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Research article

Weighted composition operators between vector-valued Bloch-type spaces on the polydisk

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Abstract: Let X and Y be two complex Banach spaces, $\mathbb{C}^N$ the N-dimensional complex Euclidean space with the inner product $\langle z, w\rangle = \sum_{l=1}^N z_l\overline{w_l}$ and $\mathbb{D}^N$ the unit polydisk in $\mathbb{C}^N$. Let φ be a holomorphic self-map of $\mathbb{D}^N$ and $u \in H(\mathbb{D}^N, \mathcal{L}(X, Y))$, where $H(\mathbb{D}^N, X)$ denotes the space of all vector-valued holomorphic functions on $\mathbb{D}^N$ and $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y. The weighted composition operator $W_{u,\varphi} : H(\mathbb{D}^N, X) \to H(\mathbb{D}^N, Y)$ is defined by

$$W_{u,\varphi}f(z) = u(z)(f(\varphi(z))).$$

The bounded and compact weighted composition operators between vector-valued Bloch-type spaces on the unit polydisk are completely characterized in the paper.

Keywords: unit polydisk; vector-valued Bloch-type space; weighted composition operator; boundedness; compactness
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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{C}$ be the complex plane. First, we give some notations. Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, by $\mathbb{C}^N$ the N-dimensional complex Euclidean space with the inner product $\langle z, w\rangle = \sum_{j=1}^N z_j\overline{w_j}$, by $\mathbb{D}^N$ the open unit polydisk in $\mathbb{C}^N$, that is, $\mathbb{D}^N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z_j| < 1, j = \overline{1, N}\}$, by $\partial\mathbb{D}^N$ the topological boundary of $\mathbb{D}^N$, by $\mathbb{B}^N$ the open unit ball in $\mathbb{C}^N$. Let $H^\infty(\mathbb{D}^N)$ be the space of all bounded holomorphic functions on $\mathbb{D}^N$ with the supremum norm $\|f\|_{H^\infty(\mathbb{D}^N)} = \sup_{z\in\mathbb{D}^N} |f(z)|$.

Let $(X, \|\cdot\|_X)$ be a complex Banach space and $H(\mathbb{D}^N, X)$ be the space of all vector-valued holomorphic functions on $\mathbb{D}^N$. It is known that if $f \in H(\mathbb{D}^N, X)$, then for each $(z_1, \dots, z_N) \in \mathbb{D}^N$ the limit

$$\lim_{w_j \to z_j} \frac{f(z^*, w_j, z_*) - f(z^*, z_j, z_*)}{w_j - z_j}$$

exists in the sense of the norm on X, usually denoted by $\frac{\partial f}{\partial z_j}(z)$.

For $p > 0$, the vector-valued Bloch-type space, usually denoted by $\mathcal{B}^p(\mathbb{D}^N, X)$, consists of all $f \in H(\mathbb{D}^N, X)$ such that

$$b(f) = \sup_{z \in \mathbb{D}^N} b_f(z) < +\infty,$$

where

$$b_f(z) = \sum_{k=1}^{N} (1 - |z_k|^2)^p \Big\| \frac{\partial f}{\partial z_k}(z) \Big\|_X.$$

With the norm

$$\|f\|_{\mathcal{B}^p(\mathbb{D}^N, X)} = \|f(0)\|_X + b(f),$$

$\mathcal{B}^p(\mathbb{D}^N, X)$ is a Banach space. We usually denote $\mathcal{B}^1(\mathbb{D}^N, X)$ by $\mathcal{B}(\mathbb{D}^N, X)$.

Vector-valued Bloch-type spaces are an important concept at the intersection of complex analysis and general functional analysis, mainly used to study the spatial structure of generalized holomorphic functions and their operator theory. Vector-valued Bloch-type spaces are also used to explore the boundary problem between bounded and unbounded functions.

If $X = \mathbb{C}$, then $\mathcal{B}^p(\mathbb{D}^N, \mathbb{C})$ is the common holomorphic Bloch-type space on $\mathbb{D}^N$, usually denoted by $\mathcal{B}^p(\mathbb{D}^N)$. For the fixed $f \in \mathcal{B}^p(\mathbb{D}^N)$ and $x \in X$, define the function $f_x : \mathbb{D}^N \to X$ by $f_x(z) = f(z)x$. Then $f_x \in \mathcal{B}^p(\mathbb{D}^N, X)$ and $\|f_x\|_{\mathcal{B}^p(\mathbb{D}^N, X)} = \|f\|_{\mathcal{B}^p(\mathbb{D}^N)} \|x\|_X$. In particular, $1_x \in \mathcal{B}^p(\mathbb{D}^N, X)$ and $\|1_x\|_{\mathcal{B}^p(\mathbb{D}^N, X)} = \|x\|_X$.

Let X and Y be two complex Banach spaces and $\mathcal{L}(X, Y)$ be the space of all bounded linear operators from X to Y. Let $\varphi(z) = (\varphi_1(z), \dots, \varphi_N(z))$ be a holomorphic self-map of $\mathbb{D}^N$ and $u \in H(\mathbb{D}^N, \mathcal{L}(X, Y))$. The weighted composition operator from $H(\mathbb{D}^N, X)$ to $H(\mathbb{D}^N, Y)$ is defined by

$$W_{u,\varphi} f(z) = u(z)(f(\varphi(z))).$$

The study of composition and weighted composition operators between Banach spaces of vector-valued holomorphic functions recently has received attention (see, for example, [4, 11–13]). Here, we first need to explain why we consider the weighted composition operators between vector-valued Bloch-type spaces on the unit polydisk $\mathbb{D}^N$. For this goal, we need to present some related definitions on $\mathbb{D}^N$.

Let $H(\mathbb{D}^N)$ be the space of all holomorphic functions on $\mathbb{D}^N$. For $p > 0$, the Bloch-type space on $\mathbb{D}^N$, usually denoted by $\mathcal{B}^p(\mathbb{D}^N)$, consists of all $f \in H(\mathbb{D}^N)$ such that

$$b(f) = \sup_{z \in \mathbb{D}^N} \sum_{k=1}^{N} (1 - |z_k|^2)^p \Big| \frac{\partial f}{\partial z_k}(z) \Big| < +\infty.$$

$\mathcal{B}^p(\mathbb{D}^N)$ is a Banach space with the norm $\|f\|_{\mathcal{B}^p(\mathbb{D}^N)} = |f(0)| + b(f)$. If $p = 1$, it is the classical Bloch space, denoted by $\mathcal{B}(\mathbb{D}^N)$. For a good reference on Bloch space, see, for example, [19]. Also see [2] for an overview of Bloch spaces and their connection to other function spaces. Bloch spaces on more general domains have been defined, such as strongly pseudo-convex domains [10] and bounded homogeneous domains [1]. However, operator theory problems are much more difficult to treat on such spaces with these domains.

Forelli in [6] proved that the isometries on Hardy space H^p (for $p \neq 2$) defined on the open unit disk $\mathbb{D}$ are certain weighted composition operators, which can be regarded as the earliest presence of the weighted composition operators. It is important to provide function-theoretic characterizations of when the symbols u and φ induce a bounded or compact weighted composition operator between various holomorphic function spaces. There have been many studies of the weighted composition operators on holomorphic function spaces. So, to make things nice and clear, here we mainly introduce the research of weighted composition operators between $\mathcal{B}^p(\mathbb{D}^N)$. It is known that the unit polydisk $\mathbb{D}^N$ is different from the unit ball $\mathbb{B}^N$ (see [16]) when $N \neq 1$, which may be one of the reasons why people pay attention to the operator theory on the holomorphic function spaces on $\mathbb{D}^N$. The authors in [18] obtained the following result, which provides the major motivation of the paper.

Let $p, q > 0$, $u \in H(\mathbb{D}^N)$ and φ be a holomorphic self-map of $\mathbb{D}^N$. Then the following statements are true.

(i) The operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N) \to \mathcal{B}^q(\mathbb{D}^N)$ is bounded if and only if

$$\sup_{z\in\mathbb{D}^N} \sum_{k=1}^{N} (1-|z_k|^2)^q \Big|\frac{\partial u}{\partial z_k}(z)\Big| G_p(\varphi(z)) < +\infty$$

and

$$\sup_{z\in\mathbb{D}^N} \sum_{j,k=1}^{N} |u(z)| \frac{(1-|z_j|^2)^q}{(1-|\varphi_k(z)|^2)^p} \Big|\frac{\partial \varphi_k}{\partial z_j}(z)\Big| < +\infty.$$

(ii) If $p \geq 1$, then the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N) \to \mathcal{B}^q(\mathbb{D}^N)$ is compact if and only if $u \in \mathcal{B}^q(\mathbb{D}^N)$, $u\varphi_k \in \mathcal{B}^q(\mathbb{D}^N)$ for each $k \in \{1, \ldots, N\}$,

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \sum_{k=1}^{N} (1-|z_k|^2)^q \Big|\frac{\partial u}{\partial z_k}(z)\Big| G_p(\varphi(z)) = 0$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \sum_{j,k=1}^{N} |u(z)| \frac{(1-|z_j|^2)^q}{(1-|\varphi_k(z)|^2)^p} \Big|\frac{\partial \varphi_k}{\partial z_j}(z)\Big| = 0.$$

(iii) If $0 < p < 1$, then the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N) \to \mathcal{B}^q(\mathbb{D}^N)$ is compact if and only if $u \in \mathcal{B}^q(\mathbb{D}^N)$, $u\varphi_k \in \mathcal{B}^q(\mathbb{D}^N)$ and

$$\lim_{|\varphi_k(z)|\to 1} \sum_{j=1}^{N} |u(z)| \frac{(1-|z_j|^2)^q}{(1-|\varphi_k(z)|^2)^p} \Big|\frac{\partial \varphi_k}{\partial z_j}(z)\Big| = 0$$

for each $k \in \{1, \ldots, N\}$.

This characterization of boundedness and compactness simplifies the results in [5], obtained by the methods used in [9, 14]. Recently, the authors in [4] have borrowed the methods and techniques in [8, 15] to characterize the bounded and compact weighted composition operators between vector-valued Bloch-type spaces on $\mathbb{D}$. Motivated by [4] and other studies previously mentioned, we naturally consider whether we can extend the related results (for example, the result from $\mathcal{B}^p(\mathbb{D}^N)$ to $\mathcal{B}^q(\mathbb{D}^N)$ previously mentioned) to vector-valued Bloch-type spaces on $\mathbb{D}^N$. In this paper, we partially use the methods in [17] to achieve the task. Namely, we completely characterize the bounded and compact weighted composition operators between vector-valued Bloch-type spaces on $\mathbb{D}^N$. Compared to the literature [4] and other related studies, the main innovation of this paper is that we generalize the results in Bloch-type spaces to vector-valued Bloch-type spaces by constructing some functions in vector-valued Bloch-type spaces.

