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Research article

Monophonic sets and rough directed topological spaces: Applications with some directed networks

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Abstract: By using the monophonic paths in the theory of directed graphs, this paper constructs a new topology, called the out mondirected topology, and characterizes the graphs that induce the indiscrete or discrete topology. We give and study some relations and properties such as the relationship between the isomorphic relation in directed graphs and the homeomorphic property in out mondirected topological spaces, compactness, $\mathbb{D}_{\pm}$-connectedness, connectedness, and $\mathbb{D}_{\pm}$-discrete properties. Finally, we apply our results of out mondirected topological spaces in the nervous system of the human body, such as in the messenger signal network, in diagrams of sensory neuron cells, and in models of two distinct nicotinic receptor types based on second messenger signal.

Keywords: directed graph; connectedness; continuous function; topology; path
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1. Introduction

The neighborhood system of vertices set or edges set in the theory of simple graphs has been a fundamental component in constructing topologies in recent studies. Several researchers have defined new classes of neighborhood systems, primarily focusing on the vertex sets or edges sets of simple graphs. Nada et al., [1], explored relationships in graph theory to establish new topologies on vertex sets of simple graphs. Amiri et al. [2] defined topology on the vertices set of simple graphs denoted by $(V, \tau_{\mathcal{A}V})$, where $\tau_{\mathcal{A}V}$ is based on the collection of neighbours $N(x)$ of points in vertices set V. For the directed graphs, the notion of pathless topological spaces on the vertices set V and its role in satisfying the connectedness of human blood circulation were introduced by [3], where it used some relations among the pathless topological spaces with E-generated subgraphs and the relative topologies. Nianga and Canoy [4] introduced a class of topologies for simple graphs via the concepts of unary and binary operations and in [5], introduced some topologies on the vertex set for simple graphs via graphical hop neighborhoods. Kiliciman and Abdu [6] describe incompatible and compatible edge topologies for directed graphs. Sari and Kopuzlu [7] used some basses introduced by [2] to generate a new class of topologies and studied the continuity of functions among these topological spaces. For the special graphs like C_n, K_n, and $K_{n,m}$, the neighborhood system of vertices and the discrete property of topologies are studied by Zomam et al. [8], who used some topologies introduced by Amiri et al. [2] to study Alexandroff space by providing some properties such as a locally finite property. Abu-Gdairi et al. [9], by using neighborhood systems, introduced and investigated some applications in the medical field by providing the role of topological visualization and rough sets. Alzubaidi et al. [10] used rough upper approximation neighborhoods to introduce some topologies. Damag et al. [11] introduced the class of out m-topological spaces for directed graphs using monophonic paths, demonstrating applications in studying some graphical properties of the human nervous system. The monophonic eccentric neighborhoods were used by [12] to introduce a way of constructing a topology on a vertex set. Atef et al. [13], introduced a new type of fuzzy topological structures in fuzzy graphs theory and studied their fundamental properties. Blake et al. [14], established some new topological spaces on the graph that represented Euler diagrams. Deka et al. [15], analyzed the graphical model of nodal voltages by using algorithms to exactly determine topology. Ahmed et al. [16], studied some fundamental properties of mathematical chemistry by using topological structures and the graphs with Atom valences and bond multiplicities are represented by vertex degrees and edge multiplicities. Timmanaikar et. al. [17], utilized graph theoretical molecular descriptors, also known as topological indices, as a numerical representation method for the chemical structures embedded in molecular graphs. Yan and Li [18], studied some topological properties with graphs concerning the decentralized gradient descent. Gamorez and Canoy [19] introduced some topologies induced by symmetric difference, Tensor product, edge corona, the corona, disjunction, and the strong product of two graphs. Shokry and Aly [20] presented some examples as applications for some topological spaces in the directed network of a human heart. In rough neighborhood systems, Yao [21] introduced the ideas of rough approximations and rough approximations of generalized rough sets by using a binary relation. Hosny et al. [22] used primal approximation spaces to explore new approaches to extending rough set theory by introducing. Atik et al. [23] used some neighborhood systems to provide approximation graphs as finite topologies. Guler [24] used the idea of an ideal family to present new generalization and different approximations and then introduced some comparisons among these generalizations. In this work, we utilize monophonic paths in simple directed graphs and a rough neighborhood system to introduce a new class of graphical neighborhood systems of vertices, termed rough approximation neighborhood systems. In Section 2, we construct a new topology on vertex sets for the class of simple directed graphs, referred to as rough directed topological spaces. We then present the discrete properties of friendship graphs F_n, generalized friendship graphs GF_n, complete bipartite graphs, and other special graphs. In Section 3, we introduce and study homeomorphically isomorphic properties, including connectedness, discreteness, monophonic discreteness, and monophonic connectedness. Section 4 discusses monophonic discreteness and connectedness within networks of the human circulatory and nervous systems. We compare our results regarding monophonic discrete properties and connectedness with those properties induced by out-mondirected topological spaces [11]. Finally, we conclude with a discussion of our findings.

A directed graph (digraph, for short) δ is a pair of a vertices set $\vartheta(\delta) \neq \emptyset$ and a directed edges set $\rho(\delta)$. The directed edge from $\alpha \in \vartheta(\delta)$ into $\beta \in \vartheta(\delta)$ is denoted by $\delta_{\alpha\beta}$ where α and β are called the initial vertex and end vertex of $\delta_{\alpha\beta}$, respectively. The directed edges $\delta_{\alpha\beta}, \delta'_{\alpha'\beta'} \in \rho(\delta)$ are called parallel if $\alpha = \alpha'$ and $\beta = \beta'$. The directed edge $\delta_{\alpha\alpha}$ is called a loop. If a graph δ does not have loops and parallel directed edges, then it is called a simple graph. The in-degree $\mathcal{D}^+_\alpha$ of $\alpha \in \vartheta(\delta)$ is the number of all arrived edges in $\rho(\delta)$ into α, and the out-degree $\mathcal{D}^-_\alpha$ of α is the number of all edges in $\rho(\delta)$ that travelled from α. The degree of α is denoted by $\mathcal{D}_\alpha$ and defined by $\mathcal{D}_\alpha = \mathcal{D}^+_\alpha + \mathcal{D}^-_\alpha$. A finite digraph δ is a digraph with finite sets $\vartheta(\delta)$ and $\rho(\delta)$. A complete digraph K_n with $n > 0$ vertices is a simple graph with $\mathcal{D}^+_\alpha = \mathcal{D}^-_\alpha = n$ for all $\alpha \in \vartheta(\delta)$. A cycle digraph C_n with $n > 0$ vertices is a simple graph with $\mathcal{D}^+_\alpha = \mathcal{D}^-_\alpha = 1$ for all $\alpha \in \vartheta(\delta)$. A friendship graph F_n with $n > 1$ is a digraph that joins n copies of the cycles C_3 with a center common vertex. A complete bipartite digraph $K_{n\to k}$ is a simple digraph that satisfies the following: (1) $\vartheta(K_{n\to k}) = V_n \cup V_k$ and $V_n \cap V_k = \emptyset$, where V_n and V_k are two sets with n vertices and k vertices, respectively; (2) every edge in $K_{n\to k}$ travels from a vertex in V_n into vertex in V_k. The open in-neighborhood $\mathcal{M}^+(\gamma)$ of $\gamma \in \vartheta(\delta)$ is the set of all vertices $\zeta \in \vartheta(\delta)$ such that there is $\delta_{\zeta\gamma} \in \rho(\delta)$, and the open out-neighborhood $\mathcal{M}^-(\gamma)$ of γ is the set of all vertices $\zeta \in \vartheta(\delta)$ such that there is $\delta_{\gamma\zeta} \in \rho(\delta)$, and the open neighborhood $N(\gamma)$ is the union set of $\mathcal{M}^-(\alpha)$ and $\mathcal{M}^+(\alpha)$. For any subset $Q \subseteq \vartheta(\delta)$, $N(Q) = \cup_{\alpha\in Q} N(\alpha)$ is called an open neighborhood of Q. A complete bipartite digraph $K_{n,k}$ is a simple digraph that satisfies: (1) $\vartheta(K_{n,k}) = V_n \cup V_k$ and $V_n \cap V_k = \emptyset$ where V_n and V_k are two sets with n vertices and k vertices, respectively; (2) $\mathcal{M}^+(\gamma) = \mathcal{M}^-(\gamma) = V_k$ and $\mathcal{M}^+(\zeta) = \mathcal{M}^-(\zeta) = V_n$ for all $\gamma \in V_n$ and $\zeta \in V_k$. Any sequence of alternative distinct on the same directed edges is a called directed path. If an undirected graph $Un(\delta)$ in digraph δ is connected, then digraph δ is called weakly connected, where $Un(\delta)$ is the graph that generated by replacing all directed edges of δ by undirected edges. A chord of a path P is a directed edge that connects two non-adjacent vertices. A monophonic path is a directed path without a chord. Recall [21] that for any subset H of a nonempty set X and for any relation $\approx$ on a set $X \neq \emptyset$, the rough approximations are given by $\underline{\approx}(H) = \{\alpha \in X : \approx_\alpha \subseteq H\}$ and $\overline{\approx}(H) = \{\alpha \in S : \approx_\alpha \cap H \neq \emptyset\}$, where $\approx_\alpha = \{\beta \in S : \alpha \approx \beta\}$. For any subgraph Q of δ, Atik et al. [23] defined the k-systems $\underline{N_k}(\vartheta(Q)) = \{\alpha \in \vartheta(\delta) : N_k(\alpha) \subseteq \vartheta(Q)\}$ and $\overline{N_k}(\vartheta(Q)) = \vartheta(Q) \cup \{\alpha \in \vartheta(\delta) : N_k(\alpha) \cap \vartheta(Q) \neq \emptyset\}$.

2. The rough monophonic system

All digraphs in our work will be assumed without isolated vertices. Let $\delta = (\vartheta(\delta), \rho(\delta))$ be any simple digraph. A subset $Q \subseteq \vartheta(\delta)$ is called a monophonic set if every vertex in $\vartheta(\delta)$ lies on monophonic paths with ends in Q. A vertex $\alpha \in \vartheta(\delta)$ is called rest monophonic if $N(\alpha)$ is a single set or $N(\alpha)$ is a monophonic set in δ. A set of all rest monophonic vertices in δ is denoted by ${}_{rm}\delta$. Define a relation $\approx$ on $\vartheta(\delta)$ by $\alpha \approx \beta$ if $\alpha \in N(\beta)$ and α is a monophonic vertex. The rest monophonic neighborhood $\mathcal{K}^{rm}_\delta(\alpha)$ of a vertex α in $\vartheta(\delta)$ is given by $\mathcal{K}^{rm}_\delta(\alpha) = \{\alpha\} \cup \{\beta \in \vartheta(\delta) : \beta \approx \alpha\}$, and $\mathcal{K}^{rm}_\delta = \{\mathcal{K}^{rm}_\delta(\alpha) : \alpha \in \vartheta(\delta)\}$ denotes the collection of all rest monophonic neighborhoods of a digraph δ. Define the operator ${}_{rm}\mathcal{H}_\delta : \vartheta(\delta) \to P(\vartheta(\delta))$ by ${}_{rm}\mathcal{H}_\delta(\alpha) = \cap^{rm}_\delta(\alpha) \cup \mathcal{K}^{rm}_\delta(\alpha)$, where $P(\vartheta(\delta))$ is a power set of $\vartheta(\delta)$ and $\cap^{rm}_\delta(\alpha)$ is the set of vertices satisfying non-empty neighborhood intersections; that is, $\cap^{rm}_\delta(\alpha) := \{\beta \in \vartheta(\delta) \setminus \{\alpha\} : \mathcal{K}^{rm}_\delta(\alpha) \cap \mathcal{K}^{rm}_\delta(\beta) \neq \emptyset\}$. The collection ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta)) = \{{}_{rm}\mathcal{H}_\delta(\alpha) :$

$\alpha \in \vartheta(\delta)\}$ is called a rough monophonic approximation neighborhood system of a digraph δ. For a subset $K \subseteq \vartheta(\delta)$, the rough monophonic approximation neighborhood $_{rm}\mathcal{H}_\delta(K)$ of K is given by $_{rm}\mathcal{H}_\delta(K) = \cap_\delta^{rm}(K) \cup \mathcal{K}_\delta^{rm}(K)$, where $\cap_\delta^{rm}(K) = \cup_{k\in K} \cap_\delta^{rm}(k)$. That is, $_{rm}\mathcal{H}_\delta(K) = \cup_{k\in K\,rm}\mathcal{H}_\delta(k)$.

