AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18770–18783.
DOI: 10.3934/math.2025838
Received: 29 April 2025
Revised: 29 July 2025
Accepted: 08 August 2025
Published: 19 August 2025

Research article

Optimal exploitation of a periodic Gompertz equation with random fluctuations

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Abstract: Taking white noise, Lévy jumps, and periodic factors in the environments into account, we formulated a periodic stochastic Gompertz model with impulsive harvesting and investigated its optimal impulsive exploitation problem. For periodic stochastic ecosystems with Lévy jumps and impulsive harvesting, two traditional approaches, the Fokker-Planck equation approach and ergodicity-based approach used to study the optimal exploitation problems were invalid because for almost all such ecosystems, one could not solve the associated Fokker-Planck equations. In addition, the ecosystems have no traditional non-boundary invariant measures. The present study utilized a different method. By exploring the associated Hamilton function, we derived an optimal exploitation strategy. An example was also introduced to illustrate the theoretical results.

Keywords: optimal exploitation; random environments; impulsive; periodic coefficients
Mathematics Subject Classification: 60H10, 92D25

1. Introduction

There are many renewable resources in nature. How to exploit them economically and sustainably has become and will continue to be one of the key issues [1, 2]. Furthermore, nature is filled with various random disturbances. In mathematical modeling, it is essential to fully account for the effects of these random disturbances [3]. As a matter of fact, neglecting the impacts of random perturbations might be one of the main reasons for the dramatic drop of many living creatures, hence many scholars focused on stochastic harvesting ecosystems, and these researches have both theoretical and practical values for decision making [4].

In recent decades, a number of scholars have devoted their efforts to stochastic exploitation ecosystems [5–14]. Early investigations focused on using explicit solutions of the associated Fokker-Planck equation (FPE) to get optimal exploitation strategies (OES) of the ecosystems [5, 6]. Nevertheless, for the vast majority of stochastic ecosystems, explicit solutions to their associated

FPEs remain unavailable. To address this challenge, the authors [7] pioneered an ergodicity-based method. This innovative method utilizes the ergodic properties of non-boundary invariant measures (NBIM) in ecosystems, thereby eliminating the requirement for explicit solutions to the associated FPEs. Now, this approach has been further developed and widely utilized [8–14].

Most stochastic harvesting studies have focused on autonomous models with continuous exploitations. Nevertheless, the natural environment often fluctuates periodically due to seasonal variations and other factors [15–17]. Additionally, exploitations in practice are often carried out impulsively [18–22]. For instance, fishermen are only allowed to harvest scallop (Pecten maximus) from November to April in the Saint Brieuc Bay (France), and only two days in a week [23]. Moreover, the growth of living creatures often encounter some abrupt fluctuations which can cause dramatic variations in the number of the creatures. For example, the Gulf of Mexico oil spill wipes out more than 30% of the laughing gulls in the area [24]. Lévy jump processes should be incorporated to model these sudden fluctuations [25–27]. Therefore, in order to make it more consistent with the actual situation, periodic stochastic ecosystems with Lévy jumps and impulsive exploitation should be investigated. However, research on this topic is still in its infancy. One reason is that stochastic ecosystems with Lévy jumps and impulsive harvesting are not continuous models, as both the FPE approach and ergodicity-based approach mentioned above are invalid. For almost all such ecosystems, one could not solve the associated FPEs; in addition, the ecosystems have no traditional NBIM. For stochastic ecosystems with Lévy jumps and impulsive harvesting, how to analyze the OES, and reveal the impacts of Lévy noise and impulsive exploitation on the OES become interesting topics.

Motivated by these factors, we consider a periodic stochastic Gompertz equation with Lévy jumps and impulsive exploitation. As said above, for our model, both the FPE method and ergodicity-based method used in [5–14] are invalid. This paper will use an innovative approach. We will show that the model has a globally attractive solution, and we shall give the explicit form of the solution. Then, by constructing an appropriate Hamilton function and using the the explicit form and the global attractiveness of the solution, we shall obtain the OES. The results reveal that both Lévy jumps and impulsive exploitation can impact the OES, and should not be neglected.

The organization of this paper is as follows. In Section 2, we formulate the model. In Section 3, by analyzing the corresponding Hamilton function, we get the OES. In Section 4, we introduce a demonstrative example. At last, the impacts of Lévy noise and impulsive exploitation on the OES are analyzed.

2. The model

As one of the most famous models in population dynamics, the Gompertz growth model has been applied with surpassing success in many areas from growth of tumors to fishery resources exploitation [15, 28]. Through curve fitting, the model has proven particularly useful for modeling a wide range of developmental phenomena [28]. The classical deterministic Gompertz model is [15]:

$$\dot{\Psi}(r) = b\Psi(r)[\ln K - \ln \Psi(r)], \tag{2.1}$$

where $\dot{\Psi}(r) = \mathrm{d}\Psi(r)/\mathrm{d}r$, $\Psi(r)$ is population's abundance, and growth rate b and carrying capacity K are constants. When the periodic factors are taken into account, the constants b and K in (2.1) are replaced by T-periodic and continuous functions $b(r)$ and $K(r)$ respectively [15], then model (2.1) becomes:

$$\dot{\Psi}(r) = b(r)\Psi(r)[\ln K(r) - \ln \Psi(r)], \tag{2.2}$$

where $\min_{r\in[0,T]}\{b(r),\ K(r)\} > 0$. This study assumes $T = 1$ to avoid complicated symbols.

