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*Research article*

# Duality of codes over non-unital rings of order six

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**Abstract:** Some basic theory on the duality of codes over two non-unital rings of order 6, namely $H_{23}$ and $H_{32}$ is presented. For a code $C$ over these rings, there is an associated binary code $C_a$ and a ternary code $C_b$. Self-orthogonal, self-dual, and quasi self-dual (QSD) codes over these rings are characterized using the associated codes $C_a$ and $C_b$, and a classification of self-orthogonal codes for short lengths is given. In addition, a building-up construction for self-orthogonal codes is presented, and cyclic and linear complementary dual (LCD) codes over the said rings are introduced.

**Keywords:** non-unital rings; self-orthogonal codes; self-dual codes; quasi self-dual codes; linear complementary dual; cyclic codes
**Mathematics Subject Classification:** 16D10, 94B05

## 1. Introduction

Coding theory was studied classically using finite fields as alphabets. However, for the last three decades, codes over rings have been used along with finite fields. In this case, linear codes are defined as modules over a ring. Most research on codes over rings takes commutative rings with unity as alphabet (see [1–3], for example). Recently, there has been interest in codes over non-unital rings (see [4–6]). One ring in particular is the non-local ring $H$ of order 4 (see [7–9]). Linear codes with complementary duals (LCD codes) and cyclic codes have attracted considerable attention due to their strong algebraic structures and practical applications in communication systems and cryptography. The study of LCD codes over non-unital rings was initiated in [10]. Similarly, the investigation of cyclic codes over non-unital rings began with the work in [11, 12], which extended the theory of cyclicity beyond traditional finite fields and unital rings. Finally, the building-up construction has proven to be an effective technique for constructing linear codes with prescribed algebraic and distance properties. The first systematic study of this method over non-unital rings was carried out in [13, 14]. The goal of this paper is to lay down the foundations of the study of duality of linear codes over two non-unital rings of order 6, denoted by $H_{23}$ and $H_{32}$ as in [15], whose properties are similar to the ring $H$. These rings are interesting because a lot of new codes can be constructed over them as alphabets. These codes also are in close connection to widely studied binary and ternary codes. For every code $C$ over $H_{23}$ or $H_{32}$, there is an associated binary code $C_a$ and a ternary code $C_b$. Thus, the properties of the code $C$ can be determined by studying the associated codes $C_a$ and $C_b$. In this paper, the conditions for self-orthogonal, self-dual, quasi self-dual, and LCD codes will be explored. Moreover, a method to construct self-orthogonal codes of longer length from codes of shorter length will be given, cyclic codes over these rings will be introduced, and a classification of self-orthogonal codes for short lengths will be presented.

This paper is organized as follows: Some basic concepts regarding linear codes over finite fields are first reviewed in Section 2. In Section 3, the rings $H_{23}$ and $H_{32}$ are defined, and their properties are explained. Then in Section 4, codes over these rings are introduced, and the notion of duality is explored. Self-orthogonal, self-dual, quasi self-dual, and LCD codes are characterized in terms of their associated binary and ternary codes. Cyclic codes over these rings are also studied. This is followed by a building-up construction, i.e., a method to construct codes from codes of smaller length in Section 5. Finally, in Section 6, a classification of self-orthogonal codes of lengths up to 7 is given before the paper is concluded in the last section.

## 2. Codes over finite fields

An $[n, k]$ linear code or an $[n, k]$-code $C$ of length $n$ and dimension $k$ is a subspace of $\mathbb{F}_q^n$ where $q$ is a prime or a power of a prime. The elements of a code are called codewords. If $q = 2$, then $C$ is called a binary code. If $q = 3$, then $C$ is called a ternary code. The (Hamming) weight $wt(\mathbf{c})$ of $\mathbf{c} \in \mathbb{F}_q^n$ is the number of nonzero coordinates in $\mathbf{c}$, and the (Hamming) distance between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{F}_q^n$, denoted by $d(\mathbf{x}, \mathbf{y})$, is the number of coordinates where $\mathbf{x}$ and $\mathbf{y}$ differ. The minimum (Hamming) distance $d(C)$ of a linear code $C$ is given by $d(C) = \min\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\} = \min\{wt(\mathbf{c}) \mid \mathbf{0} \neq \mathbf{c} \in C\}$.

We equip $\mathbb{F}_q^n$ with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$ for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$. The dual $C^\perp$ of $C$ is an $[n, n-k]$-code defined as

$$C^\perp = \{\mathbf{y} \in \mathbb{F}_q^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{x} \in C\}.$$

A linear code $C$ is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. A linear code $C$ is linear with complementary dual (LCD) if $C \cap C^\perp = \{\mathbf{0}\}$. A linear code $C$ is cyclic provided that for each codeword $\mathbf{c} = (c_1, ..., c_n) \in C$, the vector $(c_n, c_1, ..., c_{n-1})$ is also a codeword in $C$.

## 3. The rings $H_{23}$ and $H_{32}$

Consider the rings $H_{23}$ and $H_{32}$, both of order 6, given by the following presentation.

$$H_{23} = \langle a, b \mid 2a = 0, 3b = 0, a^2 = a, b^2 = 0, ab = 0 = ba \rangle,$$

and

$$H_{32} = \langle a, b \mid 2a = 0, 3b = 0, a^2 = 0, b^2 = b, ab = 0 = ba \rangle.$$

The other elements are defined as

$$\begin{aligned} c &:= a+b, \\ d &:= 2b, \\ e &:= a+2b. \end{aligned}$$

Tables 1 and 2 are the addition table for both rings and the multiplication tables for rings $H_{23}$ and $H_{32}$, respectively.

**Table 1.** Addition table for $H_{23}$ and $H_{32}$.

| $+$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | $a$ | 0 | $c$ | $b$ | $e$ | $d$ |
| $b$ | $b$ | $c$ | $d$ | $e$ | 0 | $a$ |
| $c$ | $c$ | $b$ | $e$ | $d$ | $a$ | 0 |
| $d$ | $d$ | $e$ | 0 | $a$ | $b$ | $c$ |
| $e$ | $e$ | $d$ | $a$ | 0 | $c$ | $b$ |

**Table 2.** Multiplication table for $H_{23}$ and $H_{32}$, respectively.

| $\cdot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | $a$ | 0 | $a$ |
| $b$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $c$ | 0 | $a$ | 0 | $a$ | 0 | $a$ |
| $d$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $e$ | 0 | $a$ | 0 | $a$ | 0 | $a$ |

| $\cdot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | 0 | 0 | $b$ | $b$ | $d$ | $d$ |
| $c$ | 0 | 0 | $b$ | $b$ | $d$ | $d$ |
| $d$ | 0 | 0 | $d$ | $d$ | $b$ | $b$ |
| $e$ | 0 | 0 | $d$ | $d$ | $b$ | $b$ |

From these tables, it is evident that both rings are commutative and without unity. They have two maximal ideals, $J_a = \{0, a\}$ and $J_b = \{0, b, d\}$, and hence, $H_{23}$ and $H_{32}$ are semi-local rings. From the definitions, it is easy to verify that either of the two rings can be written as the direct sum $J_a \oplus J_b$.

