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Research article

An optimal inequality for warped product submanifolds in complex space forms

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Abstract: In this work, using optimization procedures on Riemannian submanifolds, we obtained two different inequalities on the generalized normalized δ-Casorati curvatures of warped product submanifolds in complex space forms. We also quantified the conditions under which these inequalities become equalities, providing more insight into their geometric consequences. Further, we described new findings in the form of harmonic functions and Hessian functions, which offer a more general view of the interplay between curvature and analyticity.

Keywords: Casorati curvature; warped product submanifold; complex space form; harmonic function; Hessian function
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1. Introduction

The study of product manifolds plays a crucial role in both physical and geometric contexts, particularly within Hermitian geometry. In physics, spacetime in Einstein's general relativity is often modeled as the product of a three-dimensional space and a one-dimensional time, with each component possessing its own metric. Consequently, the topology of spacetime is fundamentally shaped by these metrics. Product manifolds also appear in several advanced theories, including gauge theory, Kaluza-Klein theory, and brane theory.

In 1969, R. L. Bishop and co-authors [4] introduced a generalized class of Riemannian product manifolds to explore manifolds exhibiting negative sectional curvature. These are now known as warped product (WP) manifolds. Formally, they are defined as follows.

Consider two Riemannian manifolds: N_1 of dimension k_1 with metric g_1, and N_2 of dimension k_2 with metric g_2. Let ζ be a positive differentiable function on N_1, and let $N_1 \times N_2$ denote their product manifold with projections $\iota_1 \colon N_1 \times N_2 \to N_1$ and $\iota_2 \colon N_1 \times N_2 \to N_2$. The warped product manifold

$M = N_1 \times_\zeta N_2$ is equipped with the metric

$$g(U,V) = g_1(\iota_{1*}U, \iota_{1*}V) + (\zeta \circ \iota_1)^2 g_2(\iota_{2*}U, \iota_{2*}V),$$

where U and V are vector fields on M, and $*$ denotes the tangent map [8].

Separately, Nash's embedding theorem [16] addresses the conditions under which a Riemannian manifold can be smoothly immersed in a space form. However, intrinsic invariants often impose constraints on the extrinsic geometry of submanifolds, limiting the practical applicability of Nash's result. To overcome these limitations, Chen introduced a unified framework combining intrinsic and extrinsic invariants. In 1993, Chen [6] established a fundamental inequality for submanifolds in real space forms, relating intrinsic quantities such as sectional and scalar curvature to extrinsic invariants like the squared mean curvature.

This breakthrough led to the development of the so-called δ-invariants (also known as Chen invariants), which have since found broad applications in Riemannian geometry and related fields [7].

Casorati curvature (C-curvature), introduced as the square of the normalized length of the second fundamental form, has emerged as a significant extrinsic invariant of submanifolds in Riemannian geometry. It generalizes the notion of principal directions for hypersurfaces in a Riemannian manifold [5]. Their work has since been extended in several directions by various researchers like Decu et. al [9], Zhang and Zhang [19, 20], and by others [13, 14, 18].

In 2008, S. Decu, S. Haesen, and L. Verstraelen [10] introduced the generalized normalized δ-C-curvatures, denoted by $\widehat{\delta_C}(t; k-1)$ and $\delta_C(t; k-1)$. They proved sharp inequalities involving these extrinsic measures and scalar curvature for any real number t satisfying $0 < t < k(k-1)$. Recent studies have established numerous inequalities involving normalized C-curvature and scalar curvature for various classes of submanifolds in diverse ambient spaces such as complex space forms [2], quaternionic space forms [1, 15], and other space forms [10–12].

In this paper, we derive new inequalities for $\widehat{\delta_C}(t; k-1)$ and $\delta_C(t; k-1)$ using the method of constrained extrema. This approach offers a natural framework for establishing geometric inequalities [17], particularly for warped product submanifolds in complex space forms.

2. Preliminaries

Let $\overline{M}$ be an almost Hermitian manifold, equipped with an almost complex structure J and a Riemannian metric g. Such a manifold is called a Kähler manifold if the Levi-Civita connection $\overline{\nabla}$ satisfies $\overline{\nabla}J = 0$.

An almost Hermitian manifold $\overline{M}$ is referred to as a complex space form (CSF), denoted by $\overline{M}(c)$, if its Riemannian curvature tensor $\overline{R}$ satisfies

$$\overline{R}(U,V)Z = \frac{c}{4}\{g(V,Z)U - g(U,Z)V + g(U,JZ)JV - g(V,JZ)JU + 2g(U,JV)JZ\}, \tag{2.1}$$

for all vector fields U, V, Z on $\overline{M}(c)$.

Let M be a submanifold of dimension k in a CSF $\overline{M}(c)$ of complex dimension p. The Levi-Civita connections on M and $\overline{M}(c)$ are denoted by ∇ and $\overline{\nabla}$, respectively. The Gauss and Weingarten formulas are given by:

$$\overline{\nabla}_U V = \nabla_U V + \sigma(U,V), \tag{2.2}$$

$$\overline{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.3}$$

where σ is the second fundamental form, $\nabla^{\perp}$ is the normal connection, and A denotes the shape operator of M. These quantities are related by:

$$g(\sigma(U, V), N) = g(A_N U, V), \tag{2.4}$$

where g is the induced metric on M and also the metric on $\overline{M}(c)$. If the second fundamental form σ vanishes at a point $x \in M$, then x is called a totally geodesic point. A submanifold is said to be totally geodesic if this condition holds at every point of M.