Now let us move to the impacts of environmental noises. Two classes of environmental noises are considered: white noise and Lévy noise. Minor disturbances are common in the environments, for example, the slight variations of the temperature and humidity. One could use the white noise to characterize these minor disturbances [3]. A widely accepted method for incorporating white noise is based on the assumption that it primarily influences species' per capita growth rate (see, e.g., [8–14, 29–32]), because species' per capita growth rate is the most environmentally sensitive parameter and is highly susceptible to interference from environmental noise [32]. Following this way, in model (2.2),

$$\frac{\dot{\Psi}(r)}{\Psi(r)} \to \frac{\dot{\Psi}(r)}{\Psi(r)} + \phi(r)\dot{\omega}(r),$$

where $\dot{\omega}(r)$ represents white noise; that is, $\omega(r)$ represents a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_r\}_{r\geq 0}, \mathbf{P})$ that satisfies the usual conditions, and $\phi(r)$ is a 1-periodic continuous function describing noise's intensity. Then, Eq (2.2) is replaced by:

$$\mathrm{d}\Psi(r) = b(r)\Psi(r)[\ln K(r) - \ln \Psi(r)]\mathrm{d}r + \phi(r)\Psi(r)\mathrm{d}\omega(r). \tag{2.3}$$

Besides small environmental perturbations, there are many abrupt environmental perturbations in the environments, for example, floods, droughts, and windstorms. These perturbations usually cause dramatic variations in the number of the species. One could use the Lévy noise to characterize the abrupt environmental perturbations [25]. Let ν stand for characteristic measure of a Poisson counting measure Γ satisfying $\nu(\Upsilon) < +\infty$, where $\Upsilon \subset (0, +\infty)$ is measurable. Let

$$\tilde{\Gamma}(\mathrm{d}r, \mathrm{d}\eta) = \Gamma(\mathrm{d}r, \mathrm{d}\eta) - \nu(\mathrm{d}\eta)\mathrm{d}r.$$

Following [9–12], Eq (2.3) becomes:

$$\begin{aligned} \mathrm{d}\Psi(r) &= b(r)\Psi(r)[\ln K(r) - \ln \Psi(r)]\mathrm{d}r + \phi(r)\Psi(r)\mathrm{d}\omega(r) \\ &\quad + \int_{\Upsilon} \kappa(r,\eta)\Psi(r)\tilde{\Gamma}(\mathrm{d}r, \mathrm{d}\eta). \end{aligned} \tag{2.4}$$

Here, $\kappa(r, \eta)$ is a 1-periodic and continuous function with respect to r, and $\kappa^2(r, \eta)$ measures the intensity of the abrupt environmental perturbations. Here we assume that $\kappa(r, \eta)$ is periodic because the abrupt environmental perturbations may occur periodically, for example, the Yellow River basin experiences distinct seasonal flood periods due to its latitude and monsoon climate. It is important to point out that comparing with model (2.3), due to the inclusion of the term $\int_{\Upsilon} \kappa(r,\eta)\Psi(r)\tilde{\Gamma}(\mathrm{d}r, \mathrm{d}\eta)$, the solution of model (2.4) becomes discontinuous.

Finally, let us pay attention to impulsive harvesting. Hypothesize that in $[0, 1]$, the species is exploited L times at $r = r_l$ $(l = 1, 2..., L)$ with harvesting effort h_l:

$$0 < r_1 < r_2 < \cdots < r_L < 1,\ r_{l+L} = r_l + 1,\ h_l = h(r_l) = h_{l+L},\ l \in \mathbb{N}.$$

Then Eq (2.4) is replaced by

$$\begin{cases} \mathrm{d}\Psi(r) = b(r)\Psi(r)[\ln K(r) - \ln \Psi(r)]\mathrm{d}r + \phi(r)\Psi(r)\mathrm{d}\omega(r) \\ \qquad\qquad + \int_{\Upsilon} \kappa(r,\eta)\Psi(r)\tilde{\Gamma}(\mathrm{d}r, \mathrm{d}\eta), \quad r \neq r_l, \quad l \in \mathbb{N}, \\ \Psi(r_l^+) - \Psi(r_l) = -h_l\Psi(r_l), \quad l \in \mathbb{N}. \end{cases} \tag{2.5}$$

The objective is to determine the optimal harvesting effort that maximizes yield

$$Y(h) = \liminf_{k\to+\infty,\ k\in\mathbb{N}} \sum_{l=1}^{L} h_l \mathbb{E}(\Psi(r_l + k)).$$

3. Main results

Assumption 3.1. *For $\forall r \geq 0$, $\eta \in \Upsilon$,*

$$1 + \kappa(r,\eta) > 0,$$

which means that the negative abrupt environmental perturbations cannot eliminate all the species.

