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Research article

A two-stage fourth-order modified explicit Euler/Crank-Nicolson approach for a time-variable fractional transport model

Eric Ngondiep*, **Areej A. Binsultan and Ibtisam M. Aldawish**

Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), 90950 Riyadh, Saudi Arabia

* **Correspondence:** Email: ericngondiep@gmail.com, ngondieperic@rocketmail.com.

Abstract: This paper considered a two-stage fourth-order modified explicit Euler/Crank-Nicolson numerical method for solving a time-variable fractional transport equation subject to suitable initial and boundary conditions. Both stability and error estimates of the new approach were deeply analyzed in the $L^\infty(0,T;L^2)$-norm. The analysis suggests that the proposed technique is unconditionally stable and converges with order $O(\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} + h^4)$, where $\overline{\beta}$ is a positive constant satisfying $0 < \overline{\beta} < 1$, h and k are the space step and time step, respectively. This result indicates that the two-stage fourth-order formulation is fast and efficient. Numerical experiments were performed to verify the unconditional stability and convergence rate of the developed algorithm.

Keywords: time-variable fractional transport model; explicit Euler scheme; Crank-Nicolson method; two-stage fourth-order modified explicit Euler/Crank-Nicolson approach; stability analysis; convergence rate
Mathematics Subject Classification: 65M06, 65M12

1. Introduction

In the last twenty years, fractional calculus and fractional differential equations have found a broad range of applications in hydrology [1], medical problems [2], geology [3], fluid flow [4], biology [5], electrodynamics [6], heat [7], electrostatics [8], sound [9], and finances [9]. A large class of complex models are deeply described via variable-order derivatives. Most recently, the time-variable fractional-order telegraph equation has been shown to be a suitable model for various physical phenomena. Since the equations modeled by the time fractional partial differential equations (FPDEs) are highly complex, there is no method that can compute an exact solution. The big challenge with such a set of equations is the development of fast and efficient numerical approaches in an approximate solution. In the literature, abundant numerical schemes have been proposed for solving time-variable-order (or constant-order)

FPDEs such as: finite difference schemes of first-order accuracy in time and spatial second-order convergence [10, 11], compact finite difference scheme with convergence order $O(k + h^4)$ [12–15], and some numerical procedures of higher order $O(k^{2-\gamma} + h^4)$ [16, 17]. All these techniques were one-step methods. The author of [18, 19] developed a two-stage numerical scheme with convergence order $O(k^{2-\gamma} + h^4)$ for solving the time-fractional convection-diffusion-reaction equation with a constant-order derivative. For classical integer-order ordinary/partial differential equations, such as: the Black-Scholes model [20], fractional reaction-diffusion-convection equations [21, 22], Sobolev equations [23], Benjamin-Bona-Mahony problems [24], and systems of ODEs [25], a wide set of numerical techniques have been developed and deeply analyzed. For more details, we refer the readers to [25–29] and the references therein. This class of partial differential equations (PDEs) also have a large set of applications. Furthermore, some methods developed for solving integer-order PDEs can be used to efficiently compute approximate solutions of time FPDEs with low computational costs [12–14]. In this work, we develop a two-stage modified explicit Euler/Crank-Nicolson approach in an approximate solution of a time-variable fractional mobile-immobile convection-diffusion model subject to suitable initial and boundary conditions. The proposed technique is unconditionally stable, with convergence order $O(\max\{k^{2-\bar{\beta}}, k^{\frac{3}{2}}\} + h^4)$, and is faster and more efficient than a large class of numerical schemes widely studied in the literature for the considered problem [30–33]. A time-variable fractional mobile-immobile convection-diffusion equation describes a broad range of problems in physical or mathematical systems which include ocean acoustic propagation and heat conduction through a solid [34]. In such an equation, the integer-order time derivative term is added to describe the motion of particles conveniently [34]. Furthermore, this equation lies in a class of second-order PDEs that govern continuous time random walks with heavy tailed random waiting times.

In this paper, we propose a two-stage fourth-order modified explicit Euler/Crank-Nicolson formulation for the time-variable fractional transport equation [10] described by the following initial-boundary value problem:

$$u_t(x,t) = -cD_{0t}^{\beta(x,t)}u(x,t) + u_{xx}(x,t) - u_x(x,t) + f(x,t), \quad \text{on} \quad \Omega = (L_0, L) \times (t_0, T), \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x), \quad \text{on} \quad [L_0, L], \tag{1.2}$$

and boundary conditions

$$u(L_0,t) = g_1(t) \quad \text{and} \quad u(L,t) = g_2(t), \quad \text{on} \quad [0,T], \tag{1.3}$$

where u_t, u_x, and u_{xx} denote $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$, respectively. $f = f(x,t)$ represents the source term, u_0 is the initial condition, and g_1 and g_2 designate the boundary conditions. $cD_{0t}^{\beta(x,t)}u$, where $(0 < \beta_1 \leq \beta(x,t) \leq \beta_2 < 1)$, denotes the variable-order fractional derivative which has been introduced in various physical fields and defined (for example, in [35]) as

$$cD_{0t}^{\beta(x,t)}u(x,t) = \frac{1}{\Gamma(1-\beta(x,t))}\int_0^t \frac{u_s(x,s)}{(t-s)^{\beta(x,t)}}ds. \tag{1.4}$$

It is worth mentioning that the variable-order fractional derivative given by equation (1.4) is defined in the Caputo sense and it is a modification of the Riemann-Liouville definition. Additionally, it has a great advantage of dealing with problems whose initial conditions and the corresponding integer-order derivatives are assigned in terms of the field variables according to several physical processes. We note that the goal of this work is to develop an efficient numerical method for solving the time-variable fractional equation (1.1) subject to initial condition (1.2) and boundary conditions (1.3). Specifically, the attention is focused on the following three items:

(i) a full description of a two-stage fourth-order modified explicit Euler/Crank-Nicolson approach for time-variable FPDE (1.1) with initial-boundary conditions given by (1.2) and (1.3), respectively;
(ii) analysis of the unconditional stability and convergence rate of the new technique;
(iii) a broad range of numerical evidences that confirm the theoretical results.

The outline of the paper is as follows. In Section 2, we construct a two-stage fourth-order modified explicit Euler/Crank-Nicolson numerical scheme for computing an approximate solution to the initial-boundary value problems (1.1)–(1.3). Section 3 analyzes both stability and error estimates of the new approach using the $L^{\infty}(0,T;L^2)$-norm. Some numerical examples that confirm the theoretical study are presented and discussed in Section 4. Finally, in Section 5, we draw the general conclusions and provide directions for our future investigations.

2. Development of the two-stage numerical approach

In this section, we develop a two-stage fourth-order modified explicit Euler/crank-Nicolson numerical method for solving the initial-boundary value problems (1.1)–(1.3). Starting with the explicit Euler scheme, the new approach is a two-stage implicit method which approximates the time-variable fractional operator (1.4) using both forward and backward difference approximations in each step whereas the convection and diffusion terms are approximated using the central difference formulation. To compute a numerical solution of problems (1.1)–(1.3), we introduce a uniform grid of mesh points (x_j,t_n), where $x_j = L_0 + jh$, $j = 0,1,2,...,M$, and $t_n = nk$, for $n = 0,1,2,...,N$. In this discretization, M and N are two positive integers, and $h = \frac{L-L_0}{M}$ and $k = \frac{T}{N}$ are the space step and the time step, respectively. The space of grid functions is defined as $\mathcal{U}_{hk} = \{u(x_j,t_n),\ 0 \leq j \leq M;\ 0 \leq n \leq N\}$. For the convenience of writing, we set $u(x_j,t_n) = u_j^n$. The exact solution and the computed one at the grid point (x_j,t_n) are denoted by u_j^n and U_j^n, respectively. Furthermore, the values of the functions β and f at the mesh point (x_j,t_n) are given by β_j^n and f_j^n, respectively. Finally we introduce the positive parameter $\alpha \in (0,\frac{1}{2})$.

Furthermore, we define the following operators:

$$\delta_t u_j^n = \frac{u_j^{n+\frac{1}{2}} - u_j^n}{k/2},\ \delta_x u_{j-\frac{1}{2}}^n = \frac{u_j^n - u_{j-1}^n}{h},\ \delta_x u_{j+\frac{1}{2}}^n = \frac{u_{j+1}^n - u_j^n}{h},\ \delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \tag{2.1}$$

and we introduce the given norms

$$\|u^n\|_2 = \left(h\sum_{j=2}^{M-2}(u_j^n)^2\right)^{\frac{1}{2}},\quad |||u|||_{C_D^{6,3}} = \max_{\substack{0\leq s\leq 3\\ 0\leq r\leq 6}}\left\{\max_{0\leq n\leq N}\left\|\frac{\partial^{r+s}u}{\partial x^r\partial t^s}(t_n)\right\|_{L^2}\right\},\ \text{and}\ |||u|||_{\infty,2} = \max_{0\leq n\leq N}\|u^n\|_2,$$

$$\|u^n\|_{\nabla} = \left(h\sum_{j=1}^{M-2}(\delta_x u_{j+\frac{1}{2}}^n)^2\right)^{\frac{1}{2}} = \left(h\sum_{j=2}^{M-1}(\delta_x u_{j-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}},\ \text{and}\ \|u^n\|_{\triangle} = \left(h\sum_{j=1}^{M-1}(\delta_x^2 u_j^n)^2\right)^{\frac{1}{2}}, \tag{2.2}$$

together with the inner products

$$(u^n, v^n) = h\sum_{j=2}^{M-2} u_j^n v_j^n, \quad (\delta_x u^n, v^n) = h\sum_{j=1}^{M-2}\delta_x u^n_{j+\frac{1}{2}} v_j^n = h\sum_{j=2}^{M-1}\delta_x u^n_{j-\frac{1}{2}} v_j^n,$$

$$(\delta_x u^n, \delta_x v^n) = h\sum_{j=1}^{M-1}\delta_x u^n_{j-\frac{1}{2}}\delta_x v^n_{j-\frac{1}{2}}, \quad \left(\delta_x^2 u^n, \delta_x^2 v^n\right) = h\sum_{j=1}^{M-1}\delta_x^2 u_j^n \delta_x^2 v_j^n, \tag{2.3}$$

where $D = [L_0, L] \times [0, T]$. The spaces: $L^2(L_0, L)$, $H^1(L_0, L)$, $H^2(L_0, L)$, $L^\infty(0, T; L^2)$, and $C_D^{6,3}$ are equipped with the norms $\|\cdot\|_2$, $\|\cdot\|_\nabla$, $\|\cdot\|_\triangle$, $|||\cdot|||_{\infty,2}$, and $|||\cdot|||_{C_D^{6,3}}$, respectively, whereas the Hilbert space $L^2(L_0, L)$ is endowed with the scalar product $(\cdot,\cdot)$.

The application of the Taylor series for u at the grid point $(x_j, t_{n+\frac{1}{2}+\alpha})$ with time step $\frac{k}{2}$ using forward difference representation gives

$$u_j^{n+\frac{1}{2}+\alpha} = u_j^{n+\alpha} + \frac{k}{2}u_{t,j}^{n+\alpha} + O(k^2).$$

Utilizing Eq (1.1), this becomes

$$u_j^{n+\frac{1}{2}+\alpha} = u_j^{n+\alpha} + \frac{k}{2}[-cD_{0t}^{\beta_j^{n+\alpha}} u_j^{n+\alpha} - u_{x,j}^{n+\alpha} + u_{2x,j}^{n+\alpha} + f_j^{n+\alpha}] + O(k^2). \tag{2.4}$$

Expanding the Taylor series for u at the mesh point $(x_j, t_{n+\frac{1}{2}+\alpha})$ with time step $\frac{k}{2}$ using both forward and backward difference formulations, we get

$$u_j^{n+1+\alpha} = u_j^{n+\frac{1}{2}+\alpha} + \frac{k}{2}u_{t,j}^{n+\frac{1}{2}+\alpha} + \frac{k^2}{8}u_{2t,j}^{n+\frac{1}{2}+\alpha} + O(k^3),$$

$$u_j^{n+\frac{1}{2}+\alpha} = u_j^{n+1+\alpha} - \frac{k}{2}u_{t,j}^{n+1+\alpha} + \frac{k^2}{8}u_{2t,j}^{n+1+\alpha} + O(k^3).$$

Using Eq (1.1), we obtain

$$u_j^{n+1+\alpha} = u_j^{n+\frac{1}{2}+\alpha} + \frac{k}{2}[-cD_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha} - u_{x,j}^{n+\frac{1}{2}+\alpha} + u_{2x,j}^{n+\frac{1}{2}+\alpha} + f_j^{n+\frac{1}{2}+\alpha}] + \frac{k^2}{8}u_{2t,j}^{n+\frac{1}{2}+\alpha} + O(k^3),$$

$$u_j^{n+\frac{1}{2}+\alpha} = u_j^{n+1+\alpha} - \frac{k}{2}[-cD_{0t}^{\beta_j^{n+1+\alpha}} u_j^{n+1+\alpha} - u_{x,j}^{n+1+\alpha} + u_{2x,j}^{n+1+\alpha} + f_j^{n+1+\alpha}] + \frac{k^2}{8}u_{2t,j}^{n+1+\alpha} + O(k^3).$$

Subtracting the second equation from the first one and using the mean value theorem, it is easy to see that

$$\begin{aligned} u_j^{n+1+\alpha} = {} & u_j^{n+\frac{1}{2}+\alpha} + \frac{k}{4}\Big[-(cD_{0t}^{\beta_j^{n+1+\alpha}} u_j^{n+1+\alpha} + cD_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha}) - (u_{x,j}^{n+1+\alpha} + u_{x,j}^{n+\frac{1}{2}+\alpha}) \\ & + (u_{2x,j}^{n+1+\alpha} + u_{2x,j}^{n+\frac{1}{2}+\alpha}) + (f_j^{n+1+\alpha} + f_j^{n+\frac{1}{2}+\alpha})\Big] + O(k^3). \end{aligned} \tag{2.5}$$

For the convenience of writing, we set $\gamma_j^{n+\frac{1}{2}+\alpha} = \Gamma(1-\beta_j^{n+\frac{1}{2}+\alpha})^{-1}$. So the variable-order fractional time derivative given by (1.4) at the grid point $(x_j, t_{n+\frac{1}{2}+\alpha})$ can be rewritten as

$$
\begin{aligned}
cD_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha} = {} & \gamma_j^{n+\frac{1}{2}+\alpha} \int_0^{t_{n+\frac{1}{2}+\alpha}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds = \gamma_j^{n+\frac{1}{2}+\alpha} \sum_{l=0}^{n-1} \int_{t_{l+\frac{1}{2}}}^{t_{l+\frac{3}{2}}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds \\
& + \gamma_j^{n+\frac{1}{2}+\alpha} \left[\int_0^{t_{\frac{1}{2}}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{1}{2}+\alpha}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} d\tau \right]. \qquad (2.6)
\end{aligned}
$$

Let $q_{2,j}^{1,u_j}$ be the polynomial of degree two approximating $u_j(t)$ at the mesh points (t_l, u_j^l), $(t_{l+\frac{1}{2}}, u_j^{l+\frac{1}{2}})$, and $(t_{l+\frac{3}{2}}, u_j^{l+\frac{3}{2}})$ and let $E_j^{1,u_j}(t)$ be the corresponding error. Replacing λ by $\beta_j^{n+\frac{1}{2}+\alpha}$ in Eq (53) provided in [18], we get

$$
cD_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha} = c\Delta_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha} + I_j^{\beta_j^{n+\frac{1}{2}+\alpha}}, \qquad (2.7)
$$

where $c\Delta_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha}$ and $I_j^{\beta_j^{n+\frac{1}{2}+\alpha}}$ are defined by

$$
c\Delta_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}} u_j^{n+\frac{1}{2}+\alpha} = k^{1-\beta_j^{n+\frac{1}{2}+\alpha}} (1-\beta_j^{n+\frac{1}{2}+\alpha})^{-1} \gamma_j^{n+\frac{1}{2}+\alpha} \sum_{l=0}^{n} a_{n+\frac{1}{2},l+\frac{1}{2}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}} \delta_t u_j^l, \qquad (2.8)
$$

$$
\begin{aligned}
I_j^{\beta_j^{n+\frac{1}{2}+\alpha}} = {} & \gamma_j^{n+\frac{1}{2}+\alpha} \left[\int_0^{t_{\frac{1}{2}}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{1}{2}+\alpha}} \frac{u_{s,j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds \right] \\
& + \gamma_j^{n+\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} \int_{t_{i+\frac{1}{2}}}^{t_{i+\frac{3}{2}}} \frac{E_{s,j}^{1,u_j}(s)}{(t_{n+\frac{1}{2}+\alpha}-s)^{\beta_j^{n+\frac{1}{2}+\alpha}}} ds. \qquad (2.9)
\end{aligned}
$$