Let R and $\overline{R}$ denote the curvature tensors of M and $\overline{M}(c)$, respectively. The Gauss equation, which relates these curvature tensors, is given by:

$$\overline{R}(U, V, Z, W) = R(U, V, Z, W) + g(\sigma(U, Z), \sigma(V, W)) - g(\sigma(U, W), \sigma(V, Z)), \tag{2.5}$$

for all vector fields U, V, Z, W tangent to M.

At a point $x \in M$, let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of the tangent space T_xM, and let $\{e_{k+1}, \ldots, e_{2p}\}$ be an orthonormal basis of the normal space $T^{\perp}_x M$. The mean curvature vector H at x is defined by

$$H(x) = \frac{1}{k}\sum_{i=1}^{k} \sigma(e_i, e_i). \tag{2.6}$$

The squared norm of the second fundamental form σ is given by

$$\|\sigma\|^2 = \sum_{i,j=1}^{k} g(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{2.7}$$

Let $U, V \in T_xM$ be two linearly independent vectors at a point $x \in M$. The sectional curvature $K(U, V)$, for the plane section spanned by U and V, is defined as

$$K(U, V) = \frac{g(R(U, V)V, U)}{g(U, U)g(V, V) - g(U, V)^2}. \tag{2.8}$$

The scalar curvature τ of M at the point x is defined by

$$\tau(x) = \sum_{1 \le i < j \le k} K(e_i \wedge e_j), \tag{2.9}$$

and the normalized scalar curvature ρ is expressed as

$$\rho(x) = \frac{2\tau(x)}{k(k-1)}. \tag{2.10}$$

The squared mean curvature of M in the ambient complex space form $\overline{M}(4c)$ is given by

$$\|H\|^2 = \frac{1}{k^2}\sum_{u=k+1}^{2p}\left(\sum_{i=1}^{k}\sigma^u_{ii}\right)^2, \tag{2.11}$$

and the Casorati curvature C of M is defined by

$$C = \frac{1}{k} \sum_{u=k+1}^{2p} \sum_{i,j=1}^{k} \left(\sigma_{ij}^{u}\right)^2 . \tag{2.12}$$

Let $\pi \subset T_xM$ be an s-dimensional subspace with $s \geq 2$ and orthonormal basis $\{e_1, \ldots, e_s\}$. The scalar curvature $\tau(\pi)$ of π is given by

$$\tau(\pi) = \sum_{1 \leq i < j \leq s} K(e_i \wedge e_j), \tag{2.13}$$

and the Casorati curvature of π is defined as

$$C(\pi) = \frac{1}{s} \sum_{u=k+1}^{2p} \sum_{i,j=1}^{s} \left(\sigma_{ij}^{u}\right)^2 . \tag{2.14}$$

The normalized δ-Casorati curvatures $\delta_C(k-1)$ and $\widehat{\delta_C}(k-1)$ at the point x are given by

$$[\delta_C(k-1)]_x = \frac{1}{2} C_x + \frac{k+1}{2k} \inf\{C(\pi) \mid \pi : \text{ hyperplane of } T_xM\}, \tag{2.15}$$

$$\left[\widehat{\delta_C}(k-1)\right]_x = 2C_x - \frac{2k-1}{2k} \sup\{C(\pi) \mid \pi : \text{ hyperplane of } T_xM\}. \tag{2.16}$$

The generalized normalized (GN) δ-Casorati curvatures $\delta_C(t; k-1)$ and $\widehat{\delta_C}(t; k-1)$, defined for $t > 0$ with $t \neq k(k-1)$, are expressed as

$$[\delta_C(t; k-1)]_x = tC_x + \frac{(k-1)(k+t)(k^2-k-t)}{tk} \inf\{C(\pi) \mid \pi : \text{ hyperplane of } T_xM\}, \tag{2.17}$$

$$\left[\widehat{\delta_C}(t; k-1)\right]_x = tC_x - \frac{(k-1)(k+t)(t-k^2+k)}{tk} \sup\{C(\pi) \mid \pi : \text{ hyperplane of } T_xM\}. \tag{2.18}$$

These generalized curvatures reduce to the classical ones for specific values of t, as follows:

$$[\delta_C\,(k(k-1); k-1)]_x = k(k-1)\,[\delta_C(k-1)]_x\,, \tag{2.19}$$

$$\left[\widehat{\delta_C}\,(2k(k-1); k-1)\right]_x = k(k-1)\left[\widehat{\delta_C}(k-1)\right]_x . \tag{2.20}$$

From the Gauss equation, we obtain

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{u=n+1}^{2p} \left(\sigma_{ii}^{u}\sigma_{jj}^{u} - \left(\sigma_{ij}^{u}\right)^2\right), \tag{2.21}$$

where $K(e_i \wedge e_j)$ and $\tilde{K}(e_i \wedge e_j)$ denote the sectional curvatures of the submanifold M^k and the ambient manifold $\tilde{M}^{2p}$, respectively.

