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Research article

Two regularization methods for identifying the unknown source term of space-time fractional diffusion-wave equation

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Abstract: In this paper, the inverse problem of identifying the unknown source for the space-time fractional diffusion-wave equation is researched. This problem is ill-posed and needs the regularization approach to solve this inverse problem. The fractional Tikhonov regularization method and the quasi-inverse regularization method are used to obtain the fractional Tikhonov regularization solution and the quasi-inverse regularization solution, respectively. Under the a priori and the a posteriori regularization parameter selection rules, the error estimates of the regularization solutions and the exact solution are given. Finally, we provide several numerical examples to show the effectiveness of the approach.

Keywords: space-time fractional diffusion-wave equation; ill-posed; fractional Tikhonov regularization method; quasi-inverse regularization method
Mathematics Subject Classification: 35R25, 35R30, 47A52

1. Introduction

Space-time fractional diffusion-wave equations serve as powerful tools for characterizing mechanical and physical processes with historical memory and spatial nonlocality, demonstrating broad applicability across diverse fields including fluid dynamics [1], mathematical biology [2, 3], and image processing [4, 5]. Specifically, these equations have been successfully employed not only in traditional domains such as optical and thermal systems [6], fluid mechanics [7], signal processing, and system identification [8], but have also shown remarkable effectiveness in emerging applications like image denoising [4] and super-resolution reconstruction [5]. There have been a lot of research results on the initial value problems and initial marginals of positive problems of space-time fractional wave equations [9–11]. And in the process of solving practical problems, it is often encountered that certain parameters in the space-time fractional diffusion-wave are unknown, such as the unknown source term, the unknown ratio, the unknown fractional order, and the unknown boundary conditions. It is necessary to identify these unknown parameters with the help of additional conditions, which leads to the inverse problem of the space-time fractional diffusion-wave equation. However, the inverse problem is usually ill-posed and needs to be solved by means of a regularization method. There are many regularization methods for solving the inverse problems, such as the Tikhonov regularization method [12–15], the modified quasi-inverse regularization method [16], the fractional Tikhonov regularization method [17, 18], the modified Tikhonov regularization method [19, 20], the quasi-boundary regularization method [21,22], the Landweber iterative regularization method [23,24], and the fractional Landweber iterative regularization method [25,26]. These methods provide powerful tools for solving inverse problems and ill-posed problems, with broad applications across diverse fields including medical imaging [27], computed tomography [28], high-temperature superconductivity research [29], transport theory [30], geological exploration [31, 32], and diffusion analysis in heterogeneous media. Currently, regularization methods for inverse problems remain a highly active research frontier, and further exploration of novel theories and approaches is still required to enhance the accuracy and stability of inverse problem solutions.

In this paper, we consider the space-time fractional diffusion-wave equation as follows:

$$\begin{cases} \partial_t^\alpha u(x,t) + (-\Delta)^\beta u(x,t) = f(x), & x \in \Omega, t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u_t(x,0) = \psi(x), & x \in \Omega, \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is the bounded domain in R^d ($d = 1,2,3$), $\partial\Omega$ is the smooth boundary of Ω, and $T > 0$ is the fixed time. $0 < \beta < 1$ is a constant, $\varphi(x)$ and $\psi(x)$ are the initial data defined on $L^2(\Omega)$, and $\partial_t^\alpha u(x,t)$ is the Caputo fractional derivative of order α ($1 < \alpha < 2$), defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,\tau)}{\partial \tau^2} (t-\tau)^{1-\alpha}\, d\tau, \quad 1 < \alpha < 2, \tag{1.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function and the operator $(-\Delta)^\beta$ is the Laplace operator defined by

$$(-\Delta)^\beta u(x,t) = C_{d,\beta} P.V. \int_{R^d} \frac{u(x,t) - u(z,t)}{\mid x - z \mid^{d+2\beta}} dz, \quad \beta \in (0,1), \tag{1.3}$$

where $C_{d,\beta} = \dfrac{4^\beta \Gamma(d/2+\beta)}{\pi^{d/2} \mid \Gamma(-\beta) \mid}$, $P.V.$ denotes the Cauchy principal value.

In problem (1.1), if $\varphi(x)$, $\psi(x)$, and $f(x)$ are known, this is a positive problem. But if $f(x)$ is not known, it is an inverse problem, which is to identify the source term $f(x)$ using the final value data $u(x,T) = g(x)$. In the real problem, $g(x)$ is obtained from measurements. Suppose that the measurement data $g^\delta(x)$ and the exact data $g(x)$ satisfy the following criteria:

$$\left\| g(x) - g^\delta(x) \right\| \le \delta, \tag{1.4}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the measurement error.

Current literature offers relatively limited research on solving space-time fractional diffusion-wave equations using regularization methods [22, 23, 25]. There are few papers that use the fractional

Tikhonov regularization and the quasi-inverse regularization methods to identify the unknown source of the space-time fractional diffusion equation. Therefore, this paper employs two effective regularization approaches to obtain the regularization solutions and provides both the a priori and the a posteriori convergence error estimates. Numerical examples are presented to demonstrate the effectiveness of the proposed regularization methods.

The rest of the paper is organized as follows: Section 2 presents some important definitions and lemmas. The solution of the problem and ill-posed analysis are given in Section 3. In Section 4, we give convergence error estimation based on the a priori and the a posteriori regularized parameters selection rules under the fractional order Tikhonov regularization method. Simultaneously, we use the quasi-inverse regularization method to provide convergence error estimation for both the a priori and the a posteriori regularization parameter selection rules in Section 5. In Section 6, we give several numerical examples. In Section 7, we give a brief conclusion.

2. Preliminaries

In this section, we present some important definitions and lemmas.

Definition 2.1. [33] The Mittag-Leffler function is defined by

$$E_{\alpha,\overline{\beta}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \overline{\beta})}, \quad z \in \mathbb{C}, \tag{2.1}$$

where $\alpha > 0$ and $\overline{\beta} \in \mathbb{R}$.

Definition 2.2. Let λ_n, $X_n(x) \in \Omega$ be the Dirichlet eigenvalues and eigenfunctions of the Laplace operator Δ:

$$\begin{cases} -\Delta X_n(x) = \lambda_n X_n(x), & x \in \Omega, \\ X_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and $X_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, $X_n(x)$ can be regarded as an orthonormal basis in the space $L^2(\Omega)$.

For the fractional Laplace operator $(-\Delta)^{\beta}$, we have a similar definition. The spectral decomposition of the fractional Laplace operator in a bounded domain Ω is given by

$$\begin{cases} (-\Delta)^{\beta} X_n(x) = \lambda_n^{\beta} X_n(x), & x \in \Omega, \quad 0 < \beta < 1, \\ X_n(x) = 0, & x \in \partial\Omega. \end{cases}$$

Lemma 2.1. [22] Let $1 < \alpha < 2$ and $\overline{\beta} \in \mathbb{R}$. We assume that $\overline{\mu}$ satisfies $\overline{\mu} \leq \left|\arg(z)\right| \leq \pi$, $\frac{\pi\alpha}{2} < \overline{\mu} < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C_1 > 0$, such that

$$E_{\alpha,\overline{\beta}}(z) = \frac{C_1}{1 + |z|}, \quad z \in \mathbb{C}. \tag{2.2}$$

Lemma 2.2. [22] For $1 < \alpha < 2$, $\overline{\beta} \in \mathbb{R}$, $\overline{\eta} > 0$, we have

$$E_{\alpha,\overline{\beta}}(-\overline{\eta}) = \frac{1}{\Gamma(\overline{\beta} - \alpha)\overline{\eta}} + O(\frac{1}{\overline{\eta}^2}), \quad \overline{\eta} \to \infty. \tag{2.3}$$

Lemma 2.3. [22] Let $1 < \alpha < 2$ and $\lambda_n \geq \lambda_1 > 0$; then there exists a constant $C_1 > 0$, such that

$$\frac{1}{\lambda_n T^\alpha} \leq \left|E_{\alpha,\alpha+1}(-\lambda_n T^\alpha)\right| \leq \frac{C_1}{\lambda_n T^\alpha}. \tag{2.4}$$

Lemma 2.4. For arbitrary constants $\mu > 0$, $s \geq \lambda_1^\beta > 0$, the following inequality holds:

$$F_1(s) = \frac{C_1^\gamma s}{1 + \mu s^{(\gamma+1)}} \leq C_2 \mu^{-\frac{1}{\gamma+1}}, \tag{2.5}$$

where $C_2 = (\dfrac{C_1^\gamma}{\gamma})^{\frac{1}{\gamma+1}}$.

Proof. Let $s_0 > 0$ satisfy $F_1^{'}(s_0) = 0$, then

$$F_1'(s_0) = \frac{1 + \mu s_0^{\gamma+1} - \mu s_0^\gamma (\gamma+1) s_0}{\left(1 + \mu s_0^{\gamma+1}\right)^2} = 0.$$

We have $s_0 = (\frac{1}{\mu\gamma})^{\frac{1}{\gamma+1}}$, such that

$$F_1(s) \leq F_1(s_0) = \frac{C_1^\gamma (\frac{1}{\mu\gamma})^{\frac{1}{\gamma+1}}}{1 + \frac{1}{\gamma}} \leq C_2 \mu^{-\frac{1}{\gamma+1}},$$

where $C_2 = (\frac{C_1^\gamma}{\gamma})^{\frac{1}{\gamma+1}}$.

Lemma 2.5. For arbitrary constants $p > 0$, $\mu > 0$, $s \geq \lambda_1^\beta > 0$, the following inequality holds:

$$F_2(s) = \frac{\mu s^{(\gamma+1-p)}}{1 + \mu s^{(\gamma+1)}} \leq \begin{cases} C_3 \mu^{\frac{p}{\gamma+1}}, & 0 < p < \gamma+1, \\ C_4 \mu, & p \geq \gamma+1, \end{cases} \tag{2.6}$$

where $C_3 = \frac{(\frac{\gamma+1-p}{p})^{\frac{\gamma+1-p}{\gamma+1}}}{1+\frac{\gamma+1-p}{p}}$, $C_4 = \lambda_1^{\beta(\gamma+1-p)}$.

Proof. When $0 < p < \gamma+1$, let s_0 satisfy $F_2^{'}(s_0) = 0$,

$$F_2^{'}(s_0) = \frac{(\gamma+1-p)\mu s_0^{\gamma-p}(1 + \mu s_0^{\gamma+1}) - (\gamma+1)\mu^2 s_0^{2\gamma+1-p}}{(1 + \mu s_0^{\gamma+1})^2} = 0.$$

We have $s_0 = (\frac{\gamma+1-p}{p\mu})^{\frac{1}{\gamma+1}}$, such that

$$F_2(s) \leq F_2(s_0) = \frac{\mu(\frac{\gamma+1-p}{p})^{\frac{\gamma+1-p}{\gamma+1}}(\frac{1}{\mu})^{\frac{\gamma+1-p}{\gamma+1}}}{1 + \frac{\gamma+1-p}{p}} = C_3 \mu^{\frac{p}{\gamma+1}},$$

where $C_3 = \frac{(\frac{\gamma+1-p}{p})^{\frac{\gamma+1-p}{\gamma+1}}}{1+\frac{\gamma+1-p}{p}}$.

When $p \geq \gamma + 1$,

$$F_2(s) = \frac{\mu s^{(\gamma+1-p)}}{1 + \mu s^{(\gamma+1)}} \leq \mu s^{(\gamma+1-p)} \leq \mu \lambda_1^{\beta(\gamma+1-p)} \leq C_4 \mu,$$

where $C_4 = \lambda_1^{\beta(\gamma+1-p)}$.

Lemma 2.6. For arbitrary constants $p > 0$, $\mu > 0$, $s \geq \lambda_1^{\beta} > 0$, the following inequality holds:

$$F_3(s) = \frac{C_1 \mu s^{(\gamma-p)}}{1 + \mu s^{(\gamma+1)}} \leq \begin{cases} C_5 \mu^{\frac{p+1}{\gamma+1}}, & \gamma > p > 0, \\ C_6 \mu, & p \geq \gamma, \end{cases} \tag{2.7}$$

where $C_5 = \frac{C_1 (\frac{\gamma-p}{p+1})^{\frac{1}{\gamma+1}}}{1+\frac{\gamma-p}{p+1}}$, $C_6 = C_1 \lambda_1^{\beta(\gamma-p)}$.