The summation index in Eq (2.8) varies in the range: $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., n$, whereas the summation index in (2.9) satisfies: $i = 0, 1, 2, 3, ..., n-1$. In relation (2.8), the coefficients $a_{n+\frac{1}{2},l+\frac{1}{2}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}$ form a generalized sequence defined in [18], page 11, when replacing λ with $\beta_j^{n+\frac{1}{2}+\alpha}$ by

$$
a_{\frac{1}{2},\frac{1}{2}}^{\alpha,\beta_j^{\alpha}} = \frac{1}{2}\alpha^{-\beta_j^{\alpha}}, \text{ and for } n \geq 1, \quad a_{n+\frac{1}{2},\frac{1}{2}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}} = \widetilde{f}_{n+\frac{1}{2},0}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}, \qquad (2.10)
$$

$$
a_{n+\frac{1}{2},l+\frac{1}{2}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}} = \begin{cases} \widetilde{d}_{n+\frac{1}{2},l}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}} - \widetilde{f}_{n+\frac{1}{2},l}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}, & \text{if } l = 1, 2, 3, ..., n-1, \\ \widetilde{f}_{n+\frac{1}{2},l-\frac{1}{2}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}, & \text{if } l = \frac{1}{2}, \frac{3}{2}, ..., n-\frac{1}{2}, \\ \widetilde{f}_{n+\frac{1}{2},n-1}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}} + \widetilde{f}_{n+\frac{1}{2},n}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}, & \text{if } l = n, \end{cases} \qquad (2.11)
$$

where the terms $\widetilde{f}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{\frac{1}{2},0}$, $\widetilde{\dot{\mathrm{f}}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},0}$, $\widetilde{f}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},s}$, and $\widetilde{d}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},s}$ (for $s = 0, 1, 2, ..., n$) are given in [18], pages 5 and 6, by replacing λ by $\beta_j^{n+\frac{1}{2}+\alpha}$ by

$$\widetilde{f}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},n} = \alpha^{1-\beta_j^{n+\frac{1}{2}+\alpha}},\ \widetilde{f}^{\alpha,\beta_j^{\frac{1}{2}+\alpha}}_{\frac{1}{2},0} = (\frac{1}{2}+\alpha)^{1-\beta_j^{\frac{1}{2}+\alpha}},\ \widetilde{\dot{\mathrm{f}}}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},0} = (n+\frac{1}{2}+\alpha)^{1-\beta_j^{n+\frac{1}{2}+\alpha}} - (n+\alpha)^{1-\beta_j^{n+\frac{1}{2}+\alpha}},$$

$$\widetilde{d}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},i} = (n+\alpha-i)^{1-\beta_j^{n+\frac{1}{2}+\alpha}} - (n+\alpha-i-1)^{1-\beta_j^{n+\frac{1}{2}+\alpha}},$$

$$\begin{aligned}\widetilde{f}^{\alpha,\beta_j^{n+\frac{1}{2}+\alpha}}_{n+\frac{1}{2},i} =& \frac{2}{2-\beta_j^{n+\frac{1}{2}+\alpha}}\left[(n+\alpha-i)^{2-\beta_j^{n+\frac{1}{2}+\alpha}} - (n+\alpha-i-1)^{2-\beta_j^{n+\frac{1}{2}+\alpha}}\right] \\ &- \frac{1}{2}\left[(n+\alpha-i)^{1-\beta_j^{n+\frac{1}{2}+\alpha}} + 3(n+\alpha-i-1)^{1-\beta_j^{n+\frac{1}{2}+\alpha}}\right],\end{aligned} \tag{2.12}$$

for $n \geq 1$ and $i = 0, 1, 2, ..., n-1$. Furthermore,

$$\begin{aligned} cD_{0t}^{\beta_j^\alpha} u_j^\alpha &= \frac{1}{\Gamma(1-\beta_j^\alpha)} \int_0^{t_\alpha} \frac{u_{s,j}(s)}{(t_\alpha - s)^{\beta_j^\alpha}} ds \\ &= \frac{1}{\Gamma(1-\beta_j^\alpha)} \left[\int_0^{t_\alpha} \frac{u_j^\alpha - u_j^0}{\alpha k (t_\alpha - s)^{\beta_j^\alpha}} ds + \int_0^{t_\alpha} \frac{(u_{s,j}(s) - P_{1,s}^{0,u_j}(s)}{(t_\alpha - s)^{\beta_j^\alpha} ds} \right] \\ &= c\Delta_{0t}^{\beta_j^\alpha} u_j^\alpha + J_j^{\beta_j^\alpha}, \end{aligned} \tag{2.13}$$

where

$$c\Delta_{0t}^{\beta_j^\alpha} u_j^\alpha = k^{1-\beta_j^\alpha} \Gamma(2-\beta_j^\alpha)^{-1} a^{\alpha,\beta_j^\alpha}_{\frac{1}{2},\frac{1}{2}} \delta_t^\alpha u_j^0, \tag{2.14}$$

$$\delta_t^\alpha u_j^0 = \frac{2}{k}(u_j^\alpha - u_j^0), \text{ and } J_j^{\beta_j^\alpha} = \frac{1}{\Gamma(1-\beta_j^\alpha)} \int_0^{t_\alpha} \frac{u_{s,j}(s) - P_{1,s}^{0,u_j}(s)}{(t_\alpha - s)^{\beta_j^\alpha}} ds. \tag{2.15}$$

$P_{1,s}^{0,u_j}$ is the first-order polynomial interpolating u_j at the mesh points (u_j^0, t_0) and (u_j^α, t_α).

In a similar manner, setting $\gamma_j^{n+1+\alpha} = \Gamma(1-\beta_j^{n+1+\alpha})^{-1}$ and replacing λ by $\beta_j^{n+1+\alpha}$ in the formulas obtained in [18], pages 8, 12, and 17, results in

$$cD_{0t}^{\beta_j^{n+1+\alpha}} u_j^{n+1+\alpha} = c\Delta_{0t}^{\beta_j^{n+1+\alpha}} u_j^{n+1+\alpha} + J_j^{\beta_j^{n+1+\alpha}}, \tag{2.16}$$

where

$$c\Delta_{0t}^{\beta_j^{n+1+\alpha}} u_j^{n+1+\alpha} = k^{1-\beta_j^{n+1+\alpha}} (1-\beta_j^{n+1+\alpha})^{-1} \gamma_j^{n+1+\alpha} \sum_{l=0}^{n+\frac{1}{2}} a^{\alpha,\beta_j^{n+1+\alpha}}_{n+1,l+\frac{1}{2}} \delta_t u_j^l, \tag{2.17}$$

with

$$a^{\alpha,\beta_j^{1+\alpha}}_{1,\frac{1}{2}} = \widetilde{d}^{\alpha,\beta_j^{1+\alpha}}_{1,0} - \widetilde{f}^{\alpha,\beta_j^{1+\alpha}}_{1,0},\ a^{\alpha,\beta_j^{1+\alpha}}_{1,1} = \widetilde{f}^{\alpha,\beta_j^{1+\alpha}}_{1,0} + \widetilde{f}^{\alpha,\beta_j^{1+\alpha}}_{1,1}, \tag{2.18}$$

and for $n \geq 1$,

$$a_{n+1,l+\frac{1}{2}}^{\alpha,\beta_j^{n+1+\alpha}} = \begin{cases} \widetilde{d}_{n+1,l}^{\alpha,\beta_j^{n+1+\alpha}} - \widetilde{f}_{n+1,l}^{\alpha,\beta_j^{n+1+\alpha}}, & \text{if } l = 0,1,2,3,...,n, \\ \widetilde{f}_{n+1,l-\frac{1}{2}}^{\alpha,\beta_j^{n+1+\alpha}}, & \text{if } l = \frac{1}{2}, \frac{3}{2}, ..., n-\frac{1}{2}, \\ \widetilde{f}_{n+1,n}^{\alpha,\beta_j^{n+1+\alpha}} + \widetilde{f}_{n+1,n+1}^{\alpha,\beta_j^{n+1+\alpha}}, & \text{if } l = n+\frac{1}{2}. \end{cases} \tag{2.19}$$

Here the terms $\widetilde{f}_{n+1,r}^{\alpha,\beta_j^{n+1+\alpha}}$ and $\widetilde{d}_{n+1,r}^{\alpha,\beta_j^{n+1+\alpha}}$, for $n \geq 1$ and $r = 0,1,2,...,n+1$, are obtained by replacing λ by $\beta_j^{n+1+\alpha}$ in [18], pages 8 and 9, by

$$\widetilde{d}_{n+1,i}^{\alpha,\beta_j^{n+1+\alpha}} = (n+1+\alpha-i)^{1-\beta_j^{n+1+\alpha}} - (n+\alpha-i)^{1-\beta_j^{n+1+\alpha}}, \; \widetilde{f}_{n+1,n+1}^{\alpha,\beta_j^{n+1+\alpha}} = \alpha^{1-\beta_j^{n+1+\alpha}},$$

$$\begin{aligned} \widetilde{f}_{n+1,i}^{\alpha,\beta_j^{n+1+\alpha}} &= \frac{2}{2-\beta_j^{n+1+\alpha}} \left[(n+1+\alpha-i)^{2-\beta_j^{n+1+\alpha}} - (n+\alpha-i)^{2-\beta_j^{n+1+\alpha}} \right] \\ &\quad - \frac{1}{2} \left[(n+1+\alpha-i)^{1-\beta_j^{n+1+\alpha}} + 3(n+\alpha-i)^{1-\beta_j^{n+1+\alpha}} \right]. \end{aligned} \tag{2.20}$$

We recall that in Eq (2.17) the summation index varies in the range $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., n+\frac{1}{2}$. Furthermore, in relation (52) given in [18], replacing λ with $\beta_j^{n+1+\alpha}$, we obtain

$$J_j^{\beta_j^{n+1+\alpha}} = \gamma_j^{n+1+\alpha} \left[\int_{t_{n+1}}^{t_{n+1+\alpha}} \frac{u_{s,j}(s)}{(t_{n+1+\alpha}-s)^{\beta_j^{n+1+\alpha}}} ds + \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \frac{E_{s,j}^{2,u_j}(s)}{(t_{n+1+\alpha}-s)^{\beta_j^{n+1+\alpha}}} ds \right], \tag{2.21}$$

where $E_{s,j}^{2,u_j}(s)$ is the error associated with the quadratic polynomial interpolating $u_j(t)$ at the grid points (u_j^l, t_l), $(u_j^{l+\frac{1}{2}}, t_{l+\frac{1}{2}})$, and (u_j^{l+1}, t_{l+1}).

Now, we should find the spatial fourth-order approximations of the terms $u_{x,j}^{n+s+\alpha}$ and $u_{2x,j}^{n+s+\alpha}$, for $s = 0, \frac{1}{2}$, and 1. Using the Taylor series expansion for u about the grid point $(x_j, t_{n+s+\alpha})$ with step size h using both forward and backward representations, the author of [18] has shown that

$$\begin{aligned} u_{2x,j}^{n+s+\alpha} &= \frac{1}{12h^2} \left[-u_{j+2}^{n+s+\alpha} + 16u_{j+1}^{n+s+\alpha} - 30u_j^{n+s+\alpha} + 16u_{j-1}^{n+s+\alpha} - u_{j-2}^{n+s+\alpha} \right] + O(h^4) \\ &= \delta_{2x}^4 u_j^{n+s+\alpha} + O(h^4), \end{aligned} \tag{2.22}$$

$$u_{x,j}^{n+s+\alpha} = \frac{1}{12h} \left[-u_{j+2}^{n+s+\alpha} + 8u_{j+1}^{n+s+\alpha} - 8u_{j-1}^{n+s+\alpha} + u_{j-2}^{n+s+\alpha} \right] + O(h^4) = \delta_x^4 u_j^{n+s+\alpha} + O(h^4), \tag{2.23}$$

for $s \in \{0, \frac{1}{2}, 1\}$, where

$$\delta_{2x}^4 u_j^{n+s+\alpha} = \frac{1}{12h^2} \left[-u_{j+2}^{n+s+\alpha} + 16u_{j+1}^{n+s+\alpha} - 30u_j^{n+s+\alpha} + 16u_{j-1}^{n+s+\alpha} - u_{j-2}^{n+s+\alpha} \right], \tag{2.24}$$

$$\delta_x^4 u_j^{n+s+\alpha} = \frac{1}{12h} \left[-u_{j+2}^{n+s+\alpha} + 8u_{j+1}^{n+s+\alpha} - 8u_{j-1}^{n+s+\alpha} + u_{j-2}^{n+s+\alpha} \right]. \tag{2.25}$$

For $s = 0$, replacing $n+1$ by n in (2.16) and plugging the obtained equation into (2.23), (2.22), and (2.4) yields

$$u_j^{n+\frac{1}{2}+\alpha} = u_j^{n+\alpha} + \frac{k}{2}\left[-c\Delta_{0t}^{\beta_j^{n+\alpha}} u_j^{n+\alpha} - J_j^{\beta_j^{n+\alpha}} - \delta_x^4 u_j^{n+\alpha} + \delta_{2x}^4 u_j^{n+\alpha} + f_j^{n+\alpha}\right] + O(k^2 + kh^4). \tag{2.26}$$

Replacing $n + 1$ by n in (2.17), substituting the new equation into (2.26), and rearranging terms results in

$$\begin{aligned} u_j^{n+\frac{1}{2}+\alpha} = {} & u_j^{n+\alpha} - \frac{1}{2}k^{2-\beta_j^{n+\alpha}}\Gamma(2-\beta_j^{n+\alpha})^{-1}\sum_{l=0}^{n-\frac{1}{2}} a_{n,l+\frac{1}{2}}^{\alpha,\beta_j^{n+\alpha}}\delta_t u_j^l \\ & + \frac{k}{2}\left[-\delta_x^4 u_j^{n+\alpha} + \delta_{2x}^4 u_j^{n+\alpha} + f_j^{n+\alpha}\right] - \frac{k}{2}J_j^{\beta_j^{n+\alpha}} + O(k^2 + kh^4). \end{aligned} \tag{2.27}$$

Setting $u_{0x}(x, t) = u(x, t)$ and applying the Taylor series expansion for the functions u_{mx} ($m = 0, 1, 2$) at the mesh points (x_j, t_n), $(x_j, t_{n+\frac{1}{2}+\alpha})$, and $(x_j, t_{n+1+\alpha})$ with time step $\frac{k}{2}$ gives

$$\begin{aligned} u_{mx,j}^{n+\frac{1}{2}+\alpha} &= (1+2\alpha)u_{mx,j}^{n+\frac{1}{2}} - 2\alpha u_{mx,j}^n + O(k^2) = 2\alpha u_{mx,j}^{n+1} + (1-2\alpha)u_{mx,j}^{n+\frac{1}{2}} + O(k^2), \\ u_{mx,j}^{n+1+\alpha} &= (1+2\alpha)u_{mx,j}^{n+1} - 2\alpha u_{mx,j}^{n+\frac{1}{2}} + O(k^2), \quad u_{mx,j}^{n+\alpha} = 2\alpha u_{mx,j}^{n+\frac{1}{2}} + (1-2\alpha)ku_{mx,j}^n + O(k^2). \end{aligned} \tag{2.28}$$

Combining (2.28) and (2.22)–(2.23), we get

$$\begin{aligned} \delta_{mx}^4 u_j^{n+\frac{1}{2}+\alpha} &= (1+2\alpha)\delta_{mx}^4 u_j^{n+\frac{1}{2}} - 2\alpha\delta_{mx}^4 u_j^n + O(k^2 + h^4) \\ &= 2\alpha\delta_{mx}^4 u_j^{n+1} + (1-2\alpha)\delta_{mx}^4 u_j^{n+\frac{1}{2}} + O(k^2 + h^4), \\ \delta_{mx}^4 u_j^{n+1+\alpha} &= (1+2\alpha)\delta_{mx}^4 u_j^{n+1} - 2\alpha\delta_{mx}^4 u_j^{n+\frac{1}{2}} + O(k^2 + h^4), \\ \delta_{mx}^4 u_j^{n+\alpha} &= 2\alpha\delta_{mx}^4 u_j^{n+\frac{1}{2}} + (1-2\alpha)\delta_{mx}^4 u_j^n + O(k^2 + h^4), \end{aligned} \tag{2.29}$$

where we set $\delta_{0x}^4 u_j^{n+\alpha} = u_j^{n+\alpha}$, $\delta_{0x}^4 u_j^{n+\frac{1}{2}} = u_j^{n+\frac{1}{2}}$ and $\delta_{0x}^4 u_j^n = u_j^n$. For $m = 0, 1, 2$, substituting the last equation of (2.29) into (2.27) and performing simple computations, we obtain

$$\begin{aligned} & u_j^{n+\frac{1}{2}} - \alpha k(\delta_{2x}^4 - \delta_x^4)u_j^{n+\frac{1}{2}} \\ = {} & u^n - 2^{-1}k^{2-\beta_j^{n+\alpha}}\Gamma(2-\beta_j^{n+\alpha})^{-1}\sum_{l=0}^{n-\frac{1}{2}} a_{n,l+\frac{1}{2}}^{\alpha,\beta_j^{n+\alpha}}\delta_t u_j^l \\ & + (\frac{1}{2} - \alpha)k(\delta_{2x}^4 - \delta_x^4)u_j^n + \frac{k}{2}f_j^{n+\alpha} - \frac{k}{2}J_j^{\beta_j^{n+\alpha}} + O(k^2 + k^3 + kh^4), \quad \text{for} \quad n \geq 1. \end{aligned} \tag{2.30}$$