Theorem 2.1. Let $\delta = (\vartheta(\delta), \rho(\delta))$ be any simple digraph. If $N \subseteq M$ in $\vartheta(\delta)$, then $_{rm}\mathcal{H}_\delta(N) \subseteq {_{rm}\mathcal{H}_\delta(M)}$.

Proof. Let $\alpha \in {_{rm}\mathcal{H}_\delta(N)}$. Then $\alpha \in \mathcal{K}_\delta^{rm}(N)$ or $\alpha \in \cap_\delta^{rm}(N)$. If $\alpha \in \mathcal{K}_\delta^{rm}(N)$ then $\alpha \in \mathcal{K}_\delta^{rm}(\beta)$ for some $\beta \in N$. Since $N \subseteq M$ then $\beta \in M$ and Hence, $\alpha \in \mathcal{K}_\delta^{rm}(\beta) \subseteq {_{rm}\mathcal{H}_\delta(M)}$. If $\alpha \in \cap_\delta^{rm}(N)$ then $\mathcal{K}_\delta^{rm}(\beta) \cap \mathcal{K}_\delta^{rm}(\alpha) \neq \emptyset$ for some $\beta \in N$. Since $N \subseteq M$ then $\beta \in M$ and Hence, $\alpha \in \cap_\delta^{rm}(M) \subseteq {_{rm}\mathcal{H}_\delta(M)}$. Hence, $_{rm}\mathcal{H}_\delta(N) \subseteq {_{rm}\mathcal{H}_\delta(M)}$. □

Theorem 2.2. Let $D, M \subseteq \vartheta(\delta)$ in a simple digraph $\delta = (\vartheta(\delta), \rho(\delta))$. Then $_{rm}\mathcal{H}_\delta(M \cup D) = {_{rm}\mathcal{H}_\delta(M)} \cup {_{rm}\mathcal{H}_\delta(D)}$.

Proof. Since $M, G \subseteq M \cup D$, then from Theorem 2.1, $_{rm}\mathcal{H}_\delta(M) \subseteq {_{rm}\mathcal{H}_\delta(M \cup D)}$ and $_{rm}\mathcal{H}_\delta(D) \subseteq {_{rm}\mathcal{H}_\delta(M \cup D)}$; that is, $_{rm}\mathcal{H}_\delta(M) \cup {_{rm}\mathcal{H}_\delta(D)} \subseteq {_{rm}\mathcal{H}_\delta(M \cup D)}$. On the other hand, let $\alpha \in {_{rm}\mathcal{H}_\delta(M \cup D)}$. Then $\alpha \in \mathcal{K}_\delta^{rm}(M \cup D)$ or $\alpha \in \cap_\delta^{rm}(M \cup D)$. If $\alpha \in \mathcal{K}_\delta^{rm}(M \cup D)$ then $\alpha \in \mathcal{K}_\delta^{rm}(\beta)$ for some $\beta \in M \cup D$. So in this case, if $\beta \in M$ then $\alpha \in \mathcal{K}_\delta^{rm}(\beta) \subseteq {_{rm}\mathcal{H}_\delta(M)}$ and similar if $\beta \in D$ then $\alpha \in \mathcal{K}_\delta^{rm}(\beta) \subseteq {_{rm}\mathcal{H}_\delta(D)}$. If $\alpha \in \cap_\delta^{rm}(M \cup D)$ then $\mathcal{K}_\delta^{rm}(\beta) \cap \mathcal{K}_\delta^{rm}(\alpha) \neq \emptyset$ for some $\beta \in M \cup D$. If $\beta \in M$ then $\alpha \in \cap_\delta^{rm}(\beta) \subseteq \mathcal{H}_\delta(M)$ and if $\beta \in D$ then $\alpha \in \cap_\delta^{rm}(\beta) \subseteq {_{rm}\mathcal{H}_\delta(D)}$. Therefore $_{rm}\mathcal{H}_\delta(M \cup D) \subseteq {_{rm}\mathcal{H}_\delta(M)} \cup {_{rm}\mathcal{H}_\delta(D)}$; that is, $_{rm}\mathcal{H}_\delta(M \cup D) = {_{rm}\mathcal{H}_\delta(M)} \cup {_{rm}\mathcal{H}_\delta(D)}$. □

For the intersection we obtain that $_{rm}\mathcal{H}_\delta(M \cap D) \subseteq {_{rm}\mathcal{H}_\delta(M)} \cap {_{rm}\mathcal{H}_\delta(D)}$, but it is no necessary to be $_{rm}\mathcal{H}_\delta(M \cap D) = {_{rm}\mathcal{H}_\delta(M)} \cap {_{rm}\mathcal{H}_\delta(D)}$; see in Figure 1(a), if we take $M = \{1'\}$ and $D = \{2'\}$; then

$$_{rm}\mathcal{H}_\delta(M \cap D) = \emptyset \neq \vartheta(\delta) \setminus \{7', 8'\} = \vartheta(\delta) \setminus \{8'\} \cap \vartheta(\delta) \setminus \{7'\} = {_{rm}\mathcal{H}_\delta(M)} \cap {_{rm}\mathcal{H}_\delta(D)}.$$

Theorem 2.3. The collection $\Gamma_\delta = \{_{rm}\mathcal{H}_\delta^\alpha : \alpha \in \vartheta(\delta)\}$ forms a basis in a simple digraph$\delta = (\vartheta(\delta), \rho(\delta))$ of a topology on $\vartheta(\delta)$, where $_{rm}\mathcal{H}_\delta^\alpha$ is the intersection of all rough monophonic approximation neighborhoods containing α.

Proof. It is easy to ge that $\cup_{\alpha\in\vartheta(\delta)\,rm}\mathcal{H}_\delta^\alpha \subseteq \vartheta(\delta)$ since $_{rm}\mathcal{H}_\delta^\alpha \subseteq \vartheta(\delta)$ for all $\alpha \in \vartheta(\delta)$. Since $\alpha \in \mathcal{K}_\delta^{rm}(\alpha)$ for all $\alpha \in \vartheta(\delta)$. Then $\alpha \in {_{rm}\mathcal{H}_\delta(\alpha)}$ for all $\alpha \in \vartheta(\delta)$. That is, $\vartheta(\delta) \subseteq \cup_{\alpha\in\vartheta(\delta)\,rm}\mathcal{H}_\delta^\alpha$. So we obtain that $\vartheta(\delta) = \cup_{\alpha\in\vartheta(\delta)\,rm}\mathcal{H}_\delta^\alpha$. Now we show that for any $_{rm}\mathcal{H}_\delta^\alpha, {_{rm}\mathcal{H}_\delta^\beta} \in \Gamma_\delta$, there is $M \subseteq \vartheta(\delta)$ such that $_{rm}\mathcal{H}_\delta^\alpha \cap {_{rm}\mathcal{H}_\delta^\beta} = \cup_{\gamma\in M\,rm}\mathcal{H}_\delta^\gamma$. Let $_{rm}\mathcal{H}_\delta^\alpha$ and $_{rm}\mathcal{H}_\delta^\beta$ be any two elements in Γ_δ. If $_{rm}\mathcal{H}_\delta^\alpha \cap {_{rm}\mathcal{H}_\delta^\beta} = \emptyset$ then take $M = \emptyset$ to obtain the desired. Let $_{rm}\mathcal{H}_\delta^\alpha \cap {_{rm}\mathcal{H}_\delta^\beta} \neq \emptyset$. Hence, there is $\gamma \in \vartheta(\delta)$ such that $\gamma \in {_{rm}\mathcal{H}_\delta^\alpha}$ and $\gamma \in {_{rm}\mathcal{H}_\delta^\beta}$. By $\gamma \in {_{rm}\mathcal{H}_\delta^\alpha}$ we obtain $\gamma \in {_{rm}\mathcal{H}_\delta(\alpha')}$ for all $\alpha' \in \vartheta(\delta)$ with $\alpha \in \mathcal{K}_\delta^{rm}(\alpha')$ or $\mathcal{K}_\delta^{rm}(\alpha') \cap \mathcal{K}_\delta^{rm}(\alpha) \neq \emptyset$. By $\gamma \in {_{rm}\mathcal{H}_\delta^\beta}$ we obtain that $\gamma \in {_{rm}\mathcal{H}_\delta(\beta')}$ for all $\beta' \in \vartheta(\delta)$ with $\alpha \in \mathcal{K}_\delta^{rm}(\beta')$ or $\mathcal{K}_\delta^{rm}(\beta') \cap \mathcal{K}_\delta^{rm}(\alpha) \neq \emptyset$. In two cases we obtain that $\gamma \in \cup_{\gamma\in_{rm}\mathcal{H}_\delta(\alpha')\cap_{rm}\mathcal{H}_\delta(\beta')\,rm}\mathcal{H}_\delta^\gamma$. Take $M = {_{rm}\mathcal{H}_\delta(\alpha')} \cap {_{rm}\mathcal{H}_\delta(\beta')} \subseteq \vartheta(\delta)$ to obtain $_{rm}\mathcal{H}_\delta^\alpha \cap {_{rm}\mathcal{H}_\delta^\beta} \subseteq \cup_{\gamma\in M\,rm}\mathcal{H}_\delta^\gamma$. On the other hand, let $\zeta \in \cup_{\gamma\in M\,rm}\mathcal{H}_\delta^\gamma$; that is, $\zeta \in \cup_{\gamma\in_{rm}\mathcal{H}_\delta(\alpha')\cap_{rm}\mathcal{H}_\delta(\beta')\,rm}\mathcal{H}_\delta^\gamma$. Hence, $\zeta \in {_{rm}\mathcal{H}_\delta^\gamma}$ for some $\zeta \in {_{rm}\mathcal{H}_\delta(\alpha')} \cap {_{rm}\mathcal{H}_\delta(\beta')}$. Then $\zeta \in \cap_{\gamma\in_{rm}\mathcal{H}_\delta(v)\,rm}\mathcal{H}_\delta(v)$, and so $\zeta \in {_{rm}\mathcal{H}_\delta(v)}$ for all $\gamma \in {_{rm}\mathcal{H}_\delta(v)}$. Then $\zeta \in {_{rm}\mathcal{H}_\delta(\alpha')} \cap {_{rm}\mathcal{H}_\delta(\beta')}$. Hence, $\zeta \in \cap_{\alpha\in_{rm}\mathcal{H}_\delta(\alpha')\,rm}\mathcal{H}_\delta(\alpha')$ and $\zeta \in \cap_{\beta\in_{rm}\mathcal{H}_\delta(\beta')\,rm}\mathcal{H}_\delta(\beta')$; that is, $\zeta \in {_{rm}\mathcal{H}_\delta^\alpha}$ and $\zeta \in {_{rm}\mathcal{H}_\delta^\beta}$. Hence, $\cup_{\gamma\in M\,rm}\mathcal{H}_\delta^\gamma \subseteq {_{rm}\mathcal{H}_\delta^\alpha} \cap {_{rm}\mathcal{H}_\delta^\beta}$. Therefore $_{rm}\mathcal{H}_\delta^\alpha \cap {_{rm}\mathcal{H}_\delta^\beta} = \cup_{\gamma\in M\,rm}\mathcal{H}_\delta^\gamma$. □

In a simple directed graph $\delta = (\vartheta(\delta), \rho(\delta))$, the basis Γ_δ in Theorem 2.3 induces a topology called a rough monophonic approximation topology (for short, rough directed topology) of a digraph δ and denoted by T_{Γ_δ}. In any simple digraph $\delta = (\vartheta(\delta), \rho(\delta))$ by the concepts of a neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ and the basis Γ_δ, the family ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ forms a subbasis for a rough directed topological space $(\vartheta(\delta), T_{\Gamma_\delta})$.