Using (2.21), the scalar curvatures of $N_1^{k_1}$ and $N_2^{k_2}$ decompose as follows:

$$\tau(N_1^{k_1}) = \tilde{\tau}(N_1^{k_1}) + \sum_{u=k+1}^{2p} \sum_{1 \leq i < j \leq k_1} \left(\sigma_{ii}^{u}\sigma_{jj}^{u} - \left(\sigma_{ij}^{u}\right)^2\right), \tag{2.22}$$

$$\tau(N_2^{k_2}) = \tilde{\tau}(N_2^{k_2}) + \sum_{u=k+1}^{2p} \sum_{k_1+1\le s<t\le k} \left(\sigma_{ss}^u \sigma_{tt}^u - (\sigma_{st}^u)^2\right). \tag{2.23}$$

Finally, a relation established by B.-Y. Chen is given by

$$\sum_{1\le i\le k_1} \sum_{k_1+1\le j\le k} K(e_i \wedge e_j) = k_2 \frac{\Delta\zeta}{\zeta} = k_2 \left(\Delta(\ln\zeta) - \|\nabla\zeta\|^2\right), \tag{2.24}$$

where Δ denotes the Laplacian operator.

3. Inequalities for a warped product

Consider a Riemannian submanifold M of a Riemannian manifold $(\overline{M}, g)$, and let $f : \overline{M} \to \mathbb{R}$ be a differentiable function. We examine the constrained extremum problem

$$\min_{x\in M} f(x), \tag{3.1}$$

which leads to the following result.

Lemma 3.1. *[17] If the solution to the problem* (3.1) *is $x_0 \in M$, then:*

(i) The gradient of f at x_0 lies in the normal space to M at x_0; that is,

$$(\mathrm{grad}\ f)(x_0) \in T_{x_0}^{\perp} M.$$

(ii) The bilinear form

$$\Xi : T_{x_0}M \times T_{x_0}M \to \mathbb{R}, \quad \Xi(U, V) = \mathrm{Hess}_f(U, V) + g(\sigma(U, V), \mathrm{grad}\ f(x_0))$$

is positive semi-definite, where Hess_f *denotes the Hessian of f,* grad f *is the gradient of f, and σ is the second fundamental form of M in $\overline{M}$.*

Theorem 3.2. *Let $\overline{M}^{2p}(c)$ be the CSF and $\varphi : M^k = N_1^{k_1} \times_\zeta N_2^{k_2} \to \overline{M}^{2p}(c)$ be an isometric immersion of the WP submanifold into $\tilde{M}^{2p}(c)$. Then,*

(i) The GN δ-C-curvature $\delta_C(t; k-1)$ satisfies

$$\rho \le \frac{\delta_C(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2t-t)k}{(k-1)(k^2-k-t+k_2t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\Big\{k_2\frac{\Delta\zeta}{\zeta} + \frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\Big(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\Big)\Big\}, \tag{3.2}$$

in the case of any real number t such that $0 < t < k(k-1)$.

(ii) The GN δ-C-curvature $\widehat{\delta_C}(t; k-1)$ satisfies

$$\rho \le \frac{\widehat{\delta_C}(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2t-t)k}{(k-1)(k^2-k-t+k_2t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\Big\{k_2\frac{\Delta\zeta}{\zeta} + \frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\Big(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\Big)\Big\}, \tag{3.3}$$

in the case of any real number $t > k(k-1)$.

Moreover, the equalities in Eqs (3.2) *and* (3.3) *hold if and only if the shape operators, with respect to an appropriate orthonormal basis of the tangent and normal spaces, take the following specific form.*

$$A_{e_{k+1}} = \begin{pmatrix} \sigma^{k+1}_{11} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sigma^{k+1}_{22} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma^{k+1}_{k_1 k_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \sigma^{k+1}_{k_1+1\, k_1+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \sigma^{k+1}_{k_1+2\, k_1+2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \vdots & \sigma^{k+1}_{k-1\, k-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \vdots & 0 & \sigma^{k+1}_{k k} \end{pmatrix}, \tag{3.4}$$

$$\begin{aligned} &A_{e_{k+2}} = \cdots = A_{e_{2p}} = 0, \\ &\sigma^{k+1}_{11} = \cdots = \sigma^{k+1}_{k_1 k_1} = k_1 t^2 f_1 e_{k+1}, \\ &\sigma^{k+1}_{k_1+1\, k_1+1} = \cdots = \sigma^{k+1}_{k-1\, k-1} = (k^2 - k + k_2 t - t) t f_1 e_{k+1}, \\ &\sigma_{k k} = k(k-1)(k^2 - k + k_2 t - t) f_1 e_{k+1}, \\ &\sigma_{i j} = 0, \quad i \neq j, \end{aligned}$$

where f_1 *is a function on* M^k.