For $n = 0$, utilizing relation (2.15) and substituting (2.13) into (2.4), we get

$$u_j^{\frac{1}{2}+\alpha} = u_j^\alpha + \frac{k}{2}\left[-k^{1-\beta_j^\alpha}\Gamma(2-\beta_j^\alpha)^{-1}a_{\frac{1}{2},\frac{1}{2}}^{\alpha,\beta_j^\alpha}\delta_t^\alpha u_j^0 - J_j^{\beta_j^\alpha} - u_{x,j}^\alpha + u_{2x,j}^\alpha + f_j^\alpha\right] + O(k^2). \tag{2.31}$$

Replacing n and s by 0 in (2.22)–(2.23) and (2.28)–(2.29) and m by $0, 1, 2$, in the first equations of (2.28) and (2.29), combining the obtained equations together with (2.31), and utilizing equality $\delta_t^\alpha u_j^0 = \frac{2}{k}(u_j^\alpha - u_j^0)$, it is not difficult to see that

$$u_j^{\frac{1}{2}} - \alpha k(\delta_{2x}^4 - \delta_x^4)u_j^{\frac{1}{2}} = u^0 - \alpha k\theta_{0j}^\alpha\delta_t u_j^0 + (\frac{1}{2} - \alpha)k(\delta_{2x}^4 - \delta_x^4)u_j^0 + \frac{k}{2}f_j^\alpha - \frac{k}{2}J_j^{\beta_j^\alpha} + O(k^2 + k^3 + kh^4), \tag{2.32}$$

where

$$\theta_{0j}^{\alpha} = k^{1-\beta_j^{\alpha}}\Gamma(2-\beta_j^{\alpha})^{-1}a_{\frac{1}{2},\frac{1}{2}}^{\alpha,\beta_j^{\alpha}}. \tag{2.33}$$

Suppose $U^n = (U_2^n, U_3^n, ..., U_{M-2}^n)$ is the approximate solution vector at time level n and $u^n = (u_2^n, u_3^n, ..., u_{M-2}^n)$ is the analytical one at time t_n. Truncating the error terms in both Eqs (2.30) and (2.32), we obtain the first step of the new approach

$$U_j^{\frac{1}{2}} - \alpha k(\delta_{2x}^4 - \delta_x^4)U_j^{\frac{1}{2}} = U^0 - \alpha k\theta_{0j}^{\alpha}\delta_t U_j^0 + (\frac{1}{2}-\alpha)k(\delta_{2x}^4 - \delta_x^4)U_j^0 + \frac{k}{2}f_j^{\alpha}, \tag{2.34}$$

$$\begin{aligned} U_j^{n+\frac{1}{2}} - \alpha k(\delta_{2x}^4 - \delta_x^4)U_j^{n+\frac{1}{2}} &= U^n - 2^{-1}k^{2-\beta_j^{n+\alpha}}\Gamma(2-\beta_j^{n+\alpha})^{-1}\sum_{l=0}^{n-\frac{1}{2}} a_{n,l+\frac{1}{2}}^{\alpha,\beta_j^{n+\alpha}}\delta_t U_j^l \\ &\quad + (\frac{1}{2}-\alpha)k(\delta_{2x}^4 - \delta_x^4)U_j^n + \frac{k}{2}f_j^{n+\alpha}, \quad \text{for} \quad n \geq 1. \end{aligned} \tag{2.35}$$

In addition, we set

$$f^{n+s+\alpha} = (f_2^{n+s+\alpha}, f_3^{n+s+\alpha}, ..., f_{M-2}^{n+s+\alpha}) \text{ and } \theta_{l+\frac{1}{2}}^{n+s+\alpha} = (\theta_{l+\frac{1}{2},2}^{n+s+\alpha}, \theta_{l+\frac{1}{2},3}^{n+s+\alpha}, ..., \theta_{l+\frac{1}{2},M-2}^{n+s+\alpha}), \tag{2.36}$$

where

$$\theta_{l+\frac{1}{2},j}^{n+s+\alpha} = k^{1-\beta_j^{n+s+\alpha}}\Gamma(2-\beta_j^{n+s+\alpha})^{-1}a_{n+s,l+\frac{1}{2}}^{\alpha,\beta_j^{n+s+\alpha}}, \tag{2.37}$$

$s \in \{0, 2^{-1}, 1\}$. Thus, Eqs (2.34) and (2.37) can be expressed in the matrix form as

$$A_0U^{\frac{1}{2}} + k\theta_0^{\alpha} * \delta_t U^0 = A_1U^0 + \frac{k}{2}f^{\alpha}, \tag{2.38}$$

$$A_0U^{n+\frac{1}{2}} + \frac{k}{2}\sum_{l=0}^{n-\frac{1}{2}}\theta_{l+\frac{1}{2}}^{n+\alpha} * \delta_t U^l = A_1U^n + \frac{k}{2}f^{n+\alpha}, \tag{2.39}$$

where the symbol * denotes the componentwise usual multiplication between two vectors, and A_0 and A_1 are two $(M-2)\times(M-2)$ "pentadiagonal" matrices defined by

$$A_0 = \begin{bmatrix} a_{kh}^0 & b_{kh}^0 & c_{kh}^0 & 0 & \cdots & \cdots & 0 \\ d_{kh}^0 & a_{kh}^0 & a_{kh}^0 & a_{kh}^0 & 0 & & \vdots \\ e_{kh}^0 & d_{kh}^0 & a_{kh}^0 & b_{kh}^0 & c_{kh}^0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & c_{kh}^0 \\ \vdots & & \ddots & \ddots & d_{kh}^0 & a_{kh}^0 & b_{kh}^0 \\ 0 & \cdots & \cdots & 0 & e_{kh}^0 & d_{kh}^0 & a_{kh}^0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} a_{kh}^1 & b_{kh}^1 & c_{kh}^1 & 0 & \cdots & \cdots & 0 \\ d_{kh}^1 & a_{kh}^1 & a_{kh}^1 & a_{kh}^1 & 0 & & \vdots \\ e_{kh}^1 & d_{kh}^1 & a_{kh}^1 & b_{kh}^1 & c_{kh}^1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & c_{kh}^1 \\ \vdots & & \ddots & \ddots & d_{kh}^1 & a_{kh}^1 & b_{kh}^1 \\ 0 & \cdots & \cdots & 0 & e_{kh}^1 & d_{kh}^1 & a_{kh}^1 \end{bmatrix}, \tag{2.40}$$

where

$$a_{kh}^0 = 1+\frac{5}{2}\frac{\alpha k}{h^2},\ b_{kh}^0 = \frac{2\alpha k}{3h}(1-2h^{-1}),\ c_{kh}^0 = \frac{\alpha k}{12h}(-1+h^{-1}),\ d_{kh}^0 = \frac{-2\alpha k}{3h}(1+2h^{-1}),\ e_{kh}^0 = \frac{\alpha k}{12h}(1+h^{-1}),$$

$$a_{kh}^1 = 1-\frac{5(1-2\alpha)k}{4h^2},\ b_{kh}^1 = \frac{(1-2\alpha)k}{3h}(-1+2h^{-1}),\ c_{kh}^1 = \frac{(1-2\alpha)k}{24h}(1-h^{-1}),\ d_{kh}^1 = \frac{(1-2\alpha)k}{3h}(1+2h^{-1}),$$

$$e_{kh}^{1}=\frac{(-1+2\alpha)k}{24h}(1+h^{-1}). \tag{2.41}$$

To complete the full description of the desired algorithm, we should develop the second-step of the method. Combining Eqs (2.5), (2.7), and (2.16), direct calculations give

$$\begin{aligned} u_j^{n+1+\alpha} = {} & u^{n+\frac{1}{2}+\alpha}+\frac{k}{4}\Bigg\{-\left(c\Delta_{0t}^{\beta_j^{n+1+\alpha}}u_j^{n+1+\alpha}+c\Delta_{0t}^{\beta_j^{n+\frac{1}{2}+\alpha}}u_j^{n+\frac{1}{2}+\alpha}\right)-(u_{x,j}^{n+1+\alpha}+u_{x,j}^{n+\frac{1}{2}+\alpha}) \\ & +(u_{2x,j}^{n+1+\alpha}+u_{2x,j}^{n+\frac{1}{2}+\alpha})+(f_j^{n+1+\alpha}+f_j^{n+\frac{1}{2}+\alpha})\Bigg\}-\frac{k}{4}(J_j^{\beta_j^{n+1+\alpha}}+I_j^{\beta_j^{n+\frac{1}{2}+\alpha}})+O(k^3). \end{aligned} \tag{2.42}$$

Since for $s=\frac{1}{2},1$, $\gamma_j^{n+s+\alpha}=\Gamma(1-\beta_j^{n+s+\alpha})^{-1}$, then $(1-\beta_j^{n+s+\alpha})^{-1}\gamma_j^{n+s+\alpha}=\Gamma(2-\beta_j^{n+s+\alpha})^{-1}$. For $r=\frac{1}{2}$ and $m=0,1,2$, plugging Eqs (2.8), (2.17), (2.28), (2.29), and (2.42), straightforward computations result in

$$\begin{aligned} & u_j^{n+1}-\frac{1+4\alpha}{4}k(\delta_{2x}^4-\delta_x^4)u_j^{n+1}+\frac{k}{4}\left[\sum_{l=0}^{n+\frac{1}{2}}\theta_{l+\frac{1}{2},j}^{n+1+\alpha}\delta_t u_j^l+\sum_{l=0}^{n}\theta_{l+\frac{1}{2},j}^{n+\frac{1}{2}+\alpha}\delta_t u_j^l\right] \\ = {} & u_j^{n+\frac{1}{2}}+\frac{1-4\alpha}{4}k(\delta_{2x}^4-\delta_x^4)u_j^{n+\frac{1}{2}}\frac{k}{4}(f_j^{n+1+\alpha}+f_j^{n+\frac{1}{2}+\alpha})-\frac{k}{4}(J_j^{\beta_j^{n+1+\alpha}}+I_j^{\beta_j^{n+\frac{1}{2}+\alpha}})+O(k^2+k^3+kh^4), \end{aligned} \tag{2.43}$$

where $\theta_{l+\frac{1}{2},j}^{n+s+\alpha}$ is defined by (2.37). Omitting the error terms $\frac{k}{4}(J_j^{\beta_j^{n+1+\alpha}}+I_j^{\beta_j^{n+\frac{1}{2}+\alpha}})+O(k^2+k^3+kh^4)$, Eq (2.43) can be approximated as

$$\begin{aligned} & U_j^{n+1}-\frac{1+4\alpha}{4}k(\delta_{2x}^4-\delta_x^4)U_j^{n+1}+\frac{k}{4}\left[\sum_{l=0}^{n+\frac{1}{2}}\theta_{l+\frac{1}{2},j}^{n+1+\alpha}\delta_t U_j^l+\sum_{l=0}^{n}\theta_{l+\frac{1}{2},j}^{n+\frac{1}{2}+\alpha}\delta_t U_j^l\right] \\ = {} & U_j^{n+\frac{1}{2}}+\frac{1-4\alpha}{4}k(\delta_{2x}^4-\delta_x^4)U_j^{n+\frac{1}{2}}\frac{k}{4}(f_j^{n+1+\alpha}+f_j^{n+\frac{1}{2}+\alpha}), \qquad \text{for } j=2,3,...M-2. \end{aligned} \tag{2.44}$$

We introduce the following "pentadiagonal" matrices A and A_2 of size $(M-2)\times(M-2)$ defined as

$$A=\begin{bmatrix} a_{kh} & b_{kh} & c_{kh} & 0 & \cdots & \cdots & 0 \\ d_{kh} & a_{kh} & a_{kh} & a_{kh} & 0 & & \vdots \\ e_{kh} & d_{kh} & a_{kh} & b_{kh} & c_{kh} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & c_{kh} \\ \vdots & & \ddots & \ddots & d_{kh} & a_{kh} & b_{kh} \\ 0 & \cdots & \cdots & 0 & e_{kh} & d_{kh} & a_{kh} \end{bmatrix} \text{ and } A_2=\begin{bmatrix} a_{kh}^2 & b_{kh}^2 & c_{kh}^2 & 0 & \cdots & \cdots & 0 \\ d_{kh}^2 & a_{kh}^2 & a_{kh}^2 & a_{kh}^2 & 0 & & \vdots \\ e_{kh}^2 & d_{kh}^2 & a_{kh}^2 & b_{kh}^2 & c_{kh}^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & c_{kh}^2 \\ \vdots & & \ddots & \ddots & d_{kh}^2 & a_{kh}^2 & b_{kh}^2 \\ 0 & \cdots & \cdots & 0 & e_{kh}^2 & d_{kh}^2 & a_{kh}^2 \end{bmatrix}, \tag{2.45}$$

where

$a_{kh}=1+\dfrac{5(1+4\alpha)k}{8h^2}$, $b_{kh}=\dfrac{(1+4\alpha)k}{6h}(1-2h^{-1})$, $c_{kh}=\dfrac{(1+4\alpha)k}{48h}(-1+h^{-1})$, $d_{kh}=\dfrac{-(1+4\alpha)k}{6h}(1+2h^{-1})$,

$e_{kh}=\dfrac{(1+4\alpha)k}{48h}(1+h^{-1})$, $a_{kh}^2=1-\dfrac{5(1-4\alpha)k}{8h^2}$, $b_{kh}^2=\dfrac{(1-4\alpha)k}{6h}(-1+2h^{-1})$, $c_{kh}^2=\dfrac{(1-4\alpha)k}{48h}(1-h^{-1})$,

$$d_{kh}^{2}=\frac{(1-4\alpha)k}{6h}(1+2h^{-1}),\ e_{kh}^{2}=\frac{(-1+4\alpha)k}{48h}(1+h^{-1}). \tag{2.46}$$

A combination of Eqs (2.36), (2.37), and (2.44) provides the following matrix form:

$$AU^{n+1}+\frac{k}{4(1+2\alpha)}\left[\sum_{l=0}^{n+\frac{1}{2}}\theta_{l+\frac{1}{2}}^{n+1+\alpha}*\delta_tU_j^l+\sum_{l=0}^{n}\theta_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}*\delta_tU_j^l\right]=A_2U^{n+\frac{1}{2}}+\frac{k}{4}(f^{n+1+\alpha}+f^{n+\frac{1}{2}+\alpha}), \tag{2.47}$$

where "*" represents the componentwise usual multiplication between two vectors. We recall that the summation index "l" varies in the range $l=0,\frac{1}{2},1,\frac{3}{2},2,...,n,n+\frac{1}{2}$. Furthermore, Eq (2.44) denotes the second step of the proposed two-stage fourth-order modified explicit Euler/Crank-Nicolson numerical scheme in a computed solution of the initial-boundary value problems (1.1)–(1.3).

An assembly of Eqs (2.38), (2.39) and (2.47) provides the new algorithm for solving the problems (1.1)–(1.3), that is, for $n=1,2,...,N-1$,

$$A_0U^{\frac{1}{2}}+k\theta_0^{\alpha}*\delta_tU^0=A_1U^0+\frac{k}{2}f^{\alpha}, \tag{2.48}$$

$$AU^{1}+\frac{k}{4(1+2\alpha)}\left[\sum_{l=0}^{\frac{1}{2}}\theta_{l+\frac{1}{2}}^{1+\alpha}*\delta_tU_j^l+\theta_{\frac{1}{2}}^{\frac{1}{2}+\alpha}*\delta_tU_j^0\right]=A_2U^{\frac{1}{2}}+\frac{k}{4}(f^{1+\alpha}+f^{\frac{1}{2}+\alpha}), \tag{2.49}$$

$$A_0U^{n+\frac{1}{2}}+\frac{k}{2}\sum_{l=0}^{n-\frac{1}{2}}\theta_{l+\frac{1}{2}}^{n+\alpha}*\delta_tU^l=A_1U^n+\frac{k}{2}f^{n+\alpha}, \tag{2.50}$$

$$AU^{n+1}+\frac{k}{4(1+2\alpha)}\left[\sum_{l=0}^{n+\frac{1}{2}}\theta_{l+\frac{1}{2}}^{n+1+\alpha}*\delta_tU_j^l+\sum_{l=0}^{n}\theta_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}*\delta_tU_j^l\right]=A_2U^{n+\frac{1}{2}}+\frac{k}{4}(f^{n+1+\alpha}+f^{n+\frac{1}{2}+\alpha}), \tag{2.51}$$

with initial and boundary conditions

$$U_j^0=u_j^0,\ j=0,1,...,M;\ \ U_0^n=g_1^n\ \text{and}\ U_M^n=g_2^n,\ \text{for}\ n=0,1,...,N, \tag{2.52}$$

where the matrices A_0, A_1, A, and A_2 are given by relations (2.40)–(2.41) and (2.45)–(2.46). To start the algorithm we should set $U_1^n=U_0^n$ and $U_{M-1}^n=U_M^n$, for $n=0,1,...,N$. However, the terms U_1^n and U_{M-1}^n can be obtained by using any one-step fractional approach such as the method analyzed in [36].