Define a natural action of $\mathbb{F}_2$ on $J_a$ by the rule $r0 = 0r = 0$ and $r1 = 1r = r$, for all $r \in J_a$. This action is distributive in the sense that $r(s \oplus_2 t) = rs + rt$, where $\oplus_2$ denotes the addition in $\mathbb{F}_2$, $s, t \in \mathbb{F}_2$. A similar action of $\mathbb{F}_3$ on $J_b$ can also be defined. These actions extend in the following manner. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$, define $\alpha\mathbf{x} = (\alpha x_1, \ldots, \alpha x_n) \in H_{23}^n$ for $\alpha \in J_a$. Similarly, if $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_3^n$, define $\beta\mathbf{x} = (\beta x_1, \ldots, \beta x_n) \in H_{32}^n$ for $\beta \in J_b$.

## 4. Duality of codes over $H_{23}$ and $H_{32}$

Let $z \in \{23, 32\}$. A linear code $C$ over $H_z$, or simply an $H_z$-code $C$, of length $n$ is an $H_z$-submodule of the $H_z^n$ module. Since $H_z = J_a \oplus J_b$, an $H_z$-code $C$ can be written as the direct sum $C = aC_a \oplus bC_b$, where $C_a$ is a binary code and $C_b$ is a ternary code. The (Hamming) weight of $\mathbf{c} \in H_z^n$, the (Hamming) distance between $\mathbf{x}$ and $\mathbf{y}$ in $H_z^n$, and the minimum (Hamming) distance of an $H_z$-code are defined similarly as in Section 2.

We also take the standard inner product on $H_z^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

for all $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in H_z^n$. The dual of an $H_z$-code $C$ is the submodule of $H_z^n$ defined by

$$C^\perp = \{\mathbf{y} \in H_z^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{x} \in C\}.$$

As in codes over finite fields, an $H_z$-code $C$ is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. In addition, $C$ is said to be nice if $|C||C^\perp| = 6^n$. If $C$ is a self-orthogonal code with $|C| = 6^{\frac{n}{2}}$, then $C$ is called a quasi self-dual (QSD) code.

**Theorem 4.1.** *If $C = aC_a \oplus bC_b$ is an $H_z$-code where $C_a$ is a nonzero binary code of minimum distance $d_1$ and $C_b$ is a nonzero ternary code of minimum distance $d_2$, then the minimum distance of $C$ is $d = \min\{d_1, d_2\}$.*

*Proof.* Let $\mathbf{u}_1$ be a nonzero codeword of $C_a$ of weight $d_1$ and $\mathbf{u}_2$ be a nonzero codeword of $C_b$ of weight $d_2$. Since $aC_a \subseteq C$ and $bC_b \subseteq C$, this implies that, $a\mathbf{u}_1, b\mathbf{u}_2 \in C$ with $wt(a\mathbf{u}_1) = wt(\mathbf{u}_1)$ and $wt(b\mathbf{u}_2) = wt(\mathbf{u}_2)$. Therefore, $d \leq \min\{d_1, d_2\}$. Conversely, let $\mathbf{u}$ be a nonzero codeword of $C$ of weight $d$. Then, $\mathbf{u} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in C_a$ and $\mathbf{y} \in C_b$. Since $\mathbf{u} \neq 0$, then one case of the following is true:

- If $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} \neq 0$, then $d = wt(\mathbf{u}) = wt(b\mathbf{y}) = wt(\mathbf{y}) \geq d_2$.
- If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = 0$, then $d = wt(\mathbf{u}) = wt(a\mathbf{x}) = wt(\mathbf{x}) \geq d_1$.
- If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq 0$, then $d = wt(\mathbf{u}) \geq wt(a\mathbf{y}) = wt(\mathbf{y}) \geq d_2$ and $d = wt(\mathbf{u}) \geq wt(a\mathbf{x}) = wt(\mathbf{x}) \geq d_1$.

For all three cases, we have

$$d = wt(\mathbf{u}) \geq \min\{d_1, d_2\},$$

and the result follows. □

**Theorem 4.2.** *Let $C = aC_a \oplus bC_b$ be an $H_z$-code of length $n$. Then*

*(i) for $z = 23$, $C^\perp = aC_a^\perp \oplus b\mathbb{F}_3^n$. Furthermore, $(C^\perp)^\perp = C$ if and only if $C_b = \mathbb{F}_3^n$;*
*(ii) for $z = 32$, $C^\perp = a\mathbb{F}_2^n \oplus bC_b^\perp$. Furthermore, $(C^\perp)^\perp = C$ if and only if $C_a = \mathbb{F}_2^n$.*

*Proof.* Let $C$ be an $H_{23}$-code, and suppose $\mathbf{u} \in C^\perp$. Write $\mathbf{u} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in \mathbb{F}_2^n$ and $\mathbf{y} \in \mathbb{F}_3^n$. If $\mathbf{v} \in C_a$, then $a\mathbf{v} \in C$. Thus,

$$0 = \langle \mathbf{u}, a\mathbf{v} \rangle = \langle a\mathbf{x} + b\mathbf{y}, a\,\mathbf{v} \rangle = a\langle \mathbf{v}, \mathbf{x} \rangle.$$

Since $a \neq 0$, we have $\langle \mathbf{v}, \mathbf{x} \rangle = 0$, i.e., $\mathbf{x} \in C_a^\perp$ and so $\mathbf{u} \in aC_a^\perp + b\mathbb{F}_3^n$. Consequently, $C^\perp \subseteq aC_a^\perp + b\mathbb{F}_3^n$.