We denote the norm of a linear operator $T : X \to Y$ by $\|T\|_{X\to Y}$. As usual, some positive numbers are denoted by C, and they may vary in different situations.

2. Preliminary results

For the space $\mathcal{B}^p(\mathbb{D}^N)$, the following lemma is a very common result (see [17]).

Lemma 2.1. *Let $f \in \mathcal{B}^p(\mathbb{D}^N, X)$. Then the following statements are true.*

(i) If $0 < p < 1$, then

$$\|f(z)\|_X \le C\big\|f\big\|_{\mathcal{B}^p(\mathbb{D}^N,X)}. \tag{2.1}$$

(ii) If $p = 1$, then

$$\|f(z)\|_X \le C\sum_{l=1}^{N} \ln \frac{e}{1-|z_l|^2}\|f\|_{\mathcal{B}(\mathbb{D}^N,X)}. \tag{2.2}$$

(iii) If $p > 1$, then

$$\|f(z)\|_X \le C\sum_{l=1}^{N} \frac{1}{(1-|z_l|^2)^{p-1}}\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}. \tag{2.3}$$

Proof. Let $f \in \mathcal{B}^p(\mathbb{D}^N, X)$. For each fixed $z \in \mathbb{D}^N$, by Hahn–Banach extension theorem there exists an $x^* \in X^*$ (the dual space of X) with $\|x^*\| = 1$ such that $x^*(f(z)) = \|f(z)\|_X$. Then, we have

$$\begin{aligned}
\|f(z)\|_X &= \left|x^*(f(z))\right| = \left|x^*(f(0)) + \int_0^1 \langle \nabla(x^*(f(tz))), \bar{z}\rangle dt\right| \\
&\le \left|x^*(f(0))\right| + \sum_{l=1}^{N}\int_0^1 \left|\frac{\partial x^*(f)}{\partial z_l}(tz)\right||z_l|dt \\
&\le \|f(0)\|_X + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\sum_{l=1}^{N}\int_0^1 \frac{|z_l|}{(1-t^2|z_l|^2)^p}dt \\
&\le \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\sum_{l=1}^{N}\int_0^{|z_l|} \frac{dt}{(1-t^2)^p}.
\end{aligned} \tag{2.4}$$

If $p = 1$, then it follows from (2.4) that

$$\begin{aligned}\|f(z)\|_X &\le \|f\|_{\mathcal{B}(\mathbb{D}^N,X)} + \frac{1}{2}\|f\|_{\mathcal{B}(\mathbb{D}^N,X)} \sum_{l=1}^{N} \ln \frac{1+|z_l|}{1-|z_l|} \\ &\le \|f\|_{\mathcal{B}(\mathbb{D}^N,X)} + \frac{1}{2}\|f\|_{\mathcal{B}(\mathbb{D}^N,X)} \sum_{l=1}^{N} \ln \frac{4}{1-|z_l|^2} \\ &\le \|f\|_{\mathcal{B}(\mathbb{D}^N,X)} \sum_{l=1}^{N} \ln \frac{2e^{\frac{1}{N}}}{1-|z_l|^2} \\ &\le \max\{\ln(2e^{\frac{1}{N}}), 1\}\|f\|_{\mathcal{B}(\mathbb{D}^N,X)} \sum_{l=1}^{N} \ln \frac{e}{1-|z_l|^2},\end{aligned}$$

from which (2.2) holds.

If $p \neq 1$, then it follows from (2.4) that

$$\|f(z)\|_X \le \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \sum_{l=1}^{N} \frac{1-(1-|z_l|)^{1-p}}{1-p}. \tag{2.5}$$

From (2.5), we get that if $0 < p < 1$, then

$$\|f(z)\|_X \le \Big(1 + \frac{N}{1-p}\Big)\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)},$$

from which (2.1) holds; if $p > 1$, then

$$\begin{aligned}\|f(z)\|_X &\le \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \sum_{l=1}^{N} \frac{1-(1-|z_l|)^{p-1}}{(p-1)(1-|z_l|)^{p-1}} \\ &\le \Big(\frac{1}{N} + \frac{2^{p-1}}{p-1}\Big) \sum_{l=1}^{N} \frac{1}{(1-|z_l|^2)^{p-1}} \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)},\end{aligned}$$

from which (2.3) holds. The proof is completed. □

Corollary 2.1. *If $0 < p < 1$, then $f \in \mathcal{B}^p(\mathbb{D}^N, X)$ implies $f \in H^\infty(\mathbb{D}^N, X)$. Moreover, there is a positive constant C independent of f such that $\|f\|_{H^\infty(\mathbb{D}^N,X)} \le C\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}$.*

We omit the straightforward proof of the next lemma, since the proof can be obtained by some direct calculations.

Lemma 2.2. *Let $w = (w_1, \dots, w_N) \in \mathbb{D}^N$ and $x \in X$. Then the following functions belong to $\mathcal{B}^p(\mathbb{D}^N, X)$.*

(i) If $0 < p < 1$, then for each $l \in \{1, \dots, N\}$ the function is

$$f_{l,w,x}(z) = f_{l,w}(z)x,$$

where

$$f_{l,w}(z) = \frac{1}{\overline{w}_l}\Big(\frac{1-|w_l|^2}{(1-\overline{w}_l z_l)^p} - \frac{1}{(1-\overline{w}_l z_l)^{p-1}}\Big).$$

Moreover,

$$\sup_{w\in\mathbb{D}^N} \|f_{l,w,x}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le (1+3p2^p)\|x\|_X.$$

(ii) If $p = 1$, then for each $l \in \{1, \dots, N\}$ the function is

$$g_{l,w,x}(z) = g_{l,w}(z)x,$$

where

$$g_{l,w}(z) = \ln \frac{1}{1-\overline{w}_l z_l}.$$

Moreover,

$$\sup_{w\in\mathbb{D}^N} \|g_{l,w,x}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le 2\|x\|_X.$$

(iii) If $p > 1$, then for each $l \in \{1, \dots, N\}$ the function is

$$h_{l,w,x}(z) = h_{l,w}(z)x,$$

where

$$h_{l,w}(z) = \frac{p}{(1-\overline{w}_l z_l)^{p-1}} - (p-1)\frac{1-|w_l|^2}{(1-\overline{w}_l z_l)^p}.$$

Moreover,

$$\sup_{w\in\mathbb{D}^N} \|h_{l,w,x}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le (3p(p-1)2^p + 2p - 1)\|x\|_X.$$

Remark 2.1. We assume $w_l \neq 0$ in Lemma 2.2 (i).

The next lemma extends Proposition 3.11 in [3] to the case of vector-valued functions.

Lemma 2.3. *Let φ be a holomorphic self-map of $\mathbb{D}^N$ and $u \in H(\mathbb{D}^N, \mathcal{L}(X,Y))$. Then the bounded operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact if and only if for any bounded sequence $\{f_j\}$ in $\mathcal{B}^p(\mathbb{D}^N, X)$ that converges to the zero vector in X uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$, it follows that $\|W_{u,\varphi} f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0$ as $j \to \infty$.*

Proof. Assume that the bounded operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact and $\{f_j\}$ is a sequence in $\mathcal{B}^p(\mathbb{D}^N, X)$ with $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} < +\infty$ and $f_j \to 0$ (where 0 denotes the zero vector in X) uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$. Assume that there is a subsequence $\{f_{j_k}\}$ and a $\delta_0 > 0$ such that

$$\|W_{u,\varphi} f_{j_k}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \ge \delta_0, \quad k \in \mathbb{N}. \tag{2.6}$$

From the compactness of $W_{u,\varphi}$, it follows that $\{W_{u,\varphi} f_{j_k}\}$ has a further subsequence (here we still use the same notion $\{W_{u,\varphi} f_{j_k}\}$), which converges to some g in $\mathcal{B}^q(\mathbb{D}^N, X)$. By Lemma 2.1, we have that for any compact $K \subset \mathbb{D}^N$, there is a positive constant C_K independent of f such that

$$\left\|u(z)(f_{j_k}(\varphi(z))) - g(z)\right\|_Y \le C_K \left\|W_{u,\varphi} f_{n_k} - g\right\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}, \quad z \in K. \tag{2.7}$$

Estimate (2.7) implies that $u(z)(f_{j_k}(z)) - g(z) \to 0$ uniformly on compact subsets of $\mathbb{D}^N$ as $k \to \infty$. Since for every compact subset $K \subset \mathbb{D}^N$, $\sup_{z\in K} \|u(z)\|_{X\to Y} = M_K < +\infty$, it follows that

$$\sup_{z\in K} \|u(z)(f_{j_k}(\varphi(z)))\|_Y \le M_K \sup_{w\in\varphi(K)} \|f_{j_k}(w)\|_X \to 0 \quad \text{as } k \to \infty, \tag{2.8}$$

where we have used the fact that $\varphi(K)$ is a compact subset of $\mathbb{D}^N$. Hence, the limit function g is equal to the zero vector. Therefore, we have

$$\lim_{k\to\infty} \|W_{u,\varphi} f_{j_k}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} = 0. \tag{2.9}$$

Equation (2.9) contradicts the fact that $\|W_{u,\varphi} f_{j_k}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \ge \delta_0 > 0$ for all $k \in \mathbb{N}$. Then,

$$\lim_{j\to\infty} \|W_{u,\varphi} f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} = 0.$$

Conversely, let $\{g_j\}$ be any sequence in the unit ball $B(0,1)$ of the space $\mathcal{B}^p(\mathbb{D}^N, X)$. Since $\sup_{j\in\mathbb{N}} \|g_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le 1$, by Lemma 2.1 $\{g_j\}$ is uniformly bounded on compact subsets of $\mathbb{D}^N$, and then $\{g_j\}$ is normal by Montel's theorem. Hence, we may extract a subsequence $\{g_{j_k}\}$ that converges uniformly on compact subsets of $\mathbb{D}^N$ to some $g \in H(\mathbb{D}^N, X)$. By applying Cauchy's estimate, we obtain that $\frac{\partial g_j}{\partial z_l} \to \frac{\partial g}{\partial z_l}$ uniformly on compact subsets of $\mathbb{D}^N$ for each $l \in \{1, \ldots, N\}$, which implies that $g \in \mathcal{B}^p(\mathbb{D}^N, X)$ and $\|g\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le 1$. Hence, the sequence $\{g_{j_k} - g\}$ satisfies that $\sup_{k\in\mathbb{N}} \|g_{j_k} - g\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le 2$ and converges to the zero vector on compact subsets of $\mathbb{D}^N$. By the hypothesis, we have

$$\lim_{k\to\infty} \|W_{u,\varphi} g_{j_k} - W_{u,\varphi} g\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} = 0. \tag{2.10}$$

Relation (2.10) means that the set $W_{u,\varphi}(B(0,1))$ is relatively compact, and then the proof is completed. □

From Lemma 2.3, we need to find some sequences in $\mathcal{B}^p(\mathbb{D}^N, X)$ to characterize the compactness of the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$. The next result will play such a role. The proof can be obtained by some calculations, and then we omit it.