It's worth noticing that the coefficients of the pentadiagonal matrices A and A_i, for $i=0,1,2$, come from the entries of a g-Toeplitz matrix ($T_{n,g}(f)$) or g-circulant matrix ($C_{n,g}(f)$), generated by a Lebesgue integrable function f defined over the domain $(-\pi,\pi)$, where g is a nonnegative integer. Specifically, these matrices are called band Toeplitz matrices which represent a subclass of g-Toeplitz structures. For more details about g-Toeplitz, g-circulant, and band Toeplitz matrices, we refer the readers to [37] and references therein. Furthermore, since the matrices A and A_0 are not symmetric, at time level n or $n+\frac{1}{2}$, each system of linear equations (2.48)–(2.51) can be efficiently solved using the preconditioned generalized minimal residual algorithm [38].

3. Stability analysis and error estimates of the proposed two-stage approach (2.48)–(2.52)

In this section we analyze both unconditional stability and convergence order of the new approach (2.48)–(2.52) applied to the initial-boundary value problems (1.1)–(1.3). In this study we assume that the function $\beta(x,t) := \overline{\beta}$ is constant and the parameter α satisfies $0 < \alpha < 2^{-1}$. These restrictions play crucial roles in the proof of some intermediate results (namely Lemma 3.2) and the main result of this paper (Theorem 3.1). Furthermore, the following lemmas are important in the analysis of stability and error estimates of the proposed formulation (2.48)–(2.52) for solving the time variable-order fractional transport equation (1.1) subject to suitable initial condition (1.2) and boundary conditions (1.3).

Lemma 3.1. *For any $\overline{\beta} \in (0,1)$ and $\alpha \in (0, \frac{1}{2})$, set $D = [L_0, L] \times [0, T]$ and consider a function $u \in C^{6,3}(D) := C^{6,3}_D$. Thus it holds that*

$$\max_{0\leq n\leq N-1} \|cD^{\overline{\beta}}_{0t}u^{n+s+\alpha} - c\Delta^{\overline{\beta}}_{0t}u^{n+s+\alpha}\|_2 \leq C_s k^{2-\overline{\beta}}, \tag{3.1}$$

where $s = \frac{1}{2}, 1$, $cD^{\overline{\beta}}_{0t}u^{n+s+\alpha}$, and $c\Delta^{\overline{\beta}}_{0t}u^{n+s+\alpha}$, for $0 \leq n \leq N-1$, are defined by (2.7)*,* (2.8)*, and* (2.13)–(2.17)*, and C_s are positive constants independent of the mesh size h and time step k.*

Proof. Since $0 < \overline{\beta} < 1$ is constant and $0 < \alpha < \frac{1}{2}$, the proof of this lemma can be found in [18]. □

Lemma 3.2. *[18] Assume that $0 < \overline{\beta} < \frac{2}{3}$ and consider the generalized sequences $\left(a^{\alpha,\overline{\beta}}_{n+s,l}\right)_{2^{-1}\leq l\leq n+s}$, where $s = \frac{1}{2}, 1$, defined by Eqs* (2.10) *and* (2.11) *(for $s = 2^{-1}$) and* (2.18)–(2.19) *(for $s = 1$), thus*

$$a^{\alpha,\overline{\beta}}_{n+s,l} < a^{\alpha,\overline{\beta}}_{n+s,l+\frac{1}{2}}, \quad for \quad l = \frac{1}{2}, 1, \frac{3}{2}, 2, ..., n \ \ (resp., \ \ n + \frac{1}{2}). \tag{3.2}$$

Lemma 3.3. *Let $(a^{\alpha,\overline{\beta}}_{n+s,l})_{l\leq n+s}$ be the generalized sequences defined by relations* (2.10)–(2.11) *and* (2.18)–(2.19)*. For every mesh function $u(\cdot,\cdot)$ defined on the grid space $\mathcal{U}_{hk} = \{u^n_j,\ 0 \leq j \leq M \ \text{and} \ n = 0,1,2,...,N\}$, setting $W^{\overline{\beta},l}_j = \sum_{r=0}^{l} a^{\alpha,\overline{\beta}}_{n+s,r+\frac{1}{2}}\delta_t u^r_j$ ($s = \frac{1}{2}, 1$), the following estimates hold*

$$\begin{aligned} u^{n+s}_j(c\Delta^{\overline{\beta}}_{0t}u^{n+s+\alpha}_j) =& \frac{1}{2}c\Delta^{\overline{\beta}}_{0t}(u^{n+s+\alpha}_j)^2 + \frac{k^{2-\overline{\beta}}}{4\Gamma(2-\overline{\beta})}\left\{(a^{\alpha,\overline{\beta}}_{n+s,n+s})^{-1}\left(W^{\overline{\beta},n+s-\frac{1}{2}}_j\right)^2\right. \\ &+ \left[a^{\alpha,\overline{\beta}}_{n+s,\frac{1}{2}} - (a^{\alpha,\overline{\beta}}_{n+s,1})^{-1}(a^{\alpha,\overline{\beta}}_{n+s,\frac{1}{2}})^2\right]\left(\delta_t u^0_j\right)^2 \\ &\left.+ \sum_{l=\frac{1}{2}}^{n+s-1}\left[(a^{\alpha,\overline{\beta}}_{n+s,l+\frac{1}{2}})^{-1} - (a^{\alpha,\overline{\beta}}_{n+s,l+1})^{-1}\right]\left(W^{\overline{\beta},l}_j\right)^2\right\}. \end{aligned} \tag{3.3}$$

Furthermore,

$$u^{n+s}_j(c\Delta^{\overline{\beta}}_{0t}u^{n+s+\alpha}_j) \geq \frac{1}{2}c\Delta^{\overline{\beta}}_{0t}(u^{n+s+\alpha}_j)^2. \tag{3.4}$$

Proof. The proof of (3.3) is obtained by replacing λ with $\overline{\beta}$ in the proof of Lemma 3.3 established in [18]. The proof of (3.4) is obvious since $a^{\alpha,\overline{\beta}}_{n+s,l} < a^{\alpha,\overline{\beta}}_{n+s,l+\frac{1}{2}}$, for $s = \frac{1}{2}, 1$, and $l = \frac{1}{2}, 1, \frac{3}{2}, 2, ..., n$ (resp., $n + \frac{1}{2}$). In addition, it is easy to see that $a^{\alpha,\overline{\beta}}_{n+s,\frac{1}{2}} - (a^{\alpha,\overline{\beta}}_{n+s,1})^{-1}(a^{\alpha,\overline{\beta}}_{n+s,\frac{1}{2}})^2 \geq 0$. □

Lemma 3.4. *Given $(a^{\alpha,\beta}_{n+s,l})_l$ are the generalized sequences defined by Eqs* (2.10)–(2.11) *and* (2.18)–(2.19)*, for any grid function $v(\cdot,\cdot)$ defined on the grid space $\mathcal{U}_{hk}$, it holds that*

$$\sum_{l=l_0}^{m} a^{\alpha,\beta}_{n+s,l+\frac{1}{2}}[(v_j^{l+\frac{1}{2}})^2-(v_j^l)^2] = a^{\alpha,\beta}_{n+s,m+\frac{1}{2}}(v_j^{m+\frac{1}{2}})^2 - a^{\alpha,\beta}_{n+s,l_0+\frac{1}{2}}(v_j^{l_0})^2 + \sum_{l=l_0}^{m-\frac{1}{2}}[a^{\alpha,\beta}_{n+s,l+\frac{1}{2}} - a^{\alpha,\beta}_{n+s,l+1}](v_j^{l+\frac{1}{2}})^2, \quad (3.5)$$

for $m \in \{n, n+\frac{1}{2}\}$ and $l = l_0, l_0+\frac{1}{2}, l_0+1, l_0+\frac{3}{2}, ..., m$, where l_0 is a nonnegative integer satisfying $l_0 \le m$.

Proof. Expand the left side of this equality and rearrange the terms to obtain the result. □

Lemma 3.5. *Consider the following linear operator:*

$$L_h u^q = (\delta^4_{2x} - \delta^4_x)u^q, \quad (3.6)$$

where q is any nonnegative rational number. Let $u, v \in C_D^{0,0}$ be two functions defined over the domain $D = [L_0, L] \times [0, T]$. If $u_0^q = v_0^d = 0$, $u_1^q = v_1^d = 0$, $u_{M-1}^q = v_{M-1}^d = 0$, and $u_M^q = v_M^d = 0$, for any nonnegative rational numbers q and d, then it holds that

$$|\left(L_h u^q, v^d\right)| \le \frac{4}{3}\|v^d\|_\nabla[\|u^q\|_\nabla + \|u^q\|_2], \ \ and \ \ |\left(L_h u^q, v^d\right)| \le C_p\|v^d\|_\nabla\|u^q\|_\nabla, \quad (3.7)$$

where C_p is a positive constant independent of the time step k and space step h.

Proof. In Eqs (2.24) and (2.25), replace $n+s+\alpha$ by q to obtain

$$\delta^4_{2x}u_j^q = \frac{1}{12h^2}\left[-u^q_{j+2} + 16u^q_{j+1} - 30u^q_j + 16u^q_{j-1} - u^q_{j-2}\right],$$

$$\delta^4_x u_j^q = \frac{1}{12h}\left[-u^q_{j+2} + 8u^q_{j+1} - 8u^q_{j-1} + u^q_{j-2}\right].$$

Since $\delta_x u_j^q = \frac{u_j^q - u_{j-1}^q}{h}$, direct calculations provide

$$\delta^4_{2x}u_j^q = \frac{1}{12h}\left[-(\delta_x u^q_{j-\frac{1}{2}} - \delta_x u^q_{j-\frac{3}{2}}) + 14(\delta_x u^q_{j+\frac{1}{2}} - \delta_x u^q_{j-\frac{1}{2}}) - (\delta_x u^q_{j+\frac{3}{2}} - \delta_x u^q_{j+\frac{1}{2}})\right], \quad (3.8)$$

$$\delta^4_x u_j^q = \frac{1}{12}\left[-\delta_x u^q_{j-\frac{3}{2}} + 7\delta_x u^q_{j-\frac{1}{2}} + 7\delta_x u^q_{j+\frac{1}{2}} - \delta_x u^q_{j+\frac{3}{2}}\right]. \quad (3.9)$$

Performing straightforward computations, it is not difficult to show that

$$\frac{-1}{h}\sum_{j=2}^{M-2}(\delta_x u^q_{j-\frac{1}{2}} - \delta_x u^q_{j-\frac{3}{2}})v_j^d = \frac{1}{h}(\delta_x u^q_{\frac{1}{2}} v_2^d - \delta_x u^q_{M-\frac{5}{2}} v^d_{M-2}) + \sum_{j=2}^{M-3}\delta_x u^q_{j-\frac{1}{2}}\delta_x v^d_{j+\frac{1}{2}}.$$

Utilizing the assumptions that $v_1^d = 0$ and $v^q_{M-1} = 0$, this becomes

$$\frac{-1}{h}\sum_{j=2}^{M-2}(\delta_x u^q_{j-\frac{1}{2}} - \delta_x u^q_{j-\frac{3}{2}})v_j^d = \sum_{j=1}^{M-2}\delta_x u^q_{j-\frac{1}{2}}\delta_x v^d_{j+\frac{1}{2}}. \quad (3.10)$$

$$\frac{-1}{h}\sum_{j=2}^{M-2}(\delta_x u^q_{j+\frac{3}{2}} - \delta_x u^q_{j+\frac{1}{2}})v^d_j = \frac{1}{h}(\delta_x u^q_{\frac{5}{2}} v^d_2 - \delta_x u^q_{M-\frac{1}{2}} v^d_{M-2}) + \sum_{j=2}^{M-3}\delta_x u^q_{j+\frac{3}{2}}\delta_x v^d_{j+\frac{1}{2}} = \sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{3}{2}}\delta_x v^d_{j+\frac{1}{2}}. \quad (3.11)$$

The last equality follows from the assumptions taht $v^q_1 = 0$ and $v^d_{M-1} = 0$. Analogously, one easily shows that

$$\frac{14}{h}\sum_{j=2}^{M-2}(\delta_x u^q_{j+\frac{1}{2}} - \delta_x u^q_{j-\frac{1}{2}})v^d_j = -14\sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{1}{2}}\delta_x v^d_{j+\frac{1}{2}}. \quad (3.12)$$

Plugging Eqs (3.8) and (3.10)–(3.12) yields

$$\sum_{j=2}^{M-2}(\delta^4_{2x}u^q_j)v^d_j = \frac{1}{12}\left[\sum_{j=1}^{M-2}\delta_x u^q_{j-\frac{1}{2}}\delta_x v^d_{j+\frac{1}{2}} + \sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{3}{2}}\delta_x v^d_{j+\frac{1}{2}} - 14\sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{1}{2}}\delta_x v^d_{j+\frac{1}{2}}\right].$$

Multiplying both sides of this equation by h, applying the Hölder and Cauchy-Schwarz inequalities, using the definition of L^2-norm and the scalar product $(\cdot,\cdot)$, this results in

$$\left(\delta^4_{2x}u^q, v^d\right) \le \frac{1}{12}\left[\left(h^2\sum_{j=1}^{M-2}\left(\delta_x u^q_{j-\frac{1}{2}}\right)^2\sum_{j=1}^{M-2}\left(\delta_x v^d_{j+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} + \left(h^2\sum_{j=1}^{M-2}\left(\delta_x u^q_{j+\frac{3}{2}}\right)^2\sum_{j=1}^{M-2}\left(\delta_x v^d_{j+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\right.$$
$$\left. +14\left(h^2\sum_{j=1}^{M-2}\left(\delta_x u^q_{j+\frac{1}{2}}\right)^2\sum_{j=1}^{M-2}\left(\delta_x v^d_{j+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\right] \le \frac{16}{12}\|\delta_x u^q\|_2\|\delta_x v^d\|_2. \quad (3.13)$$

In a similar manner, one easily proves that

$$\sum_{j=2}^{M-2}\delta_x u^q_{j-\frac{3}{2}}v^d_j = -\sum_{j=1}^{M-3}u^q_j\delta_x v^d_{j+\frac{3}{2}}, \quad \sum_{j=2}^{M-2}\delta_x u^q_{j+\frac{3}{2}}v^d_j = -\sum_{j=3}^{M-2}u^q_j\delta_x v^d_{j-\frac{3}{2}},$$
$$-7\sum_{j=2}^{M-2}\delta_x u^q_{j-\frac{1}{2}}v^d_j = 7\sum_{j=1}^{M-2}u^q_j\delta_x v^d_{j+\frac{1}{2}} \text{ and } -7\sum_{j=2}^{M-2}\delta_x u^q_{j+\frac{1}{2}}v^d_j = 7\sum_{j=2}^{M-2}u^q_j\delta_x v^d_{j-\frac{1}{2}}. \quad (3.14)$$

Combining Eqs (3.9) and (3.14), it is not hard to observe that

$$\sum_{j=2}^{M-2}(\delta^4_x u^q_j)v^d_j = \frac{1}{12}\left[-\sum_{j=1}^{M-3}\delta_x u^q_j\delta_x v^d_{j+\frac{3}{2}} - \sum_{j=3}^{M-2}\delta_x u^q_j\delta_x v^d_{j-\frac{3}{2}} + 7\sum_{j=1}^{M-2}\delta_x u^q_j\delta_x v^d_{j+\frac{1}{2}} + 7\sum_{j=2}^{M-2}\delta_x u^q_j\delta_x v^d_{j-\frac{1}{2}}\right].$$

Multiplying this equation by h, utilizing the Hölder and Cauchy-Schwarz inequalities together with the definitions of L^2-norm and scalar product $(\cdot,\cdot)$ to get

$$\left(\delta^4_x u^q, v^d\right) \le \frac{16}{12}\|u^q\|_2\|\delta_x v^d\|_2. \quad (3.15)$$

A combination of (3.6) and estimates (3.13) and (3.15) gives

$$\left|\left(L_h u^q, v^d\right)\right| = \left|\left((\delta^4_{2x} - \delta^4_x)u^q, v^d\right)\right| \le \left|\left(\delta^4_{2x}u^q, v^d\right)\right| + \left|\left(\delta^4_x u^q, v^d\right)\right| \le \frac{4}{3}[\|u^q\|_2\|\delta_x v^d\|_2 + \|\delta_x u^q\|_2\|\delta_x v^d\|_2].$$

The use of the gradient discrete norm given in (2.2) completes the proof of the first estimate in (3.7). The proof of the second estimate in (3.7) is obtained thanks to the Poincaré-Friedrich inequality. □

Lemma 3.6. *Suppose $u \in C_D^{0,0}$ is a function defined on $D = [L_0, L] \times [0, T]$, satisfying $u_m^q = 0$, for $m \in \{0, 1, M-1, M\}$, $q = n + \frac{1}{2}, n+1$, and $n = 0, 1, 2, ..., N-1$. So, the following estimate is satisfied:*

$$(-L_h u^q, u^q) = \|u^q\|_\nabla^2 + \frac{h^2}{12}\|u^q\|_\triangle^2, \tag{3.16}$$

where the norms $\|\cdot\|_\nabla$ and $\|\cdot\|_\triangle$ are defined in relation (2.2) *and h denotes the space step.*