Proof. From the Gauss equation and Eqs (2.1) and (2.12)–(2.14), we obtain

$$\begin{aligned} \tau(x) &= \sum_{l=1}^{k_1} \sum_{r=k_1+1}^{k} K(e_l \wedge e_r) + \sum_{1 \le i < j \le k_1} K(e_i \wedge e_j) + \sum_{k_1+1 \le s < t \le k} K(e_s \wedge e_t) \\ &= k_2 \frac{\Delta \zeta}{\zeta} + \frac{k_1(k_1-1)c}{8} + \frac{3}{4} c \|P^T\|^2_{N_1^{k_1}} + \frac{k_2(k_2-1)c}{8} + \frac{3}{4} c \|P^T\|^2_{N_2^{n_2}} \\ &\quad + \sum_{\alpha=k+1}^{2p} \sum_{1 \le i < j \le k_1} [\sigma^\alpha_{ii} \sigma^\alpha_{jj} - (\sigma^\alpha_{ij})^2] + \sum_{\alpha=k+1}^{2p} \sum_{k_1+1 \le s < t \le k} [\sigma^\alpha_{ss} \sigma^\alpha_{tt} - (\sigma^\alpha_{st})^2]. \end{aligned} \tag{3.5}$$

We define now the following quadratic polynomial denoted by $\mathbb{P}$ in the components of the second fundamental form as

$$\begin{aligned} \mathbb{P} &= tC + \frac{(k-1)(k+t)(k^2-k-t)}{kt} C(\pi) - 2\tau \\ &\quad + 2\Big\{ k_2 \frac{\Delta \zeta}{\zeta} + \frac{k_1(k_1-1)c}{8} + \frac{3}{4} c \|P^T\|^2_{N_1^{k_1}} + \frac{k_2(k_2-1)c}{8} + \frac{3}{4} c \|P^T\|^2_{N_2^{k_2}} \Big\}, \end{aligned} \tag{3.6}$$

where π is a hyperplane of T_xM. We can assume that π is spanned by $e_1, \dots, e_{k-1}$ without losing generality. Consequently, (3.6) indicates that

$$\mathbb{P} = \frac{t}{k} \sum_{\alpha} \sum_{i,j=1}^{k} (\sigma^\alpha_{ij})^2 + \frac{(k+t)(k^2-k-t)}{kt} \sum_{\alpha=k+1}^{2p} \sum_{i,j=1}^{k-1} (\sigma^\alpha_{ij})^2$$

$$
\begin{aligned}
&-2\sum_{\alpha=k+1}^{2p}\sum_{1\le i<j\le k_1}[\sigma_{ii}^{\alpha}\sigma_{jj}^{\alpha}-(\sigma_{ij}^{\alpha})^2]-2\sum_{\alpha=k+1}^{2p}\sum_{k_1+1\le s<t\le k_1}[\sigma_{ss}^{\alpha}\sigma_{tt}^{\alpha}-(\sigma_{st}^{\alpha})^2]\\
&\ge\frac{k^2+k(t-1)-2t}{t}\sum_{\alpha}\sum_{i=1}^{k-1}(\sigma_{ii}^{\alpha})^2+\frac{t}{k}\sum_{\alpha=k+1}^{2p}(\sigma_{kk}^{\alpha})^2\\
&-2\sum_{\alpha=k+1}^{2p}\sum_{1\le i<j\le k_1}\sigma_{ii}^{\alpha}\sigma_{jj}^{\alpha}-2\sum_{\alpha=k+1}^{2p}\sum_{k_1+1\le s<t\le k}\sigma_{ss}^{\alpha}\sigma_{tt}^{\alpha}. \qquad (3.7)
\end{aligned}
$$

Now, we consider the quadratic forms

$$f_{\alpha}:\mathbb{R}^k\to\mathbb{R},\quad \alpha=k+1,k+2,\dots,2p,$$

defined by

$$
\begin{aligned}
f_{\alpha}(\sigma_{11}^{\alpha},\sigma_{22}^{\alpha},\dots,\sigma_{kk}^{\alpha}) &= \frac{k^2+k(t-1)-2t}{t}\sum_{i=1}^{k-1}(\sigma_{ii}^{\alpha})^2+\frac{t}{k}(\sigma_{kk}^{\alpha})^2\\
&-2\sum_{1\le i<j\le k_1}\sigma_{ii}^{\alpha}\sigma_{jj}^{\alpha}-2\sum_{k_1+1\le s<t\le k}\sigma_{ss}^{\alpha}\sigma_{tt}^{\alpha}. \qquad (3.8)
\end{aligned}
$$

Then from (3.7) and (3.8), we deduce that

$$\mathbb{P}\ge\sum_{\alpha=k+1}^{2p}f_{\alpha}. \qquad (3.9)$$

Next, we start with the extremum problem

$$\min f_{\alpha},\quad \text{subject to}\quad \Gamma:\sigma_{11}^{\alpha}+\sigma_{22}^{\alpha}+\cdots+\sigma_{nn}^{\alpha}=r^{\alpha},$$

where r^{α} is a real constant.