Lemma 2.4. *Let x be a fixed vector in X and the sequence $\{w^j\}$ in $\mathbb{D}$ satisfy $|w^j| \to 1$ as $j \to \infty$. Then the following functions are bounded and converge to the zero vector in X uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$.*

(i) If $p \ge 1$, then we choose the functions

$$f_{j,x}(z) = f_j(z)x,$$

where

$$f_j(z) = \frac{(1 - |w^j|^2)^2}{(1 - \overline{w^j} z_l)^{p+1}} - \frac{1 - |w^j|^2}{(1 - \overline{w^j} z_l)^p},$$

and

$$\widehat{f_{j,x}}(z) = \widehat{f_j}(z)x,$$

where

$$\widehat{f_j}(z) = (z_l + 2)\Big(\frac{1 - |w^j|^2}{1 - \overline{w^j} z_l}\Big)^p.$$

(ii) If $p = 1$, then we choose the functions

$$g_{j,x}(z) = g_j(z)x,$$

where

$$g_j(z) = \frac{3}{a_j}\Big(\ln\frac{1}{1 - \overline{w^j} z_l}\Big)^2 - \frac{2}{a_j^2}\Big(\ln\frac{1}{1 - \overline{w^j} z_l}\Big)^3$$

and $a_j = -\ln(1 - |w^j|^2)$.

(iii) If $0 < p < 1$, then we choose the functions

$$h_{j,x}(z) = h_j(z)x,$$

where

$$h_j(z) = \frac{1 - |w^j|^2}{(1 - \overline{w^j} z_l)^p}.$$

Finally, inspired by the idea in [7], we obtain the following result.

Lemma 2.5. *Assume that $0 < p < 1$ and $\{f_j\} \subseteq \mathcal{B}^p(\mathbb{D}^N, X)$ is a bounded sequence satisfying $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$. Then*

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}^N} \|f_j(z)\|_X = 0. \tag{2.11}$$

And, for each $0 \le \rho < 1$ and $l \in \{1, \dots, N\}$,

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}^N, |z_l|\le\rho} \Big\|\frac{\partial f_j}{\partial z_l}(z)\Big\|_X = 0. \tag{2.12}$$

Proof. Assume that $\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le M$ for all j. For arbitrary $\varepsilon > 0$, by $\int_0^1 \frac{dr}{(1-r^2)^p} < +\infty$ (since $p \in (0, 1)$) we have some $\eta \in (0, 1)$ such that

$$\int_\eta^1 \frac{dr}{(1 - r^2)^p} < \frac{\varepsilon}{M + 1}. \tag{2.13}$$

Now, for $z = (z_1, \dots, z_l, \dots, z_N) \in \mathbb{D}^N$ with $|z_l| > \eta$, we have

$$\begin{aligned} &\Big\|f_j(z_1, \dots, z_l, \dots, z_N) - f_j(z_1, \dots, \eta\frac{z_l}{|z_l|}, \dots, z_N)\Big\|_X \\ &\le \int_\eta^{|z_l|} \Big\|\frac{\partial f_j}{\partial z_l}(r\frac{z_l}{|z_l|})\Big\|_X dr \le \|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \int_\eta^{|z_l|} \frac{dr}{(1 - r^2)^p} < \varepsilon. \end{aligned} \tag{2.14}$$

By (2.14), for any $z \in \mathbb{D}^N \setminus \eta\overline{\mathbb{D}^N}$, we have $z' \in \eta\overline{\mathbb{D}^N}$ such that

$$\big\|f_j(z) - f_j(z')\big\|_X \leq N\varepsilon. \tag{2.15}$$

For the above ε, by the hypothesis we have some positive integer N_0 such that for $j > N_0$

$$\sup_{z\in\eta\overline{\mathbb{D}^N}} \|f_j(z)\|_X < \varepsilon. \tag{2.16}$$

Combining (2.15) and (2.16), we obtain

$$\sup_{z\in\mathbb{D}^N} \|f_j(z)\|_X < (N+1)\varepsilon,$$

whenever $j > N_0$, which implies that (2.11) holds. Now, for the fixed $\rho \in (0,1)$ and $z \in \mathbb{D}^N$ with $|z_l| \leq \rho$, by applying Cauchy's estimate to one variable function $f_j(z_l) = f(z_1, \dots, z_{l-1}, z_l, \dots, z_N)$, we have

$$\Big\|\frac{\partial f_j}{\partial z_l}(z)\Big\|_X \leq C \sup_{|\zeta - z_l|<1-\rho} \|f(z_1, \dots, \zeta, \dots, z_N)\|_X \leq C \sup_{z\in\mathbb{D}^N} \|f(z)\|_X,$$

where C depends only on ρ. This together with (2.11) implies (2.12). The proof is completed. □

3. Boundedness of the operators $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$

Theorem 3.1. *Let $p, q > 0$, $\varphi(z) = (\varphi_1(z), \dots, \varphi_N(z))$ be a holomorphic self-map of $\mathbb{D}^N$ and $u \in H(\mathbb{D}^N, \mathcal{L}(X, Y))$. Then the following statements are true.*

(i) If $0 < p < 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded if and only if $u \in \mathcal{B}^q(\mathbb{D}^N, \mathcal{L}(X, Y))$ and

$$\sup_{z\in\mathbb{D}^N} \|u(z)\|_{X\to Y} \sum_{k,l=1}^{N} \frac{(1-|z_l|^2)^q}{(1-|\varphi_k(z)|^2)^p} \Big|\frac{\partial \varphi_k}{\partial z_l}(z)\Big| < +\infty. \tag{3.1}$$

(ii) If $p = 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded if and only if

$$\sup_{z\in\mathbb{D}^N} \sum_{k,l=1}^{N} (1-|z_l|^2)^q \Big\|\frac{\partial u}{\partial z_l}(z)\Big\|_{X\to Y} \ln \frac{1}{1-|\varphi_k(z)|^2} < +\infty \tag{3.2}$$

and

$$\sup_{z\in\mathbb{D}^N} \|u(z)\|_{X\to Y} \sum_{k,l=1}^{N} \frac{(1-|z_l|^2)^q}{1-|\varphi_k(z)|^2} \Big|\frac{\partial \varphi_k}{\partial z_l}(z)\Big| < +\infty. \tag{3.3}$$

(iii) If $p > 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded if and only if

$$\sup_{z\in\mathbb{D}^N} \sum_{k,l=1}^{N} \Big\|\frac{\partial u}{\partial z_l}(z)\Big\|_{X\to Y} \frac{(1-|z_l|^2)^q}{(1-|\varphi_k(z)|^2)^{p-1}} < +\infty \tag{3.4}$$

and

$$\sup_{z\in\mathbb{D}^N} \|u(z)\|_{X\to Y} \sum_{k,l=1}^{N} \frac{(1-|z_l|^2)^q}{(1-|\varphi_k(z)|^2)^p} \Big|\frac{\partial \varphi_k}{\partial z_l}(z)\Big| < +\infty. \tag{3.5}$$

Proof. We first prove (i). Necessity. Let the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ be bounded. Then, by using the function $1_x(z) = x$ defined on $\mathbb{D}^N$, we obtain

$$u(z)(x) = W_{u,\varphi}1_x(z) = u(z)(1_x(\varphi(z))) \in \mathcal{B}^q(\mathbb{D}^N, Y),$$

which shows that $u \in \mathcal{B}^q(\mathbb{D}^N, \mathcal{L}(X, Y))$.

For each fixed $l \in \{1, \dots, N\}$ and $w \in \mathbb{D}^N$, we will make use of the family of test functions $\{f_{l,\varphi_l(w),x} : |\varphi_l(w)| \neq 0\}$ defined in Lemma 2.2 by replacing w_l by $\varphi_l(w)$. From a calculation, it follows that

$$\frac{\partial f_{l,\varphi_l(w)}}{\partial z_l}(z) = \frac{p(1 - |\varphi_l(w)|^2)}{(1 - \overline{\varphi_l(w)}z_l)^{p+1}} - \frac{p-1}{(1 - \overline{\varphi_l(w)}z_l)^p}, \quad \frac{\partial f_{l,\varphi_l(w)}}{\partial z_j}(z) = 0 \text{ for } j \neq l, \tag{3.6}$$

and $|f_{l,\varphi_l(w)}(0)| = |\varphi_l(w)|$. It is clear that

$$W_{u,\varphi}f_{l,\varphi_l(w),x}(z) = u(z)(f_{l,\varphi_l(w),x}(\varphi(z))) = f_{l,\varphi_l(w)}(\varphi(z))u(z)(x),$$

from which for all $z \in \mathbb{D}^N$ we have

$$\begin{aligned}
b_{W_{u,\varphi}f_{l,\varphi_l(w),x}}(z) &= \sum_{k=1}^{N}(1 - |z_k|^2)^q \Big\| \frac{\partial W_{u,\varphi}f_{l,\varphi_l(w),x}}{\partial z_k}(z) \Big\|_Y \\
&= \sum_{k=1}^{N}(1 - |z_k|^2)^q \Big\| f_{l,\varphi_l(w)}(\varphi(z))\frac{\partial u}{\partial z_k}(z)(x) + \sum_{j=1}^{N} \frac{\partial f_{l,\varphi_l(w)}}{\partial \zeta_j}(\varphi(z))\frac{\partial \varphi_j}{\partial z_k}(z)u(z)(x) \Big\|_Y \\
&\leq \|W_{u,\varphi}f_{l,\varphi_l(w),x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \\
&\leq \|W_{u,\varphi}\|_{\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)}\|f_{l,\varphi_l(w),x}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \\
&= \|W_{u,\varphi}\|_{\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)}\|f_{l,\varphi_l(w)}\|_{\mathcal{B}^p(\mathbb{D}^N)}\|x\|_X \\
&\leq C\|x\|_X.
\end{aligned} \tag{3.7}$$

Since $f_{l,\varphi_l(w)}(\varphi(w)) = 0$ and

$$\frac{\partial f_{l,\varphi_l(w)}}{\partial \zeta_l}(\varphi(w)) = \frac{1}{(1 - |\varphi_l(w)|^2)^p},$$

it follows from replacing z by w in (3.6) and (3.7) that

$$\big\|u(w)(x)\big\|_Y \sum_{k=1}^{N} \frac{(1 - |w_k|^2)^q}{(1 - |\varphi_l(w)|^2)^p}\Big|\frac{\partial \varphi_l}{\partial w_k}(w)\Big| \leq C\|x\|_X \tag{3.8}$$

for each $l \in \{1, \dots, N\}$, which shows

$$\big\|u(z)\big\|_{X\to Y} \sum_{k=1}^{N} \frac{(1 - |z_k|^2)^q}{(1 - |\varphi_l(z)|^2)^p}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| \leq C \tag{3.9}$$

for each $l \in \{1, \dots, N\}$. By continuity the above estimate remains valid also if $\varphi_l(w) = 0$, from which (3.1) holds.