Proof. Plugging Eqs (3.8) and (3.10)–(3.12), replacing v^d with u^q, and using the scalar product defined in (2.3), it is easy to see that

$$\begin{aligned}\left(\delta_{2x}^4 u^q, u^q\right) &= \frac{h}{12}\sum_{j=1}^{M-2}\delta_x u^q_{j-\frac{1}{2}}\delta_x u^q_{j+\frac{1}{2}} - \frac{14h}{12}\sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{1}{2}}\delta_x u^q_{j+\frac{1}{2}} + \frac{h}{12}\sum_{j=1}^{M-2}\delta_x u^q_{j+\frac{3}{2}}\delta_x u^q_{j+\frac{1}{2}}\\
=&\frac{h}{12}\sum_{1=2}^{M-2}\left\{\delta_x u^q_{j-\frac{1}{2}} - 14\delta_x u^q_{j+\frac{1}{2}} + \delta_x u^q_{j+\frac{3}{2}}\right\}\delta_x u^q_{j+\frac{1}{2}}\\
=&-h\sum_{j=1}^{M-2}\left(\delta_x u^q_{j+\frac{1}{2}}\right)^2 + \frac{h}{12}\sum_{j=1}^{M-2}\left[\delta_x u^q_{j-\frac{1}{2}} - 2\delta_x u^q_{j+\frac{1}{2}} + \delta_x u^q_{j+\frac{3}{2}}\right]\delta_x u^q_{j+\frac{1}{2}},\end{aligned}$$

where the operators δ_x and δ_x^2 are defined in relation (2.1). Since $\delta_x u^q_{j-\frac{1}{2}} - 2\delta_x u^q_{j+\frac{1}{2}} + \delta_x u^q_{j+\frac{3}{2}} = (\delta_x u^q_{j+\frac{3}{2}} - \delta_x u^q_{j+\frac{1}{2}}) - (\delta_x u^q_{j+\frac{1}{2}} - \delta_x u^q_{j-\frac{1}{2}})$ and $\delta_x^2 u_p^q = \frac{1}{h}(\delta_x u^q_{p+\frac{1}{2}} - \delta_x u^q_{p-\frac{1}{2}})$, utilizing the summation by parts together with the hypothesis: $u_m^q = 0$, for any $m = 0, 1, M-1, M$, and the L^2-norm given by (2.2), to obtain

$$\left(\delta_{2x}^4 u^q, u^q\right) = -\|\delta_x u^q\|_2^2 - \frac{h^2}{12}\left[\delta_x^2 u_1^q \delta_x u^q_{\frac{3}{2}} - \delta_x^2 u^q_{M-1}\delta_x u^q_{M-\frac{3}{2}} + h\sum_{j=2}^{M-2}(\delta_x^2 u_j^q)^2\right].$$

But it comes from the hypothesis that: $u_m^q = 0$, for $m = 0, 1, M-1, M$. This fact implies $\delta_x u^q_{\frac{3}{2}} = h\delta_x^2 u_1^q$ and $\delta_x u^q_{M-\frac{3}{2}} = -h\delta_x^2 u^q_{M-1}$. Thus,

$$\left(\delta_{2x}^4 u^q, u^q\right) = -\|\delta_x u^q\|_2^2 - \frac{h^3}{12}\sum_{j=1}^{M-2}(\delta_x^2 u_j^q)^2 = -\|\delta_x u^q\|_2^2 - \frac{h^2}{12}\|\delta_x^2 u^q\|_2^2.$$

Multiplying both sides of this equation by " -1", results in

$$\left(-\delta_{2x}^4 u^q, u^q\right) = \|\delta_x u^q\|_2^2 + \frac{h^2}{12}\|\delta_x^2 u^q\|_2^2. \tag{3.17}$$

Furthermore, setting $u^q = v^d$ and using Eq (3.14), we get

$$\sum_{j=2}^{M-2}(\delta_x u^q_{j-\frac{3}{2},j})u_j^q = -\sum_{j=2}^{M-2}u_j^q\delta_x u^q_{j+\frac{3}{2}};\ \ \sum_{j=2}^{M-2}(\delta_x u^q_{j+\frac{3}{2}})u_j^q = -\sum_{j=2}^{M-2}u_j^q\delta_x u^q_{j-\frac{3}{2}};$$

$$-7\sum_{j=2}^{M-2}(\delta_x u^q_{j-\frac{1}{2}})u_j^q = 7\sum_{j=2}^{M-2}u_j^q\delta_x u^q_{j+\frac{1}{2}};\ \ -7\sum_{j=2}^{M-2}(\delta_x u^q_{j+\frac{1}{2}})u_j^q = 7\sum_{j=2}^{M-2}u_j^q\delta_x u^q_{j-\frac{1}{2}}.$$

Summing the left side (respectively, the right side) of these equations and rearranging terms to get

$$2\sum_{j=2}^{M-2}\left[(\delta_x u^q_{j-\frac{3}{2}})u^q_j - 7\left(\delta_x u^q_{j-\frac{1}{2}} + \delta_x u^q_{j+\frac{1}{2}}\right)u^q_j + (\delta_x u^q_{j+\frac{3}{2}})u^q_j\right] = 0.$$

This fact combined with Eq (3.9) and the use of the scalar product (2.3) yields

$$\left(\delta_x^4 u^q, u^q\right) = \frac{h^2}{6}\sum_{j=2}^{M-2}\left[-\delta_x u^q_{j-\frac{3}{2}} + 7\left(\delta_x u^q_{j-\frac{1}{2}} + \delta_x u^q_{j+\frac{1}{2}}\right) - \delta_x u^q_{j+\frac{3}{2}}\right](u^q_j)^2 = 0. \tag{3.18}$$

A combination of Eqs (3.6), (3.17), and (3.18) gives

$$(-L_h u^q, u^q) = \left(\delta_x^4 u^q - \delta_{2x}^4 u^q, u^q\right) = \left(\delta_x^4 u^q, u^q\right) + \left(-\delta_{2x}^4 u^q, u^q\right) = \|\delta_x u^q\|_2^2 + \frac{h^2}{12}\|\delta_x^2 u^q\|_2^2.$$

The proof of Lemma 3.6 is completed thanks to the norms $\|\cdot\|_\triangledown$ and $\|\cdot\|_\vartriangle$ defined in relation (2.2). $\square$

Armed with Lemmas 3.1–3.6, we can state and prove the main result of this work (Theorem 3.1).

Theorem 3.1. *(Unconditional stability and error estimates). Suppose U is the approximate solution provided by the proposed approach* (2.48)–(2.52) *and let u be the analytical solution of the initial-boundary value problems* (1.1)–(1.3). *Let $\beta(x,t) = \overline{\beta} \in (0,1)$, for any $(x,t) \in D$, be a positive constant function, $0 < \alpha < \frac{1}{2}$ be a parameter, and let $(a^{\alpha,\overline{\beta}}_{\cdot,l})_l$ be the generalized sequences defined by relations* (2.10), (2.11) *and* (2.18), (2.19). *Thus, the following estimates are satisfied,*

$$\max_{0\le n\le N-1}\|U^{n+\frac{1}{2}}\|_2,\ \max_{0\le n\le N}\|U^n\|_2 \le |||u|||_{\infty,2} + \sqrt{2\widehat{C}T}(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4). \tag{3.19}$$

Furthermore, denote $e = u - U$ as the error term, and it holds that

$$\max_{0\le n\le N-1}\|e^{n+\frac{1}{2}}\|_2,\ \max_{0\le n\le N}\|e^n\|_2 \le \sqrt{2\widehat{C}T}(3\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} + h^4), \tag{3.20}$$

where $\widehat{C}$ is a positive constant independent on the space size h and time step k.

We recall that estimate (3.19) suggests that the proposed technique (2.48)–(2.52) is unconditionally stable whereas inequality (3.20) shows that the developed numerical scheme is fourth-order spatial convergent and temporal accurate of order $O(k)$.

Proof. Let $e^{n+\frac{1}{2}} = u^{n+\frac{1}{2}} - U^{n+\frac{1}{2}}$ be the temporary error term and $e^{n+1} = u^{n+1} - U^{n+1}$ be the exact one at time level $n+1$. For the convenience of writing, we set $\theta^{n+s+\alpha}_{l,j} := \theta^{n+s+\alpha}_l$, for $j = 2, 3, ..., M-2$. Subtracting Eq (2.35) from (2.30) and utilizing (2.37) and (3.6) provides

$$e^{n+\frac{1}{2}}_j - e^n - \alpha k L_h e^{n+\frac{1}{2}}_j + k\sum_{l=0}^{n-\frac{1}{2}}\theta^{n+\alpha}_{l+\frac{1}{2}}\delta_t e^l_j = (\frac{1}{2} - \alpha)kL_h e^n_j - \frac{k}{2}J^{\overline{\beta}}_j + O(k^2 + k^3 + kh^4).$$

Multiplying both sides of this equation by $e^{n+\frac{1}{2}}_j$ yields

$$(e^{n+\frac{1}{2}}_j - e^n)e^{n+\frac{1}{2}}_j - \alpha k e^{n+\frac{1}{2}}_j L_h e^{n+\frac{1}{2}}_j + k e^{n+\frac{1}{2}}_j\sum_{l=0}^{n-\frac{1}{2}}\theta^{n+\alpha}_{l+\frac{1}{2}}\delta_t e^l_j$$

$$= (\frac{1}{2}-\alpha)ke_j^{n+\frac{1}{2}}L_he_j^n - \frac{k}{2}J_j^{\overline{\beta}}e_j^{n+\frac{1}{2}} + O(k^2+k^3+kh^4)e_j^{n+\frac{1}{2}}. \tag{3.21}$$

We introduce the generalized sequences $(\widehat{\theta}_l^{n+\alpha})_{l\leq n+\frac{1}{2}}$ and $(\widehat{\theta}_l^{n+\frac{1}{2}+\alpha})_{l\leq n+1}$, defined as

$$\widehat{\theta}_{n+\frac{1}{2}}^{n+\alpha} = \theta_n^{n+\alpha} \text{ and } \widehat{\theta}_l^{n+\alpha} = \theta_l^{n+\alpha}, \text{ for } l = \frac{1}{2}, 1, ..., n,$$

$$\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha} = \theta_{n+\frac{1}{2}}^{n+\frac{1}{2}+\alpha} \text{ and } \widehat{\theta}_l^{n+\frac{1}{2}+\alpha} = \theta_l^{n+\frac{1}{2}+\alpha}, \text{ for } l = \frac{1}{2}, 1, ..., n+\frac{1}{2}. \tag{3.22}$$

Thus, the sequence $(\widehat{\theta}_l^{n+s+\alpha})_{l\leq n+s+\frac{1}{2}}$ satisfies $\widehat{\theta}_l^{n+s+\alpha} \leq \widehat{\theta}_{l+\frac{1}{2}}^{n+s+\alpha}$, for $s = 0, 2^{-1}$ and $l = \frac{1}{2}, 1, ..., n+s$. Furthermore, similar to the proof of estimate (3.4) in Lemma 3.3, it is easy to show that

$$e_j^{n+s+\frac{1}{2}}\sum_{l=0}^{n+s}\widehat{\theta}_{l+\frac{1}{2}}^{n+s+\alpha}\delta_te_j^l \geq \frac{1}{2}\sum_{l=0}^{n+s}\widehat{\theta}_{l+\frac{1}{2}}^{n+s+\alpha}\delta_t(e_j^l)^2. \tag{3.23}$$

Since $\delta_te_j^n = \frac{2}{k}(e_j^{n+\frac{1}{2}} - e_j^n)$ and $k\sum_{l=0}^{n-\frac{1}{2}}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_te_j^l = k\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_te_j^l - 2\widehat{\theta}_{n+\frac{1}{2}}^{n+\alpha}(e_j^{n+\frac{1}{2}} - e_j^n)$, using this, Eq (3.21) becomes

$$\begin{aligned}
&(e_j^{n+\frac{1}{2}} - e^n)e_j^{n+\frac{1}{2}} - \alpha ke_j^{n+\frac{1}{2}}L_he_j^{n+\frac{1}{2}} + ke_j^{n+\frac{1}{2}}\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_te_j^l - 2\widehat{\theta}_{n+\frac{1}{2}}^{n+\alpha}(e_j^{n+\frac{1}{2}} - e_j^n)e_j^{n+\frac{1}{2}}\\
&= (\frac{1}{2}-\alpha)ke_j^{n+\frac{1}{2}}L_he_j^n - \frac{k}{2}J_j^{\overline{\beta}}e_j^{n+\frac{1}{2}} + O(k^2+k^3+kh^4)e_j^{n+\frac{1}{2}}.
\end{aligned} \tag{3.24}$$

But $(a-b)a = \frac{1}{2}[a^2 - b^2 + (a-b)^2]$, for all real numbers a and b. In addition, for $s = 0$, it follows from (3.23) that: $e_j^{n+\frac{1}{2}}\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_te_j^l \geq \frac{1}{2}\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_t(e_j^l)^2$. These facts, together with (3.24), result in

$$\begin{aligned}
&\frac{1}{2}\left[(e_j^{n+\frac{1}{2}})^2 + (e_j^{n+\frac{1}{2}} - e^n)^2 - (e_j^n)^2\right] - \alpha ke_j^{n+\frac{1}{2}}L_he_j^{n+\frac{1}{2}} + \frac{k}{2}\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\delta_t(e_j^l)^2\\
&-\widehat{\theta}_{n+\frac{1}{2}}^{n+\alpha}\left[(e_j^{n+\frac{1}{2}})^2 + (e_j^{n+\frac{1}{2}} - e^n)^2 - (e_j^n)^2\right]\\
&\leq (\frac{1}{2}-\alpha)ke_j^{n+\frac{1}{2}}L_he_j^n - \frac{k}{2}J_j^{\overline{\beta}}e_j^{n+\frac{1}{2}} + O(k^2+k^3+kh^4)e_j^{n+\frac{1}{2}}.
\end{aligned}$$

Setting $J^{\overline{\beta}} = (J_2^{\overline{\beta}}, J_3^{\overline{\beta}}, ..., J_{M-2}^{\overline{\beta}})$ and $\overline{O}(k^2+k^3+kh^4) = (O(k^2+k^3+kh^4), O(k^2+k^3+kh^4), ..., O(k^2+k^3+kh^4))$, multiplying both sides of this estimate by $2h$, summing the obtained estimate up from $j = 2, 3, ..., M-2$, and using the definition of the L^2-norm and scalar product given by relations (2.2) and (2.3), respectively, provides

$$\begin{aligned}
&\|e^{n+\frac{1}{2}}\|_2^2 + \|e^{n+\frac{1}{2}} - e^n\|_2^2 - \|e^n\|_2^2 + 2\alpha k\left(-L_he^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)\\
&+2\sum_{l=0}^{n}\widehat{\theta}_{l+\frac{1}{2}}^{n+\alpha}\left(\|e^{l+\frac{1}{2}}\|_2^2 - \|e^l\|_2^2\right) - 2\widehat{\theta}_{n+\frac{1}{2}}^{n+\alpha}\left[\|e^{n+\frac{1}{2}}\|_2^2 + \|e^{n+\frac{1}{2}} - e^n\|_2^2 - \|e^n\|_2^2\right]
\end{aligned}$$

$$\leq (1-2\alpha)k\left(L_h e^n, e^{n+\frac{1}{2}}\right) - k\left(J^{\overline{\beta}}, e^{n+\frac{1}{2}}\right) + \left(\overline{O}(k^2+k^3+kh^4), e^{n+\frac{1}{2}}\right). \tag{3.25}$$

Now, utilizing estimate (3.16) together with the summation by parts (3.5), it is not hard to observe that (3.25) implies

$$\begin{aligned}
&\|e^{n+\frac{1}{2}}\|_2^2 + \left(1 - 2\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}}\right)\|e^{n+\frac{1}{2}} - e^n\|_2^2 + 2\alpha k\|\delta_x e^{n+\frac{1}{2}}\|_2^2 \\
\leq\ & 2\widehat{\theta}^{n+\alpha}_{\frac{1}{2}}\|e^0\|_2^2 + \left(1 - 2\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}}\right)\|e^n\|_2^2 + 2\sum_{l=0}^{n-\frac{1}{2}}\left(\widehat{\theta}^{n+\alpha}_{l+1} - \widehat{\theta}^{n+\alpha}_{l+\frac{1}{2}}\right)\|e^{l+\frac{1}{2}}\|_2^2 \\
&+ (1-2\alpha)k\left(L_h e^n, e^{n+\frac{1}{2}}\right) - k\left(J^{\overline{\beta}}, e^{n+\frac{1}{2}}\right) + \left(\overline{O}(k^2+k^3+kh^4), e^{n+\frac{1}{2}}\right).
\end{aligned} \tag{3.26}$$