Now, suppose $\mathbf{c} \in aC_a^\perp + b\mathbb{F}_3^n$, that is, $\mathbf{c} = a\mathbf{x} + b\mathbf{y}$, where $\mathbf{x} \in C_a^\perp$ and $\mathbf{y} \in \mathbb{F}_3^n$. For any $\mathbf{u} = a\mathbf{r} + b\mathbf{t} \in C$, where $\mathbf{r} \in C_a, \mathbf{t} \in C_b$,

$$\langle \mathbf{c}, \mathbf{u} \rangle = \langle a\mathbf{x} + b\mathbf{y}, a\mathbf{r} + b\mathbf{t} \rangle = a\langle \mathbf{x}, \mathbf{r} \rangle = a \cdot 0 = 0,$$

which means $\mathbf{c} \in C^\perp$ and hence, $aC_a^\perp + b\mathbb{F}_3^n \subseteq C^\perp$. This shows $C^\perp = aC_a^\perp + b\mathbb{F}_3^n$. Clearly, the intersection of $aC_a$ and $b\mathbb{F}_3^n$ is trivial, so the sum is direct. Finally, since $(C^\perp)^\perp = (aC_a^\perp \oplus b\mathbb{F}_3^n)^\perp = aC_a \oplus b\mathbb{F}_3^n$, it follows that $(C^\perp)^\perp = C$ if and only if $C_b = \mathbb{F}_3^n$, which proves $(i)$. The case in which $C$ is an $H_{32}$ code is proved similarly. □

**Corollary 4.3.** *Let $C = a\,C_a \oplus b\,C_b$ be an $H_z$-code of length $n$. Then $C$ is nice if and only if*

*(i)* $C_b = \{\mathbf{0}\}$ *for* $z = 23$.
*(ii)* $C_a = \{\mathbf{0}\}$ *for* $z = 32$.

*Proof.* Let $C$ be an $H_{23}$ code and $k_a = \dim(C_a)$ and $k_b = \dim(C_b)$. Then $|C| = 2^{k_a}3^{k_b}$ and by Theorem 4.2 (i), $|C^{\perp}| = 2^{n-k_a}3^n$. Thus,

$$|C||C^{\perp}| = 2^n\,3^{n+k_b}.$$

Therefore, $C$ is nice if and only if $k_b = 0$.

Similarly, (*ii*) follows from Theorem 4.2 (*ii*). □

### *4.1. Self-orthogonal, self-dual and QSD codes*

In this section, the conditions so that a code $C$ over $H_{23}$ or $H_{32}$ is self-orthogonal, self-dual, and quasi self-dual are determined.

**Theorem 4.4.** *Let $C = a\,C_a \oplus b\,C_b$ be an $H_z$-code.*

*(i) If $z = 23$, then $C$ is self-orthogonal if and only if $C_a$ is a binary self-orthogonal code.*
*(ii) If $z = 32$, then $C$ is self-orthogonal if and only if $C_b$ is a ternary self-orthogonal code.*

*Proof.* Let $\mathbf{c}_1 = a\mathbf{x}_1 + b\mathbf{y}_1$ and $\mathbf{c}_2 = a\mathbf{x}_2 + b\mathbf{y}_2$ be arbitrary codewords in $C$ where $\mathbf{x}_1$, $\mathbf{x}_2 \in C_a$ and $\mathbf{y}_1$, $\mathbf{y}_2 \in C_b$. Then

$$\langle \mathbf{c}_1, \mathbf{c}_2 \rangle = a^2\langle \mathbf{x}_1, \mathbf{x}_2 \rangle + b^2\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = a\langle \mathbf{x}_1, \mathbf{x}_2 \rangle.$$

Since $a \neq 0$, it can be concluded that $C$ is self-orthogonal if and only if $C_a$ is a binary self-orthogonal code.

On the other hand, $a^2 = 0$ and $b^2 = b$ in $H_{32}$, so (ii) follows. □

**Theorem 4.5.** *Let $C_1$ be a binary linear code of length $n$ and $C_2$ be a ternary linear code of length $n$. The code $C$ defined by*

$$C = a\,C_1 \oplus b\,C_2$$

*is a self-orthogonal $H_z$-code of length $n$ where $C_a = C_1$ and $C_b = C_2$ if the following conditions are satisfied:*

*(i) For $z = 23$, $C_1$ is a binary self-orthogonal code.*
*(ii) For $z = 32$, $C_2$ is a ternary self-orthogonal.*

*Moreover, if $|C_1||C_2| = 6^{n/2}$, then $C$ is QSD.*

*Proof.* Let $z = 23$. By the linearity of $C_1$ and $C_2$, the code $C = a\,C_1 \oplus b\,C_2$ is closed under addition. For the scalar multiplication, let $\mathbf{c} = a\mathbf{x} + b\mathbf{y} \in C$ where $\mathbf{x} \in C_1$ and $\mathbf{y} \in C_2$. Then we have the following:

- $a(a\mathbf{x} + b\mathbf{y}) = a^2\mathbf{x} = a\mathbf{x} = a\mathbf{x} + b\mathbf{0} \in C$;
- $b(a\mathbf{x} + b\mathbf{y}) = \mathbf{0} = a\mathbf{0} + b\mathbf{0} \in C$.

Since $C$ is closed under addition, for any $r = ax + by \in H_{23}$ where $x \in \mathbb{F}_2$ and $y \in \mathbb{F}_3$, we have $rC \subseteq C$. Hence, $C$ is a linear $H_{23}$-code. To prove the self-orthogonality of $C$, for all $a\mathbf{x}_1 + b\mathbf{y}_1, a\mathbf{x}_2 + b\mathbf{y}_2 \in C$ where $\mathbf{x}_1, \mathbf{x}_2 \in C_1$ and $\mathbf{y}_1, \mathbf{y}_2 \in C_2$, we have

$$\langle a\mathbf{x}_1 + b\mathbf{y}_1, a\mathbf{x}_2 + b\mathbf{y}_2\rangle = a^2\langle \mathbf{x}_1, \mathbf{x}_2\rangle + b^2\langle \mathbf{y}_1, \mathbf{y}_2\rangle = a\langle \mathbf{x}_1, \mathbf{x}_2\rangle = 0,$$

since $C_1$ is a binary self-orthogonal code.

Statement (*ii*) is proved similarly.

Furthermore, the last statement holds since $|C| = |C_1||C_2|$. □

**Corollary 4.6.** *Let $C = aC_a \oplus bC_b$ be an $H_z$-code of even length $n$. Then $C$ is a QSD code if and only if the following conditions are satisfied:*

*(i) For $z = 23$, $C_a$ is a binary self-dual code, and $C_b$ is a ternary $[n, n/2]$ code.*
*(ii) For $z = 32$, $C_a$ is a binary $[n, n/2]$ code, and $C_b$ is a ternary self-dual code.*

*Proof.* The result follows from [15, Lemmas 1 and 2], Theorem 4.4, Theorem 4.5, and the definition of QSD codes. □

**Theorem 4.7.** *Let $C = aC_a \oplus bC_b$ be an $H_z$-code of even length $n$. Then $C$ is self-dual if and only if the following conditions are satisfied:*

*(i) For $z = 23$, $C_a$ is a binary self-dual code and $C_b = \mathbb{F}_3^n$.*
*(ii) For $z = 32$, $C_a = \mathbb{F}_2^n$ and $C_b$ is a ternary self-dual code.*

*Proof.* Let $C$ be an $H_{23}$-code. From the definition of self-dual codes and Theorem 4.2 (*i*), we have

$$aC_a \oplus bC_b = C = C^\perp = aC_a^\perp \oplus b\mathbb{F}_3^n.$$

Therefore, $C$ is self-dual if and only if $C_a = C_a^\perp$ and $C_b = \mathbb{F}_3^n$. Similarly, (*ii*) follows from the definition of self-dual codes and Theorem 4.2 (*ii*). □

Observe that by Corollary 4.6 and Theorem 4.7, QSD and self-dual $H_{23}$-codes have even lengths, while QSD and self-dual $H_{32}$-codes have lengths divisible by 4 since $C_b$ must be ternary self-dual codes.