Example 2.4. In a directed graph I, Figure 1(a), the digraph $\delta = (\vartheta(\delta), \rho(\delta))$ has a monophonic vertices set ${}_{rm}\delta = \{1'\}$ and a monophonic neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ which is given by

$$ {}_{rm}\mathcal{K}_\delta(1') = \{1'\},\ {}_{rm}\mathcal{K}_\delta(2') = \{1', 2'\},\ {}_{rm}\mathcal{K}_\delta(3') = \{3'\}, $$

$$ {}_{rm}\mathcal{K}_\delta(4') = \{4'\},\ {}_{rm}\mathcal{K}_\delta(5') = \{5'\},\ {}_{rm}\mathcal{K}_\delta(6') = \{6'\},\ {}_{rm}\mathcal{K}_\delta(7') = \{7'\}. $$

For the operator $\cap_\delta^{rm}$ we have

$$ \cap_\delta^{rm}(1') = \{2'\},\ \cap_\delta^{rm}(2') = \{1'\},\ \cap_\delta^{rm}(k') = \emptyset $$

for all $k = 3, 4, 5, 6, 7$. The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ is given by

$$ {}_{rm}\mathcal{H}_\delta(1') = \{1', 2'\},\ {}_{rm}\mathcal{H}_\delta(2') = \{1', 2'\}\ \ {}_{rm}\mathcal{H}_\delta(k') = \{k'\} $$

for all $k = 3, 4, 5, 6, 7$. The basis Γ_δ is given by

$$ \Gamma_\delta = \{\{1', 2'\}, \{k'\} : k \in \{3, 4, 5, 6, 7\}\}. $$

In a directed graph II, Figure 1(b), the digraph $\delta = (\vartheta(\delta), \rho(\delta))$ has a monophonic vertices set ${}_{rm}\delta = \vartheta(\delta)$ and a monophonic neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ which is given by

$$ {}_{rm}\mathcal{K}_\delta(1') = \{1', 2', 3', 5'\},\ {}_{rm}\mathcal{K}_\delta(2') = \{1', 2', 4', 6'\},\ {}_{rm}\mathcal{K}_\delta(3') = \{1', 3', 4', 7'\}, $$

$$ {}_{rm}\mathcal{K}_\delta(4') = \{2', 3', 4', 8'\},\ {}_{rm}\mathcal{K}_\delta(5') = \{1', 5', 6', 7'\},\ {}_{rm}\mathcal{K}_\delta(6') = \{2', 5', 6', 8'\}, $$

$$ {}_{rm}\mathcal{K}_\delta(7') = \{3', 5', 7', 8'\},\ {}_{rm}\mathcal{K}_\delta(8') = \{4', 6', 7', 8'\}. $$

For the operator $\cap_\delta^{rm}$ we have

$$ \cap_\delta^{rm}(1') = \cap_\delta^{rm}(8') = \vartheta(\delta) \setminus \{1', 8'\},\ \cap_\delta^{rm}(2') = \cap_\delta^{rm}(7') = \vartheta(\delta) \setminus \{2', 7'\}, $$

$$ \cap_\delta^{rm}(3') = \cap_\delta^{rm}(6') = \vartheta(\delta) \setminus \{3', 6'\},\ \cap_\delta^{rm}(4') = \cap_\delta^{rm}(5') = \vartheta(\delta) \setminus \{4', 5'\}. $$

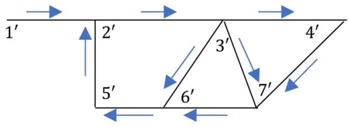
The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ is given by

$$ {}_{rm}\mathcal{H}_\delta(1') = \vartheta(\delta) \setminus \{8'\},\ {}_{rm}\mathcal{H}_\delta(2') = \vartheta(\delta) \setminus \{7'\},\ {}_{rm}\mathcal{H}_\delta(3') = \vartheta(\delta) \setminus \{6'\}, $$

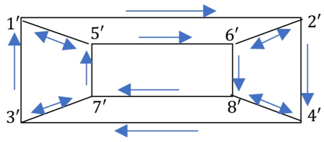
$$ {}_{rm}\mathcal{H}_\delta(4') = \vartheta(\delta) \setminus \{5'\},\ {}_{rm}\mathcal{H}_\delta(5') = \vartheta(\delta) \setminus \{4'\},\ {}_{rm}\mathcal{H}_\delta(6') = \vartheta(\delta) \setminus \{3'\}, $$

$$ {}_{rm}\mathcal{H}_\delta(7') = \vartheta(\delta) \setminus \{2'\},\ {}_{rm}\mathcal{H}_\delta(8') = \vartheta(\delta) \setminus \{1'\}. $$

The basis Γ_δ is given by $\Gamma_\delta = \{\{k'\} : k = 1, 2, 3, 4, 5, 6, 7, 8\}$. That is, the rough directed topology T_{Γ_δ} is a discrete.



(a) Directed Graph I



(b) Directed Graph II

Figure 1. Representation of a rough directed topology.

Note that ${}_{rm}\mathcal{H}_\delta(x) = \{x\}$ and ${}_{rm}\mathcal{H}_\delta(y) = \{y\}$ for any isolated edge $\delta_{xy} \in \rho(\delta)$. If a directed path $\mathcal{P}_3 : 1' \to 2' \to 3'$, then $\mathcal{P}_3$ has a rough monophonic system ${}_{rm}\mathcal{H}_\delta(\mathcal{P}_3)$ given by ${}_{rm}\mathcal{H}_{\mathcal{P}_3}(1') = \{1', 2'\}$, ${}_{rm}\mathcal{H}_{\mathcal{P}_3}(2') = \{1', 2', 3'\}$, and ${}_{rm}\mathcal{H}_{\mathcal{P}_3}(3') = \{2', 3'\}$. Similarly if we have a path $\mathcal{P}_4 : 1' \to 2' \to 3' \to 4'$, then the rough directed topological space $(\vartheta(\mathcal{P}_4), T_{\Gamma_{\mathcal{P}_4}})$ is a quasi-discrete given by $T_{\Gamma_{\mathcal{P}_4}} = \{\emptyset, \vartheta(\mathcal{P}_4), \{1', 2'\}, \{3', 4'\}\}$.

Theorem 2.5. Let $\mathcal{P}_m : 1' \to 2' \to \cdots \to m'$ be any path where $m > 4$. Then $(\vartheta(\mathcal{P}_m), T_{\Gamma_{\mathcal{P}_m}})$ has a basis

$$\Gamma_{\mathcal{P}_m} = \{\{1', 2'\}, \{(m-1)', m'\}, \{k'\} : k = 3, 4, \cdots, m-2\}.$$

Proof. Since $\mathcal{D}_{1'} = \mathcal{D}_{m'} = 1$, then $1', m' \in {}_{rm}\mathcal{P}_m$. For every $k' \in \{2', 3', \cdots, (m-1)'\}$, $(k+2)'$ does not lie on a $\alpha - \beta$ monophonic path for some $\alpha, \beta \in N(k')$. That is, ${}_{rm}\mathcal{P}_m = \{1', m'\}$. Hence, $\mathcal{P}_m$ has a monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(\vartheta(\mathcal{P}_m))$ is which is given by

$$\mathcal{K}^{rm}_{\mathcal{P}_m}(k') = \{k'\},\ \mathcal{K}^{rm}_{\mathcal{P}_m}(2') = \{1', 2'\}, \mathcal{K}^{rm}_{\mathcal{P}_m}((m-1)') = \{(m-1)', m'\}$$

for all $k = 1, 3, 4, \cdots, m-2, m$. So we obtain

$$\cap^{rm}_{\mathcal{P}_m}(1') = \{2'\}, \cap^{rm}_{\mathcal{P}_m}(2') = \{1'\},\ \cap^{rm}_{\mathcal{P}_m}(m') = \{(m-1)'\},\ \cap^{rm}_{\mathcal{P}_m}((m-1)') = \{m'\},$$

and $\cap^{rm}_{\mathcal{P}_m}(k') = \emptyset$ for all $k = 2, 3, \cdots, m-1$. That is, the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(\vartheta(\mathcal{P}_m))$ is given by

$${}_{rm}\mathcal{H}_{\mathcal{P}_m}(1') = {}_{rm}\mathcal{H}_{\mathcal{P}_m}(2') = \{1', 2'\},\ {}_{rm}\mathcal{H}_{\mathcal{P}_m}((m-1)') = {}_{rm}\mathcal{H}_{\mathcal{P}_m}(m') = \{(m-1)', m'\},$$

and ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(k') = \{k'\}$, for all $k = 1, 3, 4, \cdots, m-2, m$. Hence, from Theorem 2.3, the basis $\Gamma_{\mathcal{P}_m}$ is given by $\Gamma_{\mathcal{P}_m} = \{\{1', 2'\}, \{(m-1)', m'\}, \{k'\} : k = 3, 4, \cdots, m-2\}$. □

Theorem 2.6. For the friendship digraph F_n with $n > 1$ in the n-th directed triangular form, Figure 2(a), the rough directed topological space $(\vartheta(F_n), T_{\Gamma_{F_n}})$ is an indiscrete.

Proof. Since $N(0') = \vartheta(F_n) \setminus \{0'\}$, then $0' \in {}_{rm}F_n$. For every $m', m'' \in \vartheta(F_n) \setminus \{0'\}$, $(m+1)'$ does not lie on an $\alpha - \beta$ monophonic path for some $\alpha, \beta \in N(m')$ or $N(m'')$, respectively. That is, ${}_{rm}F_n = \{0'\}$. Hence, F_n has a monophonic neighborhood system ${}_{rm}\mathcal{H}_{F_n}(\vartheta(F_n))$ which is given by

$$\mathcal{K}^{rm}_{F_n}(0') = \{0'\},\ \mathcal{K}^{rm}_{F_n}(m') = \{0', m'\},\ \mathcal{K}^{rm}_{F_n}(m'') = \{0', m''\}$$

for all $m = 1, 2, 3, \cdots, n$. So we get $\cap^{rm}_{F_n}(0') = \vartheta(F_n) \setminus \{0'\}$, $\cap^{rm}_{F_n}(m') = \vartheta(F_n) \setminus \{m'\}$ and $\cap^{rm}_{F_n}(m'') = \vartheta(F_n) \setminus \{m''\}$ for all $m = 1, 2, 3, \cdots, n$. That is, the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{F_n}(\vartheta(F_n))$ is given by

$${}_{rm}\mathcal{H}_{F_n}(0') = {}_{rm}\mathcal{H}_{F_n}(m') = {}_{rm}\mathcal{H}_{F_n}(m'') = \vartheta(F_n)$$

for all $m = 1, 2, 3, \cdots, n$. Hence, the rough directed topological space $(\vartheta(F_n), T_{\Gamma_{F_n}})$ is an indiscrete. □

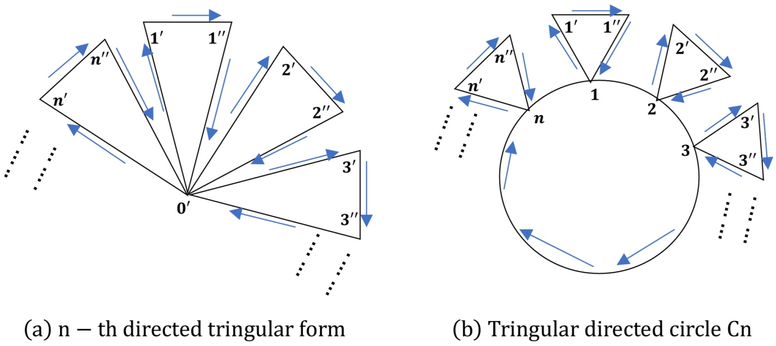


Figure 2. Friendship digraph GF_n.