It follows from the Poincaré-Friedrich inequality that $\|e^{n+\frac{1}{2}}\|_2^2 \leq C_p\|\delta_x e^{n+\frac{1}{2}}\|_2^2$, where C_p denotes a positive constant independent of k and h. Using this, estimate (3.7), and the Hölder inequality, straightforward calculations give

$$\begin{aligned}
(1-2\alpha)k\left(L_h e^n, e^{n+\frac{1}{2}}\right) \leq\ & C_p(1-2\alpha)k\|\delta_x e^n\|_2\|\delta_x e^{n+\frac{1}{2}}\|_2 \\
=\ & 2\left(\frac{1-2\alpha}{2}C_p\sqrt{\frac{4k}{\alpha}}\|\delta_x e^n\|_2\right)\left(\sqrt{\frac{\alpha k}{4}}\|\delta_x e^{n+\frac{1}{2}}\|_2\right) \\
\leq\ & \frac{(1-2\alpha)^2 C_p^2}{\alpha}k\|\delta_x e^n\|_2^2 + \frac{\alpha k}{4}\|\delta_x e^{n+\frac{1}{2}}\|_2^2.
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
-k\left(J^{\overline{\beta}}, e^{n+\frac{1}{2}}\right) \leq\ & \frac{C_p^2}{\alpha}k\|J^{\overline{\beta}}\|_2^2 + \frac{\alpha k}{4}\|\delta_x e^{n+\frac{1}{2}}\|_2^2,\ \left(\overline{O}(k^2+k^3+kh^4), e^{n+\frac{1}{2}}\right) \\
\leq\ & \frac{1}{2}\|e^{n+\frac{1}{2}}\|_2^2 + \widetilde{C}_1 k(k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2,
\end{aligned} \tag{3.28}$$

where $\widetilde{C}_1 > 0$ is a constant which does not depend on the time step k and the space step h. Setting $\widehat{C}_\alpha = \max\{\widetilde{C}_1, C_p^2\alpha^{-1}\}$, substituting estimates (3.27) and (3.28) into relation (3.26), and rearranging terms yields

$$\begin{aligned}
&\|e^{n+\frac{1}{2}}\|_2^2 + 2\left(1 - 2\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}}\right)\|e^{n+\frac{1}{2}} - e^n\|_2^2 + 3\alpha k\|\delta_x e^{n+\frac{1}{2}}\|_2^2 \\
\leq\ & 4\widehat{\theta}^{n+\alpha}_{\frac{1}{2}}\|e^0\|_2^2 + 2\left(1 - 2\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}}\right)\|e^n\|_2^2 + 4\sum_{l=0}^{n-\frac{1}{2}}\left(\widehat{\theta}^{n+\alpha}_{l+1} - \widehat{\theta}^{n+\alpha}_{l+\frac{1}{2}}\right)\|e^{l+\frac{1}{2}}\|_2^2 \\
&+ 2(1-2\alpha)^2\widehat{C}_\alpha k\|\delta_x e^n\|_2^2 + 2\widehat{C}_\alpha k[\|J^{\overline{\beta}}\|_2^2 + (k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2].
\end{aligned} \tag{3.29}$$

Estimate (3.29) is satisfied for any $0 < \alpha < 2^{-1}$. For $n \geq 1$, it comes from (2.11)–(2.12) and (2.19)–(2.20), that

$$\begin{aligned}
\alpha^{1-\overline{\beta}} = \widetilde{f}^{\alpha,\overline{\beta}}_{n,n} <\ & a^{\alpha,\overline{\beta}}_{n,n} = \widetilde{f}^{\alpha,\overline{\beta}}_{n,n-1} + \widetilde{f}^{\alpha,\overline{\beta}}_{n,n} = \widetilde{f}^{\alpha,\overline{\beta}}_{n+\frac{1}{2},n-1} + \widetilde{f}^{\alpha,\overline{\beta}}_{n+\frac{1}{2},n} = a^{\alpha,\overline{\beta}}_{n+\frac{1}{2},n+\frac{1}{2}} = \widetilde{f}^{\alpha,\overline{\beta}}_{n+1,n} + \widetilde{f}^{\alpha,\overline{\beta}}_{n+1,n+1} = a^{\alpha,\overline{\beta}}_{n+1,n+1} \\
<\ & \alpha^{1-\overline{\beta}} + \frac{2}{2-\overline{\beta}}\left[(1+\alpha)^{2-\overline{\beta}} - \alpha^{2-\overline{\beta}}\right].
\end{aligned} \tag{3.30}$$

Since $\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}} = k^{1-\overline{\beta}}\Gamma(2-\overline{\beta})^{-1}a^{\alpha,\overline{\beta}}_{n,n}$, for small values of k, it follows from (3.30) that: $1-2\widehat{\theta}^{n+\alpha}_{n+\frac{1}{2}} > 0$ and $k^{\frac{1}{2}}h^4 < h^4$. This fact and estimate (3.29) imply

$$\|e^{n+\frac{1}{2}}\|_2^2 \le \max_{0\le l\le n}\|e^l\|_2^2 + 2(1-2\alpha)^2\widehat{C}_\alpha k\|\delta_x e^n\|_2^2 + 2\widehat{C}_\alpha k[\|J^{\overline{\beta}}\|_2^2 + (k^{\frac{3}{2}} + k^{\frac{5}{2}} + h^4)^2], \tag{3.31}$$

for every $\alpha \in (0, 2^{-1})$. Set $\widehat{C}_1 = \lim_{\alpha\to\frac{1}{2}}\widehat{C}_\alpha = \lim_{\alpha\to\frac{1}{2}}\max\{\widetilde{C}_1, C_p^2\alpha^{-1}\} = \max\{\widetilde{C}_1, 2C_p^2\}$. Taking the limit in estimate (3.31) when α approaches $\frac{1}{2}$ and combining (2.16) and (3.1), it is not difficult to observe that

$$\max_{0\le l\le n+\frac{1}{2}}\|e^l\|_2^2 \le \max_{0\le l\le n}\|e^l\|_2^2 + \widehat{C}_1 k[C^2_{\frac{1}{2}}k^{4-2\overline{\beta}} + (k^{\frac{3}{2}} + k^{\frac{5}{2}} + h^4)^2] \le \max_{0\le l\le n}\|e^l\|_2^2 + \widehat{C}_2 k(k^{\frac{3}{2}} + k^{2-\overline{\beta}} + k^{\frac{5}{2}} + h^4)^2, \tag{3.32}$$

where $\widehat{C}_2 = \widehat{C}_1(1 + C^2_{\frac{1}{2}})$.

In a similar way, a combination of Eqs (2.43), (2.44) and (3.6) gives

$$\begin{aligned} & e_j^{n+1} - e^{n+\frac{1}{2}} - \frac{1+4\alpha}{4}kL_h e_j^{n+1} + \frac{k}{4}\left[\sum_{l=0}^{n+\frac{1}{2}}\theta^{n+1+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l + \sum_{l=0}^{n}\theta^{n+\frac{1}{2}+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l\right] \\ = {} & \frac{1-4\alpha}{4}kL_h e^{n+\frac{1}{2}} - \frac{k}{4}(J_j^{\overline{\beta}} + I_j^{\overline{\beta}}) + O(k^2 + k^3 + kh^4). \end{aligned} \tag{3.33}$$

Multiplying both sides of this equation by $2he_j^{n+1}$, summing this up from $j = 2, 3, ..., M-2$, and using the definition of the L^2-norm and scalar product, it is not hard to observe that (3.33) implies

$$\begin{aligned} & \|e^{n+1}\|_2^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_2^2 - \|e^{n+\frac{1}{2}}\|_2^2 + \frac{1+4\alpha}{2}k\left(-L_h e^{n+1}, e^{n+1}\right) \\ & + \frac{k}{2}h\sum_{j=2}^{M-2}\left[e^{n+1}\sum_{l=0}^{n+\frac{1}{2}}\theta^{n+1+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l + e^{n+1}\sum_{l=0}^{n}\theta^{n+\frac{1}{2}+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l\right] \\ = {} & \frac{1-4\alpha}{2}k\left(L_h e^{n+\frac{1}{2}}, e^{n+1}\right) - \frac{k}{2}\left(J^{\overline{\beta}} + I^{\overline{\beta}}, e^{n+1}\right) + \left(\overline{O}(k^2 + k^3 + kh^4), e^{n+1}\right), \end{aligned} \tag{3.34}$$

where $\overline{O}(k^2+k^3+kh^4) = (O(k^2+k^3+kh^4), ..., O(k^2+k^3+kh^4))$, $J^{\overline{\beta}} = (J_2^{\overline{\beta}}, ..., J_{M-2}^{\overline{\beta}})$, and $I^{\overline{\beta}} = (I_2^{\overline{\beta}}, ..., I_{M-2}^{\overline{\beta}})$. Utilizing the generalized sequence defined by Eq (3.22), relation (3.34) becomes

$$\begin{aligned} & \|e^{n+1}\|_2^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_2^2 - \|e^{n+\frac{1}{2}}\|_2^2 + \frac{1+4\alpha}{2}k\left(-L_h e^{n+1}, e^{n+1}\right) \\ & + \frac{k}{2}h\sum_{j=2}^{M-2}\left[e^{n+1}\sum_{l=0}^{n+\frac{1}{2}}\theta^{n+1+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l + e^{n+1}\sum_{l=0}^{n}\widehat{\theta}^{n+\frac{1}{2}+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l\right] \\ = {} & \frac{1-4\alpha}{2}k\left(L_h e^{n+\frac{1}{2}}, e^{n+1}\right) - \frac{k}{2}\left(J^{\overline{\beta}} + I^{\overline{\beta}}, e^{n+1}\right) + \left(\overline{O}(k^2 + k^3 + kh^4), e^{n+1}\right). \end{aligned} \tag{3.35}$$

For $s = 2^{-1}$, using estimate (3.23) and performing direct calculations, it holds that

$$e^{n+1}\sum_{l=0}^{n}\widehat{\theta}^{n+\frac{1}{2}+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l = e^{n+1}\sum_{l=0}^{n+\frac{1}{2}}\widehat{\theta}^{n+\frac{1}{2}+\alpha}_{l+\frac{1}{2}}\delta_t e_j^l - e_j^{n+1}\widehat{\theta}^{n+\frac{1}{2}+\alpha}_{n+1}\delta_t e_j^{n+\frac{1}{2}}$$

$$\geq \frac{1}{2}\sum_{l=0}^{n+\frac{1}{2}}\widehat{\theta}_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\delta_t(e_j^l)^2 - \frac{1}{k}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\left[(e_j^{n+1})^2 + (e_j^{n+1} - e_j^{n+\frac{1}{2}})^2 - (e_j^{n+\frac{1}{2}})^2\right]. \tag{3.36}$$

Estimate (3.36) combined with (3.35) and Lemmas 3.3 and 3.6 yield

$$\begin{aligned}
&\left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1}\|_2^2 + \left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1} - e^{n+\frac{1}{2}}\|_2^2 + \frac{1+4\alpha}{2}k\|\delta_x e^{n+1}\|_2^2 \\
&+ \frac{1}{2}\sum_{l=0}^{n+\frac{1}{2}}\left(\theta_{l+\frac{1}{2}}^{n+1+\alpha} + \widehat{\theta}_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)\left(\|e^{l+\frac{1}{2}}\|_2^2 - \|e^l\|_2^2\right) \\
\leq\ &\left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+\frac{1}{2}}\|_2^2 + \frac{1-4\alpha}{2}k\left(L_h e^{n+\frac{1}{2}}, e^{n+1}\right) - \frac{k}{2}\left(J^{\overline{\beta}} + I^{\overline{\beta}}, e^{n+1}\right) + \left(\overline{O}(k^2 + k^3 + kh^4), e^{n+1}\right).
\end{aligned}$$

Applying the summation by parts and rearranging terms, this becomes

$$\begin{aligned}
&\left(1 + \frac{1}{2}\theta_{n+1}^{n+1+\alpha}\right)\|e^{n+1}\|_2^2 + \left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1} - e^{n+\frac{1}{2}}\|_2^2 + \frac{1+4\alpha}{2}k\|\delta_x e^{n+1}\|_2^2 \\
&+ \frac{1}{2}\sum_{l=0}^{n}\left[\left(\theta_{l+\frac{1}{2}}^{n+1+\alpha} + \widehat{\theta}_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right) - \left(\theta_{l+1}^{n+1+\alpha} + \widehat{\theta}_{l+1}^{n+\frac{1}{2}+\alpha}\right)\right]\|e^{l+\frac{1}{2}}\|_2^2 \\
\leq\ &\frac{1}{2}\left(\theta_{\frac{1}{2}}^{n+1+\alpha} + \widehat{\theta}_{\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)\|e^0\|_2^2 + \left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+\frac{1}{2}}\|_2^2 + \frac{1-4\alpha}{2}k\left(L_h e^{n+\frac{1}{2}}, e^{n+1}\right) \\
&- \frac{k}{2}\left(J^{\overline{\beta}} + I^{\overline{\beta}}, e^{n+1}\right) + \left(\overline{O}(k^2 + k^3 + kh^4), e^{n+1}\right).
\end{aligned} \tag{3.37}$$

Performing direct computations, using the Hölder and Poincaré-Friedrich inequalities, Eqs (2.7) and (2.16), and the second estimate in (3.7), it is not difficult to show that

$$\frac{1-4\alpha}{2}k\left(L_h e^{n+\frac{1}{2}}, e^{n+1}\right) \leq \frac{|1-4\alpha|}{2}C_p k\|\delta_x e^{n+\frac{1}{2}}\|_2\|\delta_x e^{n+1}\|_2 = 2\left\{\frac{|1-4\alpha|}{2}C_p\sqrt{\frac{k}{1+4\alpha}}\|\delta_x e^{n+\frac{1}{2}}\|_2\right\},$$

$$\left\{\sqrt{\frac{k(1+4\alpha)}{4}}\|\delta_x e^{n+1}\|_2\right\} \leq \frac{(1-4\alpha)^2}{4(1+4\alpha)}C_p^2 k\|\delta_x e^{n+\frac{1}{2}}\|_2^2 + \frac{(1+4\alpha)k}{4}\|\delta_x e^{n+1}\|_2^2, \tag{3.38}$$

$$\begin{aligned}
-k\left(J^{\overline{\beta}} + I^{\overline{\beta}}, e^{n+1}\right) &\leq \frac{C_p}{2}k\|J^{\overline{\beta}} + I^{\overline{\beta}}\|_2\|\delta_x e^{n+1}\|_2 \\
&\leq \frac{C_p^2}{2(1+4\alpha)}k(\|J^{\overline{\beta}}\|_2^2 + \|I^{\overline{\beta}}\|_2^2) + \frac{(1+4\alpha)k}{4}\|\delta_x e^{n+1}\|_2^2,
\end{aligned} \tag{3.39}$$

$$\left(\overline{O}(k^2 + k^3 + kh^4), e^{n+1}\right) \leq \frac{1}{2}\theta_{n+1}^{n+1+\alpha}\|e^{n+1}\|_2^2 + C_2\left(\theta_{n+1}^{n+1+\alpha}\right)^{-1}k(k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2, \tag{3.40}$$

where $C_2 > 0$, is a constant independent of the time step k and mesh size h. A combination of estimates (3.37)–(3.40) results in

$$\|e^{n+1}\|_2^2 + \left(1 - \frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1} - e^{n+\frac{1}{2}}\|_2^2$$

$$\leq \frac{1}{2}\sum_{l=0}^{n}\left[\left(\theta_{l+1}^{n+1+\alpha}+\widehat{\theta}_{l+1}^{n+\frac{1}{2}+\alpha}\right)-\left(\theta_{l+\frac{1}{2}}^{n+1+\alpha}+\widehat{\theta}_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)\right]\|e^{l+\frac{1}{2}}\|_2^2$$
$$+\frac{1}{2}\left(\theta_{\frac{1}{2}}^{n+1+\alpha}+\widehat{\theta}_{\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)\|e^0\|_2^2+\left(1-\frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+\frac{1}{2}}\|_2^2+\frac{(1-4\alpha)^2}{4(1+4\alpha)}C_p^2k\|\delta_x e^{n+\frac{1}{2}}\|_2^2$$
$$+\frac{C_p^2}{2(1+4\alpha)}k(\|J^{\overline{\beta}}\|_2^2+\|I^{\overline{\beta}}\|_2^2)+C_2k(k^{\frac{3}{2}}+k^{\frac{5}{2}}+k^{\frac{1}{2}}h^4)^2. \tag{3.41}$$

Utilizing this and (3.1), relation (3.41) becomes

$$\|e^{n+1}\|_2^2+\left(1-\frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1}-e^{n+\frac{1}{2}}\|_2^2$$
$$\leq \frac{1}{2}\left\{\left(\theta_{\frac{1}{2}}^{n+1+\alpha}+\widehat{\theta}_{\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)+\sum_{l=0}^{n}\left[\left(\theta_{l+1}^{n+1+\alpha}+\widehat{\theta}_{l+1}^{n+\frac{1}{2}+\alpha}\right)\right.\right.$$
$$\left.\left.-\left(\theta_{l+\frac{1}{2}}^{n+1+\alpha}+\widehat{\theta}_{l+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}\right)\right]+2-\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right\}\max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\frac{(1-4\alpha)^2}{4(1+4\alpha)}C_p^2k\|\delta_x e^{n+\frac{1}{2}}\|_2^2$$
$$+\frac{C_p^2}{2(1+4\alpha)}(C_{\frac{1}{2}}^2+C_1^2)k^{5-2\overline{\beta}}+C_2k(k^{\frac{3}{2}}+k^{\frac{5}{2}}+k^{\frac{1}{2}}h^4)^2.$$