The next result shows that any $H_{23}$-code or $H_{32}$-code cannot be both QSD and self-dual at the same time.

**Proposition 4.8.** *Let $\mathcal{Q}$ be the set of all QSD $H_{23}$-codes of length $n$ and $\mathcal{S}$ be the set of all self-dual $H_{23}$-codes of length $n$. Then $\mathcal{Q} \cap \mathcal{S} = \emptyset$.*

*Proof.* Suppose $C$ is an $H_{23}$-code of length $n$ that is both QSD and self-dual. Then $|C| = |C^\perp| = 6^{n/2}$. Therefore, $|C||C^\perp| = 6^n$, so $C$ is a nice code. However, by Corollary 4.3, $C_b = \{\mathbf{0}\}$, which contradicts Corollary 4.6. Thus, $\mathcal{Q} \cap \mathcal{S} = \emptyset$.

The same can be said for $H_{32}$ codes.

### 4.2. Cyclic codes

Similar to the definition of cyclic codes over finite fields, a cyclic $H_z$-code $C$ of length $n$ is a linear code having the property that if $(c_1, \ldots, c_n) \in C$, then its cyclic shift $(c_n, c_1, \ldots, c_{n-1})$ is also a codeword in $C$.

**Theorem 4.9.** *Let $C = a\,C_a \oplus b\,C_b$ be an $H_z$-code of length $n$. Then $C$ is a cyclic code if and only if $C_a$ and $C_b$ are both cyclic.*

*Proof.* Suppose $C$ is a cyclic $H_z$-code of length $n$ and let $(x_1, \ldots, x_n) \in C_a$ and $(y_1, \ldots, y_n) \in C_b$. Then $(ax_1 + by_1, \ldots, ax_n + by_n) \in C$ and

$$a(x_n, x_1, \ldots, x_{n-1}) + b(y_n, y_1, \ldots, y_{n-1}) = (ax_n + by_n, ax_1 + by_1, \ldots, ax_{n-1} + by_{n-1}) \in C.$$

Therefore, $(x_n, x_1, \ldots, x_{n-1}) \in C_a$ and $(y_n, y_1, \ldots, y_{n-1}) \in C_b$, i.e., $C_a$ and $C_b$ are cyclic codes.

Conversely, since $C = a\,C_a \oplus b\,C_b$, any linear combination of codewords from $C_a$ and $C_b$ will be a codeword in $C$. Since both $C_a$ and $C_b$ are cyclic codes, it follows that $C$ is a cyclic code. □

**Corollary 4.10.** *If $C = a\,C_a \oplus b\,C_b$ is a cyclic $H_z$-code, then its dual is also cyclic.*

*Proof.* The result follows from Theorems 4.2 and 4.9. □

**Corollary 4.11.** *A QSD $H_z$-code $C = a\,C_a \oplus b\,C_b$ of even length $n$ is cyclic if and only if the following conditions are satisfied:*

*(i) For $z = 23$, $C_a$ is a binary cyclic self-dual code, and $C_b$ is a ternary cyclic $[n, n/2]-$code.*
*(ii) For $z = 32$, $C_a$ is a binary cyclic $[n, n/2]-$code and $C_b$ is a ternary cyclic self-dual code.*

*Proof.* The result follows from Corollary 4.6 and Theorem 4.9. □

### 4.3. LCD codes

An $H_z$-code $C$ is linear with complementary dual (LCD) if $C \cap C^\perp = \{\mathbf{0}\}$. The following theorem gives a characterization of LCD codes over the ring $H_z$.

**Theorem 4.12.** *An $H_z$-code $C = a\,C_a \oplus b\,C_b$ is LCD if and only if the following conditions are satisfied:*

*(i) For $z = 23$, $C_a$ is LCD and $C_b = \{\mathbf{0}\}$.*
*(ii) For $z = 32$, $C_a = \{\mathbf{0}\}$, and $C_b$ is LCD.*

*Proof.* Let $z = 23$, and suppose $C = a\,C_a \oplus b\,C_b$ is LCD, i.e., $C \cap C^\perp = \{\mathbf{0}\}$. By Theorem 4.2, $C_a \cap C_a^\perp = \{\mathbf{0}\}$ and $C_b \cap \mathbb{F}_3^n = \{\mathbf{0}\}$. This implies $C_a$ is LCD and $C_b = \{\mathbf{0}\}$.

Conversely, suppose $C_a$ is LCD and $C_b = \{\mathbf{0}\}$. By definition, $C_a \cap C_a^\perp = \{\mathbf{0}\}$, and so $C \cap C^\perp = a(C_a \cap C_a^\perp) + b(C_b \cap C_b^\perp) = \{\mathbf{0}\}$, which shows $C$ is LCD.

The case where $C$ is an $H_{32}$-code is proved similarly. □

Note that by Corollary 4.3, an $H_z$-code $C$ is LCD if it is nice and the associated nonzero code is LCD.

**Corollary 4.13.** *Suppose $C = a\,C_a \oplus b\,C_b$ is a cyclic $H_z$-code. Then $C$ is LCD if and only if the following conditions are satisfied:*

*(i) For $z = 23$, $C_a$ is an LCD cyclic code and $C_b = \{\mathbf{0}\}$.*
*(ii) For $z = 32$, $C_a = \{\mathbf{0}\}$, and $C_b$ is an LCD cyclic code.*

*Proof.* The result follows from Theorems 4.9 and 4.12. □

These results show that the existence of LCD $H_z$-codes depends on the existence of binary and ternary LCD codes. Good binary and ternary LCD codes and their characterizations can be found in many literatures, such as [16] and [17].

## 5. Building-up construction for self-orthogonal codes

In this chapter, several methods to construct self-orthogonal codes from codes of smaller length are presented. These are called the *building-up constructions*.