Lemma 3.1. *(Corollary 5.2.2 in [33]) Let*

$$\mathrm{d}Y(r) = G(r)\mathrm{d}r + \int_{\Upsilon} y(r,\eta)\widetilde{\Gamma}(\mathrm{d}\eta, \mathrm{d}r).$$

$e^{Y(r)}$ is a local martingale $\Leftrightarrow$

$$G(z) + \int_{\Upsilon} [e^{y(z,\eta)} - 1 - y(z,\eta)]\nu(\mathrm{d}\eta)\mathrm{d}z = 0$$

for almost all $z \geq 0$.

Lemma 3.2. *Under Assumption 3.1, for arbitrary initial value $\Psi_0 = \Psi(0) > 0$, Eq (2.5) possesses a global positive solution $\Psi(r)$ almost surely, which is pathwise uniqueness. Moreover, $\Psi(r)$ is globally attractive, and*

$$\begin{aligned} \Psi(r) \ &= \exp\Big\{ \ln \Psi_0 e^{-\int_0^r b(u)\mathrm{d}u} + \sum_{0\leq r_l<r} \ln(1-h_l)e^{-\int_{r_l}^r b(u)\mathrm{d}u} + \int_0^r \alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u \\ &\quad + \int_0^r \phi(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}\omega(u) + \int_0^r\int_{\Upsilon} \ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\tilde{\Gamma}(\mathrm{d}u,\mathrm{d}\eta)\Big\}, \end{aligned} \tag{3.1}$$

where

$$\alpha(r) = b(r)\ln K(r) - \frac{\phi^2(r)}{2} - \int_{\Upsilon} [\kappa(r,\eta) - \ln(1+\kappa(r,\eta))]\nu(\mathrm{d}\eta),$$

additionally,

$$\begin{aligned} \mathbb{E}(\Psi(r)) \ &= \exp\Big\{ \ln \Psi_0 e^{-\int_0^r b(u)\mathrm{d}u} + \sum_{0\leq r_l<t} \ln(1-h_l)e^{-\int_{r_l}^r b(u)\mathrm{d}u} \\ &\quad + \int_0^r \alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u + \int_0^r \frac{\phi^2(u)}{2}e^{-2\int_u^r b(z)\mathrm{d}z}\mathrm{d}u \\ &\quad + \int_0^r\int_{\Upsilon}\Big[(1+\kappa(u,\eta))^{e^{-\int_u^r b(z)\mathrm{d}z}} - 1 - \ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\Big]\nu(\mathrm{d}\eta)\mathrm{d}u\Big\}. \end{aligned} \tag{3.2}$$

Proof. Define $A(r)=\ln\Psi(r)$, and by (2.5),

$$\begin{cases} \mathrm{d}A(r)=\Big[b(r)\ln K(r)-\dfrac{\phi^2(r)}{2}-\displaystyle\int_{\Upsilon}[\kappa(r,\eta)-\ln(1+\kappa(r,\eta))]\nu(\mathrm{d}\eta)-b(r)A(r)\Big]\mathrm{d}r \\ \qquad +\phi(r)\mathrm{d}\omega(r)+\displaystyle\int_0^r\int_{\Upsilon}\ln(1+\kappa(u,\eta))\tilde{\Gamma}(\mathrm{d}u,\mathrm{d}\eta), \quad r\neq r_l, \quad l\in\mathbb{N}, \\ A(r_l^+)-A(r_l)=\ln(1-h_l), \quad l\in\mathbb{N}. \end{cases} \tag{3.3}$$

In light of Lemma 4.1 in [25], one can derive the explicit solution of (3.3):

$$\begin{aligned} A(r) &= e^{-\int_0^r b(u)\mathrm{d}u}\ln\Psi_0+\sum_{0\le r_l<t}\ln(1-h_l)e^{-\int_{r_l}^r b(u)\mathrm{d}u}+\int_0^r \alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u \\ &\quad +\int_0^r \phi(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}\omega(u)+\int_0^r\int_{\Upsilon}\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\tilde{\Gamma}(\mathrm{d}u,\mathrm{d}\eta). \end{aligned} \tag{3.4}$$

This implies that

$$\begin{aligned} \Psi(r) &= e^{A(r)} \\ &= \exp\Big\{\ln\Psi_0 e^{-\int_0^r b(u)\mathrm{d}u}+\sum_{0\le r_l<r}\ln(1-h_l)e^{-\int_{r_l}^r b(u)\mathrm{d}u}+\int_0^r \alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u \\ &\quad +\int_0^r \phi(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}\omega(u)+\int_0^r\int_{\Upsilon}\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\tilde{\Gamma}(\mathrm{d}u,\mathrm{d}\eta)\Big\}. \end{aligned}$$

Introduce $\Psi_1(r)$ and $\Psi_2(r)$ as two trajectories with distinct initial conditions, and let $A_1(r)=\ln\Psi_1(r)$ and $A_2(r)=\ln\Psi_2(r)$. For arbitrary $A_1(0)=\ln\Psi_1(0)$ and $A_2(0)=\ln\Psi_2(0)$,

$$\lim_{r\to+\infty}\mid A_1(r)-A_2(r)\mid=\lim_{r\to+\infty}e^{-\int_0^r b(u)\mathrm{d}u}\mid A_1(0)-A_2(0)\mid=0.$$

Thereby,

$$\lim_{r\to+\infty}\mid\Psi_1(r)-\Psi_2(r)\mid=0. \tag{3.5}$$

To prove (3.2), define

$$U_1(r)=\int_0^r \phi(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}\omega(u).$$