Letting $\widehat{C}_{3,\alpha}=\max\left\{\frac{C_p^2}{2(1+4\alpha)}(C_{\frac{1}{2}}^2+C_1^2),C_2\right\}$, this estimate implies

$$\|e^{n+1}\|_2^2+\left(1-\frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1}-e^{n+\frac{1}{2}}\|_2^2$$
$$\leq \left(1+\frac{1}{2}\theta_{n+1}^{n+1+\alpha}\right)\max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\frac{(1-4\alpha)^2}{4(1+4\alpha)}C_p^2k\|\delta_x e^{n+\frac{1}{2}}\|_2^2+\widehat{C}_{3,\alpha}k(k^{\frac{3}{2}}+k^{\frac{5}{2}}+k^{\frac{1}{2}}h^4)^2. \tag{3.42}$$

It follows from (2.37), (3.22) and (3.30) that $\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}=\theta_{n+1}^{n+1+\alpha}=\theta_{n+\frac{1}{2}}^{n+\frac{1}{2}+\alpha}=k^{1-\overline{\beta}}\Gamma(2-\overline{\beta})^{-1}a_{n+\frac{1}{2},n+\frac{1}{2}}^{\alpha,\overline{\beta}}$. For k sufficiently small, if $e^{n+1}=e^{n+\frac{1}{2}}$ using estimate (3.32) it is not hard to see that

$$\max_{0\leq l\leq n+1}\|e^l\|_2^2\leq \max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\widehat{C}_2k(k^{\frac{3}{2}}+k^{2-\overline{\beta}}+k^{\frac{5}{2}}+h^4)^2.$$

Otherwise, $\left(1-\frac{1}{2}\widehat{\theta}_{n+1}^{n+\frac{1}{2}+\alpha}\right)\|e^{n+1}-e^{n+\frac{1}{2}}\|_2^2>\frac{1}{2}\theta_{n+1}^{n+1+\alpha}\max\limits_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2$. This fact combined with (3.42) imply

$$\|e^{n+1}\|_2^2\leq \max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\frac{(1-4\alpha)^2}{4(1+4\alpha)}C_p^2k\|\delta_x e^{n+\frac{1}{2}}\|_2^2+\widehat{C}_{3,\alpha}k(k^{2-\overline{\beta}}+k^{\frac{3}{2}}+k^{\frac{5}{2}}+k^{\frac{1}{2}}h^4)^2. \tag{3.43}$$

Taking the limit when α approaches $\frac{1}{4}$, (3.43) provides

$$\|e^{n+1}\|_2^2\leq \max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\widehat{C}_3k(k^{2-\overline{\beta}}+k^{\frac{3}{2}}+k^{\frac{5}{2}}+h^4)^2,$$

where $\widehat{C}_3=\lim\limits_{\alpha\to\frac{1}{4}}\widehat{C}_{3,\alpha}=\lim\limits_{\alpha\to\frac{1}{4}}\max\left\{\frac{C_p^2}{1+4\alpha}(C_{\frac{1}{2}}^2+C_1^2),C_2\right\}=\max\left\{\frac{C_p^2}{2}(C_{\frac{1}{2}}^2+C_1^2),C_2\right\}$. This estimate implies

$$\max_{0\leq l\leq n+1}\|e^l\|_2^2\leq \max_{0\leq l\leq n+\frac{1}{2}}\|e^l\|_2^2+\widehat{C}_3k(k^{2-\overline{\beta}}+k^{\frac{3}{2}}+k^{\frac{5}{2}}+k^{\frac{1}{2}}h^4)^2. \tag{3.44}$$

It is worth noticing to remind that the summation index "l" varies in the range: $l = 0, \frac{1}{2}, 1, ..., n+\frac{1}{2}, n+1$. Now, setting

$$Z^q = \max_{0\le l\le q} \|e^l\|_2^2 \text{ and } \widehat{C}_4 = \max\{\widehat{C}_2, \widehat{C}_3\}, \tag{3.45}$$

estimates (3.32) and (3.44) can be rewritten as

$$Z^{n+\frac{1}{2}} \le Z^n + \widehat{C}_4 k(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + h^4)^2, \text{ and } Z^{n+1} \le Z^{n+\frac{1}{2}} + \widehat{C}_4 k(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2.$$

Substituting the first estimate into the second one gives

$$Z^{n+1} \le Z^n + 2\widehat{C}_4 k(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2.$$

Summing this up from $n = 0, 1, 2, ..., N-1$, we obtain

$$Z^N \le Z^0 + 2\widehat{C}_4 Nk(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4)^2. \tag{3.46}$$

But, it comes from the initial condition (2.52) that $e_j^0 = u_j^0 - U_j^0 = 0$, for $j = 0, 1, 2, ..., M$. Furthermore, since $k = \frac{T}{N}$, then $Nk = T$. These facts together with (3.45) and (3.46) yield

$$\max_{0\le n\le N-1} \|e^{n+\frac{1}{2}}\|_2, \quad \max_{0\le n\le N} \|e^n\|_2 \le \sqrt{2\widehat{C}_4 T}(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4).$$

These estimates imply

$$\|e^{n+\frac{1}{2}}\|_2, \quad \|e^n\|_2 \le \sqrt{2\widehat{C}_4 T}(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4), \tag{3.47}$$

for $n = 0, 1, 2, ..., N-1$ (resp., N). But $|\|u^q\|_2 - \|U^q\|_2| \le \|u^q - U^q\|_2 = \|e^q\|_2$, thus

$$\max_{0\le n\le N-1} \|U^{n+\frac{1}{2}}\|_2, \quad \max_{0\le n\le N} \|U^n\|_2 \le |||u|||_{\infty,2} + \sqrt{2\widehat{C}_4 T}(k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4).$$

This ends the proof of estimate (3.19) in Theorem 3.1. Now, since $k < 1$ so $k^{\frac{5}{2}} < \min\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\}$ and $k^{\frac{1}{2}}h^4 < h^4$, thus $k^{2-\overline{\beta}} + k^{\frac{3}{2}} + k^{\frac{5}{2}} + k^{\frac{1}{2}}h^4 \le (3\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} + h^4)$. Using this, (3.47) implies

$$\max_{0\le n\le N-1} \|e^{n+\frac{1}{2}}\|_2, \quad \max_{0\le n\le N} \|e^n\|_2 \le \sqrt{2\widehat{C}_4 T}(3\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} + h^4).$$

This completes the proof of Theorem 3.1. □

4. Numerical experiments and convergence rate

This section considers some computational results to show the unconditional stability and the convergence order of the new approach (2.48)–(2.52) applied to time-variable fractional transport equation (1.1) subject to initial and boundary value conditions (1.2) and (1.3), respectively. To demonstrate the efficiency and accuracy of the proposed algorithm, two examples are provided in [10]. We take $h \in \{2^{-i}, \ i = 1, 2, 3, 4, 5, 6\}$ and $k = 2^{-r}$, for $r = 2, 3, 4, 5, 6, 7$, and we compute the L^∞-norm of the numerical solution, U^n, the exact one, u^n, and the corresponding error, e^n, at time level n, using the following formulas:

$$|||U|||_{\infty,2} = \max_{0\le n\le N}\left(h\sum_{j=2}^{M-2}|U_j^n|^2\right)^{\frac{1}{2}}, \quad |||u|||_{\infty,2} = \max_{0\le n\le N}\left(h\sum_{j=2}^{M-2}|u_j^n|^2\right)^{\frac{1}{2}},$$

and

$$|||e_{(\cdot)}|||_{\infty,2} = \max_{0\le n\le N}\left(h\sum_{j=2}^{M-2}|u_j^n - U_j^n|^2\right)^{\frac{1}{2}}.$$

In each example, the numerical evidence is found with two different order functions: $\beta_1(x,t) = 1 - 2^{-1}e^{-xt}$ and $\beta_2(x,t) = \frac{4}{3} - 5.10^{-3}\cos(xt)\sin(xt)$. Furthermore, the convergence order, $R(h)$, in space is estimated using the formula

$$R(h) = \frac{\log\left(\frac{|||e_{2h}|||_{\infty,2}}{|||e_h|||_{\infty,2}}\right)}{\log(2)},$$

where, e_h and e_{2h} represent the spatial errors associated with the grid sizes h and $2h$, respectively, while the temporal convergence rate, $R(k)$, is calculated as follows

$$R(k) = \frac{\log\left(\frac{|||e_{2k}|||_{\infty,2}}{|||e_k|||_{\infty,2}}\right)}{\log(2)},$$

where e_{2k} and e_k denote the errors in time corresponding to time steps $2k$ and k, respectively. Finally, the numerical computations are carried out with the help of MATLAB R2007b.

Figures 1–4 suggest that the proposed two-stage technique (2.48)–(2.52) is unconditionally stable whereas Tables 1, 2, 4 and 5 indicate that the developed numerical method is temporal accurate with order $O(\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\})$ and spatial fourth-order convergent. These numerical studies confirm the theoretical results provided in Section 3, Theorem 3.1.

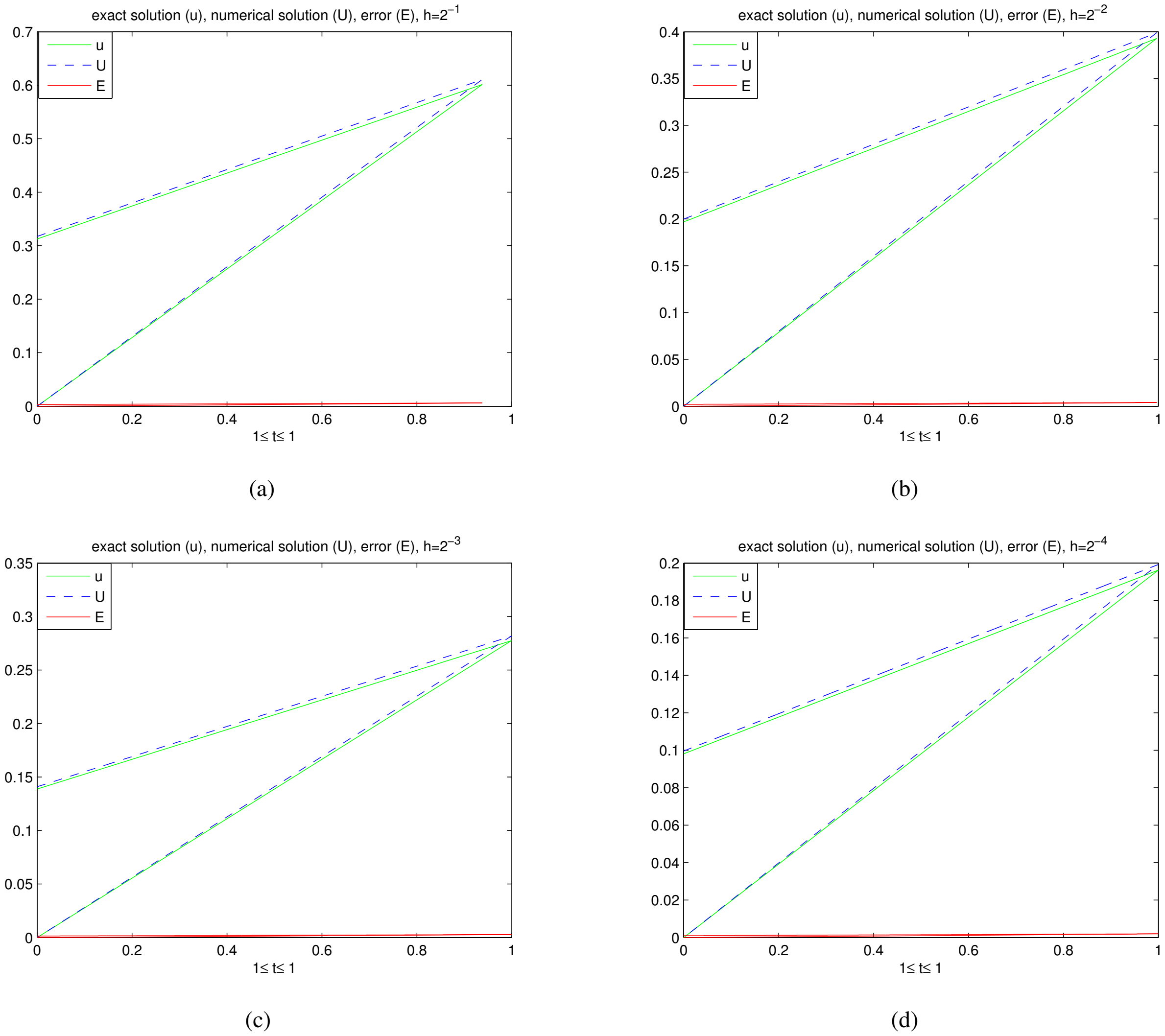

Figure 1. Analysis of stability and convergence of a two-step Euler/Crank-Nicolson technique for time-variable fractional mobile-immobile with $\alpha = 0.25$ and $k = h^4$.

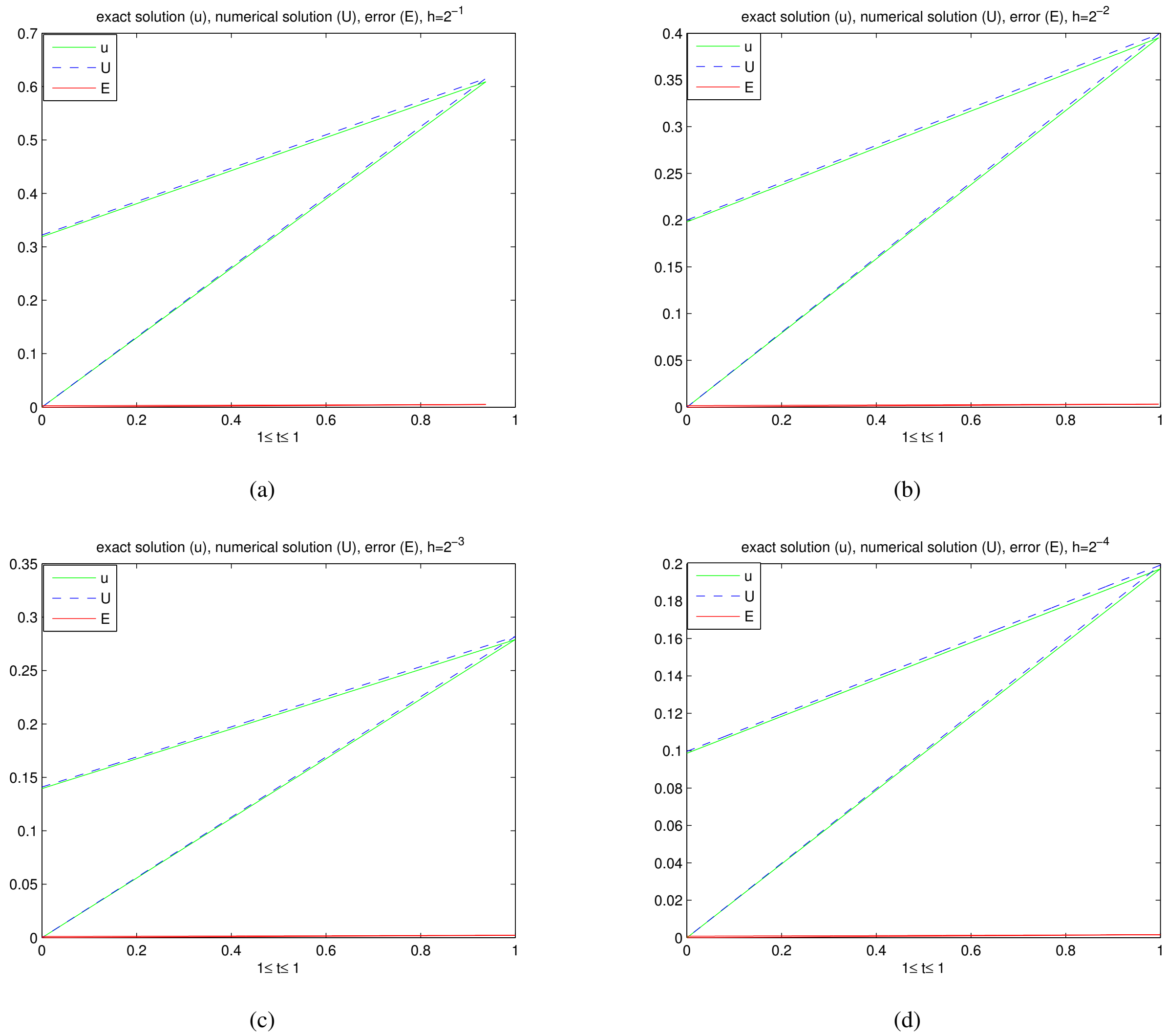

Figure 2. Stability and convergence of a two-step Euler/Crank-Nicolson approach for time-variable fractional mobile-immobile with $\alpha = 0.49$ and $k = h^4$.