Proof. When $\gamma - p > 0$, let s_0 satisfy $F_3'(s_0) = 0$,

$$F_3'(s_0) = \frac{C_1(\gamma - p)\mu s_0^{\gamma-p-1}(1 + \mu s_0^{\gamma+1}) - C_1(\gamma + 1)\mu^2 s_0^{2\gamma-p}}{(1 + \mu s_0^{\gamma+1})^2} = 0.$$

We have $s_0 = (\frac{\gamma-p}{p+1}\frac{1}{\mu})^{\frac{1}{\gamma+1}}$, such that

$$F_3(s) \leq F_3(s_0) = \frac{C_1 \mu (\frac{\gamma-p}{p+1})(\frac{1}{\mu})^{\frac{\gamma-p}{\gamma+1}}}{1 + \frac{\gamma-p}{p+1}} \leq C_5 \mu^{\frac{p+1}{\gamma+1}},$$

where $C_5 = \frac{C_1 (\frac{\gamma-p}{p+1})^{\frac{1}{\gamma+1}}}{1+\frac{\gamma-p}{p+1}}$.

When $p > \gamma$,

$$F_3(s) = \frac{C_1 \mu s^{(\gamma-p)}}{1 + \mu s^{(\gamma+1)}} \leq C_1 \mu s^{(\gamma-p)} \leq C_1 \mu \lambda_1^{\beta(\gamma-p)} \leq C_6 \mu,$$

where $C_6 = C_1 \lambda_1^{\beta(\gamma-p)}$.

Lemma 2.7. For arbitrary constants $p > 0$, $\eta > 0$, $s \geq \lambda_1^{\beta} > 0$, the following inequality holds:

$$F_4(s) = \frac{\eta s^{(1-p)}}{1 + \eta s} \leq \begin{cases} C_7 \eta^p, & 0 < p \leq 1, \\ C_8 \eta, & p > 1, \end{cases} \tag{2.8}$$

where $C_7 = (1 - p)^{1-p} p^{2-p}$, $C_8 = \lambda_1^{\beta(1-p)}$.

Proof. When $0 < p \leq 1$, let s_0 satisfy $F_4'(s_0) = 0$,

$$F_4'(s_0) = \frac{(1 - p)\eta s_0^{1-p}(1 + \eta s_0) - \eta^2 s_0^{1-p}}{(1 + \eta s_0)^2} = 0.$$

We have $s_0 = \frac{1-p}{p\eta}$, such that

$$F_4(s) \leq F_3(s_0) = \frac{\eta (\frac{1-p}{p\eta})^{1-p}}{\frac{1-p}{1+\frac{1-p}{p}}} = C_7 \eta^p,$$

where $C_7 = (1-p)^{1-p}p^{2-p}$.

When $p > 1$,

$$F_4(s) = \frac{\eta s^{(1-p)}}{1+\eta s} \le \eta s^{(1-p)} \le \eta\lambda_1^{\beta(1-p)} \le C_8\eta,$$

where $C_8 = \lambda_1^{\beta(1-p)}$.

Lemma 2.8. For arbitrary constants $p > 0$, $\eta > 0$, $s \ge \lambda_1^\beta > 0$, the following inequality holds:

$$F_5(s) = \frac{\eta\sqrt{C_1}s^{-\frac{1+p}{2}}}{\frac{1}{s}+\eta} \le \begin{cases} C_9\eta^{\frac{p+1}{2}}, & 0 < p \le 1, \\ C_{10}\eta, & p > 1, \end{cases} \tag{2.9}$$

where $C_9 = \frac{\sqrt{C_1}(\frac{1-p}{1+p})^{\frac{1-p}{2}}}{1+\frac{1-p}{1+p}}$, $C_{10} = \sqrt{C_1}\lambda_1^{\frac{\beta(1-p)}{2}}$.

Proof. When $0 < p \le 1$, let s_0 satisfy $F_5'(s_0) = 0$,

$$F_5'(s_0) = \frac{\sqrt{C_1}\frac{(1-p)}{2}\eta s_0^{-\frac{p+1}{2}}(1+\eta s_0) - \frac{1-p}{1+p}\eta^2 s_0^{\frac{1-p}{2}}}{(1+\eta s_0)^2} = 0.$$

We have $s_0 = \frac{1-p}{\eta(p+1)}$, such that

$$F_5(s) \le F_5(s_0) = \frac{\sqrt{C_1}\eta(\frac{1-p}{1+p})^{\frac{1-p}{2}}(\frac{1}{\eta})^{\frac{1-p}{2}}}{1+\frac{1-p}{1+p}} = C_9\eta^{\frac{p+1}{2}},$$

where $C_9 = \frac{\sqrt{C_1}(\frac{1-p}{1+p})^{\frac{1-p}{2}}}{1+\frac{1-p}{1+p}}$.

When $p > 1$,

$$F_5(s) = \frac{\eta\sqrt{C_1}s^{-\frac{1+p}{2}}}{\frac{1}{s}+\eta} \le \sqrt{C_1}s^{\frac{1-p}{2}}\eta \le \sqrt{C_1}\lambda_1^{\frac{\beta(1-p)}{2}}\eta \le C_{10}\eta,$$

where $C_{10} = \sqrt{C_1}\lambda_1^{\frac{\beta(1-p)}{2}}$.

3. The solution of the problem and ill-posed analysis

Using the method of separation of variables and Laplace transform of Mittag-Leffler function, we can obtain the exact solution to the problem (1.1). First, from the feature system of the Laplace operator, we can obtain

$$-\Delta X_n(x) = \lambda_n X_n(x),$$

then

$$(-\Delta)^\beta X_n(x) = \lambda_n^\beta X_n(x).$$

Let us first set

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x).$$

Substituting it into (1.1) yields the following:

$$T_n^{\alpha}(t)X_n(x) + \lambda_n^{\beta}T_n(t)X_n(x) = f_nX_n(x).$$

Using the Laplace transform, we obtain

$$P^{\alpha}T_n(p) - P^{\alpha-1}T_n(0) - P^{\alpha-2}T_n'(0) + \lambda_n^{\beta}T_n(p) = P^{-1}f_n,$$

then

$$T_n(p) = \frac{P^{\alpha-1}\varphi_n + P^{\alpha-2}\psi_n + P^{-1}f_n}{P^{\alpha} + \lambda_n^{\beta}}.$$

Using the inverse Laplace transform and the function of Mittag-leffler function, we obtain

$$T_n(t) = E_{\alpha,1}(-\lambda_n^{\beta}t^{\alpha})\varphi_n + tE_{\alpha,2}(-\lambda_n^{\beta}t^{\alpha})\psi_n + t^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n^{\beta}t^{\alpha})f_n.$$

Therefore, we can obtain the exact solution as follows:

$$u(x,t) = \sum_{n=1}^{\infty}\left(E_{\alpha,1}\left(-\lambda_n^{\beta}t^{\alpha}\right)\varphi_n + tE_{\alpha,2}\left(-\lambda_n^{\beta}t^{\alpha}\right)\psi_n + t^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n^{\beta}t^{\alpha}\right)f_n\right)X_n(x), \tag{3.1}$$

where λ_n and $X_n(x)$ are, respectively, the eigenvalues and eigenfunctions of the Laplace operator, and the sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$, $n \to \infty, \varphi_n = (\varphi(x), X_n(x))$, $\psi_n = (\psi(x), X_n(x))$ and $f_n = (f(x), X_n(x))(x)$ are Fourier coefficients.

We can get it by substituting the measurement data $g(x) = u(x,T)$,

$$g(x) = \sum_{n=1}^{\infty}\left(E_{\alpha,1}\left(-\lambda_n^{\beta}T^{\alpha}\right)\varphi_n + TE_{\alpha,2}\left(-\lambda_n^{\beta}T^{\alpha}\right)\psi_n + T^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n^{\beta}T^{\alpha}\right)f_n\right)X_n(x). \tag{3.2}$$

Theorem 3.1. If $I_1 = 0$, then there is a unique solution to the problem (1.1) in $L^2(\Omega)$, and

$$f(x) = \sum_{n=1}^{\infty}\frac{g_n - E_{\alpha,1}(-\lambda_n^{\beta}T^{\alpha})\varphi_n - TE_{\alpha,2}(-\lambda_n^{\beta}T^{\alpha})\psi_n}{T^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n^{\beta}T^{\alpha})}X_n(x) = \sum_{n=1}^{\infty}\frac{h_n}{T^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n^{\beta}T^{\alpha})}X_n(x).$$

If $I_1 \neq 0$, then there are many solutions to the source term, but there is only one nearest solution in $L^2(\Omega)$:

$$f_n = \frac{g_n - E_{\alpha,1}(-\lambda_n^{\beta}T^{\alpha})\varphi_n - TE_{\alpha,2}(-\lambda_n^{\beta}T^{\alpha})\psi_n}{T^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n^{\beta}T^{\alpha})}, \quad n \notin I_1,$$

and

$$f(x) = \sum_{n=1,\ n\notin I_1}^{\infty}\frac{g_n - E_{\alpha,1}(-\lambda_n^{\beta}T^{\alpha})\varphi_n - TE_{\alpha,2}(-\lambda_n^{\beta}T^{\alpha})\psi_n}{T^{\alpha}E_{\alpha,\alpha+1}(-\lambda_n^{\beta}T^{\alpha})}X_n(x) = \sum_{n=1,\ n\notin I_1}^{\infty}\frac{h_n}{T^{a}E_{a,a+1}(-\lambda_n^{\beta}T^{a})}X_n(x),$$

where $h_n = g_n - E_{\alpha,1}(-\lambda_n^{\beta}T^{\alpha})\varphi_n - TE_{\alpha,2}(-\lambda_n^{\beta}T^{\alpha})\psi_n$.

Substituting the measured data $g(x) = u(x,T)$, the exact solution of the problem (1.1) is obtained as

$$f(x) = \sum_{n=1,n\notin I_1}^{\infty}\frac{h_n}{T^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_n^{\beta}T^{\alpha}\right)}X_n(x), \tag{3.3}$$

where $h_n = g_n - E_{\alpha,1}(-\lambda_n^\beta T^\alpha)\varphi_n - TE_{\alpha,2}(-\lambda_n^\beta T^\alpha)\psi_n$.

The problem (1.1) becomes the following operator equation:

$$Kf(x) = h(x), \tag{3.4}$$

$$\sum_{n=1,n\notin I_1}^{\infty} T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha) f_n X_n(x) = \sum_{n=1,n\notin I_1}^{\infty} h_n X_n(x), \tag{3.5}$$

where $K : f \to h$ is a linear operator, $h_n = (h_n(x), X_n(x))$ is a Fourier coefficient. K^* is the selfadjoint operator of K, $\{X_n\}_{n=1}^\infty$ is orthogonal in $L^2(x)$, and the singular values of K are $\sigma_n = \left|T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\right|$, then

$$f(x) = \sum_{n=1,n\notin I_1}^{\infty} \frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x) = \sum_{n=1,n\notin I_1}^{\infty} \frac{h_n}{\sigma_n} X_n(x). \tag{3.6}$$

By Lemma 2.3, we can obtain

$$\frac{1}{\lambda_n^\beta} \le \sigma_n \le \frac{C_1}{\lambda_n^\beta}. \tag{3.7}$$

Due to $\lim_{n\to\infty}\lambda_n^\beta = +\infty$, then $\frac{1}{\sigma_n} \to \infty$, $h(x)$ will result in a large change in the source term $f(x)$. Therefore, the problem (1.1) is ill-posed. It is necessary to choose a suitable regularization method to solve the problem. In this paper, we will use the fractional Tikhonov regularization method and the quasi-inverse regularization method to give the fractional Tikhonov regularization solution and the quasi-inverse regularization solution of the problem (1.1).

Suppose that the exact solution $f(x)$ has the following a priori bound:

$$\| f(\cdot) \|_{D(-\Delta)^p} = \Big(\sum_{n=1,n\notin I_1}^{\infty} \lambda_n^{2\beta p} f_n^2\Big)^{\frac{1}{2}} \le E, \tag{3.8}$$

where E is a constant and $p > 0$.

4. The fractional Tikhonov regularization method

We can use the fractional Tikhonov regularization method to solve this ill-posed problem. The fractional Tikhonov regularization is the minimized Tikhonov function as follows:

$$\|Kf - h\|_W^2 + \mu\|f\|^2,$$

where $\mu > 0$ is the regularization parameter, $\|\cdot\|_W$ is a weighted semi-norm, and $\|z\|_W$, for z, we have

$$W = (K^*K)^{\frac{\gamma-1}{2}},$$

where $0 < \gamma < 1$ is a fractional parameter.