By generalized friendship digraph GF_n with $n > 3$ we mean a simple digraph that can be generated by attaching n copies of the directed cycles C_3 with a directed cycle C_n as shown in the triangular directed circle C_n, Figure 2(b).

Theorem 2.7. For all $n > 0$, the rough directed topological space $(\vartheta(GF_n), T_{\Gamma_{GF_n}})$ of GF_n is discrete space.

Proof. For every $m, m', m'' \in \vartheta(GF_n)$, $(m+1)'$ does not lie on a $\alpha - \beta$ monophonic path for some $\alpha, \beta \in N(m)$ or $N(m')$ or $N(m'')$, respectively. That is, ${}_{rm}GF_n = \emptyset$. Hence, GF_n has a monophonic neighborhood system ${}_{rm}\mathcal{H}_{GF_n}(\vartheta(GF_n))$ which is given by $\mathcal{K}^{rm}_{GF_n}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(GF_n)$. So we get $\cap^{rm}_{GF_n}(\alpha) = \emptyset$ for all $\alpha \in \vartheta(GF_n)$. That is, the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{GF_n}(\vartheta(GF_n))$ is given by ${}_{rm}\mathcal{H}_{GF_n}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(GF_n)$. Hence, from Theorem 2.3, the basis Γ_{GF_n} is given by $\Gamma_{GF_n} = \{\{\alpha\} : \alpha \in \vartheta(GF_n)\}$. That is, the rough directed topological space $(\vartheta(GF_n), T_{\Gamma_{GF_n}})$ of GF_n is discrete space for all $n > 0$. □

It is clear that the rough directed topological space $(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}})$ is indiscrete if $n = m = 1$.

Theorem 2.8. Let $m > 1$ and $\vartheta(K_{1,m}) = \vartheta_1 \cup \vartheta_m$, where $\vartheta_1 = \{0\}$ and $\vartheta_m = \{1', 2', \cdots, m'\}$. Then the rough directed topological space $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ has an indiscrete property.

Proof. Since $N(0) = \vartheta(K_{1,m}) \setminus \{0\}$, then $0 \in {}_{rm}K_{1,m}$. For every $m' \in \vartheta(K_{1,m}) \setminus \{0\}$, $N(m') = \{0\}$ is a single set. That is, ${}_{rm}K_{1,m} = \vartheta(K_{1,m})$. Hence, $K_{1,m}$ has a monophonic neighborhood system ${}_{rm}\mathcal{H}_{K_{1,m}}(\vartheta(K_{1,m}))$ which is given by

$$\mathcal{K}^{rm}_{K_{1,m}}(0) = \vartheta(K_{1,m}),\ \mathcal{K}^{rm}_{K_{1,m}}(m') = \{0, m'\}$$

for all $m = 1, 2, \cdots, m$. So we get $\cap^{rm}_{K_{1,m}}(0) = \vartheta(K_{1,m}) \setminus \{0\}$ and $\cap^{rm}_{K_{1,m}}(m') = \vartheta(K_{1,m}) \setminus \{m'\}$ for all $m = 1, 2, 3, \cdots, m$. The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{K_{1,m}}(\vartheta(K_{1,m}))$ is given by ${}_{rm}\mathcal{H}_{K_{1,m}}(0) = {}_{rm}\mathcal{H}_{K_{1,m}}(m') = \vartheta(K_{1,m})$ for all $m = 1, 2, 3, \cdots, m$. Hence, the rough directed topological space $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ has an indiscrete property. □

Theorem 2.9. For all $n, m > 1$, the space $(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}})$ has an indiscrete property.

Proof. Let $\vartheta(K_{n,m}) = \vartheta_n \cup \vartheta_m$ and $\alpha \in \vartheta(K_{n,m})$ be any vertex. Since $\vartheta_n \cap \vartheta_m = \emptyset$ then $\alpha \in \vartheta_n$ or $\alpha \in \vartheta_m$. Let $\alpha \in \vartheta_n$. Then $N(\alpha) = \vartheta_m$, and we obtain that $\alpha \in {}_{rm}K_{n,m}$. Similarly if $\alpha \in \vartheta_n$ then $\alpha \in {}_{rm}K_{n,m}$. That is, ${}_{rm}K_{n,m} = \vartheta(K_{n,m})$. Hence, a monophonic neighborhood of α is given by $\mathcal{K}^{rm}_{K_{n,m}}(\alpha) = \{\alpha\} \cup \vartheta_m$ for all $\alpha \in \vartheta_n$ and $\mathcal{K}^{rm}_{K_{n,m}}(\alpha) = \{\alpha\} \cup \vartheta_n$ for all $\alpha \in \vartheta_m$. So we get $\cap^{rm}_{K_{n,m}}(\alpha) = \vartheta(K_{n,m}) \setminus \{\alpha\}$ for all $\alpha \in \vartheta(K_{n,m})$.

The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{K_{n,m}}(\vartheta(K_{n,m}))$ is given by ${}_{rm}\mathcal{H}_{K_{n,m}}(\alpha) = \vartheta(K_{n,m})$ for all $\alpha \in \vartheta(K_{n,m})$. Hence, the space $(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}})$ of $K_{n,m}$ has indiscrete property. □

It is clear that the rough directed topological space $(\vartheta(K_{n\to K}), T_{\Gamma_{K_{n\to K}}})$ is an indiscrete if $n = k = 1$.

Theorem 2.10. Let $m > 1$ and $\vartheta(K_{1\to m}) = \vartheta_1 \cup \vartheta_m$, where $\vartheta_1 = \{0\}$ and $\vartheta_m = \{1', 2', \cdots, m'\}$. Then the basis of $(\vartheta(K_{1\to m}), T_{\Gamma_{K_{1\to m}}})$ is given by

$$\Gamma_{K_{1\to m}} = \{\{0\}, \{0, j'\} : j = 1, 2, \cdots, m\}.$$

Proof. For every $j' \in \vartheta(K_{1\to m}) \setminus \{0\}$, since $N(j') = \{0\}$ is a single set then $j' \in {}_{rm}K_{1\to m}$. Since 0 does not lie on an $\alpha - \beta$ monophonic path for some $\alpha, \beta \in N(0)$ then $0 \notin {}_{rm}K_{1\to m}$. That is, ${}_{rm}K_{1\to m} = \vartheta(K_{1\to m})\setminus\{0\}$. Hence, $K_{1\to m}$ has a monophonic neighborhood system ${}_{rm}\mathcal{H}_{K_{1\to m}}(\vartheta(K_{1\to m}))$, which is given by

$$\mathcal{K}^{rm}_{K_{1\to m}}(0) = \vartheta(K_{1\to m}),\ \ \mathcal{K}^{rm}_{K_{1\to m}}(j') = \{j'\}$$

for all $j = 1, 2, \cdots, m$. So we get $\cap^{rm}_{K_{1\to m}}(0) = \vartheta(K_{1\to m}) \setminus \{0\}$ and $\cap^{rm}_{K_{1\to m}}(j') = \{0\}$ for all $j = 1, 2, 3, \cdots, m$. The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{K_{1\to m}}(\vartheta(K_{1\to m}))$ is given by ${}_{rm}\mathcal{H}_{K_{1\to m}}(0) = \vartheta(K_{1\to m})$ and ${}_{rm}\mathcal{H}_{K_{1\to m}}(j') = \{0, j'\}$ for all $j = 1, 2, \cdots, m$. Hence, from Theorem 2.3, the basis $\Gamma_{K_{1\to m}}$ is given by $\Gamma_{K_{1\to m}} = \{\{0\}, \{0, j'\} : j = 1, 2, 3, \cdots, m\}$. □

In Theorem above if we replace $K_{m\to 1}$ instead of $K_{1\to m}$ we obtain the same basis $\Gamma_{K_{1\to m}}$ and the same rough directed topological space $(\vartheta(K_{n\to m}), T_{\Gamma_{K_{n\to m}}})$.

Theorem 2.11. For all $n, m > 1$, the space $(\vartheta(K_{n\to m}), T_{\Gamma_{K_{n\to m}}})$ has discrete property.

Proof. Let $\vartheta(K_{n\to m}) = \vartheta_n \cup \vartheta_m$ and $\alpha \in \vartheta(K_{n\to m})$ be any vertex. Since $\vartheta_n \cap \vartheta_m = \emptyset$, then $\alpha \in \vartheta_n$ or $\alpha \in \vartheta_m$. Let $\alpha \in \vartheta_n$. Then $N(\alpha) = \vartheta_m$. Since α does not lie on an $\gamma - \beta$ monophonic path for some $\gamma, \beta \in \vartheta_m$, then $\alpha \notin {}_{rm}K_{n\to m}$. Similarly if $\alpha \in \vartheta_m$, then $N(\alpha) = \vartheta_n$. Since α does not lie on an $\gamma - \beta$ monophonic path for some $\gamma, \beta \in \vartheta_n$, then $\alpha \notin {}_{rm}K_{n\to m}$. That is, ${}_{rm}K_{n\to m} = \emptyset$. Hence, a monophonic neighborhood system is given by $\mathcal{K}^{rm}_{K_{n\to m}}(\alpha) = \{\alpha\}$, and so we get $\cap^{rm}_{K_{n\to m}}(\alpha) = \emptyset$ for all $\alpha \in \vartheta(K_{n\to m})$. That is, the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{K_{n\to m}}(\vartheta(K_{n\to m}))$ is given by ${}_{rm}\mathcal{H}_{K_{n\to m}}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(K_{n\to m})$. Hence, from Theorem 2.3, the basis $\Gamma_{K_{n\to m}}$ is given by $\Gamma_{K_{n\to m}} = \{\{\alpha\} : \alpha \in \vartheta(K_{n\to m})\}$. □

In the cycle digraph $C_3 : 1' \to 2' \to 3' \to 1'$, we have ${}_{rm}\vartheta(C_3) = \emptyset$. So the space $(\vartheta(C_3), T_{\Gamma_{C_3}})$ is indiscrete.

Theorem 2.12. For all $n > 3$, the space $(\vartheta(C_m), T_{\Gamma_{C_m}})$ is discrete.

Proof. Let $C_m : 1' \to 2' \to \cdots \to m' \to 1'$, where $m > 3$. Every $i' \in \vartheta(C_m)$ lies on an $m' - 2'$ monophonic path; that is, $1' \in {}_{rm}\vartheta(C_m)$. Every $i' \in \vartheta(C_m)$ lies on an $(m-1)' - 1'$ monophonic path; that is, $m' \in {}_{rm}\vartheta(C_m)$. For every Let $k' \in \vartheta(C_m) \setminus \{1', m'\}$ and for every $i' \in \vartheta(C_m)$, we obtain that i' lies on an $(k-1)' - k'$ monophonic path; that is, $k' \in {}_{rm}\vartheta(C_m)$. Hence, ${}_{rm}C_m = \vartheta(C_m)$. Hence, a monophonic neighborhood system is given by

$$\mathcal{K}^{rm}_{C_m}(1') = \{1', m', 2'\},\ \ \mathcal{K}^{rm}_{C_m}(m') = \{1', m', (m-1)'\},\ \ \mathcal{K}^{rm}_{C_m}(k') = \{(k-1)', k', (k+1)'\}$$

for all $k = 2, 3, \cdots, m-1$. So we obtain

$$\cap^{rm}_{C_m}((m-2)') = \{(m-4)', (m-3)', (m-1)', m'\},$$

$$\cap^{rm}_{C_m}((m-1)') = \{(m-3)', (m-2)', m', 1'\},$$

$$\cap^{rm}_{C_m}(m') = \{(m-2)', (m-1)', 1', 2'\},\ \ \cap^{rm}_{C_m}(1') = \{(m-1)', m', 2', 3'\},$$

$$\cap^{rm}_{C_m}(2') = \{m', 1', 3', 4\},$$

and $\cap^{rm}_{C_m}(k') = \{(k-2)', (k-1)', (k+1)', (k+2)'\}$ for all $k = 3, 4, \cdots, m-3$. Then the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{C_m}(\vartheta(C_m))$ is given by

