Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18541–18557.
DOI: 10.3934/math.2025828
Received: 16 June 2025
Revised: 06 August 2025
Accepted: 08 August 2025
Published: 15 August 2025

Research article

Analytic solutions for a (3+1)-dimensional extended shallow water wave equation in nonlinear phenomena

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Abstract: The study of analytic solutions for high-dimensional nonlinear wave equations is crucial for comprehending complex nonlinear phenomena in fluid dynamics. In this work, several novel analytic solutions and their dynamical behaviors are investigated by the extended homoclinic test approach and the improved (G'/G)-expansion method for the (3+1)-dimensional extended shallow water wave equation. Based on the Hirota bilinear formulation, various breather wave solutions are constructed via the extended homoclinic test approach. Furthermore, multiple analytic solutions consisting of rational functions, trigonometric functions, and hyperbolic functions are derived using the improved (G'/G)-expansion method. The features of the obtained solutions are illustrated through numerical simulations, including the shape-preserving properties during propagation of the soliton solutions and the divergence characteristics of rogue wave solutions. Furthermore, the relationships between soliton stability and rogue wave generation conditions are given through parameter variation analysis.

Keywords: (3+1)-dimensional extended shallow water wave equation; extended homoclinic test approach; improved (G'/G)-expansion method; breather wave solution; rogue wave solutions
Mathematics Subject Classification: 35M33, 35Q55

1. Introduction

With the wide applications of nonlinear science in such fields as fluid mechanics, atmospheric physics, and ocean engineering, the shallow water wave equation, as a core model for describing the water wave propagation, has continued to attract widespread attention [1, 2]. Particularly in such practical problems as the ocean environment simulation, hurricane prediction, and tsunami warning, the study of analytic solutions for the higher-dimensional shallow water wave equations has held significant theoretical value and practical importance [3–5]. In recent years, with the continuous innovation of computational methods, exploring the analytic solutions for some higher-dimensional nonlinear partial differential equations has become a key approach to uncover the intrinsic mechanisms of complex wave phenomena. Existing methods for solving the nonlinear equations include the Hirota bilinear method [6], the three-wave method [7, 8], the inverse scattering method [9], the Darboux transformation method [10,11], the harmonic method [12], the variational approach [13], the homotopy perturbation method [14], and fractal-based techniques [15]. Recent advances in fractal calculus have some possibilities for modifying high-dimensional wave equations [16]. However, these methods have different limitations in application, and there is generally no universal method to solve all nonlinear partial differential equations. Therefore, the analytic solutions of the equation are mainly studied by the extended homoclinic test approach and the improved (G'/G)-expansion method, and some analyses are considered for these solutions. The combination of these two methods not only broadens the diversity of analytic solutions but also reveals hidden nonlinear phenomena such as rogue waves and blow-up solutions.

The equation studied in this paper is a (3+1)-dimensional extended shallow water wave equation in nonlinear phenomena with constant coefficients [4]

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + \alpha u_{xx} + \beta u_{yy} + \gamma u_{xy} + \delta u_{yz} = 0, \tag{1.1}$$

where the real differentiable function is $u = u(x, y, z, t)$, x, y, z, and t are the independent variables, the subscripts denote partial derivatives, and are constants. The equation was firstly introduced by Wazwaz [4] and has become a research hotspot due to its integrability and the universality of its physical background. Through Painlevé analysis, Wazwaz verified the complete integrability of this equation and derived multi-soliton and lump solutions by using the bilinear method [17]. However, other types of analytic solutions, such as periodic solutions and rogue wave solutions, as well as their dynamical properties, have not been thoroughly investigated, which limits a comprehensive understanding of complex wave phenomena.

Here, an integrated analytical framework combining the extended homoclinic test approach with an improved (G'/G)-expansion method is introduced for the (3+1)-dimensional extended shallow water wave equation (1.1). The obtained results in this paper are rarely represented, they are different from those in the reference [4].

In Section 2, fundamental concepts are firstly introduced, including the Hirota bilinear operator and the theoretical framework of the improved (G'/G)-expansion method. The auxiliary equation is established along with its solution forms under different parameter conditions, and some analyses are provided based on these results.

Section 3 is devoted to the application of the extended homoclinic test approach. Through the construction of specific auxiliary functions, two distinct types of analytic solutions are derived. These solutions include solitary wave solutions and breather wave solutions, whose behaviors are analyzed through numerical simulations. Furthermore, singularities or rogue wave phenomena are investigated under some conditions for these solutions.

In Section 4, additional analytic solutions are obtained using the improved (G'/G)-expansion method. In order to consider different ansatz forms of solutions, non-negative exponent forms with constant d, negative exponent forms without d, and hybrid forms of multiple solutions are derived. The physical implications of these solutions, including their periodic, rational, and hyperbolic characteristics, are illustrated through graphical representations.

2. Preliminaries

In order to study the analytic solutions of nonlinear partial differential equations, we introduce the following Hirota bilinear differential operator

$$D_t^m D_x^n (f \cdot g) = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n [f(t,x) g(t',x')] |_{t'=t,x'=x}, \tag{2.1}$$

where $f(t,x)$ and $g(t,x)$ are differentiable functions with respect to the variables t and x, while D_x and D_t are referred to as bilinear operators. The parameters m and n are non-negative integers.

The idea of the improved (G'/G)-expansion method is given as follows. Generally, the improved (G'/G)-expansion method relies on the following ordinary differential equation as an auxiliary equation

$$G'' - pG' + qG = 0, \tag{2.2}$$

in which p and q represent free parameters [18, 19]. When employing this technique to analyze solutions of various equations, the traveling wave solution is typically expressed in the generalized form $u(\xi) = \sum_{i=0}^{N} a_i (\frac{G'}{G})^i$,with the constraint $a_N \neq 0$ to ensure non-trivial solutions. This formulation can be further generalized to $u(\xi) = \sum_{i=-N}^{N} a_i (\frac{G'}{G})^i$, subject to the condition that either a_N or a_{-N} must be non-vanishing. Our study employs this approach to derive analytic solutions for Eq (1.1).

The methodology adopts a class of nonlinear ordinary differential equations as auxiliary equations. By applying a similar idea to the (G'/G)-expansion method and using the undetermined form $u(\xi) = \sum_{i=-N}^{N} a_i (d + \frac{G'}{G})^i$, where an additional arbitrary parameter d is introduced, while maintaining the non-zero requirement for a_N or a_{-N}. The corresponding nonlinear auxiliary equation is given by

$$aGG'' - bGG' - cG'^2 - eG^2 = 0, \tag{2.3}$$

where a, b, c, e are the undetermined coefficients. Compared with the auxiliary equation in [6] that requires hyperbolic function combinations, this generalized form (2.3) provides two key advances. On the one hand, the nonlinear terms $cG'^2 + eG^2$ enable solutions beyond classical function classes (see Section 4.3). On the other hand, the parameter d in $u(\xi) = \sum_{i=-N}^{N} e_i (d + H)^i$ allows negative-power terms, crucial for modeling singular wave patterns.

Let $\frac{G'}{G} = H(\xi)$, the expression about the solution of $H(\xi)$ can be obtained by reference [20]. To facilitate the expression of $H(\xi)$, let $M = a - c$, $\Omega = b^2 + 4eM$, $\Delta = Me$, then Eq (2.3) has the following different forms of solutions.