Following [34], there is a unique minimum function f^μ on $L^2(\Omega)$ satisfying

$$((K^*K)^{\frac{\gamma+1}{2}} + \mu I) f^\mu = (K^*K)^{\frac{\gamma-1}{2}} K^* h.$$

Especially when $\gamma = 1$, it is the classical Tikhonov regularization method, which can be found in [34]. Through the singular value of the linear operator K, a fractional Tikhonov regularization solution can be obtained without error as

$$f_\mu(x) = \sum_{n=1,n\notin I_1}^{\infty} \left(\frac{T^\alpha(E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^\gamma}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu} h_n\right) X_n(x), \quad 0<\gamma<1, \tag{4.1}$$

where $h_n = (h(x), X_n(x))$.

For noisy data,

$$h_n^\delta = g_n^\delta - E_{\alpha,1}(-\lambda_n^\beta T^\alpha)\varphi_n - TE_{\alpha,2}(-\lambda_n^\beta T^\alpha)\psi_n, \tag{4.2}$$

thus

$$\left\|h^\delta(\cdot) - h(\cdot)\right\| \le \delta.$$

We obtain the fractional Tikhonov regularization solution with error as

$$f_\mu^\delta(x) = \sum_{n=1,n\notin I_1}^{\infty} \left(\frac{T^\alpha(E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^\gamma}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu} h_n^\delta\right) X_n(x), \quad 0<\gamma<1, \tag{4.3}$$

where $h_n^\delta = (h^\delta(x), X_n(x))$ is the Fourier coefficient.

4.1. Convergence error estimation based on the a priori regularization parameter selection rule

Theorem 4.1. Let $f(x)$ be given by (3.3) and $f_\mu^\delta(x)$ be given by (4.3). Suppose that $f(x)$ satisfies the a priori bound condition (3.8) and the assumption (1.4) holds; then:

(1) If $0 < p \le \gamma+1$, the regularization parameter $\mu = (\frac{\delta}{E})^{\frac{\gamma+1}{p+1}}$ can be chosen to have the following error estimate:

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le C_{11} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{4.4}$$

where $C_{11} = C_2 + C_3$.

(2) If $p > \gamma + 1$, the regularization parameter $\mu = (\frac{\delta}{E})^{\frac{\gamma+1}{\gamma+2}}$ can be chosen to have the following error estimate:

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le C_{12} E^{\frac{1}{\gamma+2}} \delta^{\frac{\gamma+1}{\gamma+2}}, \tag{4.5}$$

where $C_{12} = C_2 + C_4$.

Proof. By triangle inequality, we have

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le \left\|f_\mu^\delta(x) - f_\mu(x)\right\| + \left\|f_\mu(x) - f(x)\right\|. \tag{4.6}$$

We give the first estimate of (4.6). From (4.1), (4.3), Lemmas 2.3 and 2.4, we have

$$\begin{aligned} \left\|f_\mu^\delta(x) - f_\mu(x)\right\| &= \left\| \sum_{n=1,n\notin I_1}^{\infty} \left(\frac{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)^\gamma}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu} (h_n^\delta - h_n) X_n(x)\right) \right\| \\ &\le \sup_{n\ge 1} \left| \frac{(\frac{C_1}{\lambda_n^\beta})^\gamma}{(\frac{1}{\lambda_n^\beta})^{\gamma+1}+\mu} \right| \delta \le \sup_{n\ge 1} \left| \frac{C_1^\gamma \lambda_n^\beta}{1+\mu\lambda_n^{\beta(\gamma+1)}} \right| \delta \\ &\le \sup_{n\ge 1} |A_1(n)|\,\delta, \end{aligned} \tag{4.7}$$

where $A_1(n) = \frac{C_1^\gamma \lambda_n^\beta}{1+\mu\lambda_n^{\beta(\gamma+1)}}$.

Let $s = \lambda_n^\beta$, from Lemma 2.4, we can infer

$$\left\|f_\mu^\delta(x) - f_\mu(x)\right\| \le C_2 \mu^{-\frac{1}{\gamma+1}}, \tag{4.8}$$

where $C_2 = (\frac{C_1^\gamma}{\gamma})^{\frac{1}{\gamma+1}}$.

For the second estimate of (4.6), using (4.1), (3.3), and Lemma 2.3, we can deduce that

$$\begin{aligned}
\left\|f_\mu(x) - f(x)\right\| &= \left\|\sum_{n=1,n\notin I_1}^{\infty} \frac{\mu}{\left((T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\right)^{\gamma+1} + \mu)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} h_n X_n(x)\right\| \\
&\le \sup_{n\ge 1}\left|\frac{\mu}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu}\lambda_n^{-\beta p}\right| E \\
&\le \sup_{n\ge 1}\left|\frac{\mu\lambda_n^{-\beta p}}{(\frac{1}{\lambda_n^\beta})^{\gamma+1} + \mu}\right| E \\
&\le \sup_{n\ge 1}|A_2(n)|E,
\end{aligned} \tag{4.9}$$

where $A_2(n) = \frac{\mu\lambda_n^{\beta(\gamma+1-p)}}{1+\mu\lambda_n^{\beta(\gamma+1)}}$.

Let $s = \lambda_n^\beta$, from Lemma 2.5, we can infer

$$\left\|f_\mu(x) - f(x)\right\| \le \begin{cases} C_3\mu^{\frac{p}{\gamma+1}}E, & 0 < p < \gamma+1, \\ C_4\mu E, & p \ge \gamma+1, \end{cases} \tag{4.10}$$

where $C_3 = \frac{(\frac{\gamma+1-p}{p})^{\frac{\gamma+1-p}{\gamma+1}}}{1+\frac{\gamma+1-p}{p}}$, $C_4 = \lambda_1^{\beta(\gamma+1-p)}$.

Combining (4.8) and (4.10) and selecting the regularization parameter μ,

$$\mu = \begin{cases} (\frac{\delta}{E})^{\frac{\gamma+1}{p+1}}, & 0 < p \le \gamma+1, \\ (\frac{\delta}{E})^{\frac{\gamma+1}{\gamma+2}}, & p > \gamma+1. \end{cases} \tag{4.11}$$

Hence,

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le \begin{cases} C_{11}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0 < p \le \gamma+1, \\ C_{12}E^{\frac{1}{\gamma+2}}\delta^{\frac{\gamma+1}{\gamma+2}}, & p \ge \gamma+1, \end{cases} \tag{4.12}$$

where $C_{11} = C_2 + C_3$, $C_{12} = C_2 + C_4$.

4.2. Convergence error estimation based on the a posteriori regularization parameter selection rule

The a posteriori regularization parameter μ is chosen using Morozov's inconsistency principle, and we obtain the error estimate between the exact solution and the regularization solution. The a posteriori regularization parameter μ is chosen to satisfy

$$\left\|Kf_\mu^\delta(x) - h^\delta(x)\right\| = \tau_1\delta, \tag{4.13}$$

where $\tau_1 > 1$ is a constant. In the following lemma, if $\left\|h^\delta\right\| > \tau_1\delta$, then a unique solution exists for (4.13).

Lemma 4.1. $\rho_1(\mu) = \left\|Kf_\mu^\delta(x) - h^\delta(x)\right\|$ establishes the following result:

(a) $\rho_1(\mu)$ is a continuous function;

(b) $lim_{\mu\to 0}\rho_1(\mu) = 0$;

(c) $\lim_{\mu\to\infty}\rho_1(\mu) = \left\|h^\delta(x)\right\|$;

(d) $\rho_1(\mu)$ is strictly monotonically increasing.

Proof.

$$\begin{aligned}\rho_1(\mu) &= \left\|Kf_\mu^\delta(x) - h^\delta(x)\right\| \\ &= \left\|\sum_{n=1,n\notin I_1}^{\infty}\left(\frac{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\right)^2 h_n^\delta}{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\right)^2 + \mu} - h_n^\delta\right)X_n(x)\right\| \\ &= \left(\sum_{n=1,n\notin I_1}^{\infty}\left(\frac{\mu h_n^\delta}{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\right)^{\gamma+1} + \mu}\right)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{4.14}$$

Obviously, the conclusions (a), (b), (c) and (d) hold.

Theorem 4.2. If the a priori bound condition (3.8) and the error assumption (1.4) hold, and the regularization parameter $\mu > 0$ is chosen by the Morozov discrepancy principle (4.13), then:

(1) If $0 < p < \gamma$, we have the following error estimates:

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le C_{13}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, \tag{4.15}$$

where $C_{13} = C_{15} + C_{17}$.

(2) If $p \ge \gamma$, the following error estimates are available:

$$\left\|f_\mu^\delta(x) - f(x)\right\| \le C_{14}(E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}} + E^{\frac{1}{p+1}}\delta^{\frac{\gamma}{\gamma+1}}), \tag{4.16}$$

where $C_{14} = C_{16} + C_{17}$.

Proof. According to triangle inequality, for (4.6) the first term of the equation is given by (4.8) and Lemma 2.3, we have

$$\left\|f_\mu^\delta(x) - f_\mu(x)\right\| \le C_2\mu^{-\frac{1}{\gamma+1}}. \tag{4.17}$$

Using (4.13) and (1.4), we can obtain

$$\begin{aligned}\tau_1\delta &= \left\|\sum_{n=1,n\notin I_1}^{\infty}\frac{\mu h_n^\delta}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu}X_n(x)\right\| \\ &\le \left\|\sum_{n=1,n\notin I_1}^{\infty}\frac{\mu(h_n^\delta - h_n)}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu}X_n(x)\right\| + \left\|\sum_{n=1,n\notin I_1}^{\infty}\frac{\mu h_n X_n(x)}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu}\right\| \\ &\le \delta + J.\end{aligned} \tag{4.18}$$

From Lemma 2.3, we have

$$
\begin{aligned}
J &= \left\| \sum_{n=1, n\notin I_1}^{\infty} \frac{\mu h_n X_n(x)}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu} \right\| \\
&= \left\| \sum_{n=1, n\notin I_1}^{\infty} \frac{\mu h_n}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu} \frac{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x) \right\| \\
&= \left\| \sum_{n=1, n\notin I_1}^{\infty} \frac{\mu T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu} f_n X_n(x) \right\| \\
&\le \sup_{n\ge1} \left| \frac{\mu T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1} + \mu} \lambda_n^{-\beta p} \right| E \\
&\le \sup_{n\ge1} \left| \frac{\mu C_1 \lambda_n^{\beta(\gamma-p)}}{1 + \mu \lambda_n^{\beta(\gamma+1)}} \right| E \\
&= \sup_{n\ge1} |A_3(n)|\, E,
\end{aligned}
\tag{4.19}
$$

where $A_3(n) = \frac{\mu C_1 \lambda_n^{\beta(\gamma-p)}}{1+\mu\lambda_n^{\beta(\gamma+1)}}$.

Let $s = \lambda_n^\beta$, from Lemma 2.6, we can obtain

$$
\tau_1 \delta \le \delta + \begin{cases} C_5 \mu^{\frac{p+1}{\gamma+1}} E, & 0 < p \le \gamma, \\ C_6 \mu E, & p > \gamma, \end{cases} \tag{4.20}
$$

where $C_5 = \frac{C_1 (\frac{\gamma-p}{p+1})^{\frac{1}{\gamma+1}}}{1+\frac{\gamma-p}{p+1}}$, $C_6 = C_1 \lambda_1^{\beta(\gamma-p)}$.

So there is

$$
\frac{1}{\mu} \le \begin{cases} C_5 (\frac{E}{(\tau_1-1)\delta})^{\frac{\gamma+1}{p+1}}, & 0 < p < \gamma, \\ C_6 \frac{E}{(\tau_1-1)\delta}, & p \ge \gamma, \end{cases} \tag{4.21}
$$

where $C_5 = \frac{C_1 (\frac{\gamma-p}{p+1})^{\frac{1}{\gamma+1}}}{1+\frac{\gamma-p}{p+1}}$, $C_6 = C_1 \lambda_1^{\beta(\gamma-p)}$.