$${}_{rm}\mathcal{H}_{C_m}((m-2)') = \{(m-4)', (m-3)', (m-2)', (m-1)', m'\},$$

$${}_{rm}\mathcal{H}_{C_m}((m-1)') = \{(m-3)', (m-2)', (m-1)', m', 1'\},$$

$${}_{rm}\mathcal{H}_{C_m}(m') = \{(m-2)', (m-1)', m', 1', 2'\},$$

$${}_{rm}\mathcal{H}_{C_m}(1') = \{(m-1)', m', 1', 2', 3'\},\ \ {}_{rm}\mathcal{H}_{C_m}(2') = \{m', 1', 2', 3', 4\},$$

and ${}_{rm}\mathcal{H}_{C_m}(k') = \{(k-2)', (k-1)', k', (k+1)', (k+2)'\}$ for all $k = 3, 4, \cdots, m-3$. Then

$${}_{rm}\mathcal{H}^{(m-2)'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(k') : k \in \{m-4, m-3, m-2, m-1, m\}\Big\} = \{(m-2)'\},$$

$${}_{rm}\mathcal{H}^{(m-1)'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(k') : k \in \{m-3, m-2, m-1, m, 1\}\Big\} = \{(m-1)'\},$$

$${}_{rm}\mathcal{H}^{m'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(k') : k \in \{m-2, m-1, m, 1, 2\}\Big\} = \{m'\},$$

$${}_{rm}\mathcal{H}^{1'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(k') : k \in \{m-1, m, 1, 2, 3\}\Big\} = \{1'\},$$

$${}_{rm}\mathcal{H}^{2'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(k') : k \in \{m, 1, 2, 3, 4\}\Big\} = \{2'\},$$

and

$${}_{rm}\mathcal{H}^{k'}_{C_m} = \cap\Big\{{}_{rm}\mathcal{H}_{C_m}(i') : i \in \{k-2, k-1, k, k+1, k+2\}\Big\} = \{k'\}$$

for all $k = 3, 4, \cdots, m-3$. Hence, from Theorem 2.3, the basis Γ_{C_m} is given by $\Gamma_{C_m} = \{\{\alpha\} : \alpha \in \vartheta(C_m)\}$. That is, the rough directed topological space $(\vartheta(C_m), T_{\Gamma_{C_m}})$ of C_m is discrete space for all $m > 3$. □

Theorem 2.13. For all $n > 0$, the space $(\vartheta(K_n), T_{\Gamma_{K_n}})$ is a discrete space.

Proof. Since $N(\alpha) = \vartheta(K_n)$ for all $\alpha \in \vartheta(K_n)$, then for every $\alpha \in \vartheta(K_n)$ dose not lie on an $\gamma - \sigma$ monophonic path in $N(\alpha)$; that is, $\alpha \notin {}_{rm}K_n$ for all $\alpha \in \vartheta(K_n)$. Then ${}_{rm}K_n = \emptyset$. Hence, a monophonic neighborhood system is given by $\mathcal{K}^{rm}_{K_n}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(K_n)$. So we get $\cap^{rm}_{K_n}(\alpha) = \emptyset$ for all $\alpha \in \vartheta(K_n)$. The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{K_n}(\vartheta(K_n))$ is given by ${}_{rm}\mathcal{H}_{K_n}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(K_n)$. Hence, from Theorem 2.3, the basis Γ_{K_n} is given by $\Gamma_{K_n} = \{\{\alpha\} : \alpha \in \vartheta(K_n)\}$. □

3. The restriction on monophonic vertices

Let $\delta = (\vartheta(\delta), \rho(\delta))$ be any simple digraph. The subdirected graph of δ induced by monophonic vertices ${}_{rm}\delta$ is denoted by $\mathbb{RM}_\delta$. A simple digraph $\delta = (\vartheta(\delta), \rho(\delta))$ is called a monophonic connected graph if $\mathbb{RM}_\delta$ has a weakly connectedness, and otherwise it is called a monophonic disconnected graph. If $\mathbb{RM}_\delta$ with topology $T_{\Gamma_\delta}|_{\vartheta(\mathbb{RM}_\delta)}$ is discrete, then we say the space $(\vartheta(\delta), T_{\Gamma_\delta})$ has a monophonic discrete property. Then we say the space $(\vartheta(\delta), T_{\Gamma_\delta})$ has a monophonic connectedness if δ is a monophonic connected and a monophonic discrete. Otherwise it is called a monophonic disconnected space. Since ${}_{rm}\vartheta(\mathcal{P}_3) = \{1', 2', 3'\}$, then $\mathcal{K}^{rm}_{\mathcal{P}_3}(1') = \{1', 2'\}$, $\mathcal{K}^{rm}_{\mathcal{P}_3}(2') = \{2'\}$, and $\mathcal{K}^{rm}_{\mathcal{P}_3}(3') = \{2', 3'\}$. So we obtain that the $(\vartheta(\mathcal{P}_3 M_e), T_{\Gamma_{\mathcal{P}_3}}|_{\vartheta(\mathcal{P}_3 M_e)})$ is not discrete; that is, $(\vartheta(\mathcal{P}_3), T_{\Gamma_{\mathcal{P}_3}})$ is not monophonic discrete.

Theorem 3.1. For all $m > 3$, the space $(\vartheta(\mathcal{P}_m), T_{\Gamma_{\mathcal{P}_m}})$ of a path $\mathcal{P}_m$ is monophonic discrete.

Proof. Let $\mathcal{P}_m : 1' \to 2' \to \cdots \to m'$ be any directed path. Let $m = 4$. Since ${}_{rm}\vartheta(\mathcal{P}_4) = \{1', 4'\}$, then we have ${}_{rm}\mathcal{P}_4(1') = {}_{rm}\mathcal{P}_4(2') = \{1', 2'\}$ and ${}_{rm}\mathcal{P}_4(3') = {}_{rm}\mathcal{P}_4(4') = \{3', 4'\}$. That is, $T_{\Gamma_{\mathcal{P}_4}} = \{\emptyset, \vartheta(\mathcal{P}_4), \{1', 2'\}, \{3', 4'\}\}$. Hence,

$$T_{\Gamma_{\mathcal{P}_4}}|_{\vartheta(\mathbb{RM}_{\mathcal{P}_4})} = \{B \cap {}_{rm}\vartheta(\mathcal{P}_4) : B \in T_{\Gamma_{\mathcal{P}_4}}\} = \{\emptyset, {}_{rm}\vartheta(\mathcal{P}_4), \{1'\}, \{4'\}\},$$

and so $T_{\Gamma_{\mathcal{P}_4}}|_{\vartheta(\mathbb{RM}_{\mathcal{P}_4})}$ is discrete; that is, the space $(\vartheta(\mathcal{P}_4), T_{\Gamma_{\mathcal{P}_4}})$ is a monophonic discrete. Let $m > 4$. Since $\mathcal{D}_{1'} = \mathcal{D}_{m'} = 1$, then $1', m' \in {}_{rm}\mathcal{P}_m$. For every $k' \in \{2', 3', \cdots, (m-1)'\}$, $(k+2)'$ does not lie on an $\alpha - \beta$ monophonic path for some $\alpha, \beta \in N(k')$. That is, ${}_{rm}\mathcal{P}_m = \{1', m'\}$. Hence, $\mathcal{P}_m$ has a monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(\vartheta(\mathcal{P}_m))$, which is given by

$$\mathcal{K}^{rm}_{\mathcal{P}_m}(k') = \{k'\},\ \mathcal{K}^{rm}_{\mathcal{P}_m}(2') = \{1', 2'\}, \mathcal{K}^{rm}_{\mathcal{P}_m}((m-1)') = \{(m-1)', m'\}$$

for all $k = 1, 3, 4, \cdots, m-2, m$. So we obtain

$$\cap^{rm}_{\mathcal{P}_m}(1') = \{2'\}, \cap^{rm}_{\mathcal{P}_m}(2') = \{1'\},\ \cap^{rm}_{\mathcal{P}_m}(m') = \{(m-1)'\},\ \cap^{rm}_{\mathcal{P}_m}((m-1)') = \{m'\},$$

and $\cap^{rm}_{\mathcal{P}_m}(k') = \emptyset$ for all $k = 2, 3, \cdots, m-1$. That is, the rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(\vartheta(\mathcal{P}_m))$ is given by

$${}_{rm}\mathcal{H}_{\mathcal{P}_m}(1') = {}_{rm}\mathcal{H}_{\mathcal{P}_m}(2') = \{1', 2'\},\ {}_{rm}\mathcal{H}_{\mathcal{P}_m}((m-1)') = {}_{rm}\mathcal{H}_{\mathcal{P}_m}(m') = \{(m-1)', m'\},$$

and ${}_{rm}\mathcal{H}_{\mathcal{P}_m}(k') = \{k'\}$, for all $k = 1, 3, 4, \cdots, m-2, m$. Hence, by Theorem 2.3, the basis $\Gamma_{\mathcal{P}_m}$ is given by $\Gamma_{\mathcal{P}_m} = \{\{1', 2'\}, \{(m-1)', m'\}, \{k'\} : k = 3, 4, \cdots, m-2\}$. Then the basis of $(\vartheta(\mathbb{RM}_{\mathcal{P}_m}), T_{\Gamma_{\mathcal{P}_m}}|_{\vartheta(\mathbb{RM}_{\mathcal{P}_m})})$ is given by

$$\Gamma_{\mathbb{RM}_{\mathcal{P}_m}} = \{B \cap \vartheta(\mathbb{RM}_{\mathcal{P}_m}) : B \in \Gamma_{\mathcal{P}_m}\} = \{\{k'\} : 2 \le k \le m-1\}.$$

Hence, $(\vartheta(\mathbb{RM}_{\mathcal{P}_m}), T_{\Gamma_{\mathcal{P}_m}}|_{\vartheta(\mathbb{RM}_{\mathcal{P}_m})})$ is discrete; that is, $(\vartheta(\mathcal{P}_m), T_{\Gamma_{\mathcal{P}_m}})$ is a monophonic discrete for all $m > 4$. □

Theorem 3.2. The space $(\vartheta(\mathcal{P}_n), T_{\Gamma_{\mathcal{P}_n}})$ of a path $\mathcal{P}_n$ is monophonic disconnected for all $n > 2$.

Proof. As shown previously $(\vartheta(\mathcal{P}_3), T_{\Gamma_{\mathcal{P}_3}})$ is not monophonic discrete, so it is monophonic disconnected. If $n > 3$, then the subgraph $\mathbb{RM}_{\mathcal{P}_n}$ of $\mathcal{P}_n$ with vertices set $\vartheta(\mathbb{RM}_{\mathcal{P}_n}) = {}_{rm}\vartheta(\mathcal{P}_n) = \{1', n'\}$ is not weakly connected. That is, the space $(\vartheta(\mathcal{P}_n), T_{\Gamma_{\mathcal{P}_n}})$ is a monophonic disconnected for all $n > 2$. □

In Figure 2(a) and by Theorem 2.6, the friendship digraph F_n has the monophonic vertices set ${}_{rm}\vartheta(F_n) = \{0'\}$. That is, F_n is a monophonic connected, and the rough directed topological space $(\vartheta(F_n), T_{\Gamma_{F_n}})$ is a monophonic connected space. In Figure 2(b) aby Theorem 2.7, the generalized friendship digraph GF_n has the monophonic vertices set ${}_{rm}\vartheta(GF_n) = \emptyset$. Hence, GF_n is a monophonic connected, and the rough directed topological space $(\vartheta(GF_n), T_{\Gamma_{GF_n}})$ is a monophonic connected space.

Theorem 3.3. For all $m > 1$, the space $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ is not monophonic discrete.

Proof. Let $\vartheta(K_{1,m}) = \vartheta_1 \cup \vartheta_m$, where $\vartheta_1 = \{0\}$, $\vartheta_m = \{1', 2', \cdots, m'\}$ and $m > 1$. By Theorem 2.8, ${}_{rm}K_{1,m} = \vartheta(K_{1,m})$ and the space $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ is indiscrete. Hence, $(\vartheta(\mathbb{RM}_{K_{1,m}}), T_{\Gamma_{K_{1,m}}}|_{\vartheta(\mathbb{RM}_{K_{1,m}})})$ is not discrete; that is, $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ is not monophonic discrete for all $m > 1$. □

Corollary 3.4. The space $(\vartheta(K_{1,m}), T_{\Gamma_{K_{1,m}}})$ is not monophonic connected for all $m > 1$.

