “The generalizing Riccati equation mapping method in nonlinear evolution equation: application to (2+1)-dimensional Boiti–Leon–Pempinelle equation”, *Chaos, Soliton. Fract.*, **39** (2009), 1493–1495. <https://doi.org/10.1016/j.chaos.2007.04.026>
18. B. Ghanbari, M. Inc, A new generalized exponential rational function method to find exact special solutions for the resonance nonlinear Schrödinger equation, *Eur. Phys. J. Plus*, **133** (2018), 142. <https://doi.org/10.1140/epjp/i2018-11984-1>

19. A. R. Seadawy, M. Bilal, M. Younis, S. T. R. Rizvi, S. Althobaiti, M. M. Makhlof, Analytical mathematical approaches for the double-chain model of DNA by a novel computational technique, *Chaos, Soliton. Fract.*, **144** (2021), 110669. <https://doi.org/10.1016/j.chaos.2021.110669>
20. S. Kumar, A. Kumar, H. Kharbanda, Abundant exact closed-form solutions and solitonic structures for the double-chain deoxyribonucleic acid (DNA) model, *Braz. J. Phys.*, **51** (2021), 1043–1068. <https://doi.org/10.1007/s13538-021-00913-8>
21. L. Ouahid, M. A. Abdou, S. Kumar, S. Owyed, S. S. Ray, A plentiful supply of soliton solutions for DNA Peyrard–Bishop equation by means of a new auxiliary equation strategy, *Int. J. Mod. Phys. B*, **35** (2021), 2150265. <https://doi.org/10.1142/s0217979221502659>
22. K. J. Wang, A fast insight into the optical solitons of the generalized third-order nonlinear Schrödinger's equation, *Results Phys.*, **40** (2022), 105872. <https://doi.org/10.1016/j.rinp.2022.105872>
23. K. J. Wang, G. D. Wang, Variational theory and new abundant solutions to the (1+2)-dimensional chiral nonlinear Schrödinger equation in optics, *Phys. Lett. A*, **412** (2021), 127588. <https://doi.org/10.1016/j.physleta.2021.127588>
24. K. J. Wang, The fractal active low-pass filter within the local fractional derivative on the Cantor set, *COMPEL*, **42** (2023), 1396–1407. <https://doi.org/10.1108/compel-09-2022-0326>
25. K. J. Wang, An effective computational approach to the local fractional low-pass electrical transmission lines model, *Alex. Eng. J.*, **110** (2025), 629–635. <https://doi.org/10.1016/j.aej.2024.07.021>
26. K. Wang, K. Yan, F. Shi, G. Li, X. Liu, Qualitative study of the (2+1)-dimensional BLMPE equation: variational principle, Hamiltonian and diverse wave solutions, *AIMS Math.*, **10** (2025), 26168–26186. <https://doi.org/10.3934/math.20251152>
27. A. Kukkar, S. Kumar, Dynamic behavior of dark and bright solitons with analytical solutions, together with other soliton forms in the double-chain DNA Model, *Mod. Phys. Lett. B*, **39** (2025), 2550203. <https://doi.org/10.1142/s0217984925502033>
28. E. Fan, J. Zhang, Applications of the Jacobi elliptic function method to special-type nonlinear equations, *Phys. Lett. A*, **305** (2002), 383–392. [https://doi.org/10.1016/s0375-9601\(02\)01516-5](https://doi.org/10.1016/s0375-9601(02)01516-5)
29. A. Akbulut, S. M. R. Islam, Study on the Biswas–Arshed equation with the beta time derivative, *Int. J. Appl. Comput. Math.*, **8** (2022), 167. <https://doi.org/10.1007/s40819-022-01350-0>
30. Z. Li, Bifurcation and traveling wave solution to fractional Biswas–Arshed equation with the beta time derivative, *Chaos, Soliton. Fract.*, **160** (2022), 112249. <https://doi.org/10.1016/j.chaos.2022.112249>
31. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. <https://doi.org/10.2298/tsci160111018a>
32. U. Younas, J. Ren, L. Akinyemi, H. Rezazadeh, On the multiple explicit exact solutions to the double-chain DNA dynamical system, *Math. Methods Appl. Sci.*, **46** (2023), 6309–6323. <https://doi.org/10.1002/mma.8904>

33. M. Z. Yousaf, M. Abbas, T. Nazir, F. A. Abdullah, A. Birhanu, H. Emadifar, Investigation of the dynamical structures of double-chain deoxyribonucleic acid model in biological sciences, *Sci. Rep.*, **14** (2024), 6410. <https://doi.org/10.1038/s41598-024-55786-z>
34. M. S. Ghayad, H. M. Ahmed, N. M. Badra, W. B. Rabie, Wave propagation analysis of the fractional generalized (3+1)-dimensional P-type equation with local M-derivative, *J. Umm Al-Qura Univ. Appl. Sci.*, 2025, 1–16. <https://doi.org/10.1007/s43994-025-00238-1>
35. S. T. Obeidat, K. K. Ahmed, H. M. Ahmed, W. W. Mohammed, M. S. Ghayad, Fractional derivative effects on exploration of soliton solutions of (3+1)-D Kadomtsev-Petviashvili-Sawada-Kotera-Ramani model using modified extended direct algebraic approach, *Contemp. Math.*, **6** (2025), 5346–5367. <https://doi.org/10.37256/cm.6420257594>



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