The partial derivatives of the function f_{α} are

$$
\begin{cases}
\frac{\partial f_{\alpha}}{\partial\sigma_{11}^{\alpha}}=\frac{2[k^2+k(t-1)-t]}{t}\sigma_{11}^{\alpha}-2\sum_{i=1}^{k_1}\sigma_{ii}^{\alpha}\\
\vdots\\
\frac{\partial f_{\alpha}}{\partial\sigma_{k_1k_1}^{\alpha}}=\frac{2[k^2+k(t-1)-t]}{t}\sigma_{k_1k_1}^{\alpha}-2\sum_{i=1}^{k_1}\sigma_{ii}^{\alpha}\\
\frac{\partial f_{\alpha}}{\partial\sigma_{k_1+1k_1+1}^{\alpha}}=\frac{2[k^2+k(t-1)-t]}{t}\sigma_{k_1+1k_1+1}^{\alpha}-2\sum_{s=k_1+1}^{k}\sigma_{ss}^{\alpha}\\
\vdots\\
\frac{\partial f_{\alpha}}{\partial\sigma_{k-1k-1}^{\alpha}}=\frac{2[k^2+k(t-1)-t]}{t}\sigma_{k-1k-1}^{\alpha}-2\sum_{s=k_1+1}\sigma_{ss}^{\alpha}\\
\frac{\partial f_{\alpha}}{\partial\sigma_{kk}^{\alpha}}=\frac{2(k+t)}{k}\sigma_{kk}^{\alpha}-2\sum_{s=k_1+1}\sigma_{ss}^{\alpha}.
\end{cases} \qquad (3.10)
$$

Lemma 3.1 states that the vector $\operatorname{grad} f_{1^\star}$ is normal to Γ, given an optimal solution $(\sigma_{11}^{1^\star}, \sigma_{22}^{1^\star}, \ldots, \sigma_{nn}^{1^\star})$ of the minimization problem described above. In other words, it is collinear with the vector $(1, 1, \ldots, 1)$.

Using Eq (3.10) and Lemma 3.1, we determine that a critical point of the problem takes the following form:

$$\begin{cases} \sigma_{11}^{\alpha} = \cdots = \sigma_{k_1 k_1}^{\alpha} = \dfrac{k_1 t^2}{(k^2 - k - t - k_2 t)^2 + k_1^2 t^2}\, r^{\alpha}, \\ \sigma_{k_1+1\, k_1+1}^{\alpha} = \cdots = \sigma_{k-1\, k-1}^{\alpha} = \dfrac{(k^2 - k + k_2 t - t)t}{(k^2 - k - t - k_2 t)^2 + k_1^2 t^2}\, r^{\alpha}, \\ \sigma_{kk}^{\alpha} = \dfrac{k(k-1)(k^2 - k + k_2 t - t)}{(k^2 - k - t - k_2 t)^2 + k_1^2 t^2}\, r^{\alpha}. \end{cases} \tag{3.11}$$

Now, let $x \in \Gamma$ be a fixed point. According to Lemma 3.1, the corresponding bilinear form $\Xi : T_x\Gamma \times T_x\Gamma \to \mathbb{R}$ is defined by

$$\Xi(U, V) = \mathrm{Hess}_{f_\alpha}(U, V) + \langle \sigma'(U, V), \operatorname{grad} f_\alpha(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, and σ' represents the second fundamental form of Γ in $\mathbb{R}^n$.

From (3.10), we get

$$\begin{cases} \frac{\partial^2 f_\alpha}{\partial (\sigma_{ii}^{\alpha})^2} = \frac{2[n^2 + n(t-1) - 2t]}{t} \\ \frac{\partial^2 f_\alpha}{\partial (\sigma_{ss}^{\alpha})^2} = \frac{2[k^2 + k(t-1) - 2t]}{t} \\ \frac{\partial^2 f_\alpha}{\partial (\sigma_{kk}^{\alpha})^2} = \frac{2t}{k} \\ \frac{\partial^2 f_\alpha}{\partial \sigma_{ii}^{\alpha} \partial \sigma_{jj}^{\alpha}} = -2 \\ \frac{\partial^2 f_\alpha}{\partial \sigma_{ss}^{\alpha} \partial \sigma_{tt}^{\alpha}} = -2 \\ \frac{\partial^2 f_\alpha}{\partial \sigma_{ii}^{\alpha} \partial \sigma_{tt}^{\alpha}} = 0. \end{cases} \tag{3.12}$$

Thus, the Hessian matrix of f_α is

$$\mathrm{Hess}_{f_\alpha} = 2 \begin{pmatrix} \frac{k^2+k(t-1)-2t}{t} & \ldots & -1 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & \ldots & \frac{k^2+k(t-1)-2t}{t} & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \frac{k^2+k(t-1)-2t}{t} & \ldots & -1 & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & -1 & \ldots & \frac{k^2+k(t-1)-2t}{t} & -1 \\ 0 & \ldots & 0 & -1 & \ldots & -1 & \frac{t}{k} \end{pmatrix}.$$

As Γ is totally geodesic in $\mathbb{R}^k$, considering a vector $U = (U_1, \ldots, U_k)$ tangent to Γ at the arbitrary point x on Γ, that is, verifying the relation $\sum_{i=1}^{k} U_i = 0$, then:

(i) For $k = 1$, we have the following possibilities:

(a) $\mathrm{Hess}_{f_\alpha} = (2t)$ (b) $\mathrm{Hess}_{f_\alpha} = (-2)$.