Sufficiency. Now, assume that $u \in \mathcal{B}^q(\mathbb{D}^N, \mathcal{L}(X, Y))$ and (3.1) holds. Let $f \in \mathcal{B}^p(\mathbb{D}^N, X)$. By Lemma 2.1,

$$\|W_{u,\varphi}f(0)\|_Y = \|u(0)(f(\varphi(0)))\|_Y \leq \|u(0)\|_{X\to Y}\|f(\varphi(0))\|_X \leq C\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}. \tag{3.10}$$

At the same time, we also have

$$
\begin{aligned}
b_{W_{u,\varphi}f}(z) &= \sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial W_{u,\varphi}f}{\partial z_k}(z)\Big\|_Y \\
&= \sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)(f(\varphi(z))) + \sum_{l=1}^{N}\frac{\partial \varphi_l}{\partial z_k}(z)u(z)\Big(\frac{\partial f}{\partial \zeta_l}(\varphi(z))\Big)\Big\|_Y \\
&\le \sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\big\|f(\varphi(z))\big\|_X \\
&\quad + \|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}(1-|z_k|^2)^q\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|\Big\|\frac{\partial f}{\partial \zeta_l}(\varphi(z))\Big\|_X \\
&\le C\|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))}\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| \\
&= \Big(C\|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))} + \|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|\Big)\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)}. \qquad (3.11)
\end{aligned}
$$

From (3.10) and (3.11), it follows that $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded.

Next, we prove (ii). Necessity. Assume that $W_{u,\varphi} : \mathcal{B}(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded. Using test functions $f_{l,\varphi_l(w),x}$, it follows that condition (3.3) holds. Now, we prove that condition (3.4) holds. For each fixed $l \in \{1, \dots, N\}$, $w \in \mathbb{D}^N$, and $x \in X$, we consider the function $g_{l,\varphi_l(w),x}$. Then from a direct calculation, we have

$$
\frac{\partial g_{l,\varphi_l(w),x}}{\partial z_k}(z) = 0 \quad \text{for } k \ne l, \text{ and } \frac{\partial g_{l,\varphi_l(w),x}}{\partial z_l}(z) = \frac{\overline{\varphi_l(w)}}{1-\overline{\varphi_l(w)}z}x. \qquad (3.12)
$$

Note that

$$
\frac{\partial g_{l,\varphi_l(w),x}}{\partial z_l}(\varphi(w)) = \frac{\overline{\varphi_l(w)}}{1-|\varphi_l(w)|^2}x \quad \text{and} \quad g_{l,\varphi_l(w),x}(\varphi(w)) = \ln\frac{1}{1-|\varphi_l(w)|^2}x. \qquad (3.13)
$$

Applying (3.11) to the function $g_{l,\varphi_l(w),x}$ with $\|x\|_X = 1$ and from (3.13), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{N}(1-|w_k|^2)^q \ln\frac{1}{1-|\varphi_l(w)|^2}\Big\|\frac{\partial u}{\partial w_k}(w)(x)\Big\|_Y \\
&\le C\big\|g_{l,\varphi_l(w),x}\big\|_{\mathcal{B}(\mathbb{D}^N,X)} + \big\|u(w)\big\|_{X\to Y}\Big|\sum_{k=1}^{N}\frac{\overline{\varphi_l(w)}}{1-|\varphi_l(w)|^2}\frac{\partial \varphi_l}{\partial w_k}(w)\Big|(1-|w_k|^2)^q \\
&\le 2C\|x\|_X + \sup_{w\in\mathbb{D}^N}\big\|u(w)\big\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|w_k|^2)^q}{1-|\varphi_l(w)|^2}\Big|\frac{\partial \varphi_l}{\partial w_k}(w)\Big| \\
&= 2C + \sup_{w\in\mathbb{D}^N}\big\|u(w)\big\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|w_k|^2)^q}{1-|\varphi_l(w)|^2}\Big|\frac{\partial \varphi_l}{\partial w_k}(w)\Big|.
\end{aligned}
$$

From this and condition (3.3), it follows that condition (3.2) holds.

Sufficiency. First note that condition (3.2) implies that $u \in \mathcal{B}^q(\mathbb{D}^N, \mathcal{L}(X,Y))$. From this, Lemma 2.1 (ii) and (3.11), it follows that

$$\begin{aligned} b_{W_{u,\varphi}f}(z) \le & \sum_{k=1}^{N}(1-|z_k|^2)^q \Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\big\|f(\varphi(z))\big\|_X \\ & + \|f\|_{\mathcal{B}(\mathbb{D}^N,X)}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|z_k|^2)^q}{1-|\varphi_l(z)|^2}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| \\ \le & C\|f\|_{\mathcal{B}(\mathbb{D}^N,X)}\|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))} \\ & + C\|f\|_{\mathcal{B}(\mathbb{D}^N,X)}\sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\sum_{l=1}^{N}\ln\frac{1}{1-|\varphi_l(z)|^2} \\ & + \|f\|_{\mathcal{B}(\mathbb{D}^N,X)}\|u(z)\|_{X\to Y}\sup_{z\in\mathbb{D}^N}\sum_{k,l=1}^{N}\frac{(1-|z_k|^2)^q}{1-|\varphi_l(z)|^2}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|. \end{aligned} \tag{3.14}$$

From (3.10) and (3.14), and using conditions (3.2) and (3.3), we obtain that the operator $W_{u,\varphi}$: $\mathcal{B}(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded.

Now, we prove (iii). Necessity. Assume that $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded. Using the same test functions $f_{l,\varphi(w),x}$ in the proof of (i), we obtain that condition (3.5) holds. In order to obtain condition (3.4), we use the function $h_{l,\varphi_l(w),x}$ defined in Lemma 2.2 (iii) by replacing w_l by $\varphi_l(w)$. Since

$$\frac{\partial h_{l,\varphi_l(w)}}{\partial z_l}(z) = p(p-1)\frac{\overline{\varphi_l(w)}}{(1-\overline{\varphi_l(w)}z_l)^p} - p(p-1)\frac{\overline{\varphi_l(w)}(1-|\varphi_l(w)|^2)}{(1-\overline{\varphi_l(w)}z_l)^{p+1}},$$

we have

$$\frac{\partial h_{l,\varphi_l(w)}}{\partial z_l}(\varphi_l(w)) = 0. \tag{3.15}$$

We also have

$$\frac{\partial h_{l,\varphi_l(w)}}{\partial z_k}(z) \ne 0 \text{ for } k \ne l, \text{ and } \quad h_{l,\varphi_l(w)}(\varphi_l(w)) = \frac{1}{(1-|\varphi_l(w)|^2)^{p-1}}. \tag{3.16}$$

Applying (3.11) to the function $h_{l,\varphi_l(w),x}$, it follows from (3.15) and (3.16) that

$$\sum_{k=1}^{N}\Big\|\frac{\partial u_w}{\partial w_k}(x)\Big\|_Y\frac{(1-|w_k|^2)^q}{(1-|\varphi_l(w)|^2)^{p-1}} \le C\|h_{l,\varphi_l(w),x}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le C(3p(p-1)2^p+2p-1)\|x\|_X \tag{3.17}$$

for each $l \in \{1,\dots,N\}$. From (3.17), it follows that

$$\sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}} \le C$$

for each $l \in \{1,\dots,N\}$, from which condition (3.4) holds.

Sufficiency. From Lemma 2.1 (iii), (3.11), and the definition of $\mathcal{B}^q(\mathbb{D}^N, Y)$, for each $f \in \mathcal{B}^p(\mathbb{D}^N, X)$ we have

$$\begin{aligned} b_{W_{u,\varphi}f}(z) \le\ & C\|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \sum_{k,l=1}^N \Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y} \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}} \\ & + \|f\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \|u(z)\|_{X\to Y} \sum_{k,l=1}^N \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} \Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|. \end{aligned} \tag{3.18}$$

Taking the supremum over $\mathbb{D}^N$ in (3.18) and using conditions (3.4), (3.5), and (3.10), the boundedness of $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ follows. The proof is completed. □

4. Compactness of the operators $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$

Theorem 4.1. *Let $p, q > 0$, $\varphi = (\varphi_1, \dots, \varphi_N)$ be a holomorphic self-map of $\mathbb{D}^N$, $u \in H(\mathbb{D}^N, \mathcal{L}(X, Y))$, and the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ be bounded. Then the following statements are true.*

(i) If $p = 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact if and only if

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \sum_{k,l=1}^N (1-|z_k|^2)^q \Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y} \ln \frac{1}{1-|\varphi_l(z)|^2} = 0 \tag{4.1}$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \|u(z)\|_{X\to Y} \sum_{k,l=1}^N \frac{(1-|z_k|^2)^q}{1-|\varphi_l(z)|^2} \Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| = 0. \tag{4.2}$$

(ii) If $p > 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact if and only if

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \sum_{k,l=1}^N \Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y} \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}} = 0 \tag{4.3}$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \|u(z)\|_{X\to Y} \sum_{k,l=1}^N \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} \Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| = 0. \tag{4.4}$$

(iii) If $0 < p < 1$, then $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact if and only if for each $l \in \{1, \dots, N\}$

$$\lim_{|\varphi_l(z)|\to 1} \|u(z)\|_{X\to Y} \sum_{k=1}^N \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} \Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big| = 0. \tag{4.5}$$

Proof. We first prove the necessity of (i) and (ii). Assume that $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact. We first prove that condition (4.4) holds when $p \ge 1$ (this means that when $p = 1$, condition (4.2) holds).