**Theorem 5.1.** *Let $C_0$ be a self-orthogonal $H_{23}$-code of length $n$ with generating set $\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}$. Fix $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ satisfying $\langle \mathbf{x}, \mathbf{x} \rangle = 1$and let $y_i = \langle \mathbf{x}, \mathbf{r}_i \rangle$, $1 \le i \le k$. Then the code $C$ with $k + 1$ generators given by*

$$(a, 0, a\mathbf{x}),\ (y_1, y_1, \mathbf{r}_1),\ \ldots,\ (y_k, y_k, \mathbf{r}_k),$$

*is a self-orthogonal $H_{23}$-code of length $n + 2$.*

*Proof.* It suffices to show that all generators are orthogonal to each other. Observe that

- The first vector is orthogonal to itself since $a^2 + a^2\langle \mathbf{x}, \mathbf{x} \rangle = a + a = 0$.
- The first vector is orthogonal to the other $k$ vectors since $ay_i + \langle a\mathbf{x}, \mathbf{r}_i \rangle = 2ay_i = 0$.
- The last $k$ vectors are orthogonal to each other and to themselves by self-orthogonality of $C_0$ since $y_i y_j + y_i y_j + \langle \mathbf{r}_i, \mathbf{r}_j \rangle = 2y_i y_j + 0 = 0$, where $y_i y_j \in J_a$ in $H_{23}$.

Hence, $C$ is a self-orthogonal $H_{23}$-code of length $n + 2$. □

**Theorem 5.2.** *Let $C_0$ be a self-orthogonal $H_{23}$-code of length $n$ with generating set $\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a fixed ternary vector. Denote $y_i = \langle \mathbf{x}, \mathbf{r}_i \rangle$, $1 \le i \le k$. Then the code $C$ with $k + 3$ generators given by*

$$(\beta, 0, 0, \beta\mathbf{x}),\ (0, \beta, 0, \beta\mathbf{x}),\ (0, 0, \beta, \beta\mathbf{x}),\ (2y_1, 2y_1, 2y_1, \mathbf{r}_1),\ \ldots,\ (2y_k, 2y_k, 2y_k, \mathbf{r}_k),$$

*where $\beta \in \{b, d\}$ is a self-orthogonal $H_{23}$-code of length $n + 3$ .*

*Proof.* The following observations show all the $k + 3$ generators of $C$ are orthogonal to each other.

- The first three vectors are orthogonal to themselves since $\beta^2 + \beta^2\langle \mathbf{x}, \mathbf{x} \rangle = 0 + 0\langle \mathbf{x}, \mathbf{x} \rangle = 0$.
- The first three vectors are orthogonal to each other as $\beta^2\langle \mathbf{x}, \mathbf{x} \rangle = 0$.
- The first three vectors are orthogonal to any of the last $k$ vectors since $2\beta y_i + \langle \beta\mathbf{x}, \mathbf{r}_i \rangle = 3\beta y_i = 0$.
- The last $k$ rows are orthogonal to each other and to themselves by self-orthogonality of $C_0$ since $4y_i y_j + 4y_i y_j + 4y_i y_j + \langle \mathbf{r}_i, \mathbf{r}_j \rangle = 12y_i y_j + 0 = 0$.

Hence, $C$ is a self-orthogonal $H_{23}$-code of length $n+3$. □

**Example 5.3.** *Consider the self-orthogonal $H_{23}$-code $C_0$ of length 2 generated by vectors*

$$\mathbf{r}_1 = (a, a) \text{ and } \mathbf{r}_2 = (b, 0).$$

*(1) Let $\mathbf{x} = (1, 0) \in \mathbb{F}_2^2$. Then $y_1 = a$ and $y_2 = b$, and by Theorem 5.1,*

$$\mathbf{r}'_1 = (a, 0, a, 0),\ \mathbf{r}'_2 = (a, a, a, a),\ \mathbf{r}'_3 = (b, b, b, 0)$$

*generate a self-orthogonal $H_{23}$-code $C_1$ of length 4.*

*(2) Consider $C_1$ with generators $\mathbf{r}'_1$, $\mathbf{r}'_2$, and $\mathbf{r}'_3$ and $\mathbf{x} = (1, 0, 1, 1) \in \mathbb{F}_2^4$. Then $y_1 = 0$, $y_2 = a$ and $y_3 = d$. Again, by Theorem 5.1, the vectors*

$$(a, 0, a, 0, a, a),\ (0, 0, a, 0, a, 0),\ (a, a, a, a, a, a), (d, d, b, b, b, 0)$$

*generate a self-orthogonal $H_{23}$-code $C_2$ of length 6.*

*(3) Let $\beta = b$ and $\mathbf{x} = (2, 1) \in \mathbb{F}_3^2$ and consider $C_0$. Then $y_1 = a$ and $y_2 = d$, and by Theorem 5.2, the vectors*

$$\mathbf{g}_1 = (b, 0, 0, d, b),\ \mathbf{g}_2 = (0, b, 0, d, b),\ \mathbf{g}_3 = (0, 0, b, d, b),\ \mathbf{g}_4 = (0, 0, 0, a, a),\ \mathbf{g}_5 = (b, b, b, b, 0)$$

*generate a self-orthogonal $H_{23}$-code $C_3$ of length 5.*

*(4) Using $C_3$ with generators $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, $\mathbf{g}_4$, and $\mathbf{g}_5$, $\mathbf{x} = (1, 0, 0, 2, 2) \in \mathbb{F}_3^5$ and $\beta = d$, we have $y_1 = b$, $y_2 = y_3 = y_4 = y_5 = 0$. By Theorem 5.2, the vectors*

$$(d, 0, 0, d, 0, 0, b, b),\ (0, d, 0, d, 0, 0, b, b),\ (0, 0, d, d, 0, 0, b, b),\ (d, d, d, b, 0, 0, d, b),$$
$$(0, 0, 0, 0, b, 0, d, b),\ (0, 0, 0, 0, 0, b, d, b),\ (0, 0, 0, 0, 0, 0, a, a),\ (0, 0, 0, b, b, b, b, 0)$$

*generate a self-orthogonal $H_{23}$-code $C_4$ of length 8.*

**Theorem 5.4.** *Let $C_0$ be a self-orthogonal $H_{32}$-code of length $n$ with generating set $\mathbf{r}_1, \ldots, \mathbf{r}_k$. Fix $\mathbf{x} \in \mathbb{F}_3^n$ and let $y_i = \langle \mathbf{x}, \mathbf{r}_i \rangle$, $1 \le i \le k$. Let $\alpha, \beta, \gamma \in H_{32}$ such that $\alpha + \beta + \gamma = 0$, with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Then the code $C$ with $k+1$ generators given by*

$$(\alpha, \beta, 0, \gamma\mathbf{x}),\ (y_1, y_1, y_1, \mathbf{r}_1),\ \ldots,\ (y_k, y_k, y_k, \mathbf{r}_k),$$

*is a self-orthogonal $H_{32}$-code of length $n+3$ if one of the following holds:*

*(i)* $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ *and* $\alpha^2 + \beta^2 + \gamma^2 = 0$,

*(ii)* $\langle \mathbf{x}, \mathbf{x} \rangle = -1$ *and* $\alpha^2 + \beta^2 - \gamma^2 = 0$.