1) When $b \neq 0$ and $\Omega > 0$,

$$H(\xi) = (\frac{G'}{G}) = \frac{b}{2M} + \frac{\sqrt{\Omega}}{2M} \frac{C_1 \sinh(\frac{\sqrt{\Omega}}{2a}\xi) - C_2 \cosh(\frac{\sqrt{\Omega}}{2a}\xi)}{C_2 \sinh(\frac{\sqrt{\Omega}}{2a}\xi) + C_1 \cosh(\frac{\sqrt{\Omega}}{2a}\xi)}. \tag{2.4}$$

2) When $b \neq 0$ and $\Omega < 0$,

$$H(\xi) = (\frac{G'}{G}) = \frac{b}{2M} + \frac{\sqrt{-\Omega}}{2M} \frac{-C_1 \sin(\frac{\sqrt{-\Omega}}{2a}\xi) + C_2 \cos(\frac{\sqrt{-\Omega}}{2a}\xi)}{C_2 \sin(\frac{\sqrt{-\Omega}}{2a}\xi) + C_1 \cos(\frac{\sqrt{-\Omega}}{2a}\xi)}. \tag{2.5}$$

3) When $b \neq 0$ and $\Omega = 0$,

$$H(\xi) = (\frac{G'}{G}) = \frac{b}{2M} + \frac{C_2}{C_1 + C_2\xi}. \tag{2.6}$$

4) When $b = 0$ and $\Delta > 0$,

$$H(\xi) = (\frac{G'}{G}) = \frac{\sqrt{\Delta}}{M}\frac{C_1 \sinh(\frac{\sqrt{\Delta}}{a}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{a}\xi)}{-C_2 \sinh(\frac{\sqrt{\Delta}}{a}\xi) + C_1 \cosh(\frac{\sqrt{\Delta}}{a}\xi)}. \tag{2.7}$$

5) When $b \neq 0$ and $\Delta < 0$,

$$H(\xi) = (\frac{G'}{G}) = \frac{\sqrt{-\Delta}}{M}\frac{-C_1 \sin(\frac{\sqrt{-\Delta}}{a}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{a}\xi)}{C_2 \sin(\frac{\sqrt{-\Delta}}{a}\xi) + C_1 \cos(\frac{\sqrt{-\Delta}}{a}\xi)}. \tag{2.8}$$

3. Extended homoclinic test approach

As an effective analytical approach, the extended homoclinic test method facilitates the construction of analytic solutions for nonlinear partial differential equations. By introducing appropriate auxiliary functions and variable transformations, Eq (1.1) has been changed into the bilinear form [18]. In this study, we employ this analytical strategy to investigate solution structures for a (3+1)-dimensional extended shallow water wave equation in nonlinear phenomena. Through careful selection of auxiliary functions and substituting them into the bilinear form, we derive a system of algebraic equations containing undetermined parameters. Through solving the algebraic equations, some analytic solutions are in explicit form. And the wave propagation mechanisms are investigated futher.

Through the transformation

$$u = -2\left[\ln f(x, y, z, t)\right]_x, \tag{3.1}$$

bilinear form for Eq (1.1) has been obtained as

$$\left(D_y D_t + D_x^3 D_y + \alpha D_x^2 + \beta D_y^2 + \gamma D_x D_y + \delta D_y D_z\right) f \cdot f = 0. \tag{3.2}$$

According to the concept of the extended homoclinic test approach [20], let the auxiliary function be defined as

$$f(x, y, z, t) = \exp(-\eta_1) + b_2 \cos(\eta_2) + b_1 \exp(\eta_1), \tag{3.3}$$

where $\eta_i = k_i(x + p_i y + q_i z + r_i t)$ and $k_i, p_i, q_i, r_i (i = 1, 2)$ are the undetermined parameters. By inserting expression (3.3) into Eq (3.1) and taking the coefficients of trigonometric-exponential terms $\cos(\eta_1)\exp(-\eta_2)$, $\cos(\eta_1)\exp(\eta_2)$, $\sin(\eta_1)\exp(-\eta_2)$, $\sin(\eta_1)\exp(\eta_2)$ to be zero, we establish an algebraic system governing the parameters $k_i, p_i, q_i, r_i (i = 1, 2)$.

$$\begin{cases} k_1^4 p_1 - \Big(3(p_1 + p_2)k_2^2 - p_1^2\beta - (q_1\delta + r_1 + \gamma)p_1 - \alpha\Big)k_1^2 - k_2^2\Big(-p_2k_2^2 + p_2^2\beta + (q_2\delta + r_2 + \gamma)p_2 + \alpha\Big) = 0, \\ 4k_1k_2b_2\Big((3p_1 + p_2)k_1^2 + (-p_1 - 3p_2)k_2^2 + (2p_2\beta + q_2\delta + r_2 + \gamma)\cdot p_1 + (q_1\delta + r_1 + \gamma)p_2 + 2\alpha\Big) = 0, \\ 2b_1b_2\Big(k_1^4 p_1 - 3(p_1 + p_2)k_2^2 + p_1^2\beta + (q_1\delta + r_1 + \gamma)p_1 + \alpha\Big)k_1^2 + p_2k_2^2 - p_2^2\beta - (q_2\delta + r_2 + \gamma)p_2 - \alpha = 0, \\ k_1k_2b_1b_2\Big((3p_1 + p_2)k_1^2 - (p_1 + 3p_2)k_2^2 + (2p_2\beta + q_2\delta + r_2 + \gamma)p_1 + (q_1\delta + r_1 + \gamma)p_2 + 2\alpha\Big) = 0, \\ b_2^2k_2^2\Big(-4p_2k_2^2 + p_2^2\beta + (q_2\delta + r_2 + \gamma)p_2 + \alpha\Big) - 8b_1k_1^2\Big(4k_1^2p_1 + p_1^2\beta + (q_1\delta + r_1 + \gamma)p_1 + \alpha\Big) = 0. \end{cases} \tag{3.4}$$

Solving (3.4), we derive the following groups of solutions.

Case I. Let

$$b_2 = 0, \quad p_2 \neq 0, \quad r_1 = -\frac{4k_1^2 p_1 + \beta p_1^2 + \alpha}{p_1} - q_1\delta - \gamma,$$
$$r_2 = \frac{3k_1^4 p_1 + 3k_1^2 k_2^2 p_1 + k_2^2\alpha}{p_2 k_2^2} + 3k_1^2 + q_2\delta + \gamma + \beta p_2 - k_2^2. \tag{3.5}$$