Theorem 3.5. The space $(\vartheta(K_{1\to m}), T_{\Gamma_{K_{1\to m}}})$ is not monophonic discrete for all $m > 1$.

Proof. Let $\vartheta(K_{1\to m}) = \vartheta_1 \cup \vartheta_m$, where $\vartheta_1 = \{0\}$, $\vartheta_m = \{1', 2', \cdots, m'\}$ and $m > 1$. By Theorem 2.10, ${}_{rm}K_{1\to m} = \vartheta(K_{1\to m}) \setminus \{0\}$ and the rough directed topological space $(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}})$ is indiscrete. Hence, $(\vartheta(\mathbb{RM}_{K_{1\to m}}), T_{\Gamma_{K_{1\to m}}}|_{\vartheta(\mathbb{RM}_{K_{1\to m}})})$ is discrete; that is, $(\vartheta(K_{1\to m}), T_{\Gamma_{K_{1\to m}}})$ is a monophonic discrete for all $m > 1$. □

Theorem 3.6. The space $(\vartheta(K_{1\to m}), T_{\Gamma_{K_{1\to m}}})$ is not monophonic disconnected for all $m > 1$.

Proof. Let $\vartheta(K_{1\to m}) = \vartheta_1 \cup \vartheta_m$, where $\vartheta_1 = \{0\}$, $\vartheta_m = \{1', 2', \cdots, m'\}$, and $m > 1$. Since ${}_{rm}K_{1\to m} = \vartheta(K_{1\to m}) \setminus \{0\}$, then the subgraph $\mathbb{RM}_{K_{1\to m}}$ of $K_{1\to m}$ with vertices set $\vartheta(\mathbb{RM}_{K_{1\to m}}) = {}_{rm}\vartheta(K_{1\to m}) = \vartheta(K_{1\to m}) \setminus \{0\}$ is not weakly connected. That is, the space $(\vartheta(K_{1\to m}), T_{\Gamma_{K_{1\to m}}})$ is a monophonic disconnected for all $m > 1$. □

From Theorems 2.9, 2.11, and 2.13, ${}_{rm}K_{n,m} = {}_{rm}K_{n\to m} = {}_{rm}K_n = \emptyset$, then the rough directed topological spaces

$$(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}}),\ (\vartheta(K_{n\to m}), T_{\Gamma_{K_{n\to m}}}) \text{ and } (\vartheta(K_n), T_{\Gamma_{K_n}})$$

are monophonic connected spaces for all $n, m > 1$. Since ${}_{rm}\vartheta(C_3) = \emptyset$, then $(\vartheta(C_3), T_{\Gamma_{C_3}})$ is not monophonic discrete.

Theorem 3.7. The space $(\vartheta(C_n), T_{\Gamma_{C_n}})$ of a path C_n is a monophonic discrete for all $n > 3$.

Proof. Let $C_n : 1' \to 2' \to \cdots \to n' \to 1'$ be any directed cycle with $m > 1$. Since ${}_{rm}\vartheta(C_n) = \vartheta(C_n)$. By Theorem 2.12, for all $n > 3$, the basis of $(\vartheta(C_n), T_{\Gamma_{C_n}})$ is given by $\Gamma_{C_n} = \{\{\alpha\} : \alpha \in \vartheta(C_n)\}$. Then the basis of $(\vartheta(\mathbb{RM}_{C_n}), T_{\Gamma_{C_n}}|_{\vartheta(\mathbb{RM}_{C_n})})$ is given by

$$\Gamma_{\mathbb{RM}_{C_n}} = \{B \cap \vartheta(\mathbb{RM}_{C_n}) : B \in \Gamma_{C_n}\} = \{\{\alpha\} : \alpha \in \vartheta(\mathbb{RM}_{C_n})\}.$$

Hence, $(\vartheta(\mathbb{RM}_{C_n}), T_{\Gamma_{C_n}}|_{\vartheta(\mathbb{RM}_{C_n})})$ is discrete; that is, $(\vartheta(C_n), T_{\Gamma_{C_n}})$ is a monophonic discrete for all $n > 3$. □

Corollary 3.8. The space $(\vartheta(C_n), T_{\Gamma_{C_n}})$ of a path C_n is a monophonic connected for all $n > 3$.

Proof. The subgraph $\mathbb{RM}_{C_n}$ of C_n with vertices set $\vartheta(\mathbb{RM}_{C_n}) = {}_{rm}\vartheta(C_n) = \vartheta(C_n)$ is weakly connected. So by Theorem 3.7 the space $(\vartheta(C_n), T_{\Gamma_{C_n}})$ is a monophonic connected for all $n > 3$. □

A function $\eta : (S_1, \tau_1) \to (S_2, \tau_2)$ of a topological space (S_1, τ_1) into a topological space (S_2, τ_2) is called continuous if $\eta^{-1}(B)$ is an open set in (S_1, τ_1) for every open set B in (S_2, τ_2). A function $\eta : (S_1, \tau_1) \to (S_2, \tau_2)$ is called an open function if $\eta(B)$ is an open set in S_2 for all open sets $B \subseteq S_1$. A function $\eta : (S_1, \tau_1) \to (S_2, \tau_2)$ is a homeomorphism if it is an open function, a continuous function and a bijective function. From Theorems 2.13, 2.9, 2.11, and 2.12, since the rough directed topological spaces $(\vartheta(K_n), T_{\Gamma_{K_n}})$, $(\vartheta(K_{n,m}), T_{\Gamma_{K_{n,m}}})$, $(\vartheta(K_{n\to m}), T_{\Gamma_{K_{n\to m}}})$ and $(\vartheta(C_n), T_{\Gamma_{C_n}})$ with $n > 0$, $n, m > 1$, $m > 1$, and $n > 3$, respectively, are discrete space then any function $\eta : (\vartheta(X), T_{\Gamma_X}) \to (\vartheta(\delta), T_{\Gamma_\delta})$ is continuous function and any function $\eta : (\vartheta(\delta), T_{\Gamma_\delta}) \to (\vartheta(X), T_{\Gamma_X})$ is open for any simple digraph $\delta = (\vartheta(\delta), \rho(\delta))$, where $X := K_n, K_{n,m}, K_{n\to m}, C_n$.

Theorem 3.9. Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be two simple digraphs. Then $\psi : (\vartheta(\delta_1), T_{\Gamma_{\delta_1}}) \to (\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ is an open function if and only if ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2} \subseteq \psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}]$ for all $\alpha \in \vartheta(\delta_1)$.

Proof. Let $\alpha \in \vartheta(\delta_1)$, and suppose that ψ is an open function. Since ${}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$ is an open set in $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ containing α and ψ is open, then $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}]$ is an open set involving $\psi(\alpha)$. By Theorem 2.3, since ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ is the smallest open involving $\psi(\alpha)$, then ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2} \subseteq \psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}]$. Now suppose that ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2} \subseteq \psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}]$ for all $\alpha \in \vartheta(\delta_1)$. Let S be any open in $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ and $\beta \in \psi(S)$. Then $\psi^{-1}(\beta) \subseteq S$. Then there exists $M \in \Gamma_{\delta_1}$ such that $\psi^{-1}(\beta) \subseteq M \subseteq S$. Since M is open involving $\psi^{-1}(\beta)$ and ${}_{rm}\mathcal{H}^{\psi^{-1}(\beta)}_{\delta_1}$ is the smallest open set in $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ containing $\psi^{-1}(\beta)$, then $\psi^{-1}(\beta) \subseteq {}_{rm}\mathcal{H}^{\psi^{-1}(\beta)}_{\delta_1} \subseteq M \subseteq S$. By the hypothesis, we obtain that ${}_{rm}\mathcal{H}^{\psi(\psi^{-1}(\beta))}_{\delta_2} \subseteq \psi[{}_{rm}\mathcal{H}^{\psi^{-1}(\beta)}_{\delta_1}] \subseteq \psi(M) \subseteq \psi(S)$; that is, ${}_{rm}\mathcal{H}^{\beta}_{\delta_2} \subseteq \psi^{-1}[{}_{rm}\mathcal{H}^{\psi^{-1}(\beta)}_{\delta_2}] \subseteq \psi(S)$. Since β is arbitrary, then $\psi(S)$ is an open set. □

Theorem 3.10. A function $\psi : (\vartheta(\delta_1), T_{\Gamma_{\delta_1}}) \to (\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ is a continuous of δ_1 into δ_2 if and only if $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}] \subseteq {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ for all $\alpha \in \vartheta(\delta_1)$.

Proof. Let $\alpha \in \vartheta(\delta_1)$ and suppose that ψ is continuous. Since ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ is an open set involving $\psi(\alpha)$ and ψ is continuous, then $\psi^{-1}[{}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}]$ is an open set involving α. By Theorem 2.3, since ${}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$ is the smallest open involving α, then ${}_{rm}\mathcal{H}^{\alpha}_{\delta_1} \subseteq \psi^{-1}[{}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}]$; that is, $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}] \subseteq {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$. Conversely, suppose that $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}] \subseteq {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ for all $\alpha \in \vartheta(\delta_1)$. We will explain that ψ is continuous. Let S be any open set and $\alpha \in \psi^{-1}(S)$. Then $\psi(\alpha) \in S$. Then there exists $M \in \Gamma_{\delta_2}$ such that $\psi(\alpha) \in M \subseteq S$. Since M is an open set involving $\psi(\alpha)$, and ${}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ is the smallest open set involving $\psi(\alpha)$ then $\psi(\alpha) \in {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2} \subseteq M \subseteq S$. By the hypothesis, we obtain that $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}] \subseteq {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2} \subseteq M \subseteq S$; that is, ${}_{rm}\mathcal{H}^{\alpha}_{\delta_1} \subseteq \psi^{-1}[{}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}] \subseteq \psi^{-1}(S)$. Since ${}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$ is an open set, then $\psi^{-1}(S)$ is open set. Hence, ψ is continuous. □

Corollary 3.11. A bijective function $\psi : (\vartheta(\delta_1), T_{\Gamma_{\delta_1}}) \to (\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ is a homeomorphism of δ_1 onto δ_2 if and only if $\psi[{}_{rm}\mathcal{H}^{\alpha}_{\delta_1}] = {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$ for all $\alpha \in \vartheta(\delta_1)$.

Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be two simple graphs. We say the digraphs δ_1 and δ_2 are isomorphic and write $\delta_1 \cong \delta_2$ if there exists a bijective function $\psi : \vartheta(\delta_1) \to \vartheta(\delta_2)$ such that $(\delta_1)_{\alpha\beta} \in \rho(\delta_1)$ if and only if $(\delta_2)_{\psi(\alpha)\psi(\beta)} \in \rho(\delta_2)$ for all $\alpha, \beta \in \vartheta(\delta_1)$.

Lemma 3.12. Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be two isomorphic simple digraphs by $\psi : \vartheta(\delta_1) \to \vartheta(\delta_2)$. Then $\psi[{}_{rm}\delta_1] = {}_{rm}\delta_2$.

Proof. Let $\alpha \in \vartheta(\delta_1)$ be any monophonic in δ_1. Since $N(\psi(\alpha)) \neq \emptyset$, then $N(\psi(\alpha))$ is a single set or not a single. If $N(\psi(\alpha))$ is a single, then $\psi(\alpha)$ is a monophonic. Let $N(\psi(\alpha))$ be not single set and $\beta' \in \vartheta(\delta_2)$ be any vertex. Since ψ is a bijective, then there is $\beta \in \vartheta(\delta_1)$ such that $\psi(\beta) = \beta'$. Since $\alpha \in \vartheta(\delta_1)$ is monophonic in δ_1, then β lies on a $\gamma - \sigma$ monophonic path in δ_1 for some $\gamma, \sigma \in N(\alpha)$. By isomorphism of ψ we obtain $\psi(\gamma) - \psi(\sigma)$ monophonic path in δ_2 such that $\psi(\gamma), \psi(\sigma) \in N(\psi(\alpha))$ and $\beta' = \psi(\beta)$ lies on this path. Since β' is arbitrary in $\vartheta(\delta_2)$, then $\psi(\alpha)$ is monophonic in δ_2. Conversely, it is clear by the first part that ψ^{-1} is also an isomorphism. □

Theorem 3.13. If δ_1 is monophonic connected and $\delta_1 \cong \delta_2$, then δ_1 is monophonic and connected.