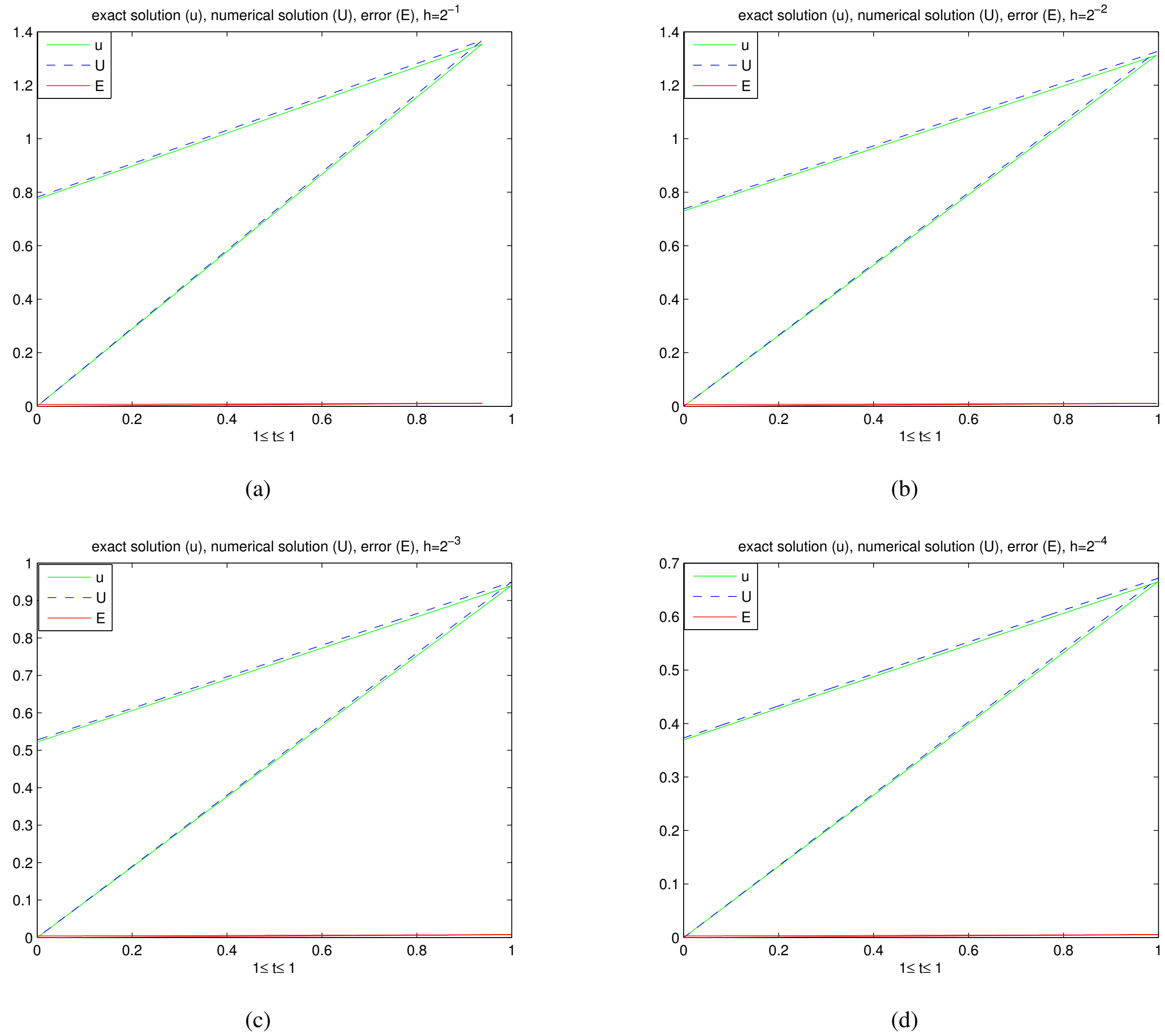

Figure 3. Stability and convergence of a two-step Euler/Crank-Nicolson approach for time-variable fractional mobile-immobile with $\lambda = 0.25$ and $k = h^4$.

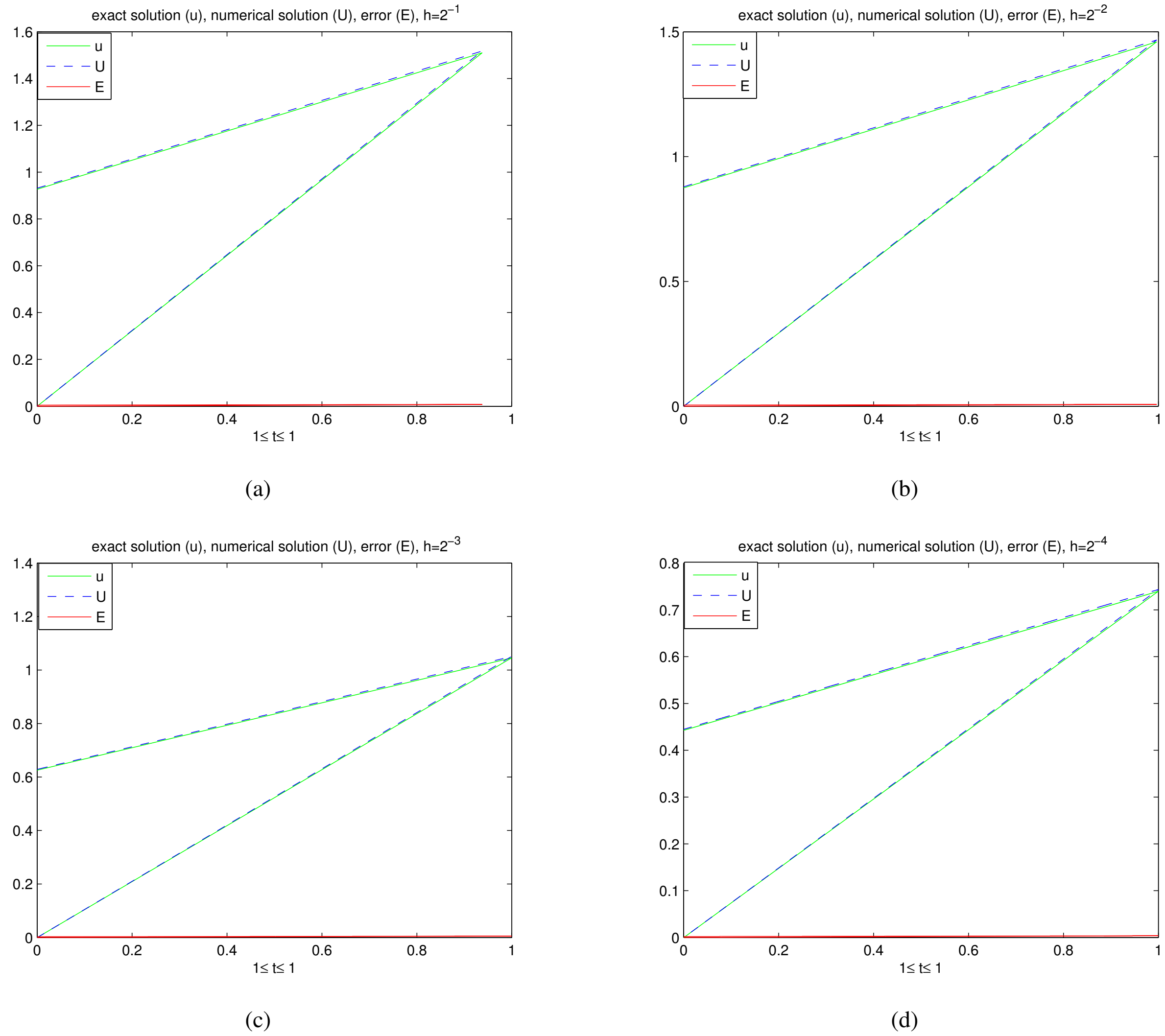

Figure 4. Analysis of stability and convergence of a two-step Euler/Crank-Nicolson numerical scheme for time-variable fractional mobile-immobile with $\alpha = 0.49$ and $k = h^4$.

Example 4.1. *Let $D = [0,1] \times [0,1]$ be the domain. The parameter $\alpha = 0.25, 0.49$ and the function β is defined as $\beta(x,t) = 1 - 2^{-1}e^{-xt}$. Consider the following time-variable fractional mobile-immobile model defined in [10] by*

$$\begin{cases} u_t(x,t) + cD_{0t}^{\beta(x,t)}u(x,t) = -u_x(x,t) + u_{2x}(x,t) + f(x,t) & \text{on } \ D, \\ u(x,0) = u_0(x) = 10x^2(1-x)^2, & \text{for } \ 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & \text{for } \ 0 \le t \le 1, \end{cases}$$

where $f(x,t) = 10x^2(1-x)^2 + \frac{10x^2(1-x)^2t^{1-\beta(x,t)}}{\Gamma(2-\beta(x,t))} + 10(1+t)(2x - 6x^2 + 4x^3) - 10(1+t)(2 - 12x + 12x^2)$. *The analytical solution u is given by*

$$u(x,t) = 10(1+t)x^2(1-x)^2.$$

Table 1. Convergence order, $R(h)$, of the two-stage fourth-order approach with varying space step h. We set $\alpha = 0.25$ and $\beta(x,t) = 1 - 2^{-1}e^{-xt}$.

h	$\|\|\|u\|\|\|_{\infty,2}$	$\|\|\|U\|\|\|_{\infty,2}$	$\|\|\|E(h)\|\|\|_{\infty,2}$	R(h)	CPU(s)
2^{-1}	6.1049×10^{-1}	6.011×10^{-1}	6.4483×10^{-2}	–	1.2238
2^{-2}	3.9871×10^{-1}	3.9279×10^{-1}	5.4102×10^{-3}	3.5671	2.3204
2^{-3}	2.8163×10^{-1}	2.7747×10^{-1}	4.2155×10^{-4}	3.6819	4.6378
2^{-4}	1.9920×10^{-1}	1.9629×10^{-1}	2.6665×10^{-5}	3.9827	9.8173
2^{-5}	1.1357×10^{-1}	1.1135×10^{-1}	1.5546×10^{-6}	4.1003	21.8159
2^{-6}	8.6737×10^{-2}	8.5693×10^{-2}	9.7115×10^{-8}	4.0007	51.4812

Table 2. Convergence rate, $R(k)$, of the new technique with varying time step k. Here we take $\alpha = 0.49$ and $\beta(x,t) = 1 - 2^{-1}e^{-xt}$.

k	$\|\|\|u\|\|\|_{\infty,2}$	$\|\|\|U\|\|\|_{\infty,2}$	$\|\|\|E(k)\|\|\|_{\infty,2}$	R(k)	CPU(s)
2^{-2}	6.1502×10^{-1}	6.0897×10^{-1}	3.9202×10^{-2}	–	0.8937
2^{-3}	3.9892×10^{-1}	3.9795×10^{-1}	1.3729×10^{-2}	1.5137	1.5099
2^{-4}	2.8176×10^{-1}	2.7888×10^{-1}	4.8597×10^{-3}	1.4983	2.7021
2^{-5}	1.9927×10^{-1}	1.9726×10^{-1}	1.7147×10^{-3}	1.5029	5.3559
2^{-6}	1.1356×10^{-1}	1.1136×10^{-1}	6.0276×10^{-4}	1.5083	10.7118
2^{-7}	8.6713×10^{-2}	8.5675×10^{-2}	2.1136×10^{-4}	1.5119	24.5235

The theoretical results (Table 3) discussed in [10] suggests that the method is temporal and spatial first-order accurate (see Theorem 2, page 698) while this table indicates that the numerical scheme is second-order convergent in both time and space, which seems to be a complete surprise. However, Tables 1 and 2 show that the new algorithm converges with order $O(k^{\frac{3}{2}} + h^4)$. This result confirms the theoretical studies obtained in Theorem 3.1. Indeed: $0.5 \leq \beta(x,t) \leq 1 - 2^{-1}e^{-1} \approx 0.8161$, $\forall (x,t) \in [0,1] \times [0,1]$. So $\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} = k^{\frac{3}{2}}$, $\forall \overline{\beta} \in [0.5, 0.8161]$.

Table 3. Method discussed in [10], where $\gamma(x,t) = 1 - 0.5e^{-xt}$.

$k = h$	$\|\|\|E(k)\|\|\|_{\infty,2}$	R(k)
1/50	9.4391×10^{-3}	—-
1/100	5.0134×10^{-3}	1.88
1/200	2.5613×10^{-3}	1.96
1/400	1.2781×10^{-3}	2.00

Example 4.2. *Let D be the bounded domain $[0,1] \times [0,1]$. We assume that the parameter $\alpha \in \{0.25, 0.49\}$ and the function β is given by $\beta(x,t) = \frac{4}{3} - 5.10^{-3}\cos(xt)\sin(xt)$. We consider the following time-variable fractional mobile-immobile convection-diffusion model defined in [10] as*

$$\begin{cases} u_t(x,t) + cD_{0t}^{\beta(x,t)}u(x,t) = -u_x(x,t) + u_{2x}(x,t) + f(x,t) \quad \text{on } D, \\ u(x,0) = u_0(x) = 5\sin(\pi x), \quad \text{for } 0 \leq x \leq 1, \\ u(0,t) = u(1,t) = 0, \quad \text{for } 0 \leq t \leq 1, \end{cases}$$

where $f(x,t) = 5(1+\pi^2(1+t))\sin(\pi x) + \frac{5\sin(\pi x)t^{1-\beta(x,t)}}{\Gamma(2-\beta(x,t))} - 5\pi(1+t)\cos(\pi x)$*. The exact solution u is defined as*

$$u(x,t) = 5(1+t)x^2\sin(\pi x).$$

We observe from Tables 4 and 5 that the proposed approach is convergent of order $O(k^{\frac{3}{2}} + h^4)$. This study confirms the theoretical result provided in Theorem 3.1. In fact: $1.333246 \leq \beta(x,t) \leq 1.333333$, $\forall(x,t) \in [0,1]\times[0,1]$. So $\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\} = k^{\frac{3}{2}}$, $\forall\overline{\beta} \in [1.333246, 1.333333]$.

Table 4. Convergence order, $R(h)$, for the two-level approach with varying spacings h. We take $\alpha = 0.25$ and $\beta(x,t) = \frac{4}{3} - 5.10^{-3}\cos(xt)\sin(xt)$.

h	$\|\|\|u\|\|\|_{\infty,2}$	$\|\|\|U\|\|\|_{\infty,2}$	$\|\|\|E(h)\|\|\|_{\infty,2}$	R(h)	CPU(s)
2^{-1}	1.3672	1.3534	1.0918×10^{-2}	–	1.1876
2^{-2}	1.3248	1.3115	7.3334×10^{-4}	3.8937	2.2512
2^{-3}	9.4892×10^{-1}	9.4356×10^{-1}	4.5602×10^{-5}	4.0073	4.4500
2^{-4}	6.7163×10^{-1}	6.6689×10^{-1}	2.5995×10^{-6}	4.1328	10.1612
2^{-5}	4.8509×10^{-1}	4.8473×10^{-1}	1.6247×10^{-7}	4.0000	23.2662
2^{-6}	3.8420×10^{-1}	3.8397×10^{-1}	9.4174×10^{-9}	4.1087	57.3092

Table 5. Convergence rate, $R(k)$, of the new technique with different time steps k. In this example, we set $\alpha = 0.49$ and $\beta(x,t) = \frac{4}{3} - 5.10^{-3}\cos(xt)\sin(xt)$.

k	$\|\|\|u\|\|\|_{\infty,2}$	$\|\|\|U\|\|\|_{\infty,2}$	$\|\|\|E(k)\|\|\|_{\infty,2}$	R(k)	CPU(s)
2^{-2}	1.5162	1.5078	6.5379×10^{-2}	–	0.8345
2^{-3}	1.4673	1.4581	2.3337×10^{-2}	1.4862	1.1264
2^{-4}	1.0512	1.0459	8.3667×10^{-3}	1.4799	2.1357
2^{-5}	7.4325×10^{-1}	7.3954×10^{-1}	2.9760×10^{-3}	1.4913	4.2711
2^{-6}	4.9639×10^{-1}	4.9124×10^{-1}	1.0519×10^{-3}	1.5004	9.0013
2^{-7}	3.0667×10^{-1}	3.0247×10^{-1}	3.6732×10^{-4}	1.5179	20.8246

5. General conclusions and future works

This paper has developed a two-stage fourth-order modified explicit Euler/Crank-Nicolson formulation for solving a time-variable fractional transport equation subject to suitable initial and boundary value conditions. Both stability and error estimates of the new technique have been deeply analyzed in the $L^\infty(0,T;L^2)$-norm. The theory has shown that the proposed approach is unconditionally stable, with temporal convergence of order $O(\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\})$, and fourth-order accurate in space (see Theorem 3.1). This theoretical analysis is confirmed by some numerical examples. Especially, the graphs (Figures 1–4) show that the new procedure is unconditionally stable whereas Tables 1, 2, 4 and 5 indicate the convergence rate of the developed algorithm (convergence with order $O(\max\{k^{2-\overline{\beta}}, k^{\frac{3}{2}}\})$ in time and fourth-order accurate in space). Furthermore, the theoretical and numerical studies suggest that the constructed numerical scheme (2.48)–(2.52) is faster and more efficient than a wide set of numerical methods [30–33] developed for the considered problem (1.1)–(1.3). Our future works will construct a high-order two-stage explicit/implicit numerical technique for solving a multi-dimensional transport equation with time-variable fractional derivatives.

Author contributions

Eric Ngondiep: Conceptualization, writing-review & editing, numerical experiments; Areej A. Binsultan: Investigation, writing-original draft; Ibtisam M. Aldawish: Formal analysis, methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interests.

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