Since $U_1 = 0$, in these two cases, $\mathrm{Hess}_{f_\alpha}(U, U) = 0$.

(ii) For $k = 2$, we have the following possibilities:

(a) $\mathrm{Hess}_{f_\alpha} = \begin{pmatrix} \frac{4}{t} & -2 \\ -2 & \frac{4}{t} \end{pmatrix}$ (b) $\mathrm{Hess}_{f_\alpha} = \begin{pmatrix} \frac{4}{t} & 0 \\ 0 & t \end{pmatrix}$

(c) $\mathrm{Hess}_{f_\alpha} = \begin{pmatrix} \frac{4}{t} & -2-2t \end{pmatrix}$.

Clearly, for (a) and (c), we have $\mathrm{Hess}_{f_\alpha} \geq 0$ and for (b), $\mathrm{Hess}_{f_\alpha} > 0$.

(iii) For $k \geq 3$:

(a) When $n = m$, we have

$$\begin{aligned} \mathrm{Hess}_{f_\alpha} &= \frac{2(k+t)(k-1)}{t}(\sum_{i=1}^{k} U_i^2) - 2(\sum_{i=1}^{k} U_i)^2 \\ &= \frac{2(k+t)(k-1)}{t}(\sum_{i=1}^{k} U_i^2) \\ &\geq 0. \end{aligned}$$

(b) When $n > m$, let

$$A = \begin{pmatrix} \frac{2[k^2+k(t-1)-2t]}{t} & -2 & \cdots & -2 \\ -2 & \frac{2[k^2+k(t-1)-2t]}{t} & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & \frac{2[k^2+k(t-1)-2t]}{t} \end{pmatrix} \text{and}$$

$$B = \begin{pmatrix} \frac{2[k^2+k(t-1)-2t]}{t} & -2 & \cdots & -2 & -2 \\ -2 & \frac{2[k^2+k(t-1)-2t]}{t} & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & \frac{2[k^2+k(t-1)-2t]}{t} & -2 \\ -2 & -2 & \cdots & -2 & \frac{2t}{k} \end{pmatrix}.$$

We notice that $|\lambda E - A| = 0$ implies all the eigenvalues of A are greater than 0, i.e., A is positive definite.

Since $k > k_1$, when $k - k_1 = k_2 = 1$, we get

$$B = \left(\frac{2t}{k}\right),$$

that is, B is positive definite.

When $k - k_1 = k_2 \geq 21$, we find

$$0 = |\lambda E - B|$$

$$= \Big[\lambda - \frac{2(k+t)(k-1)}{t}\Big]^{k_2-2}\Big[\lambda^2 - \Big(\frac{2t}{k} + \frac{2(k+t)(k-1) - 2(k_2-1)t}{t}\Big)\lambda + \frac{4(k+t)}{t} \times \frac{k_1 t}{k}\Big].$$

Since

$$\lambda^2 - \Big(\frac{2t}{k} + \frac{2(k+t)(k-1) - 2(k_2-1)t}{t}\Big)\lambda + \frac{4(k+t)}{t} \times \frac{k_1 t}{k} = 0,$$

then, we have

$$\lambda_{k-1} \cdot \lambda_k = \frac{4(k+t)}{k} \times \frac{k_1 t}{k} \geq 0,$$
$$\lambda_{k-1} + \lambda_k = \frac{2t}{k} + \frac{2(k+t)(k-1) + 2k_1 t}{t} > 0,$$

that is, either $\lambda_{k_1} > 0, \lambda_k \geq 0$ or $\lambda_{k_1} \geq 0, \lambda_k > 0$.

Hence, all the eigenvalues of B are ≥ 0, i.e., B is positive definite. Thus, it is clear that $\mathrm{Hess}_{f_\alpha}(U, U) \geq 0$. This implies $\Xi(U, U) \geq 0$.

Hence from (3.11), the point $(\sigma_{11}^{\alpha}, \sigma_{22}^{\alpha}, \ldots, \sigma_{kk}^{\alpha})$ is a global minimum point. Therefore using (3.11) in f_α, a lengthy but straightforward computation yields

$$f_\alpha \geq \frac{k_1 t(k^2 - k + k_2 t - t)}{(k^2 - k - t + k_2 t)^2 k_1^2 t^2}(r^\alpha)^2. \tag{3.13}$$

As $0 < t < k(k-1)$ or $t > k(k-1)$, we now split the theorem's proof into two major situations.