Substituting (4.21) into (4.17), we obtain

$$
\left\| f_\mu^\delta(x) - f_\mu(x) \right\| \le C_2 \mu^{-\frac{1}{\gamma+1}} \delta \le \begin{cases} C_{15} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p < \gamma, \\ C_{16} E^{\frac{1}{\gamma+1}} \delta^{\frac{\gamma}{\gamma+1}}, & p \ge \gamma, \end{cases} \tag{4.22}
$$

where $C_{15} = C_2 C_5^{\frac{1}{\gamma+1}} (\frac{1}{\tau_1-1})^{\frac{1}{p+1}}$, $C_{16} = C_2 C_6^{\frac{1}{\gamma+1}} (\frac{1}{\tau_1-1})^{\frac{1}{\gamma+1}}$.

For the second estimate of (4.6) , from (3.3) , (4.1) and Lemma 2.3, we can obtain

$$\left\|f_\mu(x)-f(x)\right\|^2=\left\|\sum_{n=1,n\notin I_1}^{\infty}\frac{\mu h_n}{\Big((T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}X_n(x)\right\|^2$$
$$=\sum_{n=1,n\notin I_1}^{\infty}[\frac{\mu h_n}{\Big(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\Big)^{\gamma+1}+\mu)}]^{\frac{2p}{p+1}}$$
$$[\frac{\mu h_n}{\Big((T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\Big)^{\gamma+1}+\mu)(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}}]^{\frac{2}{p+1}}$$
$$\leq 2\sum_{n=1,n\notin I_1}^{\infty}(\frac{\mu(h_n-h_n^\delta)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu})^2+2\sum_{n=1,n\notin I_1}^{\infty}(\frac{\mu h_n^\delta}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu})^2]^{\frac{p}{p+1}}$$
$$[\sum_{n=1,n\notin I_1}^{\infty}(\frac{f_n}{\Big((T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{\gamma+1}+\mu)(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\Big)^p})^2]^{\frac{1}{p+1}}$$
$$\leq(2\delta^2+2\tau^2\delta^2)^{\frac{p}{p+1}}(\sum_{n=1}^{\infty}(f_n(\frac{1}{\lambda_n^\beta})^{-p})^2)^{\frac{1}{p+1}}$$
$$\leq(2+2\tau^2)^{\frac{p}{p+1}}\delta^{\frac{2p}{p+1}}(E^2)^{\frac{1}{p+1}}$$
$$=C_{17}E^{\frac{2}{p+1}}\delta^{\frac{2p}{p+1}},\tag{4.23}$$

where $C_{17}=(2+2\tau^2)^{\frac{p}{p+1}}$.

Hence,

$$\left\|f_\mu(x)-f(x)\right\|\leq C_{15}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}.\tag{4.24}$$

Combining (4.22) with (4.24), we obtain (4.15) and (4.16).

The proof is complete.

5. Quasi-inverse regularization method

The quasi-inverse regularization method is used to solve the ill-posed problem and obtain the regularization solution. Under the a priori regularization and the a posteriori regularization parameter selection rules, we all obtain the error estimators between the exact solution and the regularization solutions, respectively.

Let $u_\eta^\delta(x,t)$ be the proposed inverse regularization solution to the following problem, and η is the regularization parameter.

$$\begin{cases}\partial_t^\alpha u_\eta^\delta(x,t)+(-\Delta)^\beta u_\eta^\delta(x,t)=f_\eta^\delta(x)+\eta(-\Delta)^\beta f_\eta^\delta(x), & x\in\Omega, t\in(0,T),\\ u_\eta^\delta(x,t)=0, & x\in\partial\Omega, t\in(0,T),\\ u_\eta^\delta(x,0)=\varphi^\delta(x), & x\in\Omega,\\ u_{\eta,t}^\delta(x,0)=\psi^\delta(x), & x\in\Omega,\\ u_\eta^\delta(x,T)=g^\delta(x), & x\in\Omega.\end{cases}\tag{5.1}$$

An exact solution to the problem (5.1) can be obtained using the separation of variable method and the Laplace transform of Mittag-Leffler function as follows:

$$u_\eta^\delta(x,t)=\sum_{n=1}^{\infty}(E_{\alpha,1}(-\lambda_n^\beta t^\alpha)\varphi_n^\delta+tE_{\alpha,2}(-\lambda_n^\beta t^\alpha)\psi_n^\delta+t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta t^\alpha)(1+\eta\lambda_n^\beta)f_{\eta,n}^\delta)X_n(x), \tag{5.2}$$

where λ_n and $X_n(x)$ are, respectively, the eigenvalues and eigenfunctions of the Laplace operator, and the sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies $\lambda_1\le\lambda_2\le\cdots\le\lambda_n\to\infty,\ n\to\infty, \varphi_n=(\varphi(x),X_n(x)), \psi_n=(\psi(x),X_n(x))$ and $f_n=(f(x),X_n(x))(x)$ are Fourier coefficients.

Using $u_\eta^\delta(x,T)=g^\delta(x)$, we obtain

$$g^\delta(x)=\sum_{n=1,n\in I_1}^{\infty}(E_{\alpha,1}(-\lambda_n^\beta t^\alpha)\varphi_n^\delta+tE_{\alpha,2}(-\lambda_n^\beta t^\alpha)\psi_n^\delta+t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta t^\alpha)(1+\eta\lambda_n^\beta)f_{\eta,n}^\delta)X_n(x), \tag{5.3}$$

then

$$g_n^\delta=\sum_{n=1,n\in I_1}^{\infty}E_{\alpha,1}(-\lambda_n^\beta t^\alpha)\varphi_n^\delta+tE_{\alpha,2}(-\lambda_n^\beta t^\alpha)\psi_n^\delta+t^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta t^\alpha)(1+\eta\lambda_n^\beta)f_{\eta,n}^\delta), \tag{5.4}$$

and we can get

$$f_{\eta,n}^\delta=\frac{g_n^\delta-E_{\alpha,1}(-\lambda_n^\beta T^\alpha)\varphi_n^\delta-TE_{\alpha,2}(-\lambda_n^\beta T^\alpha)\psi_n^\delta}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}. \tag{5.5}$$

Let

$$h_n^\delta:=g_n^\delta-E_{\alpha,1}(-\lambda_n^\beta T^\alpha)\varphi_n^\delta-TE_{\alpha,2}(-\lambda_n^\beta T^\alpha)\psi_n^\delta. \tag{5.6}$$

Thus,

$$\left\|h^\delta(\cdot)-h(\cdot)\right\|\le\delta.$$

We obtain the quasi-inverse regular solution with error as

$$f_\eta^\delta(x)=\sum_{n=1,n\notin I_1}^{\infty}\frac{h_n^\delta}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}X_n(x), \tag{5.7}$$

where $h_n^\delta=(h^\delta(x),X_n(x))$.

And the quasi-inverse regular solution without error as

$$f_\eta(x)=\sum_{n=1,n\notin I_1}^{\infty}\frac{h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}X_n(x), \tag{5.8}$$

where $h_n=(h(x),X_n(x))$.

5.1. Convergence error estimation based on the a priori regularization parameter selection rule

Theorem 5.1. Let $f(x)$ be given by (3.3) and $f_\eta^\delta(x)$ be given by (5.7). Assume $f(x)$ satisfies the a priori bound condition (3.8) and let (1.4) hold; then:

(1) If $0<p\le1$, the regularization parameter $\eta=(\frac{\delta}{E})^{\frac{1}{p+1}}$ can be chosen to have the following error estimate:

$$\left\|f_\eta^\delta(x)-f(x)\right\|\le C_{18}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, \tag{5.9}$$

where $C_{18} = 1 + C_7$.

(2) If $p > 1$, the regularization parameter $\eta = (\frac{\delta}{E})^{\frac{1}{2}}$ can be chosen to have the following error estimate:

$$\left\| f_\eta^\delta(x) - f(x) \right\| \le C_{19} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, \tag{5.10}$$

where $C_{19} = 1 + C_8$.

Proof. Based on triangle inequality, we obtain

$$\left\| f_\eta^\delta(x) - f(x) \right\| \le \left\| f_\eta^\delta(x) - f_\eta(x) \right\| + \left\| f_\eta(x) - f(x) \right\|. \tag{5.11}$$

Using Lemma 2.3, we obtain

$$\begin{aligned} \left\| f_\eta^\delta(x) - f_\eta(x) \right\| &= \left\| \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^\delta - h_n}{(1 + \eta\lambda_n^\beta) T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x) \right\| \\ &\le \left\| \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^\delta - h_n}{(1 + \eta\lambda_n^\beta) \frac{1}{\lambda_n^\beta}} X_n(x) \right\| \\ &\le \left\| \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^\delta - h_n}{\lambda_n^\beta + \eta} X_n(x) \right\| \\ &\le \frac{\delta}{\eta}. \end{aligned} \tag{5.12}$$

Using (3.3), (5.8) and Lemma 2.3, we have

$$\begin{aligned} \left\| f_\eta(x) - f(x) \right\| &= \left\| \sum_{n=1, n \notin I_1}^{\infty} \left[\frac{h_n}{(1 + \eta\lambda_n^\beta) T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha) + \eta} - \frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} \right] X_n(x) \right\| \\ &= \left\| \sum_{n=1, n \notin I_1}^{\infty} \frac{\eta\lambda_n^\beta h_n}{(1 + \eta\lambda_n^\beta) T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x) \right\| \\ &= \left\| \sum_{n=1, n \notin I_1}^{\infty} \frac{\eta\lambda_n^\beta}{(1 + \eta\lambda_n^\beta)} f_n \lambda_n^{\beta p} \lambda_n^{-\beta p} X_n(x) \right\| \\ &\le \sup_{n \ge 1} \left| \frac{\eta\lambda_n^{(1-p)\beta}}{1 + \eta\lambda_n^\beta} \right| E = \sup |A_4(n)| E, \end{aligned} \tag{5.13}$$

where $A_4(n) = \frac{\eta\lambda_n^{\beta(1-p)}}{1+\eta\lambda_n^\beta}$.

Let $s = \lambda_n^\beta$, from Lemma 2.7, we can infer

$$\left\| f_\eta(x) - f(x) \right\| \le \begin{cases} C_7 \eta^p E, & 0 < p \le 1, \\ C_8 \eta E, & p > 1, \end{cases} \tag{5.14}$$

where $C_7 = (1 - p)^{1-p} p^{2-p}$, $C_8 = \lambda_1^{\beta(1-p)}$.

Combining (5.12) and (5.14), the regularization parameter is selected by η.

$$\eta = \begin{cases} (\frac{\delta}{E})^{\frac{1}{p+1}}, & 0 < p \le 1, \\ (\frac{\delta}{E})^{\frac{1}{2}}, & p > 1. \end{cases} \tag{5.15}$$

Hence,

$$\left\|f_\eta^\delta(x)-f(x)\right\|\leq\begin{cases}C_{18}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}, & 0<p<2,\\ C_{19}E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\geq 2,\end{cases}\tag{5.16}$$

where $C_{18}=1+C_7$, $C_{19}=1+C_8$.

5.2. Convergence error estimation based on the a posteriori regularization parameter selection rule

The a posteriori regularization parameter η can be chosen by Morozov's inconsistency principle to obtain the posteriori convergence error estimate of the source term. The Morozov's inconsistency principle for the a posteriori regularization parameter η is chosen as

$$\left\|\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}[Kf_\eta^\delta(x)-h^\delta(x)]\right\|=\tau_2\delta,\tag{5.17}$$

where $\tau_2>1$ is a contstant. If $\left\|h^\delta\right\|>\tau_2\delta$, then a unique solution exists for (5.17).

Lemma 5.1. $\rho_2(\eta)=\left\|\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}[Kf_\eta^\delta(x)-h^\delta(x)]\right\|$ has the following result:

(a) $\rho_2(\eta)$ is a continuous function;

(b) $\lim_{\eta\to0}\rho_2(\eta)=0$;

(c) $\lim_{\eta\to\infty}\rho_2(\eta)=\|h^\delta(x)\|$;

(d) $\rho_2(\eta)$ is strictly monotonically increasing.

Proof.

$$\begin{aligned}\rho_2(\eta)&=\left\|\sum_{n=1,n\notin I_1}^{\infty}\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}[T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\frac{h_n^\delta}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)+\eta}-h_n^\delta]X_n(x)\right\|\\&=\left\|\sum_{n=1,n\notin I_1}^{\infty}(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2h_n^\delta X_n(x)\right\|\\&=(\sum_{n=1,n\notin I_1}^{\infty}((\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2h_n^\delta X_n(x))^2)^{\frac{1}{2}}.\end{aligned}\tag{5.18}$$

Obviously, the conclusions (a), (b), (c) and (d) hold.