*Proof.* Similar to the previous theorems, the following observations show that the generators are orthogonal to each other.

(i) If $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ and $\alpha^2 + \beta^2 + \gamma^2 = 0$, then

- The first vector is orthogonal to itself since $\alpha^2 + \beta^2 + \gamma^2 \langle \mathbf{x}, \mathbf{x} \rangle = \alpha^2 + \beta^2 + \gamma^2 = 0$.

- The first vector is orthogonal to any of the other $k$ vectors since $\alpha y_i+\beta y_i+\gamma y_i = (\alpha+\beta+\gamma)y_i = 0$.
- The last $k$ vectors are orthogonal to each other and to themselves by self-orthogonality of $C_0$ since $3y_iy_j + \langle \mathbf{r}_i, \mathbf{r}_j\rangle = 0$, where $y_iy_j \in J_b$ in $H_{32}$.

(ii) If $(\mathbf{x}, \mathbf{x}) = -1$ and $\alpha^2 + \beta^2 - \gamma^2 = 0$, then

- The first vector is orthogonal to itself since $\alpha^2 + \beta^2 + \gamma^2\langle \mathbf{x}, \mathbf{x}\rangle = \alpha^2 + \beta^2 - \gamma^2 = 0$.
- The first vector is orthogonal to any of the other $k$ vectors since $\alpha y_i+\beta y_i+\gamma y_i = (\alpha+\beta+\gamma)y_i = 0$.
- The last $k$ vectors are orthogonal to each other and to themselves by self-orthogonality of $C_0$ since $3y_iy_j + \langle \mathbf{r}_i, \mathbf{r}_j\rangle = 0$, where $y_iy_j \in J_b$ in $H_{32}$.

Hence $C$ is a self-orthogonal $H_{32}$-code of length $n + 3$. □

**Example 5.5.** *Consider the self-orthogonal $H_{32}$-code $C_0$ of length 3 generated by*

$$\mathbf{r}_1 = (a, 0, 0) \text{ and } \mathbf{r}_2 = (b, b, b).$$

*(1) Let $\alpha = a$, $\beta = b$, $\gamma = e$, and $\mathbf{x} = (0, 1, 1) \in \mathbb{F}_3^3$. Then $y_1 = 0$ and $y_2 = d$. By Theorem 5.4 (ii), the vectors*

$$\mathbf{r}'_1 = (a, b, 0, 0, e, e),\ \mathbf{r}'_2 = (0, 0, 0, a, 0, 0),\ \mathbf{r}'_3 = (d, d, d, b, b, b)$$

*generate a self-orthogonal $H_{32}$-code $C_1$ of length 6.*

*(2) Consider $C_1$ with generators $\mathbf{r}'_1$, $\mathbf{r}'_2$, and $\mathbf{r}'_3$, and let $\mathbf{x} = (2, 2, 0, 1, 1, 0) \in \mathbb{F}_3^6$, $\alpha = e$, $\beta = e$, and $\gamma = d$. Then $y_1 = c$, $y_2 = a$, and $y_3 = b$. By Theorem 5.4 (i), the vectors*

$$(e, e, 0, b, b, 0, d, d, 0),\ (c, c, c, a, b, 0, 0, e, e),\ (a, a, a, 0, 0, 0, a, 0, 0),\ (b, b, b, d, d, d, b, b, b)$$

*generate a self-orthogonal $H_{32}$-code $C_2$ of length 9.*

## 6. Numerical results

In this section, a classification of self-orthogonal codes over $H_{23}$ and $H_{32}$ of lengths up to 7 will be given, using a method similar to [15]. For each pair $(C_a, C_b)$, we find self-orthogonal codes that are permutation equivalent to $C = aC_a \oplus bC_b$. This can be accomplished by double coset decompositions of the symmetry group $S_n$ by automorphism groups of codes $C_a$ and codes $C_b$ satisfying conditions given in Theorem 4.5. Then the System of Distinct Representatives (SDR), i.e., a set of representatives of the double cosets in the group $S_n$ are obtained. Here is an analog of Theorem 2 in [15].

**Theorem 6.1.** *Let $(C_a, C_b)$ be a pair of codes as defined in Theorem 4.5. Let $Aut(C_a)$ and $Aut(C_b)$ denote permutation groups of $C_a$ and $C_b$, respectively. Then the set*

$$S_{C_a,C_b} := \{aC_a \oplus b\,\sigma(C_b) \mid \sigma \text{ runs over an SDR of } Aut(C_a)\backslash S_n/Aut(C_b)\}$$

*forms a set of permutation inequivalent self-orthogonal $H_z$-codes. Moreover, $|S_{C_a,C_b}| = |Aut(C_a)\backslash S_n/Aut(C_b)|$.*

As in [15], the notation $Aut(C_a)\backslash S_n/Aut(C_b)$ refers to the set of all $(Aut(C_a), Aut(C_b))$-double cosets of $S_n$. To carry out the classification, note the following consequence of Theorem 6.1, similar to Corollary 1 and Corollary 2 in [15].

**Corollary 6.2.** *Let $L_a$ be the set of all inequivalent binary codes of length n and $L_b$ be the set of all (permutation) inequivalent ternary codes that satisfy Theorem 4.5. Then the set of all self-orthogonal $H_z$-codes of length n is the disjoint union*

$$\bigcup_{C_a\in La,\, C_b\in L_b} S_{C_a,C_b}.$$

Based on these results, the following is the classification algorithm for self-orthogonal $H_z$-codes of length $n$.

(1) If $z = 23$,

  (a) write a list $L_a$ of inequivalent binary self-orthogonal $[n, k_a]$-codes;
  (b) write a list $L_b$ of permutation inequivalent ternary $[n, k_b]$-codes.

(2) If $z = 32$,

  (a) write a list $L_a$ of inequivalent binary $[n, k_a]$-codes;
  (b) write a list $L_b$ of permutation inequivalent ternary self-orthogonal $[n, k_b]$-codes.

(3) For every pair $(C_a, C_b) \in L_a \times L_b$,

  (a) compute the automorphism groups $Aut(C_a)$ and $Aut(C_b)$,
  (b) determine a list $\sigma_1, \ldots, \sigma_r$ of representatives of $Aut(C_a)\backslash S_n/Aut(C_b)$,
  (c) for $i = 1, \ldots, r$, compute $a\,C_a \oplus b\,\sigma_i(C_b)$.