The quadratic variation of $U_1(r)$ is

$$\langle U_1(r),U_1(r)\rangle=\int_0^r \phi^2(u)e^{-2\int_u^r b(z)\mathrm{d}z}\mathrm{d}u.$$

By virtue of Gardiner's relation [34], one gets

$$\mathbb{E}\{e^{U_1(r)}\}=e^{0.5\langle U_1(r),U_1(r)\rangle}=\exp\Big\{\int_0^r\frac{\phi^2(u)}{2}e^{-2\int_u^r b(z)\mathrm{d}z}\mathrm{d}u\Big\}.$$

Define

$$U_2(r)=\int_0^r\int_{\Upsilon}\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\tilde{\Gamma}(\mathrm{d}u,\mathrm{d}\eta).$$

According to Lemma 3.1,

$$\exp\Big\{-\int_0^r\int_\Upsilon\Big[(1+\kappa(u,\eta))^{e^{-\int_u^r b(z)\mathrm{d}z}}-1-\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\Big]\nu(\mathrm{d}\eta)\mathrm{d}u+U_2(r)\Big\}$$

is a local martingale. It follows that,

$$\mathbb{E}[\exp\{U_2(r)\}]=\exp\Big\{\int_0^r\int_\Upsilon\Big[(1+\kappa(u,\eta))^{e^{-\int_u^r b(z)\mathrm{d}z}}-1-\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\Big]\nu(\mathrm{d}\eta)\mathrm{d}u\Big\}.$$

Then, the required assertion (3.2) follows from (3.1). □

Define

$$\begin{aligned}
&\lambda(r_{l+1})=e^{-\int_{r_l}^{r_{l+1}} b(z)\mathrm{d}z},\\
&\sigma(r)=\int_{r_l}^r\int_\Upsilon\Big[(1+\kappa(u,\eta))^{e^{-\int_u^r b(z)\mathrm{d}z}}-1-\ln(1+\kappa(u,\eta))e^{-\int_u^r b(z)\mathrm{d}z}\Big]\nu(\mathrm{d}\eta)\mathrm{d}u,\\
&\rho(r_{l+1})=\int_{r_l}^{r_{l+1}}\alpha(u)e^{-\int_u^{r_{l+1}} b(z)\mathrm{d}z}\mathrm{d}u+\int_{r_l}^{r_{l+1}}\frac{\phi^2(u)}{2}e^{-2\int_u^{r_{l+1}} b(z)\mathrm{d}z}\mathrm{d}u+\sigma(r_{l+1}),\\
&\pi_l=1-\big[\lambda(r_{l+1})\big]^{\frac{1}{1-\lambda(r_{l+1})}}\big[\lambda(r_l)\big]^{\frac{\lambda(r_l)}{\lambda(r_l)-1}}e^{\frac{\rho(r_{l+1})}{1-\lambda(r_{l+1})}-\frac{\rho(r_l)}{1-\lambda(r_l)}},\ 1\le l\le L.
\end{aligned}$$

Theorem 1. *Let Assumption 3.1 hold. If $\pi_l>0,\ 1\le l\le L$, then model (2.5) admits a unique optimal capture effort $h^*=\pi$, the associated output Y^* is*

$$Y^*=\sum_{l=1}^L e^{\frac{\rho(r_l)}{1-\lambda(r_l)}}\big[\lambda(r_l)\big]^{\frac{\lambda(r_l)}{1-\lambda(r_l)}}\Big\{1-\lambda(r_l)\Big\},\tag{3.6}$$

and the mean of optimal population level $\mathbb{E}(\Psi^*(r))$ is a 1-periodic function and has the following form: for $t\in(r_l,r_{l+1}]$,

$$\begin{aligned}\mathbb{E}(\Psi^*(r))\ &=\exp\Big\{\frac{\ln\lambda(r_{l+1})+\rho(r_{l+1})}{1-\lambda(r_{l+1})}e^{-\int_{r_l}^r b(z)\mathrm{d}z}+\int_{r_l}^r\alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u\\
&\quad+\int_{r_l}^r\frac{\phi^2(u)}{2}e^{-2\int_u^r b(z)\mathrm{d}z}\mathrm{d}u+\sigma(r)\Big\}.\end{aligned}\tag{3.7}$$

Proof. According to (3.2), for arbitrary $r\in(r_l,r_{l+1}]$,

$$\begin{aligned}\mathbb{E}(\Psi(r))\ &=\exp\Big\{\ln\Big[(1-h_l)\mathbb{E}(\Psi(r_l))\Big]e^{-\int_{r_l}^r b(z)\mathrm{d}z}+\int_{r_l}^r\alpha(u)e^{-\int_u^r b(z)\mathrm{d}z}\mathrm{d}u\\
&\quad+\int_{r_l}^r\frac{\phi^2(u)}{2}e^{-2\int_u^r b(z)\mathrm{d}z}\mathrm{d}u+\sigma(r)\Big\}.\end{aligned}\tag{3.8}$$