Theorem 5.2. Let $f(x)$ be given by (3.3) and $f_\eta^\delta(x)$ be given by (5.7). Assume $f(x)$ satisfies the a priori bound condition (3.8) and let the noise assumption (1.4) hold, and the a posteriori regularization parameter is given by the Morozov inconsistency principle (5.17), then:

(1) If $0<p\leq1$, the following error estimates are available:

$$\left\|f_\eta^\delta(x)-f(x)\right\|\leq C_{20}E^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}},\tag{5.19}$$

where $C_{20}=(\frac{C_9^2}{(\tau_2-1)})^{\frac{1}{p+1}}$.

(2) If $p>1$, the following error estimates are available:

$$\left\|f_\mu^\delta(x)-f(x)\right\|\leq C_{21}E^{\frac{1}{2}}\delta^{\frac{1}{2}},\tag{5.20}$$

where $C_{21}=(\frac{C_{10}^2}{(\tau_2-1)})^{\frac{1}{2}}$.

Proof. Based on triangle inequality, we obtain

$$\left\|f_\eta^\delta(x) - f(x)\right\| \le \left\|f_\eta^\delta(x) - f_\eta(x)\right\| + \left\|f_\eta(x) - f(x)\right\|. \tag{5.21}$$

The right hand of the first term for the equation for (5.21) is still given by (5.12), i.e.,

$$\left\|f_\eta^\delta(x) - f_\eta(x)\right\| \le \frac{\delta}{\eta}. \tag{5.22}$$

Using (1.4) and (5.17), we can obtain

$$\begin{aligned}
\tau_2\delta &= \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 h_n^\delta X_n(x)\right\| \\
&= \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 (h_n^\delta - h_n + h_n) X_n(x)\right\| \\
&\le \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 (h_n^\delta - h_n) X_n(x)\right\| + \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 h_n X_n(x)\right\| \\
&\le \delta + J.
\end{aligned} \tag{5.23}$$

From Lemma 2.3, we have

$$\begin{aligned}
\mathrm{J} &= \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 h_n X_n(x)\right\| \\
&= \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 h_n \frac{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x)\right\| \\
&= \left\|\sum_{n=1,n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 f_n T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\lambda_n^{-\beta p}\lambda_n^{\beta p} X_n(x)\right\| \\
&\le \sup_{n\ge1}\left|(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\lambda_n^{-\beta p}\right| E \\
&\le \sup_{n\ge1}\left|(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 \frac{C_1}{\lambda_n^\beta}\lambda_n^{-\beta p}\right| E \\
&= \sup_{n\ge1}|A_5^2(n)|E,
\end{aligned} \tag{5.24}$$

where $A_5(n) = \frac{\eta\sqrt{C_1}\lambda_n^{\frac{\beta(1-p)}{2}}}{1+\eta\lambda_n^\beta}$.

Let $s = \lambda_n^\beta$, from Lemma 2.8, we have

$$\tau_2\delta \le \begin{cases} \delta + C_9^2\eta^{p+1}E, & 0 < p \le 1, \\ \delta + C_{10}^2\eta^2 E, & p > 1, \end{cases} \tag{5.25}$$

where $C_9 = \frac{\sqrt{C_1}(\frac{1-p}{1+p})^{\frac{1-p}{2}}}{1+\frac{1-p}{1+p}}$, $C_{10} = \sqrt{C_1}\lambda_1^{\frac{\beta(1-p)}{2}}$.

Therefore, we have

$$\frac{1}{\eta} \leq \begin{cases} (\frac{C_9^2}{(\tau_2-1)})^{\frac{1}{p+1}} E^{\frac{1}{p+1}} \delta^{-\frac{1}{p+1}}, & 0 < p \leq 1, \\ (\frac{C_{10}^2}{(\tau_2-1)})^{\frac{1}{2}} E^{\frac{1}{2}} \delta^{-\frac{1}{2}}, & p > 1, \end{cases} \tag{5.26}$$

where $C_9 = \frac{\sqrt{C_1}(\frac{1-p}{1+p})^{\frac{1-p}{2}}}{1+\frac{1-p}{1+p}}$, $C_{10} = \sqrt{C_1}\lambda_1^{\frac{\beta(1-p)}{2}}$.

Substituting (5.26) into (5.22), we can obtain

$$\left\|f_\eta^\delta(x) - f_\eta(x)\right\| \leq \frac{\delta}{\eta} \leq \begin{cases} C_{20} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p \leq 1, \\ C_{21} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p > 1, \end{cases} \tag{5.27}$$

where $C_{20} = (\frac{C_9^2}{(\tau_2-1)})^{\frac{1}{p+1}}$, $C_{21} = (\frac{C_{10}^2}{(\tau_2-1)})^{\frac{1}{2}}$.

When $0 < p \leq 1$, we first estimate the right hand of the second term for (5.21):

$$\begin{aligned} \left\|f_\eta(x) - f(x)\right\|^2 &= \left\|\sum_{n=1, n\notin I_1}^{\infty} [\frac{h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} - \frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}] X_n(x)\right\|^2 \\ &= \left\|\sum_{n=1, n\notin I_1}^{\infty} \frac{\eta\lambda_n^\beta h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x)\right\|^2 \\ &= \sum_{n=1, n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)})^2 \\ &= \sum_{n=1, n\notin I_1}^{\infty} [(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^{\frac{4p}{p+1}} h_n^{\frac{2p}{p+1}} (\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^{\frac{p-2p}{p+1}} \frac{h_n}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{p+1}}]^{\frac{2}{p+1}} \\ &\leq \sum_{n=1, n\notin I_1}^{\infty} ((\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta})^2 h_n)^{\frac{2p}{p+1}} (\frac{h_n}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{p+1}})^{\frac{2}{p+1}} \\ &\leq (\sum_{n=1, n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{(1+\eta\lambda_n^\beta)})^2 h_n^2)^{\frac{p}{p+1}} (\sum_{n=1, n\notin I_1}^{\infty} (\frac{h_n}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha))^{p+1}})^2)^{\frac{1}{p+1}} \\ &\leq (2\sum_{n=1, n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{(1+\eta\lambda_n^\beta)})^2 (h_n - h_n^\delta)^2)^2 + (2\sum_{n=1, n\notin I_1}^{\infty} (\frac{\eta\lambda_n^\beta}{(1+\eta\lambda_n^\beta)})^2 (h_n^\delta)^2)^{\frac{p}{p+1}} (\sum_{n=1, n\notin I_1}^{\infty} (f_n\lambda_n^{\beta p})^2)^{\frac{1}{p+1}} \\ &\leq (2\delta^2 + 2\tau_2^2\delta^2)^{\frac{p}{p+1}} E^{\frac{2}{p+1}} \\ &\leq C_{22}^2 \delta^{\frac{2p}{p+1}} E^{\frac{2}{p+1}}, \end{aligned} \tag{5.28}$$

where $C_{22} = (2 + 2\tau_2^2)^{\frac{p}{p+1}}$.

And when $p > 1$,

$$
\begin{aligned}
\left\|f_\eta(x) - f(x)\right\|^2 &= \left\|\sum_{n=1,n\notin I_1}^{\infty} \frac{\eta\lambda_n^\beta h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)} X_n(x)\right\|^2 \\
&= \sum_{n=1,n\notin I_1}^{\infty} \Big(\frac{\eta\lambda_n^\beta h_n}{(1+\eta\lambda_n^\beta)T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}\Big)^2 \\
&= \sum_{n=1,n\notin I_1}^{\infty} \Big(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}\Big)^2 h_n \frac{h_n}{\Big(T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)\Big)^2} \\
&\le \Big(\sum_{n=1,n\notin I_1}^{\infty} \Big(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}\Big)^2 h_n^2\Big)^{\frac{1}{2}} \Big(\sum_{n=1,n\notin I_1}^{\infty} \Big(\frac{h_n}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_n^\beta T^\alpha)}\Big)^2\Big)^{\frac{1}{2}} \\
&\le \Big(2\sum_{n=1,n\in I_1}^{\infty} \Big(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}\Big)^2 (h_n - h_n^\delta)^2\Big)^{\frac{1}{2}} + \Big(2\sum_{n=1,n\notin I_1}^{\infty} \Big(\frac{\eta\lambda_n^\beta}{1+\eta\lambda_n^\beta}\Big)^2 h_n^{2\delta}\Big)^{\frac{1}{2}} \Big(\sum_{n=1,n\notin I_1}^{\infty} (f_n\lambda_n^\beta\lambda_n^{-p\beta}\lambda_n^{p\beta})^2\Big)^{\frac{1}{2}} \\
&\le (2\delta^2 + 2\tau_2^2\delta^2)^{\frac{1}{2}} \lambda_n^{\beta(1-p)} E \\
&\le (2\delta^2 + 2\tau_2^2\delta^2)^{\frac{1}{2}} \lambda_1^{\beta(1-p)} E \\
&\le C_{23}^2 E\delta,
\end{aligned}
\tag{5.29}
$$

where $C_{23} = (2 + 2\tau_2^2)^{\frac{1}{2}} \lambda_1^{\beta(1-p)}$.

Hence,

$$
\left\|f_\eta(x) - f(x)\right\| \le \begin{cases} C_{22} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, & 0 < p \le 1, \\ C_{23} E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p > 1, \end{cases}
\tag{5.30}
$$

where $C_{22} = (2 + 2\tau_2^2)^{\frac{1}{2}}$, $C_{23} = (2 + 2\tau_2^2)^{\frac{1}{4}} \lambda_1^{\frac{\beta(1-p)}{2}}$.

Combining (5.27) and (5.30), we obtain (5.19) and (5.20).

6. Numerical implementation

In this section, we are going to use numerical examples and software to verify the efficiency of our method. Let $d = 1$, $\Omega = (0, \pi)$, $a_{i,j} = 1$, and $T = 1$ in (1.1). Then the eigenvalues and eigenfunctions of the negative Laplace operator are

$$
\lambda_n = n^2, \quad X_n = \sqrt{\frac{2}{\pi}} \sin(nx), \quad n = 1, 2, \cdots .
$$

Consider the following direct problem:

$$
\begin{cases}
\partial_t^\alpha u(x,t) + (-\Delta)^\beta u(x,t) = f(x), & x \in (0,\pi), t \in (0,1], \\
u(0,t) = u(\pi,t) = 0, & t \in (0,1], \\
u(x,0) = \varphi(x), & x \in (0,\pi), \\
u_t(x,0) = \psi(x), & x \in (0,\pi), \\
u(x,T) = g(x), & x \in (0,\pi),
\end{cases}
\tag{6.1}
$$

where $f(x)$, $\varphi(x)$, $\psi(x)$ are given.

We define

$$x_i = i\Delta x \ \ (i = 0, 1, 2, \cdots, M), \quad t_m = m\Delta t \ \ (m = 0, 1, 2, \cdots N), \tag{6.2}$$

where $\Delta x = \frac{\pi}{M}$ is the step size of space and $\Delta t = \frac{T}{N}$ is the step size of time. $u_i^m = u(x_i, t_m)$ is the value of each grid point.

The discrete difference format for space fractional derivatives is

$$\frac{\partial^2 u(x_i, t_n)}{\partial^2 x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}. \tag{6.3}$$

The discrete difference format for Caputo fractional derivatives is [35]

$$\partial_t^\alpha u(x_i, t_m) \approx \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}(\frac{b_0}{\Delta t}(u_i^m - u_i^{m-1})) - \sum_{j=1}^{n-1} \frac{b_{n-j-1} - b_{n-j}}{\Delta t}(u_i^j - u_i^{j-1}) - b_{m-1}\psi(x_i)), \tag{6.4}$$

where $i = 1, 2, \cdots, M-1$, $m = 1, 2, \cdots, N$, $b_j = (j+1)^{2-\alpha} - j^{2-\alpha}$.