Clearly, the zero code of length $n$ is self-orthogonal. The following results are for nonzero codes. All computations are done with the help of MAGMA [18].

### 6.1. Self-orthogonal codes over $H_{23}$

If $(k_a, k_b) = (0, n)$, then we have the self-orthogonal code $C = b\mathbb{F}_3^n$. If $(k_a, k_b) = (0, k_b)$, then the number of inequivalent self-orthogonal $H_{23}$-codes for a given length $n$ is equal to the number of permutation inequivalent ternary codes of dimension $k_b$. If $(k_a, k_b) = (k_a, 0)$ or $(k_a, k_b) = (k_a, n)$, then the number of inequivalent self-orthogonal $H_{23}$-codes for a given length $n$ is equal to the number of inequivalent binary self-orthogonal codes of dimension $k_a$. For the other cases, refer to Table 3.

**Table 3.** Number of nonzero inequivalent self-orthogonal $H_{23}$-codes of lengths $n$ up to 7.

| $n$ | $(k_a, k_b)$ | $\lvert L_a\rvert$ | $\lvert L_b\rvert$ | #Codes | Remark | $n$ | $(k_a, k_b)$ | $\lvert L_a\rvert$ | $\lvert L_b\rvert$ | #Codes | Remark |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | (1, 1) | 1 | 3 | 3 | QSD | 6 | (2, 4) | 3 | 87 | 2003 | |
| 3 | (1, 1) | 1 | 5 | 9 | | 6 | (2, 5) | 3 | 15 | 145 | |
| 3 | (1, 2) | 1 | 5 | 9 | | 6 | (3, 1) | 1 | 15 | 31 | |
| 4 | (1, 1) | 2 | 8 | 27 | | 6 | (3, 2) | 1 | 87 | 404 | |
| 4 | (1, 2) | 2 | 16 | 66 | | 6 | (3, 3) | 1 | 168 | 1032 | QSD |
| 4 | (1, 3) | 2 | 8 | 27 | | 6 | (3, 4) | 1 | 87 | 404 | |
| 4 | (2, 1) | 1 | 8 | 12 | | 6 | (3, 5) | 1 | 15 | 31 | |
| 4 | (2, 2) | 1 | 16 | 30 | QSD | 7 | (1, 1) | 3 | 19 | 185 | |
| 4 | (2, 3) | 1 | 8 | 12 | | 7 | (1, 2) | 3 | 176 | 3822 | |
| 5 | (1, 1) | 2 | 11 | 54 | | 7 | (1, 3) | 3 | 644 | 20458 | |
| 5 | (1, 2) | 2 | 39 | 289 | | 7 | (1, 4) | 3 | 644 | 20458 | |
| 5 | (1, 3) | 2 | 39 | 289 | | 7 | (1, 5) | 3 | 176 | 3822 | |
| 5 | (1, 4) | 2 | 11 | 54 | | 7 | (1, 6) | 3 | 19 | 185 | |
| 5 | (2, 1) | 1 | 11 | 33 | | 7 | (2, 1) | 3 | 19 | 333 | |
| 5 | (2, 2) | 1 | 39 | 220 | | 7 | (2, 2) | 3 | 176 | 11400 | |
| 5 | (2, 3) | 1 | 39 | 220 | | 7 | (2, 3) | 3 | 644 | 76968 | |
| 5 | (2, 4) | 1 | 11 | 33 | | 7 | (2, 4) | 3 | 644 | 76968 | |
| 6 | (1, 1) | 3 | 15 | 109 | | 7 | (2, 5) | 3 | 176 | 11400 | |
| 6 | (1, 2) | 3 | 87 | 1143 | | 7 | (2, 6) | 3 | 19 | 333 | |
| 6 | (1, 3) | 3 | 168 | 2640 | | 7 | (3, 1) | 2 | 19 | 118 | |
| 6 | (1, 4) | 3 | 87 | 1143 | | 7 | (3, 2) | 2 | 176 | 4074 | |
| 6 | (1, 5) | 3 | 15 | 109 | | 7 | (3, 3) | 2 | 644 | 29998 | |
| 6 | (2, 1) | 3 | 15 | 145 | | 7 | (3, 4) | 2 | 644 | 29998 | |
| 6 | (2, 2) | 3 | 87 | 2003 | | 7 | (3, 5) | 2 | 176 | 4074 | |
| 6 | (2, 3) | 3 | 168 | 5096 | | 7 | (3, 6) | 2 | 19 | 118 | |

There is a unique binary self-dual code of length 2 with generator matrix

$$\begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Up to equivalence, there are three QSD $H_{23}$-codes of length 2 and each of their associated codes $C_a$ and $\sigma_i(C_b)$ are listed below. All these codes are cyclic by Corollary 4.11.

| $C_a$ | $\sigma_i(C_b)$ | Remark |
|---|---|---|
| | $[1, 0]$ | cyclic |
| $[1, 1]$ | $[1, 1]$ | cyclic |
| | $[1, 2]$ | cyclic |

There is a unique binary self-dual code $C_a$ of length 4 with generator matrix

$$G_a = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Up to equivalence, we find 30 inequivalent QSD $H_{23}$-codes of length 4 and associated ternary codes $\sigma_i(C_b)$ has generator matrices listed below:

$$\begin{bmatrix} 1&2&2&0\\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&2&1&0\\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&2&0&0\\ 0&0&1&2 \end{bmatrix}, \begin{bmatrix} 1&0&2&0\\ 0&1&0&2 \end{bmatrix}, \begin{bmatrix} 1&0&0&2\\ 0&1&0&2 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&2&0\\ 0&1&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\ 0&1&2&0 \end{bmatrix}, \begin{bmatrix} 1&0&2&2\\ 0&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1\\ 0&1&2&2 \end{bmatrix}, \begin{bmatrix} 1&0&1&2\\ 0&1&1&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&2&0&1\\ 0&0&1&2 \end{bmatrix}, \begin{bmatrix} 1&0&2&0\\ 0&1&2&2 \end{bmatrix}, \begin{bmatrix} 1&0&1&1\\ 0&1&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&1&0\\ 0&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1\\ 0&1&1&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&0&2\\ 0&0&1&1 \end{bmatrix}, \begin{bmatrix} 0&1&0&2\\ 0&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&2&0\\ 0&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1\\ 0&1&2&0 \end{bmatrix}, \begin{bmatrix} 1&0&1&1\\ 0&1&2&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&2&0\\ 0&1&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\ 0&0&1&1 \end{bmatrix}, \begin{bmatrix} 1&0&1&0\\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&1&0\\ 0&1&2&1 \end{bmatrix}, \begin{bmatrix} 1&0&2&1\\ 0&1&1&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&2&1\\ 0&1&2&0 \end{bmatrix}, \begin{bmatrix} 1&0&2&0\\ 0&1&2&1 \end{bmatrix}, \begin{bmatrix} 1&0&2&2\\ 0&1&1&2 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\ 0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\ 0&0&0&1 \end{bmatrix}.$$

The ternary code $C'_b$ with generator matrix $G_a$ is cyclic, so the $H_{23}$-code $a\,C_a \oplus b\,C'_b$ is a cyclic code by Corollary 4.11.