That is to say,

$$\begin{aligned}\mathbb{E}(\Psi(r_{l+1}))\ &=\exp\Big\{\ln\Big[(1-h_l)\mathbb{E}(\Psi(r_l))\Big]\lambda(r_{l+1})+\sigma(r_{l+1})\\
&\quad+\int_{r_l}^{r_{l+1}}\alpha(u)e^{-\int_u^{r_{l+1}} b(z)\mathrm{d}z}\mathrm{d}u+\int_{r_l}^{r_{l+1}}\frac{\phi^2(u)}{2}e^{-2\int_u^{r_{l+1}} b(z)\mathrm{d}z}\mathrm{d}u\Big\}.\end{aligned}\tag{3.9}$$

Thus,

$$\mathbb{E}(\Psi(r_{l+1})) = \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\}. \tag{3.10}$$

As a result,

$$\mathbb{E}(\Psi(r_{l+1})) - \mathbb{E}(\Psi(r_l)) = \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\} - \mathbb{E}(\Psi(r_l)).$$

Define

$$\widetilde{Y}(h) = \sum_{l=1}^{L} h_l \mathbb{E}(\Psi(r_l)).$$

Our objective is to search for $h^* = (h_1^*, ..., h_L^*)$ which maximizes $\widetilde{Y}(h)$. For this purpose, we define the following Hamilton function:

$$H\Big(\mathbb{E}(\Psi(r_l)), h_l, \beta_l, r_l\Big) = h_l \mathbb{E}(\Psi(r_l)) + \beta_{l+1}\Big(\mathbb{E}(\Psi(r_{l+1})) - \mathbb{E}(\Psi(r_l))\Big), \tag{3.11}$$

where β_l obeys

$$\beta_{l+1} - \beta_l = -h_l - \beta_{l+1}\left\{ \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\} \frac{\lambda(r_{l+1})}{\mathbb{E}(\Psi(r_l))} - 1\right\}.$$

As a result,

$$\beta_l = h_l + \beta_{l+1} \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\} \frac{\lambda(r_{l+1})}{\mathbb{E}(\Psi(r_l))}. \tag{3.12}$$

Differentiating (3.11) leads to

$$\frac{\partial H}{\partial h_l} = \mathbb{E}(\Psi(r_l)) - \beta_{l+1}\left\{ \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\} \frac{\lambda(r_{l+1})}{1-h_l}\right\}.$$

Let $\frac{\partial H}{\partial h_l} = 0$, and one can see that

$$(1-h_l)\mathbb{E}(\Psi(r_l)) = \beta_{l+1} \exp\left\{ \ln\left[(1-h_l)\mathbb{E}(\Psi(r_l))\right]\lambda(r_{l+1}) + \rho(r_{l+1})\right\}\lambda(r_{l+1}).$$

Then, by (3.12), we have $\beta_l = 1$. When $\beta_l = 1$ is used in (3.12), we obtain

$$\lambda(r_{l+1}) = \frac{(1-h_l)\mathbb{E}(\Psi(r_l))}{\mathbb{E}(\Psi(r_{l+1}))}. \tag{3.13}$$

Moreover, we deduce from (3.10) that

$$(1-h_l)\mathbb{E}(\Psi(r_l)) = \left[\mathbb{E}(\Psi(r_{l+1}))\right]^{\frac{1}{\lambda(r_{l+1})}} e^{-\frac{\rho(r_{l+1})}{\lambda(r_{l+1})}}.$$

Combining with (3.13) leads to

$$\mathbb{E}(\Psi^*(r_{l+1})) = \left[\lambda(r_{l+1})\right]^{\frac{\lambda(r_{l+1})}{1-\lambda(r_{l+1})}} e^{\frac{\rho(r_{l+1})}{1-\lambda(r_{l+1})}}.$$

Consequently,

$$\mathbb{E}(\Psi^*(r_l)) = [\lambda(r_l)]^{\frac{\lambda(r_l)}{1-\lambda(r_l)}} e^{\frac{\rho(r_l)}{1-\lambda(r_l)}}. \tag{3.14}$$

Substituting this equality into (3.13) gives

$$h_l^* = 1 - [\lambda(r_{l+1})]^{\frac{1}{1-\lambda(r_{l+1})}} [\lambda(r_l)]^{\frac{\lambda(r_l)}{\lambda(r_l)-1}} e^{\frac{\rho(r_{l+1})}{1-\lambda(r_{l+1})} - \frac{\rho(r_l)}{1-\lambda(r_l)}},\ 1 \le l \le L. \tag{3.15}$$

By (3.13)–(3.15), h_l is unique, hence it is a global maximum. Thereby, if $h_l^* > 0$ $(1 \le l \le L)$, notice that $\lambda(r_{L+1}) = \lambda(r_1 + 1) = \lambda(r_1)$, and $\rho(r_{L+1}) = \rho(r_1 + 1) = \rho(r_1)$, and one has