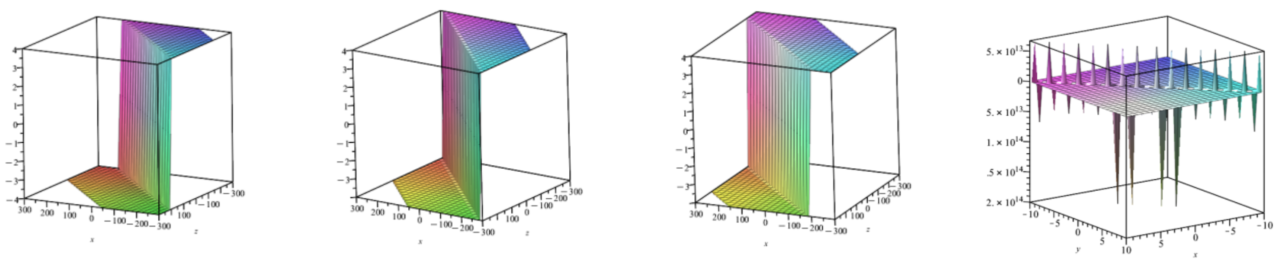
Substituting (3.5) into (3.1), we derive the solution of (1.1).

$$f_1(x, y, z, t) = \exp(-\eta_1) + b_1 \exp(\eta_1), \tag{3.6}$$

$$u_1 = \frac{2k_1 \exp(-\eta_1) - 2b_1 k_1 \exp(\eta_1)}{\exp(-\eta_1) + b_1 \exp(\eta_1)}, \tag{3.7}$$

where $\eta_1 = k_1(x + p_1 y + q_1 z - \frac{4{k_1}^2 p_1 + \beta {p_1}^2 + p_1 q_1 \delta + p_1 \gamma + \alpha}{p_1} t)$.

Assigning values to the undetermined parameters, the visualization graphs of u_1 for $t = -5, 0, 5$ are shown in Figure 1(a)–(c), respectively. When $b_1 < 0$, the visualization graph of u_1 is shown in Figure 1(d).



(a) **(b)** **(c)** **(d)**

Figure 1. u_1 evolutionary behaviors when $b_1 = k_2 = p_1 = p_2 = q_1 = q_2 = 1$, $k_1 = 2$, $y = 0$, $\alpha = \beta = \gamma = \delta = 1$. (a) $t = -5$ (b) $t = 0$ (c) $t = 5$ (d) $b_1 < 0$.

Case II. Let

$$b_2 = p_2 = 0, p_1 = -\frac{k_2^2\alpha}{3k_1^2(k_1^2 + k_2^2)},$$
$$r_1 = \frac{9k_1^8 + k_2^4\alpha\beta}{3k_1^2 k_2^2(k_1^2 + k_2^2)} + \frac{2k_1^4 - k_1^2}{k_1^2 + k_2^2} - q_1\delta - \gamma. \tag{3.8}$$

Substituting (3.8) into (3.1), we derive the solution of (1.1).

$$f_2(x, y, z, t) = \exp(-\eta_1) + b_1 \exp(\eta_1), \tag{3.9}$$

$$u_2 = \frac{2k_1 \exp(-\eta_1) - 2b_1 k_1 \exp(\eta_1)}{\exp(-\eta_1) + b_1 \exp(\eta_1)}. \tag{3.10}$$

When $b_1 \geq 0$,

$$u_2 = -2k_1 \tanh\left(\eta_1 + \ln\sqrt{b_1}\right),$$

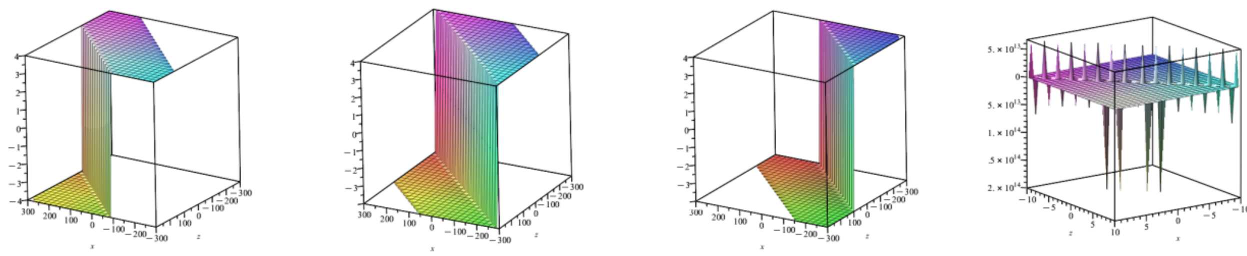
and when $b_1 < 0$,

$$u_2 = -2k_1 \coth\left(\eta_1 + \ln\sqrt{b_1}\right),$$

where

$$\eta_1 = k_1 x + k_1 t\left(\frac{9k_1^8 + k_2^4\alpha\beta}{3k_1^2k_2^2(k_1^2+k_2^2)} + \frac{2k_1^4 - k_1^2k_2^2}{k_1^2+k_2^2} - q_1\delta - \gamma\right) - \frac{k_2^2\alpha y}{3k_1(k_1^2+k_2^2)} + k_1 q_1 z.$$

Assigning values to the undetermined parameters, the visualization graphs of u_2 for $t = -5, 0, 5$ are shown in Figure 2(a)–(c), respectively. When $b_1 < 0$, the visualization graph of u_2 is shown in Figure 2(d).



(a) **(b)** **(c)** **(d)**

Figure 2. u_2 evolutionary behaviors when $b_1 = k_2 = p_1 = p_2 = q_1 = q_2 = 1$, $k_1 = 2$, $y = 0$, $\alpha = \beta = \gamma = \delta = 1$. (a) $t = -5$ (b) $t = 0$ (c) $t = 5$ (d) $b_1 < 0$.

Case III. Let

$$\begin{aligned}
r_2 =& \frac{3b_2^2k_1^2k_2^2 + 24b_1k_1^4 + 12b_1k_1^2k_2^2}{\left(-b_2^2+4b_1\right)k_2^2} - (q_1\delta+\gamma) + k_2^2,\\
r_1 =& \frac{b_1b_2^2\left(k_1^4 - 2k_1^2k_2^2 - 3k_2^4\right)}{\left(4b_1 - b_2^2\right)\left(k_1^2+k_2^2\right)} + \frac{4k_1^2b_1^2\left(3k_1^4 + 2k_1^2k_2^2 - k_2^2\right)}{\left(4b_1-b_2^2\right)k_2^2\left(k_1^2+k_2^2\right)}\\
&+ \frac{k_2^2\left(4b_1 - b_2^2\right)\alpha\beta}{12k_1^2\left(k_1^2+k_2^2\right)} + b_1(q_1\delta+\gamma),\\
p_1 =& -\frac{k_2^2\alpha(-b_2^2+4b_1)}{12b_1k_1^2\left(k_1^2+k_2^2\right)}, \quad p_2 = 0.
\end{aligned} \tag{3.11}$$

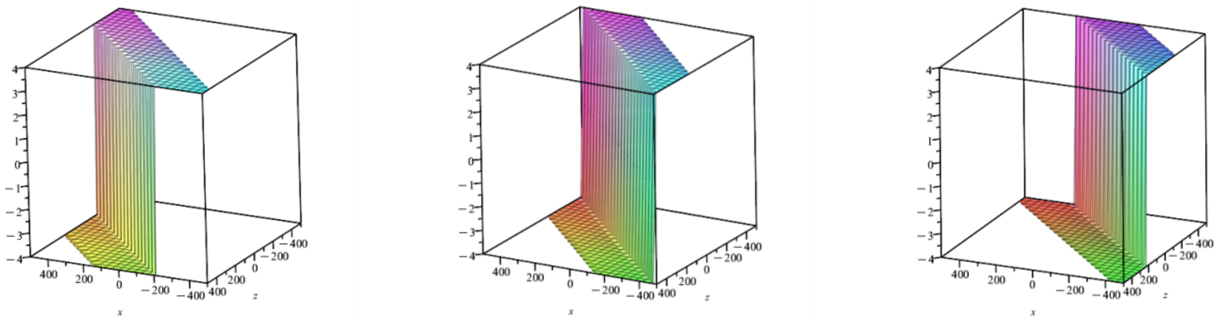
Substituting Eq (3.11) into Eq (3.1), we derive the solution of Eq (1.1).