Proof. Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be any two isomorphic simple digraphs, and let δ_1 be monophonic and connected. We prove that δ_2 is a monophonic and connected. Since $\delta_1 \cong \delta_2$, then there is a bijective function $\psi : \vartheta(\delta_1) \to \vartheta(\delta_2)$ such that $(\delta_1)_{\alpha\beta} \in \rho(\delta_1)$ if and only if $(\delta_2)_{\psi(\alpha)\psi(\beta)} \in \rho(\delta_2)$ for all $\alpha, \beta \in \vartheta(\delta_1)$. Since δ_1 is monophonic connected then $\mathbb{RM}_{\delta_1}$ is weakly connected. By Lemma 3.12, $\psi[{}_{rm}\delta_1] = {}_{rm}\delta_2$. Hence, $\mathbb{RM}_{\delta_2}$ is weakly connected; that is, δ_2 is monophonic connected. □

Theorem 3.14. The monophonic discrete is a topological property.

Proof. Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be two simple digraphs and $\psi : \vartheta(\delta_1) \to \vartheta(\delta_2)$ be a homeomorphism. Let $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ be a monophonic discrete. We prove that $(\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ is a monophonic discrete. Let $x' \in \vartheta(\mathbb{RM}_{\delta_2})$ be any vertex. Since ψ is bijective, then there exists $x \in \vartheta(\mathbb{RM}_{\delta_1})$ such that $x' = \psi(x)$. Since $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ is a monophonic discrete, then $\{x\}$ is an open set in $(\vartheta(\mathbb{RM}_{\delta_1}), T_{\Gamma_{\delta_1}}|_{\vartheta(\mathbb{RM}_{\delta_1})})$. Since ψ is bijective and open, then $\psi(\{x\}) = \{\psi(x)\} = \{x'\}$ is an open set in

$$(\vartheta(\mathbb{RM}_{\delta_2}), T_{\Gamma_{\delta_2}}|_{\vartheta(\mathbb{RM}_{\delta_2})}).$$

That is, $(\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ is a monophonic discrete. □

Corollary 3.15. In the class of roughly directed topological spaces, monophonic connectedness is a homeomorphic-isomorphic property.

Proof. From Theorems 3.14 and 3.13. □

Theorem 3.16. Let $\delta_1 = (\vartheta(\delta_1), \rho(\delta_1))$ and $\delta_2 = (\vartheta(\delta_2), \rho(\delta_2))$ be two simple graphs. Then $(\vartheta(\delta_1), T_{\Gamma_{\delta_1}})$ and $(\vartheta(\delta_2), T_{\Gamma_{\delta_2}})$ are homeomorphic if δ_1 and δ_2 are isomorphic graphs.

Proof. Since δ_1 and δ_2 are isomorphic, then there exists a bijective $\psi : \vartheta(\delta_1) \to \vartheta(\delta_2)$ such that $(\delta_1)_{\alpha\beta} \in \rho(\delta_1)$ if and only if $(\delta_2)_{\psi(\alpha)\psi(\beta)} \in \rho(\delta_2)$ for all $\alpha, \beta \in \vartheta(\delta_1)$. By Corollary 3.11, it is enough to prove that $\psi[{}_{rm}\mathcal{H}^{\gamma}_{\delta_1}] = {}_{rm}\mathcal{H}^{\psi(\gamma)}_{\delta_2}$ for all $\gamma \in \vartheta(\delta_1)$. Let $\gamma \in \vartheta(\delta_1)$ be any vertex and $\gamma' \in \psi[{}_{rm}\mathcal{H}^{\gamma}_{\delta_1}]$. Since ψ is a injective, then there exists only one vertex $\zeta \in {}_{rm}\mathcal{H}^{\gamma}_{\delta_1}$ such that $\gamma' = \psi(\zeta)$. Hence, $\zeta \in {}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$ for all $\gamma \in {}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$; that is, $\zeta \in \mathcal{K}^{rm}_{\delta_1}(\alpha)$ or $\mathcal{K}^{rm}_{\delta_1}(\alpha) \cap \mathcal{K}^{rm}_{\delta_1}(\zeta) \neq \emptyset$ for all $\gamma \in {}_{rm}\mathcal{H}^{\alpha}_{\delta_1}$. By the isomorphism of δ_1 and δ_2 we obtain that $\gamma' = \psi(\zeta) \in \mathcal{K}^{rm}_{\delta_2}(\psi(\alpha))$ or $\mathcal{K}^{rm}_{\delta_2}(\psi(\alpha)) \cap \mathcal{K}^{rm}_{\delta_2}(\psi(\zeta)) \neq \emptyset$ for all $\psi(\gamma) \in {}_{rm}\mathcal{H}^{\psi(\alpha)}_{\delta_2}$. That is, $\gamma' \in {}_{rm}\mathcal{H}^{\psi(\gamma)}_{\delta_2}$ and Hence, $\psi[{}_{rm}\mathcal{H}^{\gamma}_{\delta_1}] \subseteq {}_{rm}\mathcal{H}^{\psi(\gamma)}_{\delta_2}$. For the other side, let $\gamma' \in {}_{rm}\mathcal{H}^{\psi(\gamma)}_{\delta_2}$. Since ψ is injective, then there is only one vertex $\zeta \in {}_{rm}\mathcal{H}^{\gamma}_{\delta_1}$ such that $\gamma' = \psi(\zeta)$. Hence, $\gamma' \in {}_{rm}\mathcal{H}^{\beta}_{\delta_2}$ for all $\psi(\gamma) \in {}_{rm}\mathcal{H}^{\beta}_{\delta_2}$; that is, $\gamma' \in \mathcal{K}^{rm}_{\delta_2}(\beta)$ or $\mathcal{K}^{rm}_{\delta_2}(\beta) \cap \mathcal{K}^{rm}_{\delta_2}(\gamma') \neq \emptyset$ for all $\psi(\gamma) \in {}_{rm}\mathcal{H}^{\beta}_{\delta_2}$. Since ψ is injective, then there is only one vertex $\sigma \in {}_{rm}\mathcal{H}^{\gamma}_{\delta_1}$ such that $\beta = \psi(\sigma)$. By the condition of isomorphism of δ_1

and δ_2, we obtain that $\zeta \in \mathcal{K}^{rm}_{\delta_1}(\sigma)$ or $\mathcal{K}^{rm}_{\delta_1}(\sigma) \cap \mathcal{K}^{rm}_{\delta_1}(\zeta) \neq \emptyset$ for all $\gamma \in {}_{rm}\mathcal{H}^{\sigma}_{\delta_1}$. That is, $\zeta \in {}_{rm}\mathcal{H}^{\gamma}_{\delta_1}$, and Hence, $\gamma' = \psi(\zeta) \in \psi[{}_{rm}\mathcal{H}^{\gamma}_{\delta_1}]$. So we obtain ${}_{rm}\mathcal{H}^{\psi(\gamma)}_{\delta_2} \subseteq \psi[{}_{rm}\mathcal{H}^{\gamma}_{\delta_1}]$. □

The digraphs K_n and C_n are not isomorphic for all $n > 3$, while by Theorems 2.12 and 2.13, the graphical topological spaces $(\vartheta(K_n), T_{\Gamma_{K_n}})$ and $(\vartheta(C_n), T_{\Gamma_{C_n}})$ are discrete spaces for all $n \geq 3$, and Hence, they $(\vartheta(K_n), T_{\Gamma_{K_n}})$ and $(\vartheta(C_n), T_{\Gamma_{C_n}})$ are homeomorphic.

Theorem 3.17. A simple digraph $\delta = (\vartheta(\delta), \rho(\delta))$ is a weakly connected digraph if the rough directed topological space $(\vartheta(\delta), T_{\Gamma_\delta})$ is a connected space.

Proof. Suppose that $\delta = (\vartheta(\delta), \rho(\delta))$ is disconnected. Then the undirected graph $Un(\delta)$ is disconnected. Then we can put $\eta := \{\eta_n : n \in M\}$ as the family of all components in $Un(\delta)$ where $\eta_n = (\vartheta(\eta_n), \rho(\eta_n))$ for all $n \in M$. Now for all $n \in M$, $\vartheta(\eta_n) = \cup_{x \in \vartheta(\eta_n) rm}\mathcal{H}_\delta(x)$. Then $M := \vartheta(\eta_{n_o})$ is a nonempty proper open subset of $\vartheta(\delta)$ where $n_o \in M$. Then $[\vartheta(\eta_n)]^c = \cup_{n \in M \setminus \{n_o\}} \vartheta(\eta_n)$ is also a nonempty proper open subset of $\vartheta(\delta)$. That is, $(\vartheta(\delta), T_{\Gamma_\delta})$ is a disconnected space, and this is a contradiction. □

By Theorem 2.13, the rough directed topological space $(\vartheta(K_n), T_{\Gamma_{K_n}})$ is disconnected while K_n is a connected graph.

4. Some applications

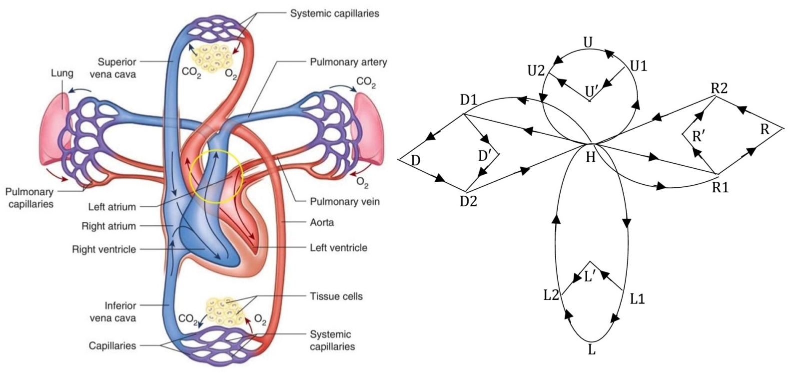
In this section, we examine and verify satisfying the monophonic discrete and monophonic connectedness properties in the general graphical networks concerning the human circulatory and nervous systems in the human body. Furthermore, we will provide some comparisons of these properties as influenced by out-directed topological spaces [11].

For the human circulatory system, consider Figure 3(a) [25] as the representation network. In the corresponding directed graph $\delta = (\vartheta(\delta), \rho(\delta))$, in Figure 3(b), every vertex $\alpha \in \vartheta(\delta)$ is rest monophonic with monophonic set $N(\alpha)$; that is, ${}_{rm}\delta = \vartheta(\delta)$. The monophonic neighborhood system is given by

$$\mathcal{K}^{rm}_\delta(H) = \vartheta(\delta), \ \mathcal{K}^{rm}_\delta(K) = \{H, K, K1, K2\}, \ \mathcal{K}^{rm}_\delta(K') = \{H, K', K1, K2\},$$
$$\mathcal{K}^{rm}_\delta(K1) = \mathcal{K}^{rm}_\delta(K2) = \{H, K, K1, K2, K'\},$$

for $K = U, D, L, R$. So we obtain

$$\cap^{rm}_\delta(K) = \vartheta(\delta) \setminus \{K\}, \ \text{for all } K \in \vartheta(\delta).$$



(a) Human circulatory system (b) General network of circularity system

Figure 3. Human circulatory system and its network.