$$\begin{aligned}
\widetilde{Y}^* = \sum_{l=1}^{L} h_l^* \mathbb{E}(\Psi^*(r_l)) &= \sum_{l=1}^{L} \Big\{ [\lambda(r_l)]^{\frac{\lambda(r_l)}{1-\lambda(r_l)}} e^{\frac{\rho(r_l)}{1-\lambda(r_l)}} - [\lambda(r_{l+1})]^{\frac{1}{1-\lambda(r_{l+1})}} e^{\frac{\rho(r_{l+1})}{1-\lambda(r_{l+1})}} \Big\} \\
&= [\lambda(r_1)]^{\frac{\lambda(r_1)}{1-\lambda(r_1)}} e^{\frac{\rho(r_1)}{1-\lambda(r_1)}} - [\lambda(r_2)]^{\frac{1}{1-\lambda(r_2)}} e^{\frac{\rho(r_2)}{1-\lambda(r_2)}} \\
&\quad + [\lambda(r_2)]^{\frac{\lambda(r_2)}{1-\lambda(r_2)}} e^{\frac{\rho(r_2)}{1-\lambda(r_2)}} - [\lambda(r_3)]^{\frac{1}{1-\lambda(r_3)}} e^{\frac{\rho(r_3)}{1-\lambda(r_3)}} \\
&\quad + \cdots\cdots\cdots\cdots \\
&\quad + [\lambda(r_L)]^{\frac{\lambda(r_L)}{1-\lambda(r_L)}} e^{\frac{\rho(r_L)}{1-\lambda(r_L)}} - [\lambda(r_{L+1})]^{\frac{1}{1-\lambda(r_{L+1})}} e^{\frac{\rho(r_{L+1})}{1-\lambda(r_{L+1})}} \\
&= \sum_{l=1}^{L} e^{\frac{\rho(r_l)}{1-\lambda(r_l)}} \Big\{ [\lambda(r_l)]^{\frac{\lambda(r_l)}{1-\lambda(r_l)}} - [\lambda(r_l)]^{\frac{1}{1-\lambda(r_l)}} \Big\} \\
&= \sum_{l=1}^{L} e^{\frac{\rho(r_l)}{1-\lambda(r_l)}} [\lambda(r_l)]^{\frac{\lambda(r_l)}{1-\lambda(r_l)}} \Big\{ 1 - \lambda(r_l) \Big\}.
\end{aligned}$$

When (3.14) and (3.15) are utilized in (3.8), for $r \in (r_l, r_{l+1}]$,

$$\begin{aligned}
\mathbb{E}(\Psi^*(r)) &= \exp\Big\{ \frac{\ln \lambda(r_{l+1}) + \rho(r_{l+1})}{1 - \lambda(r_{l+1})} e^{-\int_{r_l}^{r} b(z)\mathrm{d}z} \\
&\quad + \int_{r_l}^{r} \alpha(u) e^{-\int_u^r b(z)\mathrm{d}z} \mathrm{d}u + \int_{r_l}^{r} \frac{\phi^2(u)}{2} e^{-2\int_u^r b(z)\mathrm{d}z} \mathrm{d}u + \sigma(r) \Big\}.
\end{aligned}$$

We then deduce from $\lambda(r_{l+1} + 1) = \lambda(r_{l+1})$, $\rho(r_{l+1} + 1) = \rho(r_{l+1})$, $b(r + 1) = b(r)$, $\alpha(r + 1) = \alpha(r)$, $\kappa(r + 1, \eta) = \kappa(r, \eta)$ and $\phi(r + 1) = \phi(r)$ that

$$\begin{aligned}
&\mathbb{E}(\Psi^*(r + 1)) \\
&= \exp\Big\{ \frac{\ln \lambda(r_{l+1} + 1) + \rho(r_{l+1} + 1)}{1 - \lambda(r_{l+1} + 1)} e^{-\int_{r_l+1}^{r+1} b(z)\mathrm{d}z} \\
&\quad + \int_{r_l+1}^{r+1} \alpha(u) e^{-\int_u^{r+1} b(z)\mathrm{d}z} \mathrm{d}u + \int_{r_l+1}^{r+1} \frac{\phi^2(u)}{2} e^{-2\int_u^{r+1} b(z)\mathrm{d}z} \mathrm{d}u \\
&\quad + \int_{r_l+1}^{r+1} \int_{\Upsilon} \Big[(1 + \kappa(u, \eta))^{e^{-\int_u^{r+1} b(z)\mathrm{d}z}} - 1 - \ln(1 + \kappa(u, \eta)) e^{-\int_u^{r+1} b(z)\mathrm{d}z} \Big] \nu(\mathrm{d}\eta)\mathrm{d}u \Big\} \\
&= \exp\Big\{ \frac{\ln \lambda(r_{l+1}) + \rho(r_{l+1})}{1 - \lambda(r_{l+1})} e^{-\int_{r_l}^{r} b(z)\mathrm{d}z} \\
&\quad + \int_{r_l}^{r} \alpha(u) e^{-\int_u^r b(z)\mathrm{d}z} \mathrm{d}u + \int_{r_l}^{r} \frac{\phi^2(u)}{2} e^{-2\int_u^r b(z)\mathrm{d}z} \mathrm{d}u \\
&\quad + \int_{r_l}^{r} \int_{\Upsilon} \Big[(1 + \kappa(u, \eta))^{e^{-\int_u^r b(z)\mathrm{d}z}} - 1 - \ln(1 + \kappa(u, \eta)) e^{-\int_u^r b(z)\mathrm{d}z} \Big] \nu(\mathrm{d}\eta)\mathrm{d}u \Big\} \\
&= \mathbb{E}(\Psi^*(r)).
\end{aligned}$$