$$f_3(x,y,z,t) = \exp(-\eta_1) + b_1\cos(\eta_2) + b_1\exp(\eta_1), \tag{3.12}$$

$$u_3 = \frac{2\left[k_1\exp(-\eta_1) + b_2k_2\sin(\eta_2) - b_1k_1\exp(\eta_1)\right]}{\exp(-\eta_1) + b_2\cos(\eta_2) + b_1\exp(\eta_1)}, \tag{3.13}$$

where $\eta_1 = k_1(x + p_1y + q_1z + r_1t)$, $\eta_2 = k_2(x + q_2z + r_2t)$.

Assigning values to the undetermined parameters, the visualization graphs of u_3 for $t = -5, 0, 5$ are shown in Figure 3(a)–(c), respectively.



(a) (b) (c)

Figure 3. u_3 evolutionary behaviors when $b_1 = k_2 = p_1 = p_2 = q_1 = q_2 = 1$, $k_1 = 2$, $y = z = 0$, $\alpha = \beta = \gamma = \delta = 1$. (a) $t = -5$ (b) $t = 0$ (c) $t = 5$.

Case IV. Let

$$
\begin{aligned}
b_1 &= -\frac{k_2^2 b_2^2(3k_2^4p_2^3 + 3k_1^4p_1^2p_2 + 3k_1^2k_2^2p_1^2p_2 + 3k_1^2k_2^2p_2^3 - k_1^2\alpha p_1^2 + 2k_1^2\alpha p_1p_2 - k_1^2\alpha p_2^2)}{4k_1^2\left(3k_1^4p_1^3 + 3k_1^2k_2^2p_1^3 + 3k_1^2k_2^2p_1p_2^2 + k_2^4p_1p_2^2 + k_2^2\alpha p_1^2 - 2k_2^2\alpha p_1p_2 + k_2^2\alpha p_2^2\right)},\\
q_2 &= \frac{2k_1^2\alpha p_1 - k_1^2\alpha p_2 + k_2^2\alpha p_2}{\delta(k_1^2p_1^2 + k_2^2p_2^2)} - \frac{\beta p_2 + \gamma + r_2 - k_2^2 + 3k_1^2}{\delta},\\
r_1 &= -\frac{k_1^2\alpha p_1 - k_2^2\alpha p_1 + 2k_2^2\alpha p_2}{k_1^2p_1^2 + k_2^2{p_2}^2} - k_1^2 + 3k_2^2 - \beta p_1 - \gamma - q_1\delta.
\end{aligned}
\tag{3.14}
$$

Substituting (3.14) into (3.1), we derive the solution of (1.1).

$$f_4(x, y, z, t) = \exp(-\eta_1) + b_2\cos(\eta_2) + b_1\exp(\eta_1), \tag{3.15}$$

$$u_4 = \frac{2[k_1\exp(-\eta_1) + b_2k_2\sin(\eta_2) - b_1k_1\exp(\eta_1)]}{\exp(-\eta_1) + b_2\cos(\eta_2) + b_1\exp(\eta_1)}. \tag{3.16}$$

When $b_1 \geq 0$,

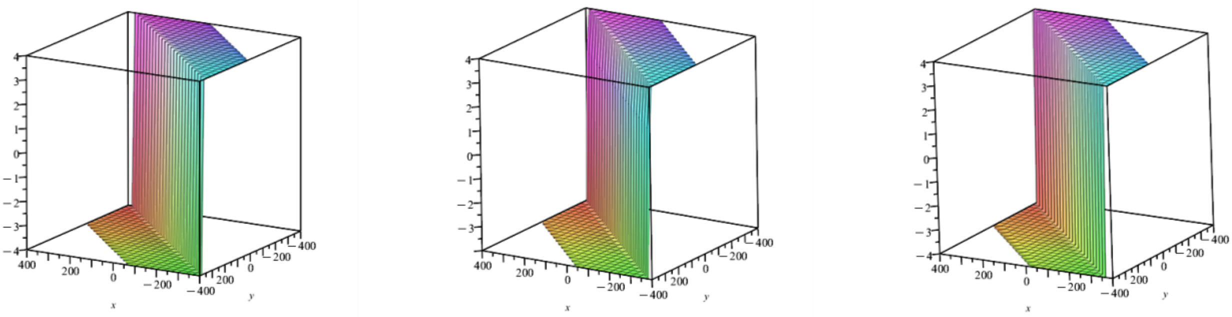
$$u_4 = -2k_1\frac{2\sqrt{b_1}\sinh\left(\eta_1 + \ln\sqrt{b_1}\right) - \frac{b_2k_2}{k_1}\sin\eta_2}{2\sqrt{b_1}\cosh\left(\eta_1 + \ln\sqrt{b_1}\right) + b_2\cos\eta_2},$$

and when $b_1 < 0$,

$$u_4 = -2k_1\frac{2\sqrt{-b_1}\cosh\left(\eta_1 + \ln\sqrt{-b_1}\right) - \frac{b_2k_2}{k_1}\sin\eta_2}{2\sqrt{-b_1}\sinh\left(\eta_1 + \ln\sqrt{-b_1}\right) + b_2\cos\eta_2},$$

where $\eta_1 = k_1(x + p_1y + q_1z + r_1t), \eta_2 = k_2(x + p_2y + q_2z + r_2t)$.

Assigning values to the undetermined parameters, the visualization graphs of u_4 for $t = -5, 0, 5$ are shown in Figure 4(a)–(c), respectively.



(a) (b) (c)

Figure 4. u_4 evolutionary behaviors when $b_1 = k_2 = p_1 = p_2 = q_1 = q_2 = 1$, $k_1 = 2$, $y = z = 0$, $\alpha = \beta = \gamma = \delta = 1$. (a) $t = -5$ (b) $t = 0$ (c) $t = 5$.

From the expressions (3.7), (3.10), (3.13), and (3.16), we observe that the two-parameter solutions u_1 and u_2 belong to the first type of solutions, while u_3 and u_4 belong to the second type. The mathematical structure of the first type of solutions can be analyzed based on the sign of the parameter b_1. When $b_1 \geq 0$, the solution reduces to $u_1 = -2k_1 \tanh\left(\eta_1 + \ln\sqrt{b_1}\right)$, which demonstrates the smoothness, boundedness, and monotonic asymptotic properties of the hyperbolic tangent function, corresponding to the classical solitary wave solution, as shown in Figures 1(a)–(c) and 2(a)–(c). The waveform evolves with t and propagates continuously along the negative half-axis of the z-axis, showing shape preservation during propagation.