Using [36], we obtain

$$(-\Delta)^\beta U = C_{1,2\beta} BU, \tag{6.5}$$

where $C_{1,2\beta} = \frac{4^\beta \Gamma(d/2+\beta)}{\pi^{d/2}|\Gamma(-\beta)|}$, B is a strictly diagonally dominant and symmetric positive definite matrix. $B \triangleq (h)_{i,p=1}^{M-1}$ and

$$h_{i,p} = \begin{cases} -(Z_1(i, p+1) + Z_2(i, p))\frac{1}{i-p}, & 1 \le p \le i-2, \\ -\frac{h^{-2\beta}}{2-2\beta} - Z_2(i, i-1), & p = i-1, \\ -\frac{h^{-2\beta}}{2-2\beta} - Z_3(i, i+2), & p = i+1, \\ -(Z_3(i, p+1) + Z_4(i, p))\frac{1}{p-i}, & i+2 \le p \le M, \end{cases} \tag{6.6}$$

$h_{i,i}$ satisfied

$$h_{i,i} + \sum_{p=1, p\neq i}^{M} h_{i,p} - Y_1(i) - Y_2(i) = \begin{cases} \frac{h^{-2\beta}}{2-2\beta} + \frac{Z_4(i, M+1)}{M+1-i}, & i = 1, \\ \frac{Z_1(i,1)}{i} + \frac{Z_4(i, M+1)}{M+1-i}, & 2 \le i \le M-1, \\ \frac{h^{-2\beta}}{2-2\beta} + \frac{Z_1(i,1)}{i}, & i = M, \end{cases} \tag{6.7}$$

where $Z_1(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k}(x_k - y)(x_i - y)^{-2\beta}dy$, $Z_2(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k}(y - x_k)(x_i - y)^{-2\beta}dy$, $Z_3(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k}(x_k - y)(y - x_i)^{-2\beta}dy$, $Z_4(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k}(y - x_k)(y - x_i)^{-2\beta}dy$, and $Y_1(i) = \frac{1}{h^2}\int_{-\infty}^{0}\frac{1}{(x_i - y)^{1+2\beta}}dy$, $Y_2(i) = \frac{1}{h^2}\int_{1}^{\infty}\frac{1}{(y - x_i)^{1+2\beta}}dy$.

By a simple calculation, when $\beta \in (0,1)$,

$$\begin{aligned} & Z_1(i, p+1) + Z_2(i, p) \\ = & Z_3(i, p+1) + Z_4(i, p) \\ = & \begin{cases} \frac{h^{-2\beta}}{(2\beta-1)(2-2\beta)}[2 \mid i-p \mid^{2-2\beta} - (\mid i-p \mid -1)^{2-2\beta} - (\mid i-p \mid +1)^{2-2\beta}], & \beta \neq 0.5, \\ \frac{1}{h}[-2 \mid i-p \mid \ln \mid i-p \mid + (\mid i-p \mid +1)\ln(\mid i-p \mid +1) + (\mid i-p \mid -1)\ln(\mid i-j \mid -1)], & \beta = 0.5, \end{cases} \end{aligned} \tag{6.8}$$

$$Z_2(i,i-1)=Z_3(i,i+2)=\begin{cases}\frac{h^{-2\beta}}{(2\beta-1)(2-2\beta)}(3-2\beta-2^{2-2\beta}), & \beta\neq 0.5,\\ \frac{1}{h}(2\ln 2-1), & \beta=0.5,\end{cases} \tag{6.9}$$

$$Z_1(i,1)=\begin{cases}\frac{h^{-2\beta}}{(2\beta-1)(2-2\beta)}(i^{2-2\beta}-(i-1)^{2-2\beta}-(2-2\beta)i^{1-2\beta}), & \beta\neq 0.5,\\ \frac{1}{h}[(1-i)\ln(\frac{i}{i-1})+1], & \beta=0.5,\end{cases} \tag{6.10}$$

$$Z_4(i,M+1)=\begin{cases}\frac{h^{-2\beta}}{(2\beta-1)(2-2\beta)}((M+1-i)^{2-2\beta}-(M-i)^{2-2\beta}-(2-2\beta)(M+1-i)^{1-2\beta}), & \beta\neq 0.5,\\ \frac{1}{h}[(i-M)\ln(\frac{M+1-i}{M-i})+1], & \beta=0.5,\end{cases} \tag{6.11}$$

$$Y_1(i)=\frac{(x_i)^{-2\beta}}{2\beta}, Y_2(i)=\frac{(1-x_i)^{-2\beta}}{2\beta}, \tag{6.12}$$

sorting the above equations yields the matrix B.

Let

$$U^m=((u_1^m),(u_2^m),\cdots,(u_{M-1}^m))^T,\ \varphi=(\varphi(x_1),\varphi(x_2),\cdots\varphi(x_{M-1}))^T,$$
$$\psi=(\psi(x_1),\psi(x_2),\cdots\psi(x_{M-1}))^T, F=(f(x_1),f(x_2),\cdots f(x_{M-1}))^T.$$

We can obtain the discrete format of the fractional Tikhonov regularization method

$$\begin{aligned}&AU^1=F^{1,\delta}+I^*d(\varphi+\Delta t\psi),\\&AU^m=F^{\mu,\delta}+I^*d((b_1-2b_0)U^{m-1}+\sum_{j=2}^{n-1}(b_{j-2}+b_j-2b_{j-1})U^{n-j}+(b_{n-2}-b_{n-1})\varphi-\Delta t b_{n-1}\psi),\end{aligned} \tag{6.13}$$

where I is a unit matrix, $d=\frac{(\Delta t)^{-\alpha}}{\Gamma(3-\alpha)}$, $A=A_1+C_{1,2\beta}B$, A_1 is the tridiagonal matrix, and

$$\left(A_1\right)_{(M-1)\times(M-1)}=\begin{pmatrix} r+\frac{2}{h^2} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & r+\frac{2}{h^2} & -\frac{1}{h^2} & & \\ & & \ddots & \ddots & -\frac{1}{h^2} \\ & & & -\frac{1}{h^2} & r+\frac{2}{h^2}\end{pmatrix}, \tag{6.14}$$

$$B_{(M-1)\times(M-1)}=\begin{pmatrix} h_{1,1} & \cdots & h_{1,M-1}\\ \vdots & \cdots & \vdots \\ h_{M-1,1} & \cdots & h_{M-1,M-1}\end{pmatrix}. \tag{6.15}$$

For the quasi-inverse regularization, let $V^n=(u_\eta^\delta(x_1,t_n),\cdots,u_\eta^\delta(x_{M-1},t_n))$, we obtain the following format:

$$AV^n=C_n+D_nf^{\eta,\delta}\quad(n=1,2,\cdots,N), \tag{6.16}$$

where C_n is a vector of the specific form:

$$\begin{aligned}&C_1=F^{1,\delta},\\&C_n=F^{\eta,\delta}+d(b_1C_{n-1}+b_2C_{n-2}+\cdots+b_{n-1}C_1)\quad(n=2,3,\cdots,N).\end{aligned} \tag{6.17}$$

D is a matrix of the following:

$$\begin{aligned}&D_1=H_1,\\&D_n=H_n+d(b_1D_{n-1}+b_2D_{n-2}+\cdots+b_{n-1}D_1)\quad(n=2,3,\cdots,N),\end{aligned} \tag{6.18}$$

where $H_l = (h_{i,j})_{(M-1)\times(M-1)},\ l = 2, 3, \cdots, N$,

$$(h_{i,j})_{(M-1)\times(M-1)} = \begin{pmatrix} r+\frac{2\eta}{h^2} & -\frac{\eta}{h^2} & & & \\ -\frac{\eta}{h^2} & r+\frac{2\eta}{h^2} & -\frac{\eta}{h^2} & & \\ & & \ddots & \ddots & -\frac{\eta}{h^2} \\ & & & -\frac{\eta}{h^2} & r+\frac{2\eta}{h^2} \end{pmatrix}. \tag{6.19}$$

Through the boundary of Eq (5.1), we obtain

$$V^n = G^\delta, \tag{6.20}$$

where $G^\delta = (g^\delta(x_1), \cdots, g^\delta(x_{M-1}))$. Then, we can get that the initial function satisfies

$$AG^\delta = C_N + D_N f^{\eta,\delta}. \tag{6.21}$$

In this numerical calculation, we generate noise date g^δ by adding disturbance to $g(x)$, respectively. Let the noise data g^δ be

$$g^\delta = g + \epsilon \cdot randn(size(g)), \tag{6.22}$$

where $size(g)$ gives the dimensions of g in space, the function $randn(\cdot)$ produces a list of random numbers with a mean of 0 and a variance of 1, and ϵ reflects the noise level.

The absolute error level is

$$\delta = \sqrt{\frac{1}{M+1}\sum_{i=1}^{M+1}(g_i - g_i^\delta)^2}. \tag{6.23}$$

The relative error level is

$$e_{r_1} = \frac{\sqrt{\sum_{i=1}^{M+1}(f - f_\mu^\delta)^2}}{\sqrt{\sum_{i=1}^{M+1}(f)^2}}, \quad e_{r_2} = \frac{\sqrt{\sum_{i=1}^{M+1}(f - f_\eta^\delta)^2}}{\sqrt{\sum_{i=1}^{M+1}(f)^2}}. \tag{6.24}$$

In practical problem, we know that it is difficult to obtain a priori boundary condition E in practical applications. So the a priori regularization parameter which is based on the a priori boundary can not be obtained exactly. So, we only give the numerical results under the a posteriori regularization parameter choice rule. The regularization parameter is obtained through Eq (5.17), and $\tau_1 = \tau_2 = 1.1$, $T = 1$.

Here, we give three numerical examples.

Example 6.1. We consider the following functions:

$$f(x) = \sin(\pi x), \quad \varphi(x) = x\sin(x), \quad \psi(x) = 3\sin(x), \quad x \in [0, \pi]. \tag{6.25}$$

Figure 1(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 1(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$. Figure 2(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 2(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$.

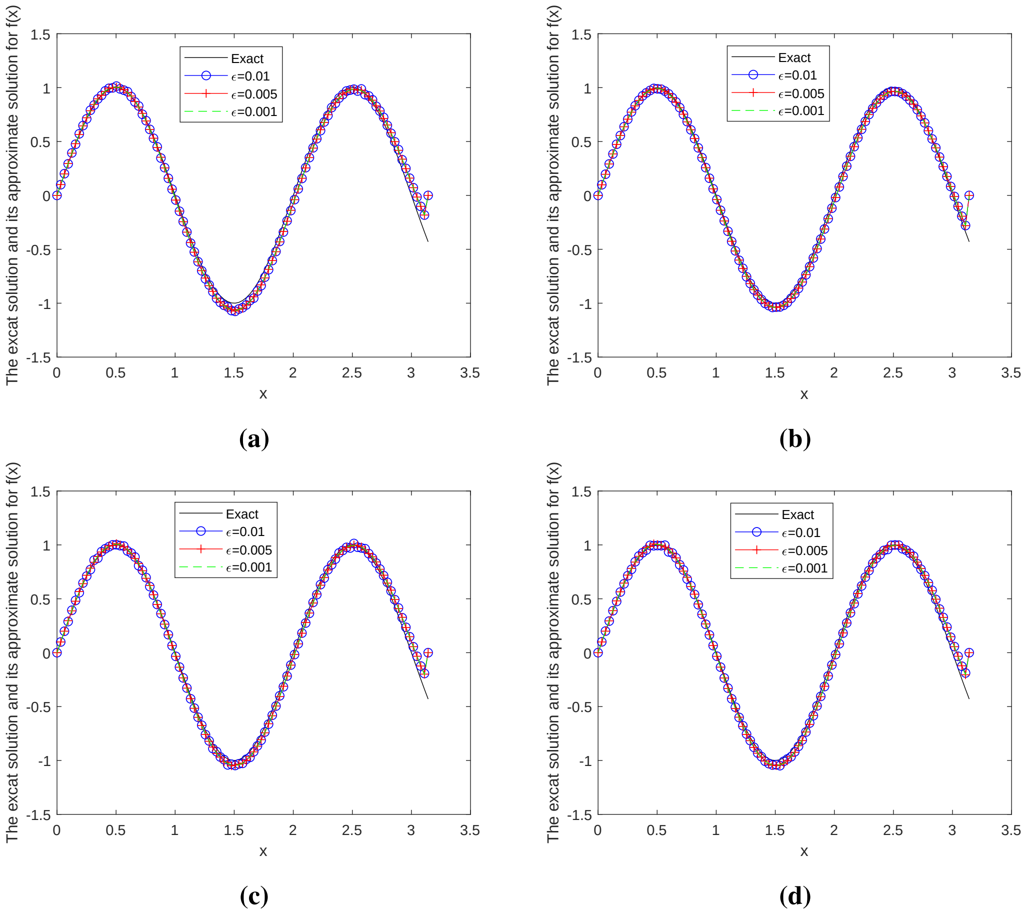

Figure 1. The exact solution and the fractional Tikhonov regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.1: (a) $s = 0.6$, $\alpha = 1.4$; (b) $s = 0.6$, $\alpha = 1.9$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

We can see from Figures 1 and 2, and Table 1 that with increasing noise levels, the error levels at various values also grow. When the space parameter s is fixed, as the time parameter α increases, the fitting effect of the image improves and the relative root mean square error decreases. This is because when α increases, the ill-posed weakens. On the other hand, both the change of the image and the relative root mean square error are not significant, which indicates that the two regularization methods we adopted are effective. When the time parameter α is fixed, as the space parameter s increases, the fitting effect of the image deteriorates and the relative root mean square error increases. This is because when s increases, the ill-posed intensifies. Since both changes are not obvious, it means that the adopted regularization methods are effective.