Case (i). $0 < t < k(k-1)$. In this case, using (3.9) in (3.13), we find

$$\begin{aligned}
\mathbb{P} &\geq \sum_\alpha \frac{k_1 t(k^2 - k + k_2 t - t)}{(k^2 - k - t + k_2 t)^2 k_1^2 t^2}(r^\alpha)^2 \\
&= \frac{k_1 t(k^2 - k + k_2 t - t)}{(k^2 - k - t + k_2 t)^2 k_1^2 t^2} \sum_\alpha (r^\alpha)^2 \\
&= \frac{k_1 t(k^2 - k + k_2 t - t)k^2}{(k^2 - k - t + k_2 t)^2 k_1^2 t^2} \|H\|^2.
\end{aligned} \tag{3.14}$$

Combining (3.6) and (3.14), we derive

$$\begin{aligned}
& tC + \frac{(k-1)(k+t)(k^2-k-t)}{kt} C(\pi) - 2\tau \\
& + 2\Big\{k_2 \frac{\Delta\zeta}{\zeta} + \frac{k_1(k_1-1)c}{8} + \frac{3}{4}c\|P^T\|^2_{N_1^{k_1}} + \frac{k_2(k_2-1)c}{8} + \frac{3}{4}c\|P^T\|^2_{N_2^{k_2}}\Big\} \\
& \geq \frac{k_1 t(k^2 - k + k_2 t - t)k^2}{(k^2 - k - t + k_2 t)^2 k_1^2 t^2} \|H\|^2.
\end{aligned} \tag{3.15}$$

Calculating the infimum of all tangent hyperplanes π of $T_x M$ in (3.15), we obtain

$$\frac{\delta_C(t; k-1)}{k(k-1)} \geq \rho + \frac{k_1 t(k^2 - k + k_2 t - t)k}{(k-1)(k^2 - k - t + k_2 t)^2 k_1^2 t^2} \|H\|^2$$

$$-\frac{2}{k(k-1)}\Big\{k_2\frac{\Delta\zeta}{\zeta}+\frac{k_1(k_1-1)c}{8}+\frac{3}{4}c\|P^T\|^2_{N_1^{k_1}}+\frac{k_2(k_2-1)c}{8}+\frac{3}{4}c\|P^T\|^2_{N_2^{k_2}}\Big\}, \tag{3.16}$$

and (3.16) instantly leads to the inequality (4.4).

Case (ii). $t > k(k-1)$. Then with the same method, it is easy to see that the following inequality holds:

$$\frac{\widehat{\delta}_C(t;k-1)}{k(k-1)} \geq \rho + \frac{k_1 t(k^2-k+k_2t-t)k}{(k-1)(k^2-k-t+k_2t)^2k_1^2t^2}\|H\|^2$$
$$-\frac{2}{k(k-1)}\Big\{k_2\frac{\Delta\zeta}{\zeta}+\frac{k_1(k_1-1)c}{8}+\frac{3}{4}c\|P^T\|^2_{N_1^{k_1}}+\frac{k_2(k_2-1)c}{8}+\frac{3}{4}c\|P^T\|^2_{N_2^{k_2}}\Big\}. \tag{3.17}$$

□

The equalities hold in (3.16) and (3.17) at a point if and only if inequalities (3.7) and (3.13) become equalities. Thus, we have

$$\begin{cases}\sigma^\alpha_{11}=\cdots=\sigma^\alpha_{k_1k_1}=\frac{k_1t^2}{(k^2-k-t-k_2t)^2+k_1^2t^2}r^\alpha\\ \sigma^\alpha_{k_1+1k_1+1}=\cdots=\sigma^\alpha_{k-1k-1}=\frac{(k^2-k+k_2t-t)t}{(k^2-k-t-k_2t)^2+k_1^2t^2}r^\alpha\\ \sigma^\alpha_{kk}=\frac{k(k-1)(k^2-k+k_2t-t)}{(k^2-k-t-k_2t)^2+k_1^2t^2}r^\alpha\\ \sigma^\alpha_{ij}=0,\quad i\neq 0.\end{cases} \tag{3.18}$$

By selecting an orthonormal basis that aligns e_{k+1} with the mean curvature vector, we obtain

$$\sigma(e_1,e_1)=\cdots=\sigma(e_{k_1},e_{k_1})=k_1t^2f_1e_{k+1}$$
$$\sigma(e_{k_1+1},e_{k_1+1})=\cdots=\sigma(e_{k-1},e_{k-1})=(k^2-k+k_2t-t)tf_1e_{k+1}$$
$$\sigma(e_k,e_k)=k(k-1)(k^2-k+k_2t-t)f_1e_{k+1}$$
$$\sigma(e_i,e_j)=0,\quad i\neq j,$$

where $f_1=\frac{r^{k+1}}{(k^2-k+k_2t-t)^2+k_1^2t^2}$ is a function of M^k.

4. Some consequences

Let ϕ be a smooth (C^∞) positive function on M^k. We define the symmetric 2-covariant tensor field, known as the Hessian of ϕ, as

$$\mathbb{H}^\phi : \mathfrak{U}(M)\times\mathfrak{U}(M)\to\mathcal{F}(M), \tag{4.1}$$

such that

$$\mathbb{H}^\phi(U,V)=\mathbb{H}^\phi_{ij}U^iV^j, \tag{4.2}$$

for any $U,V\in\mathfrak{U}(M)$, where the local components $\mathbb{H}^\phi_{ij}$ are given by

$$\mathbb{H}^\phi_{ij}=\frac{\partial^2\phi}{\partial x_i\partial x_j}-\Gamma^k_{ij}\frac{\partial\phi}{\partial x_k}. \tag{4.3}$$

In particular, let us assume $\phi=\log\zeta$.