When $b_1 < 0$, the solution is expressed as $u_1 = -2k_1 \coth\left(\eta_1 + \ln\sqrt{b_1}\right)$, where the singularity of the solution is determined by the poles of the hyperbolic cotangent function. When the denominator term $\sinh\left(\eta_1 + \ln\sqrt{b_1}\right)$ approaches zero, the solution exhibits divergent behavior, with a critical point $\eta_1^* = \frac{1}{2}\ln\left(\frac{-1}{b_1}\right)$, where the denominator becomes zero, leading to a blow-up phenomenon and forming rogue waves with highly localized energy concentration. The gradient of such solutions increases sharply near the critical point, revealing the nonlinear focusing effect of rogue waves, yielding a singular solution (Figures 1(d) and 2(d)). Therefore, to obtain smooth solutions, it is necessary to satisfy $\eta_1 \neq \frac{1}{2}\ln\left(-\frac{1}{b_1}\right)$.

The second type of solutions, u_3 and u_4, contain both exponential and trigonometric terms in their expressions, characterizing them as breather wave solutions. The distinctive feature of breather waves lies in their periodic oscillatory behavior during propagation, where the energy is localized but not entirely concentrated at a single point. When $b_1 \geq 0$, the solution form of u_4 simplifies to a combination of hyperbolic and trigonometric functions. When $b_1 < 0$, zeros may appear in the denominator of the solution, leading to enhanced singularities and the formation of transient rogue wave patterns with high gradients. Therefore, the second type of solutions must satisfy $b_1 \neq \frac{\exp(-\eta_1)+b_2\cos(\eta_2)}{\exp(\eta_1)}$ to avoid the denominator becoming zero.

Both types of solutions essentially belong to hyperbolic function-type solutions, and their differences come from the influence of the parameter b_1 on the existence of denominator zeros, establishing a theoretical basis for controlling the singularity of nonlinear equation solutions. By analyzing the solution forms of u_1 and u_2, we see that they are both composed of hyperbolic functions, and as $t \to +\infty$, the solutions approach the steady-state value $-2k_1$. For the second type of solutions, u_3 and u_4, although their expressions include trigonometric terms, the contribution of these periodic oscillations weakens as $t \to +\infty$. Therefore, the asymptotic behavior of u_3 and u_4 is also determined by the dominant exponential term, ultimately approaching $-2k_1$. This theoretical analysis shows excellent agreement with the numerical simulation results.

4. Improved (G'/G)-expansion method

Firstly, perform a wave transformation on Eq (1.1). Let $\xi = r_1x+r_2y+r_3z-r_4t$, where $r_1, r_2, r_3,$ *and* r_4 are the undetermined coefficients. Thus, the unknown function is transformed as follows:

$$u(\xi) = u(r_1x + r_2y + r_3z - r_4t). \tag{4.1}$$

Substituting (4.1) into Eq (1.1), and by wave transformation, we obtain the equation

$$r_2r_4u'' - r_1^3r_2u''' + 6r_1^2r_2u'u'' - \alpha r_1^2u'' - \beta r_2^2u'' - \gamma r_1r_2u'' - \delta r_2r_3u'' = 0. \tag{4.2}$$

Integrating both sides of Eq (4.2) yields

$$r_1^3r_2u''' - 3r_1^2r_2(u')^2 + (\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4)u' + m = 0. \tag{4.3}$$

Consider the solution of Eq (4.3), which can be represented in the following form:

$$u(\xi) = \sum_{i=-N}^{N} e_i\,(d + H(\xi))^i\,, \tag{4.4}$$

where $H(\xi) = \frac{G'(\xi)}{G(\xi)}$ are the undetermined coefficients, $i = 0, \pm1, \pm2, \ldots, \pm N$, $G(\xi)$ satisfies the new auxiliary equation (2.3). By the method of repeated balance, we can determine the value of N. From Eq (4.3), it is known that the highest power of u with respect to $(d + H)$ is N, the highest power of $(u')^2$ with respect to $(d + H)$ is $2(N + 1)$, and the highest power of u''' with respect to $(d + H)$ is $N + 3$. By the method of repeated balance, balancing the highest order nonlinear term $(u')^2$ with the linear term u''', we get $2(N + 1) = N + 3$. It follows that $N = 1$. When $N = 1$, Eq (4.4) can be expressed as

$$u(\xi) = e_{-1}(d + H)^{-1} + e_0 + e_1(d + H)^1. \tag{4.5}$$

Next, we will derive the exact solution expressions for Eq (4.5) when $d = 0$ and when $d \neq 0$, and then analyze the form of the solutions under different circumstances through numerical simulation and plotting of the solution graphs.

4.1. Case with non-negative exponent and a constant d

Using the auxiliary equation (2.3), the solution is computed by removing the negative exponential power, when $u(\xi)$ can be expressed as

$$u(\xi) = e_0 + e_1(d + H(\xi)), \tag{4.6}$$

where e_0, e_1 are arbitrary constants and $H(\xi) = \frac{G'(\xi)}{G(\xi)}$. The equation can be transformed into a polynomial on $(d+H)^i (i = 0,1,2,3,4)$. By equating the coefficients of various $H(\xi)$ power terms to zero, we derive a system of algebraic equations that includes the following components:

$$\begin{cases} r_1^2 r_2 e_1 M^2 (2r_1 M + ae_1) = 0, \\ r_1^2 r_2 (12d + 6b)(2r_1 M + ae_1) = 0, \\ \left(-36d^2M^3 - 36dbM^2 + 8eM^2 - 7b^2M\right) e_1 r_1 r_2 - 3e_1^2 a(6d^2M^2 + 6adMb + b^2 - 2Me) \\ \quad - (\alpha r_1^2 + \beta r_2^2 + \gamma r_1 r_2 + \delta r_2 r_3 - r_2 r_4) e_1 a^2 M = 0, \\ \left(24d^3M^3 + 36d^2M^2b - 16dM^2e + 14dMb^2 - 8Mbe + e_1 b^3\right) e_1 r_1^3 r_2 \\ \quad - 3\left(-4e^2d^3aM^2 - 6ae^2d^2Mb - 2e_1^2 dab^2 + 4ae_1^2 dMe + 2e_1^2 abe\right) r_1^2 r_2 \\ \quad + (\alpha r_1^2 + \beta r_2^2 + \gamma r_1 r_2 + \delta r_2 r_3 - r_2 r_4)\left(2e_1 da^2 M + e_1 a^2 b\right) = 0, \\ e_1\left(-6d^4M^3 - 12d^3M^2b + 8d^2M^2e - 7d^2Mb - 8e^2dMb - db^3 - 2Me^2 + b^2e\right) \\ \quad - 3e_1^2 a(d^4M^2 + 2d^3Mb + d^2b^2 - 2d^2Me - 2dbe + e^2) r_1^2 r_2 \\ \quad + e_1 a^2((\alpha r_1^2 + \beta r_2^2 + \gamma r_1 r_2 + \delta r_2 r_3 - r_2 r_4)\left(-d^2M - db + e\right) + a^3 m = 0. \end{cases}$$

By solving the above algebraic equations by Maple, a set of coefficient relations that are consistent with the solution can be obtained.