There is a unique binary self-dual code $C_a$ of length 6 with generator matrix

$$\begin{bmatrix} 1&0&0&0&0&1\\ 0&1&0&0&1&0\\ 0&0&1&1&0&0 \end{bmatrix}.$$

Up to equivalence, we find 1032 inequivalent QSD $H_{23}$-codes of length 6. We found more QSD $H_{23}$-codes for lengths $n = 2, 4, 6$ than [15].

### 6.2. *Self-orthogonal codes over $H_{32}$*

If $(k_a, k_b) = (n, 0)$, then we have the self-orthogonal code $C = a\mathbb{F}_2^n$. If $(k_a, k_b) = (0, k_b)$ or $(k_a, k_b) = (n, k_b)$, then the number of inequivalent self-orthogonal $H_{32}$-codes for a given length $n$ is equal to the number of permutation inequivalent self-orthogonal ternary codes of dimension $k_b$. If $(k_a, k_b) = (k_a, 0)$, then the number of inequivalent self-orthogonal $H_{32}$-codes for a given length $n$ is equal to the number of inequivalent binary codes of dimension $k_a$. For the other cases, refer to Table 4.

**Table 4.** Number of nonzero inequivalent self-orthogonal $H_{32}$-codes of lengths $n$ up to 7.

| $n$ | $(k_a, k_b)$ | $\lvert L_a \rvert$ | $\lvert L_b \rvert$ | #Codes | Remark | $n$ | $(k_a, k_b)$ | $\lvert L_a \rvert$ | $\lvert L_b \rvert$ | #Codes | Remark |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | (1, 1) | 3 | 2 | 8 | | 6 | (2, 2) | 16 | 4 | 469 | |
| 3 | (2, 1) | 3 | 2 | 8 | | 6 | (3, 2) | 22 | 4 | 866 | |
| 4 | (1, 1) | 4 | 2 | 18 | | 6 | (4, 2) | 16 | 4 | 469 | |
| 4 | (2, 1) | 6 | 2 | 35 | | 6 | (5, 2) | 6 | 4 | 75 | |
| 4 | (3, 1) | 4 | 2 | 18 | | 7 | (1, 1) | 7 | 6 | 132 | |
| 4 | (1, 2) | 4 | 1 | 7 | | 7 | (2, 1) | 23 | 6 | 963 | |
| 4 | (2, 2) | 6 | 1 | 13 | QSD | 7 | (3, 1) | 43 | 6 | 2623 | |
| 4 | (3, 2) | 4 | 1 | 7 | | 7 | (4, 1) | 43 | 6 | 2623 | |
| 5 | (1, 1) | 5 | 2 | 28 | | 7 | (5, 1) | 23 | 6 | 963 | |
| 5 | (2, 1) | 10 | 2 | 99 | | 7 | (6, 1) | 7 | 6 | 132 | |
| 5 | (3, 1) | 10 | 2 | 99 | | 7 | (1, 2) | 7 | 10 | 354 | |
| 5 | (4, 1) | 5 | 2 | 28 | | 7 | (2, 2) | 23 | 10 | 3999 | |
| 5 | (1, 2) | 5 | 1 | 15 | | 7 | (3, 2) | 43 | 10 | 13675 | |
| 5 | (2, 2) | 10 | 1 | 57 | | 7 | (4, 2) | 43 | 10 | 13675 | |
| 5 | (3, 2) | 10 | 1 | 57 | | 7 | (5, 2) | 23 | 10 | 3999 | |
| 5 | (4, 2) | 5 | 1 | 15 | | 7 | (6, 2) | 7 | 10 | 354 | |
| 6 | (1, 1) | 6 | 6 | 78 | | 7 | (1, 3) | 7 | 2 | 78 | |
| 6 | (2, 1) | 16 | 6 | 360 | | 7 | (2, 3) | 23 | 2 | 939 | |
| 6 | (3, 1) | 22 | 6 | 603 | | 7 | (3, 3) | 43 | 2 | 3405 | |
| 6 | (4, 1) | 16 | 6 | 360 | | 7 | (4, 3) | 43 | 2 | 3405 | |
| 6 | (5, 1) | 6 | 6 | 78 | | 7 | (5, 3) | 23 | 2 | 939 | |
| 6 | (1, 2) | 6 | 4 | 75 | | 7 | (6, 3) | 7 | 2 | 78 | |

There is a unique ternary self-dual code up to permutation equivalence of length 4 with generator matrix

$$C_b = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Up to equivalence, we find 13 QSD $H_{32}$-codes of length 4. This result agrees with [15], and by Corollary 4.11, none of these codes is a cyclic code. The pairs $(C_a, \sigma_i(C_b))$ that correspond to the code $a\,C_a \oplus b\,\sigma_i(C_b)$ are listed in Table 5.

**Table 5.** The 13 inequivalent self-orthogonal $H_{32}$-codes of length $n = 4$.

| $C_a$ | $\sigma_i(C_b)$ | $C_a$ | $\sigma_i(C_b)$ |
|---|---|---|---|
| $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ |
| $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ | | |

Generator matrices of the pairs $(C_a, \sigma_i(C_b))$ that corresponds to $H_z$-codes in Tables 3 and 4 may be obtained by the interested reader from the authors.

## 7. Conclusions

In this paper, we developed a basic framework to study the duality of linear codes over two non-unital rings of order six, namely, $H_{23}$ and $H_{32}$. We use the fact that every code over these rings can be associated with a binary code and a ternary code, which helps to understand when these codes are self-orthogonal, self-dual, quasi self-dual, or have a complementary dual (LCD). We introduced building-up construction methods that provide a systematic way to construct longer self-orthogonal codes from shorter ones. In addition, we discussed conditions for when an $H_z$-code is cyclic, for $z \in \{23, 32\}$. Our numerical computations confirmed the results from previous studies and revealed new results in the classification of inequivalent self-orthogonal codes for short lengths.

## Author contributions

Altaf Alshuhail: Conceptualization, methodology, formal analysis, writing—original draft preparation, supervision; Rowena Alma Betty: Methodology, formal analysis, writing—original draft preparation, writing—review and editing; Lucky Galvez: Methodology, formal analysis, writing—original draft preparation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

All authors declare no conflicts of interest in this paper.

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