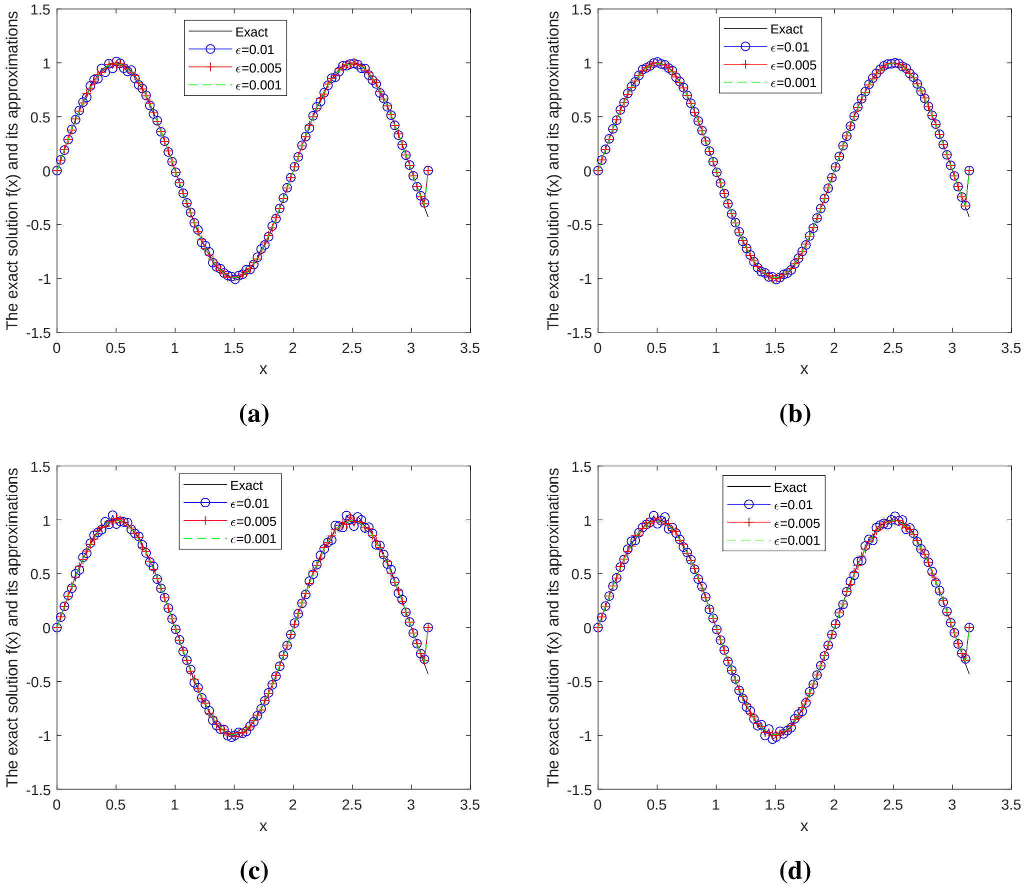

Figure 2. The exact solution and the quasi-inverse regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.1: (a) $s = 0.6$, $\alpha = 1.1$; (b) $s = 0.6$, $\alpha = 1.6$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

Table 1. When $\varepsilon = 0.01, 0.005, 0.001$, the error level of Example 6.1 for different values of α and s.

ε			0.01	0.005	0.001
$s = 0.6$	$\alpha = 1.4$	e_{r_1}	0.1012	0.1010	0.1011
	$\alpha = 1.9$	e_{r_1}	0.0780	0.0779	0.0781
	$\alpha = 1.4$	e_{r_2}	0.0225	0.0119	0.0080
	$\alpha = 1.9$	e_{r_2}	0.0072	0.0053	0.0063
$\alpha = 1.3$	$s = 0.2$	e_{r_1}	0.0880	0.0878	0.0878
	$s = 0.75$	e_{r_1}	0.0877	0.0880	0.0878
	$s = 0.2$	e_{r_2}	0.0266	0.0150	0.0083
	$s = 0.75$	e_{r_2}	0.0267	0.0143	0.0087

As can be seen from Figures 1 and 2, and Tables 1 and 2, both regularization methods are effective. However, the CPU time required by the fractional Tikhonov regularization method is less than that of the quasi-inverse regularization method.

Table 2. Comparison of CPU time between the fractional Tikhonov regularization method and the quasi-inverse regularization method in Example 6.1.

α ($s = 0.6$)		1.4	1.9
CPU-Time	Fractional Tikhonov	0.4390	0.4856
	Quasi-inverse	2.0476	2.0721
s ($\alpha = 1.3$)		0.2	0.75
CPU-Time	Fractional Tikhonov	0.4255	0.4232
	Quasi-inverse	2.0423	2.0954

Example 6.2. We consider the following functions:

$$f(x) = e^x \sin x, \quad \varphi(x) = \begin{cases} 0, & x \in [0, \frac{\pi}{3}), \\ 1, & x \in [\frac{\pi}{3}, \frac{2\pi}{3}), \\ 0, & x \in [\frac{2\pi}{3}, \pi], \end{cases} \quad \psi(x) = \begin{cases} 2x, & x \in [0, \frac{\pi}{2}), \\ 2(\pi - x), & x \in [\frac{\pi}{2}, \pi]. \end{cases} \tag{6.26}$$

Figure 3(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 3(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$. Figure 4(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 4(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$.

We can see from Figures 3 and 4, and Table 3 that with increasing noise levels, the error levels at various values also grow. When the space parameter s is fixed, as the time parameter α increases, the fitting effect of the image improves and the relative root mean square error decreases. This is because when α increases, the ill-posed weakens. On the other hand, both the change of the image and the relative root mean square error are not significant, which indicates that the two regularization methods we adopted are effective. When the time parameter α is fixed, as the space parameter s increases, the fitting effect of the image deteriorates and the relative root mean square error increases. This is because when s increases, the ill-posed intensifies. Since both changes are not obvious, it means that the adopted regularization methods are effective.

As can be seen from Figures 3 and 4, and Tables 3 and 4, both regularization methods are effective. However, the CPU time required by the fractional Tikhonov regularization method is less than that of the quasi-inverse regularization method.

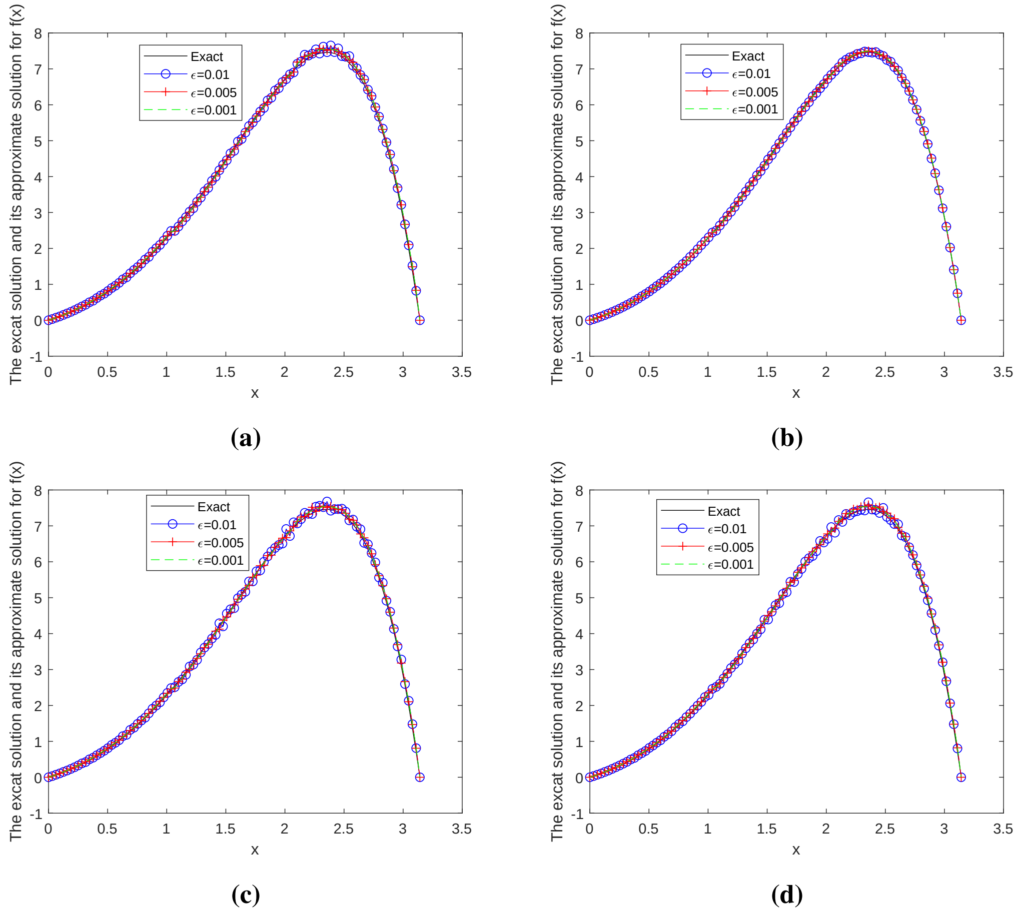

Figure 3. The exact solution and the fractional Tikhonov regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.2: (a) $s = 0.6$, $\alpha = 1.4$; (b) $s = 0.6$, $\alpha = 1.9$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

Table 3. When $\varepsilon = 0.01, 0.005, 0.001$, the error level of Example 6.2 for different values of α and s.

ε			0.01	0.005	0.001
$s = 0.6$	$\alpha = 1.4$	e_{r_1}	0.0162	0.0145	0.0149
	$\alpha = 1.9$	e_{r_1}	0.0064	0.0062	0.0062
	$\alpha = 1.4$	e_{r_2}	0.0174	0.0117	0.0102
	$\alpha = 1.9$	e_{r_2}	0.0143	0.0131	0.0110
$\alpha = 1.3$	$s = 0.2$	e_{r_1}	0.0173	0.0134	0.0125
	$s = 0.75$	e_{r_1}	0.0143	0.0139	0.0124
	$s = 0.2$	e_{r_2}	0.0188	0.0113	0.0107
	$s = 0.75$	e_{r_2}	0.0290	0.0144	0.0108

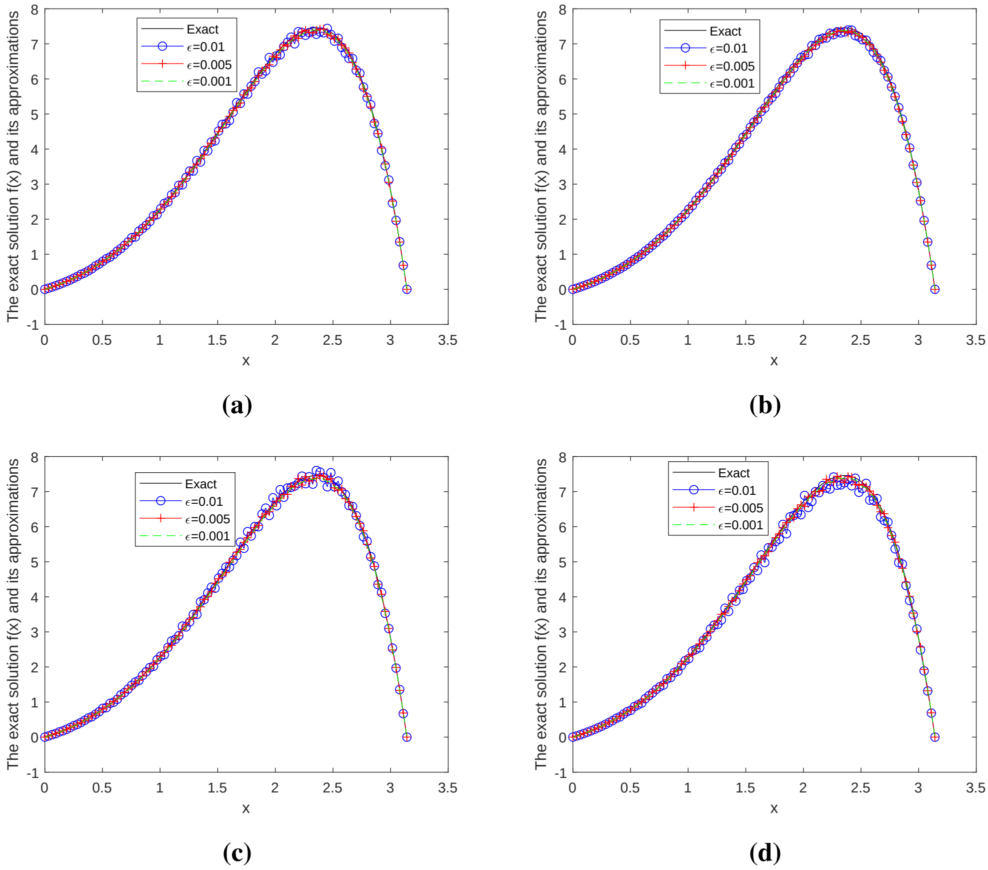

Figure 4. The exact solution and the quasi-inverse regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.2: (a) $s = 0.6$, $\alpha = 1.4$; (b) $s = 0.6$, $\alpha = 1.9$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

Table 4. Comparison of CPU time between the fractional Tikhonov regularization method and the quasi-inverse regularization method in Example 6.2.