If condition (4.4) fails, then there is a sequence $\{z^j\}$ in $\mathbb{D}^N$ such that $w^j = \varphi(z^j) \to \partial\mathbb{D}^N$ as $j \to \infty$, and $\varepsilon_0 > 0$ such that

$$\|u(z^j)\|_{X\to Y} \sum_{k,l=1}^{N} \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^p} \Big|\frac{\partial\varphi_l}{\partial z_k}(z^j)\Big| \geq \varepsilon_0 \tag{4.6}$$

for all $j \in \mathbb{N}$. Since the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded, by Theorem 3.1 condition (3.5) (correspondingly, condition (3.3) when $p = 1$) holds. So, there is a subsequence of $\{z^j\}$ (here we shall use the same notion $\{z^j\}$) such that there is a finite limit

$$\lim_{j\to\infty} \|u(z^j)\|_{X\to Y} \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^p} \Big|\frac{\partial\varphi_l}{\partial z_k}(z^j)\Big| \tag{4.7}$$

for all $k, l \in \{1, \ldots, N\}$. Also, we may assume that for every $l \in \{1, \ldots, N\}$ there is a finite limit

$$\lim_{j\to\infty} |w_l^j| = \lim_{j\to\infty} |\varphi_l(z^j)|.$$

From (4.6), without loss of generality, we may assume that $l = 1$ and

$$\|u(z^j)\|_{X\to Y} \frac{(1-|z_{k_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^p} \Big|\frac{\partial\varphi_1}{\partial z_{k_0}}(z^j)\Big| = \varepsilon_1 > 0 \tag{4.8}$$

for some $k_0 \in \{1, \ldots, N\}$. By the definition of the norm of a linear operator, for small enough $\varepsilon > 0$, there is a unit vector $x_\varepsilon \in X$ such that

$$\|u(z^j)(x_\varepsilon)\|_Y \frac{(1-|z_{k_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^p} \Big|\frac{\partial\varphi_1}{\partial z_{k_0}}(z^j)\Big| \geq \varepsilon_1 - \varepsilon > 0. \tag{4.9}$$

We shall construct a sequence of functions $\{f_j\}$ in $\mathcal{B}^p(\mathbb{D}^N, X)$ satisfying the following conditions:
(a) $\{f_j\}$ is a bounded sequence in $\mathcal{B}^p(\mathbb{D}^N, X)$;
(b) $\{f_j\}$ converges to zero vector in X uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$;
(c) $\|W_{u,\varphi} f_j\|_{\mathcal{B}^q(Y,\mathbb{D}^N)} \nrightarrow 0$ as $j \to \infty$.
This will contradict the compactness of the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ in view of Lemma 2.3, which implies that (4.4) is necessary for the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ to be compact.

Case 1. First assume that $|w_1^j| \to 1$ as $j \to \infty$. For each $j \in \mathbb{N}$, we consider the functions $f_{j,x}(z) = f_j(z)x$ in Lemma 2.4 by replacing w^j by w_1^j. Then

$$\frac{\partial f_j}{\partial z_l}(z) = 0, \quad l \neq 1,$$

and

$$\frac{\partial f_j}{\partial z_1}(z) = (p+1)\overline{w_1^j} \frac{(1-|w_1^j|^2)^2}{(1-\overline{w_1^j}z_1)^{p+2}} - p\overline{w_1^j} \frac{1-|w_1^j|^2}{(1-\overline{w_1^j}z_1)^{p+2}},$$

and consequently,

$$\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N)} = |f_j(0)| + \sup_{z\in\mathbb{D}^N}\sum_{k=1}^{N}(1-|z_k|^2)^p\Big|\frac{\partial f_j}{\partial z_k}(z)\Big| \le 1+(3p+2)2^{p+1}. \tag{4.10}$$

For previous mentioned unit vector $x_\varepsilon \in X$, using the function f_{j,x_ε}, we will show that $\|W_{u,\varphi}f_{j,x_\varepsilon}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \nrightarrow 0$ as $j\to\infty$.

Let

$$J(z^j) = \big\|u(z^j)(x_\varepsilon)\big\|_Y \sum_{k=1}^{N}(1-|z_k|^2)^q\Big|\frac{\partial f_j}{\partial z_1}(\varphi(z^j))\Big|\Big|\frac{\partial \varphi_1}{\partial z_k}(z^j)\Big|.$$

Then, we have

$$\begin{aligned}\|W_{u,\varphi}f_{j,x_\varepsilon}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \ge J(z^j) &= \big|w_1^j\big|\big\|u(z^j)(x_\varepsilon)\big\|_Y \sum_{k=1}^{N}\frac{(1-|z_k^j|^2)^q}{(1-|w_1^j|^2)^p}\Big|\frac{\partial\varphi_1}{\partial z_k}(z^j)\Big| \\ &\ge \big|w_1^j\big|\big\|u(z^j)(x_\varepsilon)\big\|_Y \frac{(1-|z_{k_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^p}\Big|\frac{\partial\varphi_1}{\partial z_{k_0}}(z^j)\Big| \\ &\ge \frac{1}{2}(\varepsilon_1-\varepsilon) > 0 \end{aligned} \tag{4.11}$$

for large enough j, since $|w_1^j| \to 1$ as $j\to\infty$. From (4.11), the result follows in this case.

Case 2. Assume that $|w_1^j| \to r_0$ as $j\to\infty$. Since $w^j \to \partial\mathbb{D}^N$, there is an $l_0 \in \{1,\dots,N\}$ such that $|w_{l_0}^j| \to 1$ as $j\to\infty$. If there is a $k_1 \in \{1,\dots,N\}$ and $\varepsilon_2 > 0$ such that

$$\|u(z^j)\|_{X\to Y}\frac{(1-|z_{k_1}^j|^2)^q}{(1-|\varphi_{l_0}(z^j)|^2)^p}\Big|\frac{\partial\varphi_{l_0}}{\partial z_{k_1}}(z^j)\Big| = \varepsilon_2,$$

we also obtain a contradiction similar to Case 1. Hence, we may assume that for the l_0 chosen above such that

$$\lim_{j\to\infty}\|u(z^j)\|_{X\to Y}\frac{(1-|z_k^j|^2)^q}{(1-|\varphi_{l_0}(z^j)|^2)^p}\Big|\frac{\partial\varphi_{l_0}}{\partial z_k}(z^j)\Big| = 0$$

for each $k\in\{1,\dots,N\}$. Let $\widehat{f_{j,x}}$ be functions defined in Lemma 2.4 by replacing w^j by $w_{l_0}^j$. Clearly, we have

$$\frac{\partial\widehat{f_j}}{\partial z_l} = 0,\ \ l\ne 1, l_0,\ \frac{\partial\widehat{f_j}}{\partial z_1}(z) = \Big(\frac{1-|w_{l_0}^j|^2}{1-\overline{w_{l_0}^j}z_{l_0}}\Big)^p \text{ and } \frac{\partial\widehat{f_j}}{\partial z_{l_0}}(z) = p(z_1+2)\overline{w_{l_0}^j}\frac{(1-|w_{l_0}^j|^2)^p}{(1-\overline{w_{l_0}^j}z_{l_0})^{p+1}}.$$

From Lemma 2.4, we know that $\{\widehat{f_{j,x}}\}$ is bounded and $\widehat{f_{j,x}} \to 0$ uniformly on the compact subset of $\mathbb{D}^N$ as $j\to\infty$. We prove that $\|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^p(\mathbb{D}^N,Y)} \nrightarrow 0$ as $j\to\infty$. Note that

$$\frac{\partial\widehat{f_j}}{\partial\zeta_1}(\varphi(z^j)) = 1,\ \ \frac{\partial\widehat{f_j}}{\partial\zeta_{l_0}}(\varphi(z^j)) = p\frac{(w_1^j+2)\overline{w_{l_0}^j}}{1-|w_{l_0}^j|^2}. \tag{4.12}$$

Since the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact, it follows that

$$\lim_{j\to\infty} \|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} = 0. \tag{4.13}$$

Since this operator is bounded, from Theorem 3.1 it follows that (3.2) or (3.4) holds. Therefore, we obtain

$$\lim_{\varphi(z)\to\partial\mathbb{D}^N} \sum_{k=1}^{N} \Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}(1-|z_k|^2)^q = 0.$$

In particular, we have

$$\lim_{j\to\infty} \sum_{k=1}^{N} \Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}(1-|z_k^j|^2)^q = 0. \tag{4.14}$$

From (3.11), and (4.12)–(4.14), it follows that

$$\begin{aligned}
L_j &= \|u(z^j)(x)\|_Y \sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big|\frac{\partial \widehat{f_{j,x}}}{\partial \zeta_1}(\varphi(z^j))\frac{\partial \varphi_1}{\partial z_k}(z^j)\Big| \\
&\le \|u(z^j)(x)\|_Y \sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big|\frac{\partial \widehat{f_{j,x}}}{\partial \zeta_{l_0}}(\varphi(z^j))\frac{\partial \varphi_{l_0}}{\partial z_k}(z^j)\Big| \\
&\quad + \|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} + \|\widehat{f_{j,x}}(\varphi(z^j))\|_X \sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}(1-|z_k^j|^2)^q \\
&\le 3p\|u(z^j)(x)\|_Y \sum_{k=1}^{N}\frac{(1-|z_k^j|^2)^q}{1-|\varphi_{l_0}(z^j)|^2}\Big|\frac{\partial \varphi_{l_0}}{\partial z_k}(z^j)\Big| \\
&\quad + \|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} + \|\widehat{f_{j,x}}\|_{H^\infty(\mathbb{D}^N,X)} \sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}(1-|z_k^j|^2)^q \\
&\le 3p\|u(z^j)(x)\|_Y \sum_{k=1}^{N}\frac{(1-|z_k^j|^2)^q}{(1-|\varphi_{l_0}(z^j)|^2)^p}\Big|\frac{\partial \varphi_{l_0}}{\partial z_k}(z^j)\Big| \\
&\quad + \|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} + 3\cdot 2^p \sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}(1-|z_k^j|^2)^q.
\end{aligned} \tag{4.15}$$

Suppose to the contrary that $\|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} = 0$ as $j\to\infty$. Letting $j\to\infty$ in (4.15), we obtain

$$\lim_{j\to\infty} L_j = \lim_{j\to\infty} \|u(z^j)(x)\|_Y \sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big|\frac{\partial \varphi_1}{\partial z_k}(z^j)\Big| = 0. \tag{4.16}$$

Since $|w_1^j| \le \delta < 1$, the following estimates hold for sufficiently large j:

$$0 < C_\delta(\varepsilon_1-\varepsilon) \le C_\delta\|u(z^j)(x_\varepsilon)\|_Y \frac{(1-|z_{k_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^p}\Big|\frac{\partial \varphi_1}{\partial z_{k_0}}(z^j)\Big|$$

$$\leq \|u(z^j)(x_\varepsilon)\|_Y \sum_{k=1}^{N}(1-|z_k|^2)^q\Big|\frac{\partial\varphi_1}{\partial z_k}(z^j)\Big|, \tag{4.17}$$

where we can choose, for example, $C_\delta = (1-\delta^2)^p$. It is clear that (4.17) contradicts the convergence $\lim_{j\to\infty} L_j = 0$. Hence,

$$\lim_{j\to\infty} \|u(z^j)\|_{X\to Y}\frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^p}\Big|\frac{\partial\varphi_l}{\partial z_k}(z^j)\Big| = 0. \tag{4.18}$$

It follows from the above that condition (4.4) holds for every $p \geq 1$.