Case I. Let e_0 be an arbitrary constant, $e_1 = -\frac{2r_1 M}{a}$,

$$\begin{aligned} r_3 = {} & \frac{36dr_1M(Mdr_2 + Md + ab - br_2)}{a^2\delta r_2} - \frac{\alpha r_1^2 + \beta r_2^2 + r_1 r_2 \gamma}{\delta r_2} \\ & + \frac{8Mer_1 + a^2 r_4 - 7b^2 r_1}{a^2 \delta r_1} - \frac{12Mer_1 - 6b^2 r_1}{a^2 \delta r_2}, \end{aligned}$$

$$\begin{aligned} m = {} & \frac{14bMr_1^2 r_2(b^2 d - e)}{a^4} + \frac{24M^3 r_1^4 r_2(bd^3 - d^2 e)}{a^4} + \frac{12M^2 r_1^4 r_2(b^2d^2 - 2bde + e^2)}{a^4} \\ & + \frac{8M^3 r_1^2 r_2(18bd^3 - 11d^2 e)}{a^4} + \frac{12M^2 r_1^2(6abde - 6ab^2d^2 - b^2d^2 - 2e^2)}{a^4} \\ & + \frac{24M^3 r_1^2(3abd^3 - 3bd^3 + 2d^2 e)}{a^4} + \frac{2M^2 r_1^2 r_2(43b^2d^2 - 44bde + 8e^2)}{a^4} \\ & + \frac{2b^2 M r_1(e - 2bd) + 8Mabde^2}{a^4} + \frac{12M^4 d^4 r_1(r_1^3 r_2 + 6r_1 r_2 - 6r_1 - 1)}{a^4} \\ & + \frac{8M^3 r_1(2d^2 e - 3bd^3)}{a^4} + \frac{2M^2 r_1(-2e^2 + 12bder_1 - 7bd^2)}{a^4} \\ & + \frac{12Mb^2 r_1^2(e - bd)}{a^4}, \end{aligned}$$

where $M = a - c$.

From Case I, when $e_0 \neq 0, e_1 \neq 0$, and conditions (2.4)–(2.8) are satisfied, the expression of the solution is

$$u_{1j}(\xi) = e_0 + e_1(d + H(\xi)), \quad j = 1,2,3,4,5.$$

Taking the expressions of $H(\xi)$ from (2.4)–(2.8), when $C_1 = 0, C_2 \neq 0$, the corresponding particular solutions for the above five cases can be obtained, where $u_{1j1}(j = 1,2,3,4,5)$ are the particular solutions corresponding to $u_{1j}(j = 1,2,3,4,5)$, as shown below

$$
\begin{aligned}
u_{111}(\xi) &= e_0 + e_1\left(d + \frac{b}{2M} + \frac{\sqrt{\Omega}}{2M}\coth\left(\frac{\sqrt{\Omega}}{2a}\xi\right)\right),\\
u_{121}(\xi) &= e_0 + e_1\left(d + \frac{b}{2M} + \frac{\sqrt{-\Omega}}{2M}\cot\left(\frac{\sqrt{-\Omega}}{2a}\xi\right)\right),\\
u_{131}(\xi) &= e_0 + e_1\left(d + \frac{b}{2M} + \frac{1}{\xi}\right) + e_2\left(d + \frac{b}{2M} + \frac{1}{\xi}\right)^2,\\
u_{141}(\xi) &= e_0 + e_1\left(d + \frac{\sqrt{\Delta}}{M}\coth\left(\frac{\sqrt{\Delta}}{a}\xi\right)\right),\\
u_{151}(\xi) &= e_0 + e_1\left(d + \frac{\sqrt{-\Delta}}{M}\cot\left(\frac{\sqrt{-\Delta}}{a}\xi\right)\right).
\end{aligned}
$$

4.2. Case with negative exponent and no constant d

Using the auxiliary equation (2.3), compute the solution without the constant d, when $u(\xi)$ can be expressed as

$$u(\xi) = e_{-1}\left(H(\xi)\right)^{-1} + e_0 + e_1\left(H(\xi)\right), \tag{4.7}$$

where e_{-1}, e_0, e_1 are arbitrary constants and $H(\xi) = \frac{G'(\xi)}{G(\xi)}$. The equation can be transformed into a polynomial on $(H(\xi))^i$ $(i = -4, \ldots, 4)$, by equating the coefficients of various $H(\xi)$ power terms to zero, we derive a system of algebraic equations that includes the following components:

$$
\begin{cases}
-3e_1e^2r_1^2r_2\left(2r_1e + e_{-1}a\right) = 0,\\
-6e_1ebr_1^2r_2\left(2r_1e + e_{-1}a\right) = 0,\\
\left(8e_{-1}Me - 7e_{-1}b^2\right)er_1^3r_2 - 3\left(e_{-1}b^2 - 2e_1e^2 - 2e_{-1}eM\right)ae_{-1}r_1^2r_2\\
\qquad - e_{-1}e\cdot a^2(\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4) = 0,\\
\left(8e_1ebM - e_{-1}b^3\right)r_1^3r_2 + 6be_{-1}ar_1^2r_2\left(e_{-1}M + 2e_1e\right)\\
\qquad - e_{-1}ba^2(\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4) = 0,\\
\left(-2e_1Me^2 + e_1b^2e - 2e_{-1}eM^2 + e_1b^2M\right)r_1^3r_2\\
\qquad - 3\left(e_1^2e^2 + e_{-1}^2M^2 + 4e_{-1}e_1\cdot eM - 2e_1e_{-1}b^2\right)ar_1^2r_2\\
\qquad + a^2(\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4)\left(e_1e + e_{-1}M\right) + a^3m = 0,\\
e_1r_1^3r_2b\left(-8eM + b^2\right) - 3\left(2e_1e + 4e_{-1}M\right)e_1abr_1^2r_2\\
\qquad + (\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4)e_1a^2b = 0,\\
e_1r_1^3r_2M\left(8eM - 7b^2\right) - 3\left(e_1b^2 - 2e_{-1}M^2 - 2e_1eM\right)ae_1r_1^2r_2\\
\qquad - (\alpha r_1^2 + \beta r_2^2 + \gamma r_1r_2 + \delta r_2r_3 - r_2r_4)e_1Ma^2 = 0,\\
16e_1bMr_1^2r_2\left(2Mr_1 + a\right) = 0,\\
-3r_1^2r_2M^2e_1\left(2Mr_1 + a\right) = 0.
\end{cases}
$$

By solving the above algebraic equations by Maple, two sets of coefficient relations that are consistent with the solution can be obtained.

Case II. Let e_0 be an arbitrary constant, $e_{-1} = -\frac{2r_1 e}{a}$, $e_1 = 0$

$$r_3 = \frac{-7b^2er_1^3 - 4Mer_1^3 + 6b^2r_1^3}{a^2\delta} - \frac{\alpha r_1^2 + \beta r_2^2 + \lambda r_1 r_2}{\delta r_2} - \frac{r_1^3(-7b^2e + 8Me + 7b^2)}{2a^2\delta},$$

$$r_4 = \frac{r_1^3(-7b^2e + 8Me + 7b^2)}{2a^2},\ m = -\frac{2eb^2r_1^4r_2M(7e-6)}{a^4},$$

where $M = a - c$.