α ($s = 0.6$)		1.4	1.9
CPU-Time	Fractional order Tikhonov	0.4368	0.4870
	Quasi-inverse	2.0942	2.1168
s ($\alpha = 1.3$)		0.2	0.75
CPU-Time	Fractional order Tikhonov	0.4489	0.4295
	Quasi-inverse	2.0965	2.0824

Example 6.3. Consider the following functions:

$$f(x) = \begin{cases} 2x, & x \in [0, \frac{\pi}{2}), \\ 2(\pi - x), & x \in [\frac{\pi}{2}, \pi], \end{cases} \quad \varphi(x) = x \sin x, \quad \psi(x) = e^x \sin x. \tag{6.27}$$

Figure 5(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 5(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the fractional Tikhonov regularization solution $f_\mu^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$. Figure 6(a), (b) shows that when the space parameter s is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different time parameters α while $s = 0.6, \alpha = 1.4$; $s = 0.6, \alpha = 1.9$. Figure 6(c), (d) shows that when the time parameter α is fixed and the noise levels $\epsilon = 0.01, 0.005, 0.001$ are taken separately, the comparison between the exact solution *f(x)* and the quasi-inverse regularization solution $f_\eta^\delta(x)$ for different space parameters s while $\alpha = 1.3, s = 0.2$; $\alpha = 1.3, s = 0.75$.

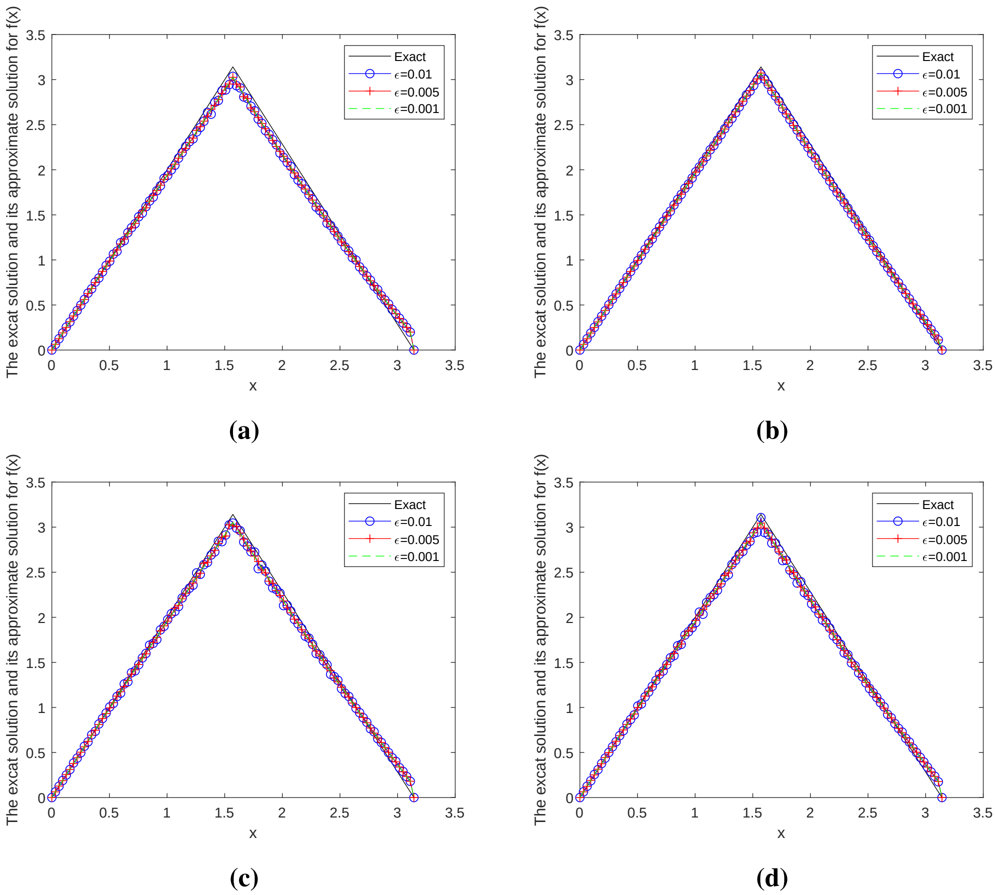

Figure 5. The exact solution and the fractional Tikhonov regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.3: (a) $s = 0.6$, $\alpha = 1.4$; (b) $s = 0.6$, $\alpha = 1.9$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

We can see from Figures 5, 6, and Table 5 that with increasing noise levels, the error levels at various values also grow. When the space parameter s is fixed, as the time parameter α increases, the fitting effect of the image improves and the relative root mean square error decreases. This is because when α increases, the ill-posed weakens. On the other hand, both the change of the image and the relative root mean square error are not significant, which indicates that the two regularization methods we adopted are effective. When the time parameter α is fixed, as the space parameter s increases, the fitting effect of the image deteriorates and the relative root mean square error increases. This is because when s increases, the ill-posed intensifies. Since both changes are not obvious, it means that the adopted regularization methods are effective.

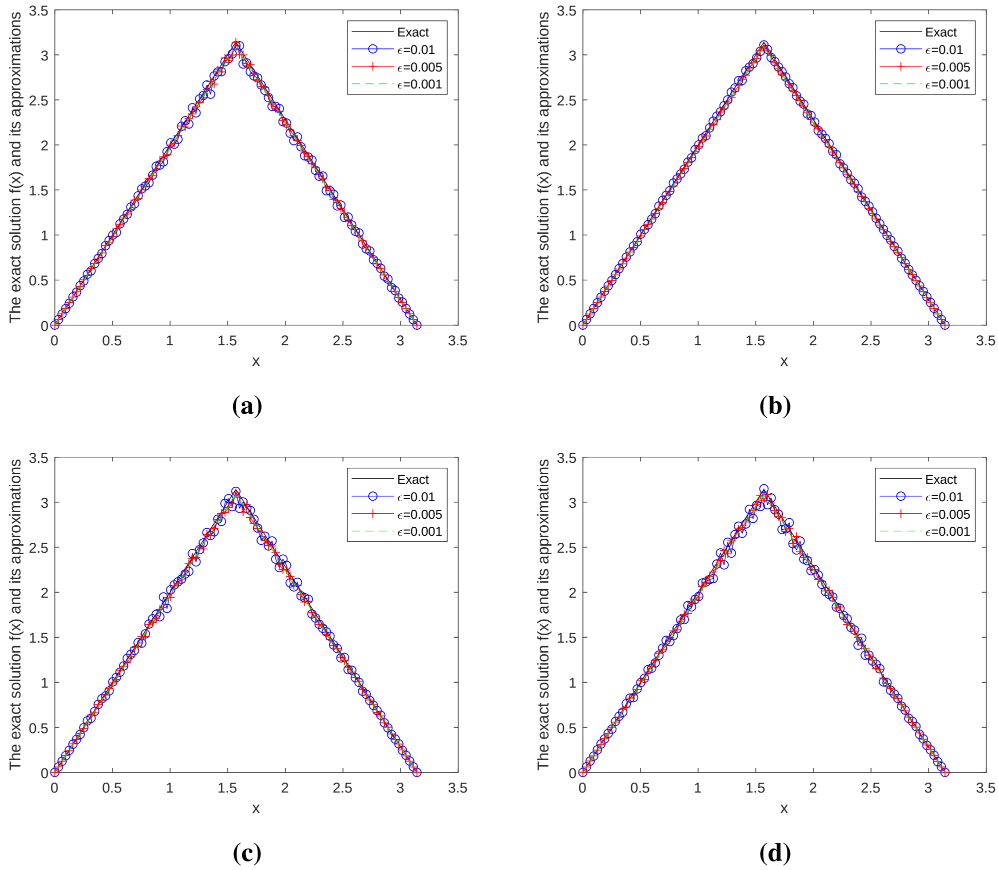

Figure 6. The exact solution and the quasi-inverse regularization solution with $\epsilon = 0.01, 0.005, 0.001$ in Example 6.3: (a) $s = 0.6$, $\alpha = 1.1$; (b) $s = 0.6$, $\alpha = 1.6$; (c) $\alpha = 1.3$, $s = 0.2$; (d) $\alpha = 1.3$, $s = 0.75$.

Table 5. When $\varepsilon = 0.01, 0.005, 0.001$, the error level of Example 6.3 for different values of α and s.

ε			0.01	0.005	0.001
$s = 0.6$	$\alpha = 1.4$	e_{r_1}	0.0446	0.0437	0.0430
	$\alpha = 1.9$	e_{r_1}	0.0274	0.0270	0.0271
	$\alpha = 1.4$	e_{r_2}	0.0228	0.0174	0.0172
	$\alpha = 1.9$	e_{r_2}	0.0198	0.0140	0.0155
$\alpha = 1.3$	$s = 0.2$	e_{r_1}	0.0348	0.0335	0.0329
	$s = 0.75$	e_{r_1}	0.0369	0.0340	0.0329
	$s = 0.2$	e_{r_2}	0.0283	0.0188	0.0169
	$s = 0.75$	e_{r_2}	0.0265	0.212	0.0175

As can be seen from Figures 5, 6, and Tables 5 and 6, both regularization methods are effective. However, the CPU time required by the fractional order Tikhonov regularization method is less than that of the quasi-inverse regularization method.

Table 6. Comparison of CPU time between the fractional Tikhonov regularization method and the quasi-inverse regularization method in Example 6.3.

α ($s = 0.6$)		1.4	1.9
CPU-Time	Fractional order Tikhonov	0.4547	0.4538
	Quasi-inverse	2.0788	2.0631
s ($\alpha = 1.3$)		0.2	0.75
CPU-Time	Fractional order Tikhonov	0.4972	0.4082
	Quasi-inverse	2.1256	2.0749

7. Conclusions

This paper solves the inverse problem of recovering the source term for a space-time fractional diffusion-wave equation. By utilizing final time measurement data, we obtain the solution to the inverse problem and analyze its ill-posed . To overcome this ill-posed, we employ both the fractional Tikhonov regularization method and the quasi-inverse regularization method. Convergence error estimates are derived under both the a priori and the a posteriori parameter choice rules. These error estimates demonstrate that both regularization methods effectively restore solution stability. Finally, several numerical examples are provided to verify the validity and feasibility of the proposed approaches. However, this paper is limited to the inverse source problem for the space-time fractional diffusion-wave equation. Therefore, future research will investigate the initial value problem, as well as the simultaneous identification of both source terms and initial values. Additionally, the numerical experiments in this paper only consider one-dimensional cases. Subsequent studies will extend the analysis to two- and three-dimensional scenarios to further validate the effectiveness of the proposed methods.

Author contributions

Hui-Ming Heng: Writing–original draft, Methodology; Fan Yang: Methodology, Supervision; Xiao-Xiao Li: Data curation; Zhen-Ji Tian: Software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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