Now, we begin to prove that conditions (4.1) and (4.3) hold. Assume that $p > 1$ and that condition (4.3) fails. Then there is a sequence $\{z^j\}$ in $\mathbb{D}^N$ such that $\varphi(z^j) \to \partial\mathbb{D}^N$ as $j \to \infty$, and $\varepsilon_3 > 0$ such that

$$\sum_{k,l=1}^{N}\Big\|\frac{\partial u}{\partial z_l}(z^j)\Big\|_{X\to Y}\frac{(1-|z_l^j|^2)^q}{(1-|\varphi_k(z^j)|^2)^{p-1}} \geq \varepsilon_3 \quad \text{for each } j\in\mathbb{N}. \tag{4.19}$$

Since the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded, from Theorem 3.1 it follows that (3.4) holds. Hence, there is a subsequence of $\{z^j\}$ (we still use the same notation $\{z^j\}$) such that for all $k, l \in \{1, \ldots, N\}$ such that

$$\Big\|\frac{\partial u}{\partial z_l}(z^j)\Big\|_{X\to Y}\frac{(1-|z_l^j|^2)^q}{(1-|\varphi_k(z^j)|^2)^{p-1}}$$

converges to a finite number as $j \to \infty$, and so that for every $k \in \{1, \ldots, N\}$ there is a finite limit $\lim_{j\to\infty} |w_k^j| = \lim_{j\to\infty} |\varphi_k(z^j)|$. Without loss of generality, assume that $k = 1$ and

$$\Big\|\frac{\partial u}{\partial z_{l_0}}(z^j)\Big\|_{X\to Y}\frac{(1-|z_{l_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^{p-1}} = \varepsilon_4 \tag{4.20}$$

for some $l_0 \in \{1, \ldots, N\}$. Then, from the definition of norms of the bounded linear operators, for small enough $\varepsilon > 0$ there is a unit vector $y_\varepsilon \in X$ such that

$$\Big\|\frac{\partial u}{\partial z_{l_0}}(z^j)(y_\varepsilon)\Big\|_{Y}\frac{(1-|z_{l_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^{p-1}} \geq \varepsilon_4 - \varepsilon. \tag{4.21}$$

As above, we still construct a sequence of functions that converges to zero uniformly on compact subsets of $\mathbb{D}^N$ in order to obtain a contradiction by Lemma 2.3. We have to consider the following cases.

Case 1. Let $|w_1^j| \to 1$ as $j \to \infty$. First, we define the following functions

$$\widehat{h}_{w_1^j}(z) = (p+1)\frac{1-|w_1^j|^2}{(1-\overline{w_1^j}z_1)^p} - p\frac{(1-|w_1^j|^2)^2}{(1-\overline{w_1^j}z_1)^{p+1}}.$$

Then

$$\frac{\partial \widehat{h}_{w_1^j}}{\partial z_k}(z) = 0 \text{ for } k \neq 1, \quad \frac{\partial \widehat{h}_{w_1^j}}{\partial z_1}(\varphi(z^j)) = 0 \quad \text{and} \quad \widehat{h}_{w_1^j}(\varphi(z^j)) = \frac{1}{(1-|w_1^j|^2)^{p-1}}. \tag{4.22}$$

For unit vector y_ε in X, we define the function $\widehat{h}_{w_1^j,y_\varepsilon}(z) = \widehat{h}_{w_1^j}(z)y_\varepsilon$. Since from a direct calculation it follows that

$$\|\widehat{h}_{w_1^j,y_\varepsilon}\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \leq 2p + 1 + 3p(p+1)2^{p+1},$$

we obtain that the sequence $\{\widehat{h}_{w_1^j,y_\varepsilon}\}$ is uniformly bounded. Also it is easy to see that $\{\widehat{h}_{w_1^j,y_\varepsilon}\}$ converges to the zero vector uniformly on compact subsets of $\mathbb{D}^N$. From Lemma 2.3 it follows that $\|W_{u,\varphi}\widehat{h}_{w_1^j,y_\varepsilon}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0$ as $j \to \infty$. Hence, by (4.22) we have

$$\begin{aligned} C\|W_{u,\varphi}\widehat{h}_{w_1^j,y_\varepsilon}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} &\geq \sum_{k=1}^N (1-|z_k^j|^2)^q \Big\| \widehat{h}_{w_1^j}(\varphi(z^j))\frac{\partial u}{\partial z_k}(z^j)(y_\varepsilon) + \sum_{l=1}^N \frac{\partial \widehat{h}_{w_1^j}}{\partial \zeta_l}(\varphi(z^j))\frac{\partial \varphi_l}{\partial z_k}(z^j)u(z^j)(y_\varepsilon)\Big\|_Y \\ &= \sum_{k=1}^N \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^{p-1}}\Big\|\frac{\partial u}{\partial z_k}(z^j)(y_\varepsilon)\Big\|_Y \geq \varepsilon_4 - \varepsilon > 0, \end{aligned}$$

which is a contradiction.

Case 2. Assume that $|w_1^j| \to \rho$ as $j \to \infty$. Since $w^j \to \partial\mathbb{D}^N$, there is an $l \in \{1,\dots,N\}$ such that $|w_l^j| \to 1$ as $j \to \infty$. If there is a $k_0 \in \{1,\dots,N\}$ and $\varepsilon_5 > 0$ such that

$$\lim_{j\to\infty} \Big\|\frac{\partial u}{\partial z_{k_0}}(z^j)\Big\|_{X\to Y} \frac{(1-|z_{k_0}^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^{p-1}} = \varepsilon_5, \tag{4.23}$$

then we obtain a contradiction by replacing w_1^j by w_l^j in the function $\widehat{h}_{w_1^j,y_\varepsilon}$ as in Case 1. Hence, for this l, we may assume that

$$\lim_{j\to\infty} \Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y} \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^{p-1}} = 0 \tag{4.24}$$

for each $k \in \{1,\dots,N\}$. By using the above-defined functions $\widehat{f}_{j,x}$, the boundedness of the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N,X) \to \mathcal{B}^q(\mathbb{D}^N,Y)$, it follows that

$$\begin{aligned} &|w_1^j+2| \sum_{k=1}^N \Big\|\frac{\partial u}{\partial z_k}(z^j)(x)\Big\|_Y (1-|z_k^j|^2)^q \\ &\quad \leq C\|W_{u,\varphi}\widehat{f}_{j,x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} + \|u(z^j)(x)\|_Y \sum_{k=1}^N (1-|z_k^j|^2)^q \Big|\frac{\partial \varphi_1}{\partial z_k}(z^j)\Big| \\ &\qquad + 3p\|u(z^j)(x)\|_Y \sum_{k=1}^N \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^{p-1}}\Big|\frac{\partial \varphi_l}{\partial z_k}(z^j)\Big| \end{aligned}$$

$$\leq C\|W_{u,\varphi}\widehat{f_{j,x}}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} + \|u(z^j)(x)\|_Y \sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big|\frac{\partial\varphi_1}{\partial z_k}(z^j)\Big|$$

$$+ 3p\|u(z^j)(x)\|_Y \sum_{k=1}^{N} \frac{(1-|z_k^j|^2)^q}{(1-|\varphi_l(z^j)|^2)^{p-1}}\Big|\frac{\partial\varphi_l}{\partial z_k}(z^j)\Big| \to 0 \tag{4.25}$$

as $j \to \infty$, by condition (3.7).

On the other hand, since $|w_1^j| \leq \rho < 1$ and $p > 1$, by (4.20) we have

$$\frac{\varepsilon_4}{2} < \Big\|\frac{\partial u}{\partial z_{l_0}}(z^j)\Big\|_{X\to Y}\frac{(1-|z_{l_0}^j|^2)^q}{(1-|\varphi_1(z^j)|^2)^{p-1}} \leq C\sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big\|\frac{\partial u}{\partial z_{l_0}}(z^j)\Big\|_{X\to Y}$$

for sufficiently large j, which contradicts (4.25).

Assume now that $p = 1$. We prove that condition (4.1) holds. Assume the contrary, that is, condition (4.1) fails. Then there is a sequence $\{z^j\}$ in $\mathbb{D}^N$ such that $w^j = \varphi(z^j) \to \partial\mathbb{D}^N$ as $j \to \infty$, and $\varepsilon_5 > 0$ such that

$$\sum_{k,l=1}^{N}(1-|z_k^j|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_l(z^j)|^2} \geq \varepsilon_5 \tag{4.26}$$

for each $j \in \mathbb{N}$. Since the operator $W_{u,\varphi} : \mathcal{B}(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded, by Theorem 3.1 condition (3.2) holds. Hence, for any $k, l \in \{1,\dots,N\}$ there is a subsequence of $\{z^j\}$ (here we still use the same notation $\{z^j\}$) such that the limits

$$\lim_{j\to\infty}(1-|z_k^j|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_l(z^j)|^2}$$

are finite for all $k, l \in \{1,\dots,N\}$ and for each $l \in \{1,\dots,N\}$ there is a finite limit

$$\lim_{j\to\infty}|w_l^j| = \lim_{j\to\infty}|\varphi_l(z^j)|.$$

Without loss of generality, we may assume that $l = 1$ and

$$\lim_{j\to\infty}(1-|z_{k_0}^j|^2)^q\Big\|\frac{\partial u}{\partial z_{k_0}}(z^j)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_1(z^j)|^2} = \varepsilon_6 > 0$$

from some $k_0 \in \{1,\dots,N\}$. Hence, for sufficiently large j,

$$(1-|z_{k_0}^j|^2)^q\Big\|\frac{\partial u}{\partial z_{k_0}}(z^j)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_1(z^j)|^2} > \frac{\varepsilon_6}{2}.$$

By the definition of the norm of the bounded linear operators, there is some unit vector x_0 in X such that

$$(1-|z_{k_0}^j|^2)^q\Big\|\frac{\partial u}{\partial z_{k_0}}(z^j)(x_0)\Big\|_Y\ln\frac{1}{1-|\varphi_1(z^j)|^2} > \frac{\varepsilon_6}{2}. \tag{4.27}$$

We have to consider the following subcases.