This implies that $\mathbb{E}(\Psi^*(r))$ is 1-periodic. Thereby,

$$Y^* = \max_h \left\{ \liminf_{k\to+\infty} \sum_{l=1}^{L} h_l \mathbb{E}(\Psi(r_l + k)) \right\} \geq \sum_{l=1}^{L} h_l^* \mathbb{E}(\Psi^*(r_l)) = \widetilde{Y}^*.$$

On the other hand, for arbitrary $n \in \mathbb{N}$ and $h_l > 0$,

$$\sum_{l=1}^{L} h_l \mathbb{E}(\Psi(r_l + k)) = \sum_{l=1}^{L} h(r_l + k)\mathbb{E}(\Psi(r_l + k)) \leq \sum_{l=1}^{L} h_l^* \mathbb{E}(\Psi^*(r_l)) = \widetilde{Y}^*.$$

As a consequence,

$$Y^* \leq \widetilde{Y}^*,$$

Thus,

$$Y^* = \widetilde{Y}^*,$$

which completes the proof. □

Remark 3.1. *The approach used in Theorem 1 can avoid solving the associated FPEs and does not need to use the traditional NBIM. This approach could be used to study other models, for example, logistic model (details are left to the reader). Hence, the approach may provide a new way to study the OES for periodic stochastic ecosystems with Lévy jumps and impulsive exploitation.*

Remark 3.2. *Liu [16] considered the optimal exploitation of the following model:*

$$d\Psi(r) = b(r)\Psi(r)[\ln K(r) - \ln \Psi(r)]dr - h(r)\Psi(r)dr + \phi(r)\Psi(r)d\omega(r), \tag{3.16}$$

where $h(r)$ is a 1-periodic continuous function. The author obtained the optimal capture effort and the associated output. Clearly, Eq (3.16) is a model with continuous harvesting, while our system (2.5) is a model with noncontinuous harvesting. In addition, our system (2.5) considers the effects of Lévy jumps that are not considered in Eq (3.16).

4. An example

Now let us introduce a demonstrative example. Pay attention to the following equation where the coefficients are hypothesized:

$$\begin{cases} d\Psi(r) = \Big(0.3 + 0.1\cos(2\pi r)\Big)\Psi(r)\Big[\ln(5 + 0.2\sin(2\pi r)) - \ln\Psi(r)\Big]dr \\ \qquad + \Big(0.1 + 0.05\cos(2\pi r)\Big)\Psi(r)d\omega(r) \\ \qquad + \Psi(r)\displaystyle\int_{\mathbb{R}_+}\Big(0.05 + 0.06\cos(2\pi r)\Big)\tilde{\Gamma}(dr, d\eta), \quad r \neq r_l, \quad l \in \mathbb{N}, \\ \Psi(r_l^+) - \Psi(r_l) = -h_l\Psi(r_l), \quad l \in \mathbb{N}. \end{cases} \tag{4.1}$$

Assume that each year, people exploit twice at the moments 0.4, 0.6, i.e., $r_1 = 0.4$, $r_2 = 0.6$, $r_3 = 1.4$, $r_4 = 1.6$,.... Notice that

$$b(r) = 0.3 + 0.1\cos(2\pi r),\ K(r) = 5 + 0.2\sin(2\pi r),\ \phi(r) = 0.1 + 0.05\cos(2\pi r),$$

$$\Upsilon = \mathbb{R}_+,\ \kappa(r,\eta) = 0.05 + 0.06\cos(2\pi r),\ \nu(\Upsilon) = 1,$$

hence

$$\lambda(r_1) = \lambda(r_3) = e^{-\int_{r_2}^{r_3} b(z)\mathrm{d}z} = 0.7720,\ \lambda(r_2) = e^{-\int_{r_1}^{r_2} b(z)\mathrm{d}z} = 0.9596,$$

$$\begin{aligned}
\rho(r_1) &= \rho(r_3) = \int_{r_2}^{r_3}\left[b(u)\ln K(u) - \frac{\phi^2(u)}{2}\right]e^{-\int_u^{r_3} b(z)\mathrm{d}z}\mathrm{d}u \\
&\quad + \int_{r_2}^{r_3}\frac{\phi^2(u)}{2}e^{-2\int_u^{r_3} b(z)\mathrm{d}z}\mathrm{d}u \\
&\quad + \int_{r_2}^{r_3}\int_{\Upsilon}\left[(1+\kappa(u,\eta))^{e^{-\int_u^{r_3} b(z)\mathrm{d}z}} - 1 - \ln(1+\kappa(u,\eta))e^{-\int_u^{r_3} b(z)\mathrm{d}z}\right]\nu(\mathrm{d}\eta)\mathrm{d}u \\
&= 0.3647,
\end{aligned}$$

$$\begin{aligned}
\rho(r_2) &= \int_{r_1}^{r_2}\left[b(u)\ln K(u) - \frac{\phi^2(u)}{2}\right]e^{-\int_u^{r_2} b(z)\mathrm{d}z}\mathrm{d}u \\
&\quad + \int_{r_1}^{r_2}\frac{\phi^2(u)}{2}e^{-2\int_u^{r_2} b(z)\mathrm{d}z}\mathrm{d}u \\
&\quad + \int_{r_1}^{r_2}\int_{\Upsilon}\left[(1+\kappa(u,\eta))^{e^{-\int_u^{r_2} b(z)\mathrm{d}z}} - 1 - \ln(1+\kappa(u,\eta))e^{-\int_u^{r_2} b(z)\mathrm{d}z}\right]\nu(\mathrm{d}\eta)\mathrm{d}u \\
&= 0.0650.
\end{aligned}$$