Using this definition together with Theorem 3.2, we obtain the following result:

Corollary 4.1. *Let $\overline{M}^{2p}(c)$ be a complex space form, and let $\varphi : M^k = N_1^{k_1} \times_\zeta N_2^{k_2} \to \overline{M}^{2p}(c)$ be an isometric immersion of a warped product submanifold into $\overline{M}^{2p}(c)$. Then:*

(i) The generalized normalized δ-Casorati curvature $\delta_C(t; k-1)$ satisfies

$$\rho \leq \frac{\delta_C(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2 t-t)k}{(k-1)(k^2-k-t+k_2 t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\left\{k_2 \frac{\operatorname{tr}\mathbb{H}^\phi}{\zeta} + \frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\right)\right\} \tag{4.4}$$

for all real numbers t such that $0 < t < k(k-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_C}(t; k-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta_C}(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2 t-t)k}{(k-1)(k^2-k-t+k_2 t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\left\{k_2 \frac{\operatorname{tr}\mathbb{H}^\phi}{\zeta} + \frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\right)\right\} \tag{4.5}$$

for all real numbers $t > k(k-1)$.

Now, if the warping function ζ is harmonic, then $\Delta\zeta = 0$. Hence, the inequality simplifies as follows:

Corollary 4.2. *Let $\overline{M}^{2p}(c)$ be a complex space form, and let $\varphi : M^k = N_1^{k_1} \times_\zeta N_2^{k_2} \to \overline{M}^{2p}(c)$ be an isometric immersion of a warped product submanifold. If the warping function ζ is harmonic, then:*

(i) The GN δ-Casorati curvature $\delta_C(t; k-1)$ satisfies

$$\rho \leq \frac{\delta_C(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2 t-t)k}{(k-1)(k^2-k-t+k_2 t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\left\{\frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\right)\right\}. \tag{4.6}$$

(ii) The GN δ-Casorati curvature $\widehat{\delta_C}(t; k-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta_C}(t;k-1)}{k(k-1)} - \frac{k_1 t(k^2-k+k_2 t-t)k}{(k-1)(k^2-k-t+k_2 t)^2 k_1^2 t^2}\|H\|^2 + \frac{2}{k(k-1)}\left\{\frac{k(k-1)-2k_1k_2}{8}c + \frac{3}{4}c\left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}}\right)\right\} \tag{4.7}$$

for all real numbers $t > k(k-1)$.

Finally, if the submanifold is minimal, then the mean curvature vector $H = 0$, leading to the following conclusion:

Corollary 4.3. *Let M be a k-dimensional minimal warped product submanifold immersed in a complex space form $\overline{M}^{2p}(c)$. Then:*

(i) The GN δ-Casorati curvature $\delta_C(t; k-1)$ satisfies

$$\rho \leq \frac{\delta_C(t; k-1)}{k(k-1)} + \frac{2}{k(k-1)} \left\{ k_2 \frac{\Delta \zeta}{\zeta} + \frac{k(k-1) - 2k_1 k_2}{8} c + \frac{3}{4} c \left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}} \right) \right\}. \tag{4.8}$$

(ii) The GN δ-Casorati curvature $\widehat{\delta_C}(t; k-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta_C}(t; k-1)}{k(k-1)} + \frac{2}{k(k-1)} \left\{ k_2 \frac{\Delta \zeta}{\zeta} + \frac{k(k-1) - 2k_1 k_2}{8} c + \frac{3}{4} c \left(\|P^T\|^2_{N_1^{k_1}} + \|P^T\|^2_{N_2^{k_2}} \right) \right\} \tag{4.9}$$

for all real numbers $t > k(k-1)$.

5. Conclusions

In this study, we have established sharp geometric inequalities involving the generalized normalized δ-Casorati curvatures for warped product submanifolds in complex space forms. Utilizing constrained optimization techniques on Riemannian submanifolds, we derived upper bounds that capture the interplay between intrinsic and extrinsic curvature invariants. The characterization of equality cases adds further geometric insight and strengthens the theoretical framework. Moreover, the introduction of Hessian and harmonic function components has provided an analytical extension, linking curvature behavior to the smooth structure of warping functions.

These results not only generalize previously known inequalities but also suggest several interesting directions for future research. One open problem is to investigate whether similar optimal inequalities can be formulated for warped product submanifolds in other ambient geometries, such as nearly Kähler manifolds, quaternionic space forms, or pseudo-Riemannian settings. Another important question is whether the role of the warping function can be further quantified through function-theoretic or PDE methods, especially in the presence of additional geometric constraints such as minimality or slant structures.

Furthermore, exploring stability phenomena and rigidity results under the derived inequalities may lead to classification theorems for equality cases. Potential applications to mathematical physics, particularly in spacetimes modeled by warped products and in the geometry of field equations, represent another promising direction. The integration of techniques from geometric analysis, such as eigenvalue estimates and heat kernel methods, may also enrich the understanding of the δ-Casorati framework.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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