Subcase 1. Let $|w_1^j| \to 1$ as $j \to \infty$. We use the functions g_{j,x_0} in Lemma 2.4 by replacing w^j by w_1^j. Then

$$\frac{\partial g_j}{\partial z_1}(z) = \frac{6\overline{w_1^j}}{1-\overline{w_1^j}z_1}\Big[\frac{1}{a_j}\Big(\ln\frac{1}{1-\overline{w_1^j}z_1}\Big) - \frac{1}{a_j^2}\Big(\ln\frac{1}{1-\overline{w_1^j}z_1}\Big)^2\Big]$$

and for $k \neq 1$

$$\frac{\partial g_j}{\partial z_k}(z) = 0.$$

At the same time, we also have

$$\frac{\partial g_j}{\partial z_1}(\varphi(z)) = 0, \quad g_j(\varphi(z)) = \ln\frac{1}{1-|\varphi_1(z^j)|^2}.$$

From Lemma 2.4 and the boundedness of $W_{u,\varphi}$, it follows that

$$\begin{aligned}\|W_{u,\varphi}g_{j,x_0}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} &\geq \sum_{k=1}^N (1-|z_k^j|^2)^q \Big\| \frac{\partial u}{\partial z_k}(z^j)(x_0) g_j(\varphi(z^j)) \\ &\quad + u(z^j)(x_0)\sum_{l=1}^N \frac{\partial g_j}{\partial \zeta_l}(\varphi(z^j))\frac{\partial \varphi_l}{\partial z_k}(z^j)\Big\|_Y \\ &= \sum_{k=1}^N \Big\|\frac{\partial u}{\partial z_k}(z^j)(x_0)\Big\|_Y (1-|z_k^j|^2)^q \ln\frac{1}{1-|\varphi_1(z^j)|^2} \geq \frac{\varepsilon_6}{2} \end{aligned} \tag{4.28}$$

for sufficiently large j. Hence, we get $\|W_{u,\varphi}g_{j,x_0}\| \nrightarrow 0$ as $j \to \infty$, from which the result follows in this case.

Subcase 2. Assume that $|w_1^j| \to \rho < 1$ as $j \to \infty$. Since $w^j = \varphi(z^j) \to \partial\mathbb{D}^N$ as $j \to \infty$, there is some $l \in \{1, \ldots, N\}$ such that $|w_l^j| \to 1$ as $j \to \infty$. If there is a $k_1 \in \{1, \ldots, N\}$ and $\varepsilon_7 > 0$ such that

$$\lim_{j\to\infty}\Big\|\frac{\partial u}{\partial z_{k_1}}(z^j)\Big\|_{X\to Y}(1-|z_{k_1}^j|^2)^q \ln\frac{1}{1-|\varphi_l(z^j)|^2} = \varepsilon_7,$$

then we can also obtain a contradiction similar to Case 1. Hence, for the l chosen above, we may assume that

$$\lim_{j\to\infty}\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}(1-|z_k^j|^2)^q \ln\frac{1}{1-|\varphi_l(z^j)|^2} = 0 \quad \text{for each } k \in \{1, \ldots, N\}. \tag{4.29}$$

By using the functions defined in the proof of the case $p > 1$, we obtain that estimate (4.25) holds with $p = 1$. On the other hand, since $|w_1^j| \leq \rho < 1$, we have

$$\frac{\varepsilon_6}{2} < (1-|z_{k_0}^j|^2)^q \Big\|\frac{\partial u}{\partial z_{k_0}}(z^j)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_1(z^j)|^2} \leq C\sum_{k=1}^N (1-|z_k^j|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z^j)\Big\|_{X\to Y}$$

for sufficiently large j, which contradicts (4.25).

Now, we prove the sufficiency of (i). First assume that conditions (4.1) and (4.2) hold. Assume that a sequence $\{f_j\}$ is such that $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le M$ and $\{f_j\}$ converges to the zero vector in X uniformly on compact subsets of $\mathbb{D}^N$. We need to prove that $\|W_{u,\varphi} f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0$ as $j \to \infty$. From (4.1) and (4.2) it follows that for every $\varepsilon > 0$ there exists an r $(0 < r < 1/(2e))$ such that

$$\sum_{k,l=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\ln\frac{1}{1-|\varphi_l(z)|^2} < \varepsilon \tag{4.30}$$

and

$$\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|z_l|^2)^q}{1-|\varphi_k(z)|^2}\Big|\frac{\partial \varphi_k}{\partial z_l}(z)\Big| < \varepsilon, \tag{4.31}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N) < r$.

Since $0 < r < 1/(2e)$, it follows that

$$1 \le \sum_{l=1}^{N}\ln\frac{1}{1-|\varphi_l(z)|^2} \tag{4.32}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N) < r$. So, it follows from (4.30) and (4.32) that

$$\sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y} < \varepsilon \tag{4.33}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N) < r$. From (3.7), Lemma 2.1, (4.30), (4.31), and (4.33), it follows that

$$\begin{aligned}
b_{W_{u,\varphi}f_j}(z) \le{}& C\|f_j\|_{\mathcal{B}(\mathbb{D}^N,X)}\sum_{k,l=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}(1-|z_k|^2)^q\ln\frac{e}{1-|\varphi_l(z)|^2}\\
&+ C\|f_j\|_{\mathcal{B}(\mathbb{D}^N,X)}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|\frac{(1-|z_k|^2)^q}{1-|\varphi_l(z)|^2}\\
\le{}& (N+2)C\varepsilon\|f_j\|_{\mathcal{B}(\mathbb{D}^N,X)} \le (N+2)CM\varepsilon
\end{aligned} \tag{4.34}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N) < r$.

Let

$$G_r = \{z \in \mathbb{D}^N : \operatorname{dist}(z,\partial\mathbb{D}^N) \ge r\}.$$

Clearly, G_r is a compact subset of $\mathbb{D}^N$. Hence $f_j \to 0$ uniformly on G_r as $j \to \infty$. By Cauchy's estimate it follows that $\frac{\partial f_j}{\partial z_k} \to 0$ uniformly on G_r as $j \to \infty$ for each $k \in \{1,\dots,N\}$. Since the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N,X) \to \mathcal{B}^q(\mathbb{D}^N,Y)$ is bounded, it follows that $u \in \mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))$. On the other hand, by Theorem 3.1 we have that condition (3.3) holds. By using these facts, we obtain

$$b_{W_{u,\varphi}f_j}(z) \le \sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}(1-|z_k|^2)^q\|f_j(\varphi(z))\|_X$$

$$+\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big\|\frac{\partial f_j}{\partial\zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q$$

$$\le\|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))}\sup_{w\in G_r}\|f_j(w)\|_X$$

$$+\Big(\sup_{z\in\mathbb{D}^N}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|\frac{(1-|z_k|^2)^q}{1-|\varphi_k(z)|^2}\Big)\max_{l=\overline{1,N}}\sup_{w\in G_r}\Big\|\frac{\partial f_j}{\partial\zeta_l}(w)\Big\|_X\to 0, \tag{4.35}$$

as $j\to\infty$, whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N)\ge r$. Since $u(0)f_j(\varphi(0))\to 0$ as $j\to\infty$, combining (4.34) and (4.35) we have

$$\lim_{j\to\infty}\|W_{u,\varphi}f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}=0,$$

which shows that the operator $W_{u,\varphi}:\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)$ is compact.

Next, we prove the sufficiency of (ii). First assume that conditions (4.3) and (4.4) hold. We want to prove that the operator $W_{u,\varphi}:\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)$ is compact. Assume that a sequence $\{f_j\}$ is such that $\sup_{j\in\mathbb{N}}\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\le M$ and $\{f_j\}$ converges to the zero vector in X uniformly on compact subsets of $\mathbb{D}^N$. We need to prove that $\|W_{u,\varphi}f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}\to 0$ as $j\to\infty$. From (4.3) and (4.4) it follows that for every $\varepsilon>0$ there exists an r $(0<r<1))$ such that

$$\sum_{k,l=1}^{N}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}<\varepsilon \tag{4.36}$$

and

$$\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\frac{(1-|z_l|^2)^q}{(1-|\varphi_k(z)|^2)^p}\Big|\frac{\partial\varphi_k}{\partial z_l}(z)\Big|<\varepsilon, \tag{4.37}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N)<r$. Hence, from (3.7), Lemma 2.1, (4.36), and (4.37), it follows that for sufficiently large j

$$b_{W_{u,\varphi}f_j}(z)\le C\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\sum_{k,l=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}}$$

$$+C\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p}$$

$$\le 2C\varepsilon\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\le 2CM\varepsilon \tag{4.38}$$

whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N)<r$.

As the proof of (i), we see that $f_j\to 0$ uniformly on G_r as $j\to\infty$. By Cauchy's estimate it follows that $\frac{\partial f_j}{\partial z_k}\to 0$ uniformly on G_r as $j\to\infty$ for each $k\in\{1,\dots,N\}$. Since the operator $W_{u,\varphi}:\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)$ is bounded, it follows that $u\in\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))$. On the other hand, by Theorem 3.1 we have that condition (3.5) holds. By using these facts, we obtain

$$b_{W_{u,\varphi}f_j}(z)\le\sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}(1-|z_k|^2)^q\|f_j(\varphi(z))\|_X$$

$$
\begin{aligned}
&+\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q\\
&\le \|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))}\sup_{w\in G_r}\|f_j(w)\|_X\\
&+\Big(\sup_{z\in\mathbb{D}^N}\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|\frac{(1-|z_k|^2)^q}{(1-|\varphi_k(z)|^2)^p}\Big)\max_{l=\overline{1,N}}\sup_{w\in G_r}\Big\|\frac{\partial f_j}{\partial \zeta_l}(w)\Big\|_X\to 0,
\end{aligned}
\tag{4.39}
$$

as $j\to\infty$, whenever $\operatorname{dist}(\varphi(z),\partial\mathbb{D}^N)\ge r$. Since $u(0)f_j(\varphi(0))\to 0$ as $j\to\infty$, combining (4.38) and (4.39) we have

$$
\lim_{j\to\infty}\|W_{u,\varphi}f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}=0,
$$

which shows that the operator $W_{u,\varphi}:\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)$ is compact.

Finally, we prove (iii). Sufficiency. First assume that $W_{u,\varphi}:\mathcal{B}^p(\mathbb{D}^N,X)\to\mathcal{B}^q(\mathbb{D}^N,Y)$ is bounded and (4.5) holds. Assume $\{f_j\}$ is a bounded sequence in $\mathcal{B}^p(\mathbb{D}^N,X)$, say $\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\le M$, and $f_j(z)\to 0$ uniformly on compact subsets of $\mathbb{D}^N$. We prove that $\|W_{u,\varphi}f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}\to 0$ as $j\to\infty$.