As a result,

$$h_1^* = 1 - [\lambda(r_2)]^{\frac{1}{1-\lambda(r_2)}}[\lambda(r_1)]^{\frac{\lambda(r_1)}{\lambda(r_1)-1}}e^{\frac{\rho(r_2)}{1-\lambda(r_2)} - \frac{\rho(r_1)}{1-\lambda(r_1)}} = 0.1277,$$

$$h_2^* = 1 - [\lambda(r_3)]^{\frac{1}{1-\lambda(r_3)}}[\lambda(r_2)]^{\frac{\lambda(r_2)}{\lambda(r_2)-1}}e^{\frac{\rho(r_3)}{1-\lambda(r_3)} - \frac{\rho(r_2)}{1-\lambda(r_2)}} = 0.1507,$$

$$Y^* = e^{\frac{\rho(r_1)}{1-\lambda(r_1)}}[\lambda(r_1)]^{\frac{\lambda(r_1)}{1-\lambda(r_1)}}\{1-\lambda(r_1)\} + e^{\frac{\rho(r_2)}{1-\lambda(r_2)}}[\lambda(r_2)]^{\frac{\lambda(r_2)}{1-\lambda(r_2)}}\{1-\lambda(r_2)\} = 0.5459.$$

5. Discussions and conclusions

Revealing the influences of random noises and impulsive exploitation on the OES of ecosystems has significant implications. Gompertz growth model can depict a wide range of developmental phenomena [28]; in addition, abrupt fluctuations often occur in the real word, therefore we study a periodic Gompertz equation with impulsive exploitation and Lévy jumps. Theorem 1 is the main result, which gives the OES of the model.

Two traditional approaches, the FPE method and ergodicity-based method, do not work for periodic stochastic ecosystems with Lévy jumps and impulsive harvesting. In this paper, we used a different approach to derive the OES, which may provide a new way to study the OES for periodic stochastic ecosystems with Lévy jumps and impulsive exploitation.

The research findings indicate that OES is closely dependent on Lévy jumps and impulsive exploitation. According to Theorem 1, optimal capture effort h^* and the associated annual output Y^* depend only on the values of $\theta(r_i)$ and $\rho(r_i)$. By the definitions of $\theta(r_i)$ and $\rho(r_i)$, one can see that the intensity of Lévy jumps γ and the impulsive harvesting time r_i influence the values of $\theta(r_i)$ and $\rho(r_i)$. Due to the complicated expressions of h^* and Y^*, the impacts of r_i on h^* and Y^* cannot be given clearly. The impacts of γ on h^* depend on the values of $\theta(r_i)$ $(1 \le i \le L)$, which cannot be given clearly either. However, the impacts of γ on Y^* are clear. In fact, when $\gamma > 0$, one can see that

$$\frac{\partial \rho(r_i)}{\partial \gamma} > 0, \quad \frac{\partial Y^*}{\partial \rho(r_i)} > 0.$$

That is to say, when $\gamma > 0$, with the increasing of γ, one can get more annual yield Y^*. Biologically, $\gamma > 0$ means that the population size increases suddenly (for example, releasing fish fry). Hence, in this case, one can get more annual yield.

Compared to most existed researches (see, e.g., [6–12]), this study makes the following key contributions.

(i) This study considers the effects of white noise, Lévy jumps, periodic environment, and impulsive harvesting simultaneously, and the model is more realistic. In fact, this letter represents the initial study to investigate OES of models under Lévy noises and noncontinuous harvesting.
(ii) This study establishes *explicit forms* of optimal capture effort and associated output which are easy to utilize in reality.
(iii) Our results reveal that both Lévy jumps and impulsive exploitation can impact the OES, and should not be neglected.

To finish this paper, we would like to point out that this paper only considers a single-species model, and it is interesting to test multi-species models or epidemiological coupled systems [35]. Additionally, this paper does not consider the effects of time delay [36], and it is interesting to test models with time delay. Third, our model is with classical Brownian motion, and it is interesting to test models with G-Brownian motion [37]. Finally, this paper doesn't illustrate numerical approach because there is white noise, Lévy jumps, and impulsive exploitation in model (2.5), and how to numerically illustrate the results is still not solved; specifically, the appropriate discretization schemes are still unknown. These problems are left for further consideration.

Author contributions

Weiming Ji finished the establishment of the model, Weiming Ji and Meng Liu finished the investigation of model. All authors read and approved the final manuscript.

Acknowledgments

We are very grateful to the anonymous referees for their careful reading and valuable comments, which led to an improvement of our paper. This work was supported by the Natural Science Foundation of Jiangsu Province (No. BK20231361), "333 Project" Research Foundation and "333 High-level Personnel Training Project" of Jiangsu Province.

Conflict of interest

Meng Liu is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article.

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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