From Case II, when $e_0 = 0$, and conditions (2.4)–(2.8) are satisfied, the expression for the solution is

$$u_{2j}(\xi) = e_{-1}H(\xi) + e_0,\ j = 1, 2, 3, 4, 5.$$

Taking the expressions of $H(\xi)$ from (2.4)–(2.8), when $C_1 = 0, C_2 \neq 0$, the corresponding particular solutions for the above five cases can be obtained, where $u_{2j1}(j = 1, 2, 3, 4, 5)$ are the particular solutions corresponding to $u_{2j}(j = 1, 2, 3, 4, 5)$, as shown below

$$\begin{aligned}
u_{211}(\xi) &= e_0 + e_{-1}(\frac{b}{2M} + \frac{\sqrt{\Omega}}{2M}\coth(\frac{\sqrt{\Omega}}{2a}\xi))^{-1},\\
u_{221}(\xi) &= e_0 + e_{-1}(\frac{b}{2M} + \frac{\sqrt{-\Omega}}{2M}\cot(\frac{\sqrt{-\Omega}}{2a}\xi))^{-1},\\
u_{231}(\xi) &= e_0 + e_{-1}(\frac{b}{2M} + \frac{1}{\xi})^{-1},\\
u_{241}(\xi) &= e_0 + e_{-1}(\frac{\sqrt{\Delta}}{M}\coth(\frac{\sqrt{\Delta}}{a}\xi))^{-1},\\
u_{251}(\xi) &= e_0 + e_{-1}(\frac{\sqrt{-\Delta}}{M}\cot(\frac{\sqrt{-\Delta}}{a}\xi))^{-1}.
\end{aligned}$$

4.3. Case with negative exponent and a constant d

Using the auxiliary equation (2.3), compute the solution without the constant d, when $u(\xi)$ can be expressed as

$$u(\xi) = e_1\left(d + H(\xi)\right)^{-1} + e_0 + e_1\left(d + H(\xi)\right), \tag{4.8}$$

where e_{-1}, e_0, e_1 are arbitrary constants and $H(\xi) = \frac{G'(\xi)}{G(\xi)}$. The equation can be transformed into a polynomial on $(d + H(\xi))^i (i = -4, \ldots, 4)$, by equating the coefficients of various $H(\xi)$ power terms to zero, we derive a system of algebraic equations that includes the following components

$$\begin{cases} -3e_{-1}r_1^2r_2s^2(2r_1s+ae_{-1})=0,\\ sh(2r_1s+ae_{-1})=0,\\ r_1^2r_2e_{-1}\left(7h^2s-8r_1Ms^2+3e_{-1}ah^2-6Mas\right)-6e_1e_{-1}r_2as^2+a^2e_{-1}(\alpha r_1^2\\ \qquad +\beta r_2^2+\gamma r_1r_2+\delta r_2r_3-r_2r_4)s=0,\\ r_1^2r_2e_{-1}h\left(r_1h^2-8Ms-6e_{-1}aM-12e_1as\right)+e_{-1}a^2h(\alpha r_1^2+\beta r_2^2+\gamma r_1r_2\\ \qquad +\delta r_2r_3-r_2r_4)=0,\\ r_1^2r_2\left(r_1e_1h^3+8r_1e_1M^2s-6e_1^2ahs-12e_{-1}e_1aMh\right)+a^2e_1(\alpha r_1^2+\beta r_2^2+\gamma r_1\cdot\\ \qquad r_2+\delta r_2r_3-r_2r_4)h=0,\\ r_1^2r_2e_1M\left(-7r_1Mh^2+8r_1M^2s-3e_1ah^2+6e_{-1}aM^2+6e_1aMs\right)-e_1a^2(\alpha r_1^2\\ \qquad +\beta r_2^2+\gamma r_1r_2+\delta r_2r_3-r_2r_4)=0,\\ e_1r_1^3r_2Mh^2-2e_1r_1^3r_2M^2s+e_1r_1^3r_2h^2s-2e_1r_1^3r_2Ms^2-3e_1^2r_1^2r_2as^2-3e_{-1}r_1^2\cdot\\ r_2aM^2-12e_{-1}e_1r_1^2r_2aMs+6e_{-1}e_1r_1^2r_2ah^2+e_1a^2(\alpha r_1^2+\beta r_2^2+\gamma r_1r_2-r_2r_4)\cdot\\ \qquad s+e_1a^2(\alpha r_1^2+\beta r_2^2+\gamma r_1r_2+\delta r_2r_3-r_2r_4)M+a^3m=0,\\ Mh(-2r_1M-e_1a)=0,\\ e_1r_1^2r_2M^2(2r_1M+e_1a)=0. \end{cases}$$

By solving the above algebraic equations by Maple, two sets of coefficient relations that are consistent with the solution can be obtained.

Case III. Let e_0 be an arbitrary constant, $e_{-1}=-\frac{2r_1s}{a}$, $e_1=0$,

$$r_3=-\frac{4Msr_1^3+h^2r_1^3}{a^2\delta}-\frac{\alpha r_1^2+\beta r_2^2+\gamma r_1r_2-r_2r_4}{\delta r_2},$$

$$m=\frac{3r_1^2r_2(2M^2e_1r_1s+2Me_1r_1s^2)}{a^3}+\frac{3r_1^2r_2(Me_2r_1-ce_2r_1)}{a}+\frac{3r_1^2r_2\left(4Me_1e_2r_1s+c^2e_2r_1-2h^2e_1e_2+e_1^2s^2\right)}{a^2},$$

where $s=-d^2M-bd+e$, $\quad h=b+2dM$.

From Case III, when $e_0=0$, and conditions (2.4)–(2.8) are satisfied, the expression for the solution is

$$u_{3j}(\xi)=e_{-1}(d+H(\xi))^{-1}+e_0,\ j=1,2,3,4,5.$$

Taking the expressions of $H(\xi)$ from (2.4)–(2.8), when $C_1=0, C_2\neq 0$, the corresponding particular solutions for the above five cases can be obtained, where $u_{3j1}(j=1,2,3,4,5)$ are the particular solutions corresponding to $u_{3j}(j=1,2,3,4,5)$, as shown below

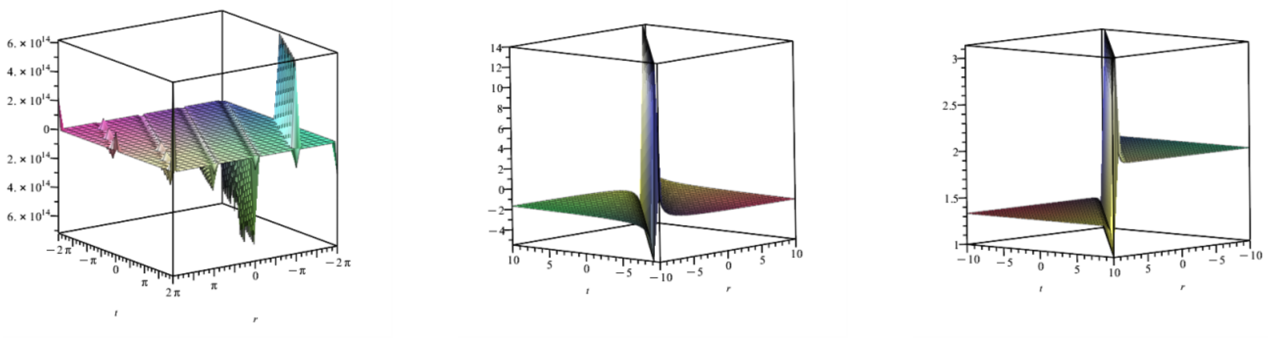
$$u_{311}(\xi)=e_0+e_{-1}(d+\frac{b}{2M}+\frac{\sqrt{\Omega}}{2M}\coth(\frac{\sqrt{\Omega}}{2a}\xi))^{-1},$$