Note that for $f_l(z)=z_lx\in\mathcal{B}^p(\mathbb{D}^N,X)$, $l\in\{1,\dots,N\}$, where x is a unit vector in X, we obtain

$$
\sup_{z\in\mathbb{D}^N}\sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)(x)\varphi_l(z)+u(z)(x)\frac{\partial \varphi_l}{\partial z_k}(z)\Big\|_Y:=\widehat{M}<+\infty,
$$

and consequently for each $l\in\{1,\dots,N\}$

$$
\begin{aligned}
\sup_{z\in\mathbb{D}^N}\|u(z)(x)\|_Y\sum_{k=1}^{N}(1-|z_k|^2)^q\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|
&\le \sup_{z\in\mathbb{D}^N}\Big(\sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)(x)\Big\|_Y\Big)\|\varphi_l\|_\infty+M\\
&\le \sup_{z\in\mathbb{D}^N}\Big(\sum_{k=1}^{N}(1-|z_k|^2)^q\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}\Big)\|\varphi_l\|_\infty+M\\
&<+\infty,
\end{aligned}
\tag{4.40}
$$

since $u\in\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))$, which implies that

$$
\sup_{z\in\mathbb{D}^N}\|u(z)\|_{X\to Y}\sum_{k=1}^{N}(1-|z_k|^2)^q\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|<+\infty.
\tag{4.41}
$$

From (4.5), we have some $\rho\in(0,1)$ such that for all $k,l\in\{1,\dots,N\}$

$$
\|u(z)\|_{X\to Y}\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|<\varepsilon,
\tag{4.42}
$$

if $|\varphi_l(z)|>\rho$.

Since for each $k\in\{1,\dots,N\}$

$$
b_{W_{u,\varphi}f_j}(z)\le\sum_{k=1}^{N}\Big\|\frac{\partial u}{\partial z_k}(z)\Big\|_{X\to Y}(1-|z_k|^2)^q\|f_j(\varphi(z))\|_X
$$

$$+ \|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q$$

and $u \in \mathcal{B}^q(\mathbb{D}^N, \mathcal{L}(X, Y))$, Since from Lemma 2.5 it follows that $\|f_j(\varphi(z))\|_X \to 0$ as $j \to \infty$, it is enough to estimate the expression

$$\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q$$

for each $l \in \{1, \dots, N\}$.

From (4.42) and $\|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)} \le M$, we have

$$\begin{aligned}&\|u(z)\|_{X\to Y}\sum_{k,l=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q\\ &\le \|f_j\|_{\mathcal{B}^p(\mathbb{D}^N,X)}\sum_{k=1}^{N}\|u(z)\|_{X\to Y}\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|\frac{(1-|z_k|^2)^q}{(1-|\varphi_k(z)|^2)^p} \le 2NM\varepsilon,\end{aligned} \tag{4.43}$$

if $|\varphi_l(z)| > \rho$, for each $l \in \{1, \dots, N\}$.

On the other hand, from (4.41) and (4.43) we have that for each $l \in \{1, \dots, N\}$

$$\sup_{|\varphi_l(z)|\le\rho}\|u(z)\|_{X\to Y}\sum_{k=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q \le C\sup_{|\varphi_l(z)|\le\rho}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X. \tag{4.44}$$

By Cauchy's estimate applied to the function $g(z_l) = f(z_1, \dots, z_N)$ of the one variable, if $|z_l| \le \rho$, we have

$$\Big\|\frac{\partial f}{\partial z_l}(z)\Big\|_X \le C\sup_{|z_l|\le(1+\rho)/2}\|f(z)\|_X \le C\sup_{z\in\mathbb{D}^N}\|f(z)\|_X \tag{4.45}$$

for some positive constant C independent of f. By applying (4.45) to f_j, and from (4.44), we have

$$\sup_{|\varphi_l(z)|\le\rho}\|u(z)\|_{X\to Y}\sum_{k=1}^{N}\Big\|\frac{\partial f_j}{\partial \zeta_l}(\varphi(z))\Big\|_X\Big|\frac{\partial \varphi_l}{\partial z_k}(z)\Big|(1-|z_k|^2)^q \to 0, \tag{4.46}$$

as $j \to \infty$, for each $l \in \{1, \dots, N\}$. From (4.43) and (4.46), since $\|u(0)f_j(\varphi(0))\|_X \to 0$ as $j \to \infty$, it follows that $\|W_{u,\varphi}f_j\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0$ as $j \to \infty$, as desired.

Necessity. Now suppose that the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is compact. Since the operator $W_{u,\varphi} : \mathcal{B}^p(\mathbb{D}^N, X) \to \mathcal{B}^q(\mathbb{D}^N, Y)$ is bounded, from Theorem 3.1 we know that (3.1) holds. Now we prove that condition (4.5) holds for each $l \in \{1, \dots, N\}$. Let $\{z^j\}$ be a sequence in $\mathbb{D}^N$ and $w^j = \varphi(z^j) = (w_1^j, \dots, w_N^j)$ such that $|w_l^j| \to 1$ as $j \to \infty$.

Let

$$h_j(z) = \frac{1-|w_l^j|^2}{(1-\overline{w_l^j}z_l)^p}.$$

It is easy to see that $h_j \in \mathcal{B}^p(\mathbb{D}^N)$ and $\|h_j\|_{\mathcal{B}^p(\mathbb{D}^N)} \leq 1 + 2^{p+1}p$. Let x be a unit vector in X. By using the functions h_j, we define the functions $h_{j,x}(z) = h_j(z)x$. Then $\{h_{j,x}\}$ is a bounded sequence in $\mathcal{B}^p(\mathbb{D}^N, X)$ and $\{h_{j,x}\}$ converges to the zero vector uniformly on compact subsets of $\mathbb{D}^N$ as $j \to \infty$. Therefore, by Lemma 2.3, $\|W_{u,\varphi}h_{j,x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0$ as $j \to \infty$, and consequently as $j \to \infty$

$$\sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big\|u(z^j)(x)\sum_{l=1}^{N}\frac{\partial h_j}{\partial \zeta_l}(\varphi(z^j))\frac{\partial \varphi_l}{\partial z_k}(z^j)+h_j(\varphi(z^j))\frac{\partial u}{\partial z_k}(z^j)(x)\Big\|_Y \leq \|W_{u,\varphi}h_{j,x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0. \tag{4.47}$$

Hence

$$\begin{aligned}
&\|u(z^j)(x)\|_Y\sum_{k=1}^{N}\frac{(1-|z_k^j|^2)^q}{(1-|w_l^j|^2)|)^p}\Big|\frac{\partial \varphi_l}{\partial z_k}(z^j)\Big|\\
&\leq \frac{1}{p|w_l^j|}(1-|w_l^j|^2)^{1-p}\sum_{k=1}^{N}(1-|z_k^j|^2)^q\Big\|\frac{\partial u_{z^j}}{\partial z_k}(x)\Big\|+\frac{1}{p|w_l^j|}\|W_{u,\varphi}h_{j,x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)}\\
&\leq \frac{1}{p|w_l^j|}(1-|w_l^j|^2)^{1-p}\|u\|_{\mathcal{B}^q(\mathbb{D}^N,\mathcal{L}(X,Y))}+\frac{1}{p|w_l^j|}\|W_{u,\varphi}h_{j,x}\|_{\mathcal{B}^q(\mathbb{D}^N,Y)} \to 0,
\end{aligned}$$

as $j \to \infty$. The proof is completed. □

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no competing interests.

References

1. R. F. Allen, F. Colonna, Weighted composition operators on the Bloch space of a bounded homogeneous domain, In: *Topics in operator theory*, Basel: Birkhäuser, 2010, 11–37. https://doi.org/10.1007/978-3-0346-0158-0_2
2. J. A. Cima, The basic properties of Bloch functions, *Int. J. Math. Math. Sci.*, **2** (1979), 369–413. https://doi.org/10.1155/S0161171279000314
3. C. C. Cowen, B. D. Maccluer, *Composition operators on spaces of analytic functions*, New York: CRC Press, 1995. https://doi.org/10.1201/9781315139920

4. K. Esmaeili, H. Mahyar, Weighted composition operators between vector-valued Bloch-type spaces, *Turk. J. Math.*, **43** (2019), 151–171. https://doi.org/10.3906/mat-1711-91
5. Z.-S. Fang, Z.-H. Zhou, New characterizations of the weighted composition operators between Bloch type spaces in the polydisk, *Can. Math. Bull.*, **57** (2014), 794–802. http://doi.org/10.4153/CMB-2013-043-4
6. F. Forelli, The isometries of H^p, *Can. J. Math.*, **16** (1964), 721–728. https://doi.org/10.4153/CJM-1964-068-3
7. Z. J. Hu, Composition operators between Bloch-type spaces in polydisk, *Science in China. Series A, Mathematics, Physics, Astronomy*, **48** (2005), 268–282. https://doi.org/10.1360/ya2005-V48-IS1-P268-282
8. O. Hyv̈arinen, I. Nieminen, Weighted composition followed by differentiation between Bloch-type spaces, *Rev. Mat. Complut.*, **27** (2014), 641–656. https://doi.org/10.1007/s13163-013-0138-y
9. O. Hyv̈arinen, M. Lindström, Estimates of essential norms of weighted composition operators between Bloch-type spaces, *J. Math. Anal. Appl.*, **393** (2012), 38–44. https://doi.org/10.1016/j.jmaa.2012.03.059
10. S. G. Krantz, D. Ma, Bloch functions on strongly pseudoconvex domains, *Indiana Univ. Math. J.*, **37** (1988), 145–163. https://doi.org/10.1512/iumj.1988.37.37007
11. J. Laitila, Weakly compact composition operators on vector-valued BMOA, *J. Math. Anal. Appl.*, **308** (2005), 730–745. https://doi.org/10.1016/j.jmaa.2004.12.002
12. J. Laitila, H.-O. Tylli, Operator-weighted composition operators on vector-valued analytic function spaces, *Illinois J. Math.*, **53** (2009), 1019–1032. https://doi.org/10.1215/ijm/1290435336
13. J. Laitila, H.-O. Tylli, M. Wang, Composition operators from weak to strong spaces of vector-valued analytic functions. *J. Operat. Theor.*, **62** (2009), 281–295.
14. J. S. Manhas, R. Zhao, New estimates of essential norms of weighted composition operators between Bloch type spaces, *J. Math. Anal. Appl.*, **389** (2012), 32–47. https://doi.org/10.1016/j.jmaa.2011.11.039
15. B. D. Maccluer, R. Zhao, Essential norm of weighted composition operators between Bloch-type spaces, *Rocky Mountain J. Math.*, **33** (2003), 1437–1458. https://doi.org/10.1216/rmjm/1181075473
16. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Berlin: Springer, 2008. https://doi.org/10.1007/978-3-540-68276-9
17. S. Stević, R. Chen, Z. Zhou, Weighted composition operators between Bloch-type spaces in the polydisk, *Sb. Math.*, **201** (2010), 289–319. https://doi.org/10.1070/SM2010v201n02ABEH004073
18. H. Xu, T. Liu, Weighted composition operators between different Bloch-type spaces on the unit polydisk, (Chinese), *Chinese Annals of Mathematics Series B*, **26A** (2005), 61–72.
19. K. H. Zhu, *Spaces of holomorphic functions in the unit ball*, New York: Springer, 2005. https://doi.org/10.1007/0-387-27539-8