$$u_{321}(\xi) = e_0 + e_{-1}(d + \frac{b}{2M} + \frac{\sqrt{-\Omega}}{2M}\cot(\frac{\sqrt{-\Omega}}{2a}\xi))^{-1},$$
$$u_{331}(\xi) = e_0 + e_{-1}(d + \frac{b}{2M} + \frac{1}{\xi})^{-1},$$
$$u_{341}(\xi) = e_0 + e_{-1}(d + \frac{\sqrt{\Delta}}{M}\coth(\frac{\sqrt{\Delta}}{a}\xi))^{-1},$$
$$u_{351}(\xi) = e_0 + e_{-1}(d + \frac{\sqrt{-\Delta}}{M}\cot(\frac{\sqrt{-\Delta}}{a}\xi))^{-1}.$$

From the solution expressions under different cases, it can be observed that there are significant differences between the expressions containing negative power terms of d and those without negative power terms of d. Thereby some new solutions are obtained. Taking cases 1, 2, and 3 as examples, using Matlab software with two parameter sets ($r_i = 2$ and $r_i = 1$), the exact solution simulation graphs of u_{151} (Figures 5(a) and 6(a)), u_{231} (Figures 5(b) and 6(b)), and u_{341} (Figures 5(c) and 6(c)) were plotted with $r = x + y + z$ and t as the independent variables.

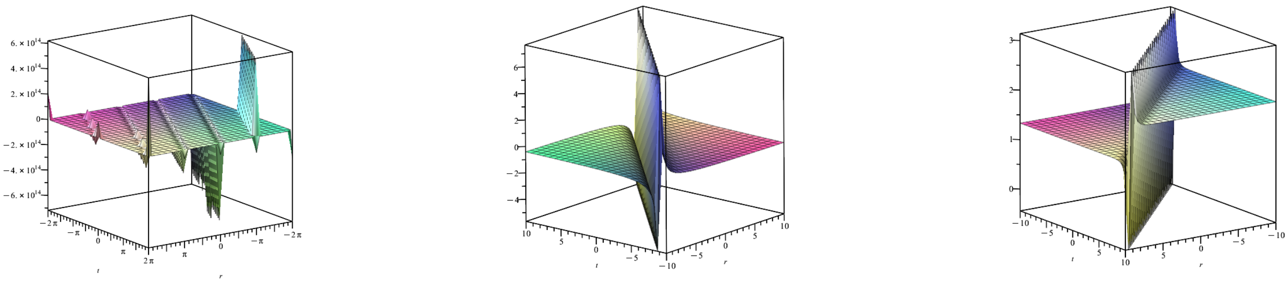
- The trigonometric solution u_{151} forms a regular oscillatory pattern through the periodic characteristics of the cotangent function (Figures 5(a) and 6(a)). The consistent spike structure across parameter variations confirms the solution's inherent periodicity, with only minor amplitude modulation when r_i changes from 2 to 1.
- The rational solution u_{231} exhibits an asymmetric sharp peak due to its linear singularity term $\frac{1}{\xi}$ (Figures 5(b) and 6(b)). The preserved singularity structure under parameter scaling demonstrates its fundamental algebraic nature.
- The hyperbolic solution u_{341} shows a smooth solitary wave profile with rapid energy decay (Figures 5(c) and 6(c)). The stable pulse shape and energy distribution confirm the solution's robustness against parameter variations.

This comprehensive analysis demonstrates that all three solution types maintain their fundamental characteristics when parameters r_i are changed, validating the structural stability of solutions obtained through the improved (G'/G)-expansion method.



(a) (b) (c)

Figure 5. Solution profiles under standardized parameters $r_1 = r_2 = r_3 = r_4 = 2$, $a = 2$, $c = 1$, $M = 1$, $C_1 = 0$, $C_2 = 1$, $\alpha = \beta = 1$ demonstrating three distinct solution types: (a) Trigonometric solution u_{151} (b) Rational solution u_{231} (c) Hyperbolic solution u_{341}.



(a) (b) (c)

Figure 6. Solution profiles under standardized parameters $r_1 = r_2 = r_3 = r_4 = 1$, $a = 2$, $c = 1$, $M = 1$, $C_1 = 0$, $C_2 = 1$, $\alpha = \beta = 1$ demonstrating three distinct solution types: (a) Trigonometric solution u_{151} (b) Rational solution u_{231} (c) Hyperbolic solution u_{341}.

5. Conclusions

This study systematically investigates various analytic solutions and their dynamics characteristics of the (3+1)-dimensional extended shallow water wave equation in nonlinear phenomena by combining the extended homoclinic test approach and the improved (G'/G)-expansion method. Note that these methods yield solutions that were previously unreported for this equation, such as hybrid trigonometric-hyperbolic breather waves and rational-rogue wave combinations, significantly enriching the diversity of known solutions. The complementary advantages of the two methods are demonstrated in solving high-dimensional nonlinear equations.

Through the extended homoclinic test approach, two types of analytic solutions are successfully constructed. One is a hyperbolic-function solitary wave solution, which is characterized by waveform invariance during propagation. The other is a breather wave solution combining hyperbolic and trigonometric functions, which is not a class of solutions obtained generally by conventional techniques like Hirota's bilinear method or inverse scattering transforms. These solutions exhibit periodic oscillations and energy localization. By analyzing parameters, the critical conditions for rogue wave generation are revealed.

In addition, three types of analytical solutions are obtained using the improved (G'/G)-expansion method, including trigonometric function solutions, rational fractional solutions, and hyperbolic function solutions. Among these, the rational fractional solutions demonstrate significant nonlinear singularity under specific parameter conditions, revealing the divergence behavior of the equation near critical points, which is a type of solution that cannot be obtained by the homoclinic test approach.

The combination of these two methods provides an idea for analyzing high-dimensional nonlinear wave dynamics, demonstrating potential applications in understanding physical phenomena such as oceanic rogue waves and plasma solitons. Future research may extend these methods to other high-dimensional nonlinear partial differential equations to uncover more complex physical phenomena.

Author contributions

Xixi Wu: Method, software, writing-original draft, writing-review and editing; Minkun Xiong: Method, writing-original draft, writing-review and editing; Yanfeng Guo: Supervision, method, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in this article.

Acknowledgments

The work was supported by National Natural Science Foundation of China No. 12261053; the Natural Science Foundation of Hubei Soliton Research Association No. 2025HBSRA02; the University Student Independent Innovation Funding Program Grant No. 2024XLB33 and the National College Student Innovation and Entrepreneurship Training Program Grant No. S202410491225.

Conflict of interest

The authors declare no conflicts of interest.

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