The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ is given by

$${}_{rm}\mathcal{H}_\delta(K) = \vartheta(\delta), \text{ for all } K \in \vartheta(\delta).$$

The rough space $(\vartheta(\delta), T_{\Gamma_\delta})$ of δ is discrete space and connected space. The basis of $(\vartheta(\mathbb{RM}_\delta), T_{\Gamma_\delta}|_{\vartheta(\mathbb{RM}_\delta)})$ is given by

$$\Gamma_{\mathbb{RM}_\delta} = \{B \cap \vartheta(\mathbb{RM}_\delta) : B \in \Gamma_\delta\} = \{\vartheta(\delta)\}.$$

Hence, $(\vartheta(\mathbb{RM}_\delta), T_{\Gamma_\delta}|_{\vartheta(\mathbb{RM}_\delta)})$ is indiscrete; that is, $(\vartheta(\delta), T_{\Gamma_\delta})$ is not monophonic discrete, and δ is monophonic connected.

For the human nervous network, consider Figure 4(b) as the representation network. In the corresponding directed graph $\delta = (\vartheta(\delta), \rho(\delta))$, Figure 4(b), the open neighborhoods of vertices $c_1, c_2, m_1, m_2, m_3, m_4, m_5$ are single sets; that is, $c_1, c_2, m_1, m_2, m_3, m_4, m_5$ are rest monophonic vertices, and the other vertices are not rest monophonic as we easily check. Hence, ${}_{rm}\delta = \{c_1, c_2, m_1, m_2, m_3, m_4, m_5\}$. The monophonic neighborhood system is given by

$$\mathcal{K}_\delta^{rm}(v_1) = \{v_1, c_1, c_2\},\ \mathcal{K}_\delta^{rm}(v_4) = \{v_4, m_1, m_2\},\ \mathcal{K}_\delta^{rm}(v_5) = \{v_5, m_3, m_4\},$$

$\mathcal{K}_\delta^{rm}(t_3) = \{t_3, m_5\}$ and $\mathcal{K}_\delta^{rm}(\alpha) = \{\alpha\}$ for all $\alpha \in \vartheta(\delta) \setminus \{v_1, v_4, v_5, t_3\}$. So we obtain

$$\cap_\delta^{rm}(v_1) = \{c_1, c_2\},\ \cap_\delta^{rm}(c_1) = \cap_\delta^{rm}(c_2) = \{v_1,\ \cap_\delta^{rm}(t_3) = \{m_5\},\ \cap_\delta^{rm}(v_4) = \{m_1, m_2\},$$

$$\cap_\delta^{rm}(v_5) = \{m_3, m_4\},\ \cap_\delta^{rm}(m_1) = \cap_\delta^{rm}(m_2) = \{v_4\},\ \cap_\delta^{rm}(m_3) = \cap_\delta^{rm}(m_4) = \{v_5\},$$

$\cap_\delta^{rm}(m_5) = \{t_3\}$ and $\cap_\delta^{rm}(\alpha) = \emptyset$ for all $\alpha \in \{b, s_1, s_2, s_3, t_1, t_2, v_2, v_3\}$. The rough monophonic approximation neighborhood system ${}_{rm}\mathcal{H}_\delta(\vartheta(\delta))$ is given by

$${}_{rm}\mathcal{H}_\delta(v_1) = \{v_1, c_1, c_2\},\ {}_{rm}\mathcal{H}_\delta(c_1) = \{c_1, v_1\},\ {}_{rm}\mathcal{H}_\delta(c_2) = \{c_2, v_1\},$$

$${}_{rm}\mathcal{H}_\delta(t_3) = \{t_3, m_5\},\ {}_{rm}\mathcal{H}_\delta(v_4) = \{v_4, m_1, m_2\},\ {}_{rm}\mathcal{H}_\delta(v_5) = \{v_5, m_3, m_4\},$$

$${}_{rm}\mathcal{H}_\delta(m_1) = \{m_1, v_4\},\ {}_{rm}\mathcal{H}_\delta(m_2) = \{m_2, v_4\},\ {}_{rm}\mathcal{H}_\delta(m_3) = \{m_3, v_5\},$$

$${}_{rm}\mathcal{H}_\delta(m_4) = \{m_4, v_5\},\ {}_{rm}\mathcal{H}_\delta(m_5) = \{m_5, t_3\},$$

and ${}_{rm}\mathcal{H}_\delta(\alpha) = \{\alpha\}$ for all $\alpha \in \{b, s_1, s_2, s_3, t_1, t_2, v_2, v_3\}$. The basis Γ_δ is given by

$$\Gamma_\delta = \{\{\alpha\}, \{v_1, c_1\}, \{v_1, c_2\}, \{t_3, m_5\}, \{m_1, v_4\}, \{m_2, v_4\}, \{m_3, v_5\}, \{m_4, v_5\} : \alpha \in J\},$$

where $J = \vartheta(\delta) \setminus \{c_1, c_2, t_3, m_1, m_2, m_3, m_4, m_5\}$. The rough directed topological space $(\vartheta(\delta), T_{\Gamma_\delta})$ of δ is not a discrete space and is disconnected by the closed-open set $\{m_1, m_2, v_4\}$. The basis of $(\vartheta(\mathbb{RM}_\delta), T_{\Gamma_\delta}|_{\vartheta(\mathbb{RM}_\delta)})$ is given by

$$\Gamma_{\mathbb{RM}_\delta} = \{B \cap \vartheta(\mathbb{RM}_\delta) : B \in \Gamma_\delta\} = \{\{\alpha\} : \alpha \in \vartheta(\mathbb{RM}_\delta)\}.$$

Hence, $(\vartheta(\mathbb{RM}_\delta), T_{\Gamma_\delta}|_{\vartheta(\mathbb{RM}_\delta)})$ is discrete; that is, $(\vartheta(\delta), T_{\Gamma_\delta})$ is a monophonic discrete while the out mondirected topological space [11] does not have the monophonic discrete property. The graph δ is not monophonic connected, and so $(\vartheta(\delta), T_{\Gamma_\delta})$ is a monophonic disconnection. For more details, see Table 1, while the out mondirected topological space in [11] is a monophonic connected space.

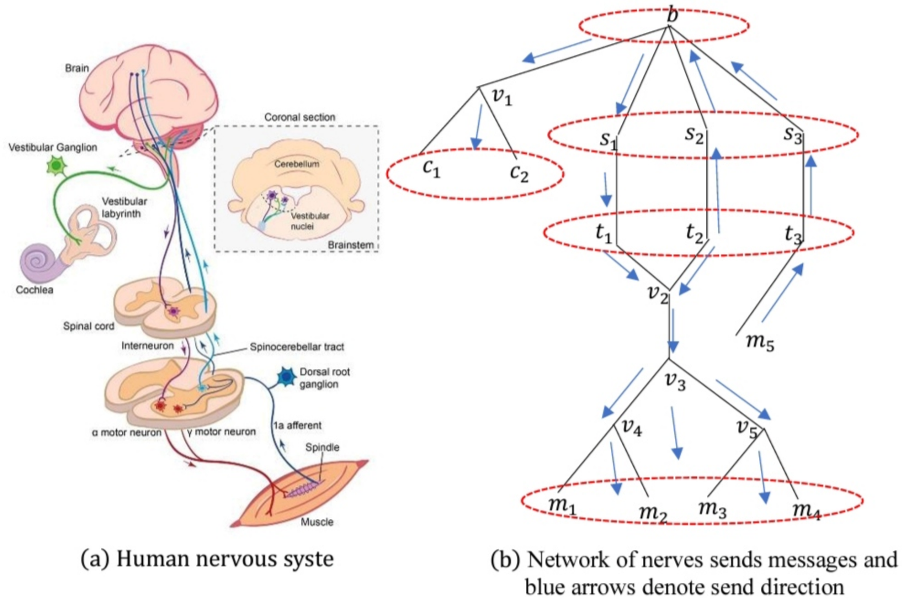


Figure 4. Somatosensory and vestibular sensory pathways.

Table 1. A rough monophonic approximation neighborhood system in Figure 4(b). $x_{ijk\cdots}$ denotes the points $x_i, x_j, x_k, \cdots$ for short.

α	$\mathcal{K}_\delta^{rm}(\alpha)$	$\cap_\delta^{rm}(\alpha)$	${}_{rm}\mathcal{H}_\delta(\alpha)$	${}_{rm}\mathcal{H}_\delta^\alpha$	$\mathcal{M}^e(\alpha)$ [11]
b	$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\{t_2, v_{345}, m_{1234}, c_{12}\}$
s_1	$\{s_1\}$	$\emptyset$	$\{s_1\}$	$\{s_1\}$	$\{t_2, v_{345}, m_{1234}\}$
s_2	$\{s_2\}$	$\emptyset$	$\{s_2\}$	$\{s_2\}$	$\{t_2, v_{1345}, m_{1234}, c_{12}\}$
s_3	$\{s_3\}$	$\emptyset$	$\{s_3\}$	$\{s_3\}$	$\{c_{12}, m_{1234}, v_{1345}\}$
t_1	$\{t_1\}$	$\emptyset$	$\{t_1\}$	$\{t_1\}$	$\{t_2, v_{345}, m_{1234}\}$
t_2	$\{t_2\}$	$\emptyset$	$\{t_2\}$	$\{t_2\}$	$\{m_{1234}, t_1, b, s_1, v_{1345}, c_{12}\}$
v_2	$\{v_2\}$	$\emptyset$	$\{v_2\}$	$\{v_2\}$	$\{m_{1234}, v_{45}\}$
v_3	$\{v_3\}$	$\emptyset$	$\{v_3\}$	$\{v_3\}$	$\{m_{1234}, t_2\}$
v_1	$\{v_1, c_1, c_2\}$	$\{c_1, c_2\}$	$\{v_1, c_1, c_2\}$	$\{v_1\}$	$\{t_{23}, s_{23}, m_5\}$
c_1	$\{c_1\}$	$\{v_1\}$	$\{v_1, c_1\}$	$\{v_1, c_1\}$	$\{b, t_{23}, s_{23}, m_5\}$
c_2	$\{c_2\}$	$\{v_1\}$	$\{v_1, c_2\}$	$\{v_1, c_2\}$	$\{b, t_{23}, s_{23}, m_5\}$
t_3	$\{t_3, m_5\}$	$\{m_5\}$	$\{t_3, m_5\}$	$\{t_3, m_5\}$	$\{c_{12}, v_{1345}, m_{1234}\}$
v_4	$\{v_4, m_1, m_2\}$	$\{m_1, m_2\}$	$\{v_4, m_1, m_2\}$	$\{v_4\}$	$\{b, t_{123}, s_{123}, v_2, m_5\}$
v_5	$\{v_5, m_3, m_4\}$	$\{m_3, m_4\}$	$\{v_5, m_3, m_4\}$	$\{v_5\}$	$\{b, t_{123}, s_{123}, v_2, m_5\}$
m_1	$\{m_1\}$	$\{v_4\}$	$\{m_1, v_4\}$	$\{v_4, m_1\}$	$\{b, m_5, s_{123}, t_{123}, v_{23}\}$
m_2	$\{m_2\}$	$\{v_4\}$	$\{m_2, v_4\}$	$\{v_4, m_2\}$	$\{b, m_5, t_{123}, v_{23}, s_{123}\}$
m_3	$\{m_3\}$	$\{v_5\}$	$\{m_3, v_5\}$	$\{v_5, m_3\}$	$\{b, m_5, t_{123}, v_{23}, s_{123}\}$
m_4	$\{m_4\}$	$\{v_5\}$	$\{m_4, v_5\}$	$\{v_5, m_4\}$	$\{b, m_5, t_{123}, v_{23}, s_{123}\}$
m_5	$\{m_5\}$	$\{t_3\}$	$\{t_3, m_5\}$	$\{t_3, m_5\}$	$\{c_{12}, v_{1345}, m_{1234}\}$

5. Conclusions

The class of monophonic paths is an important class in graph theory. The monophonic paths are used together with rough approximation sets to define a rough approximation neighborhood system, and next this system played as important role in introducing the notion of rough directed topological spaces. We noticed that there are analogical correspondences between the graphical properties and topological properties, such as the connectedness and discrete properties. Also, there are analogical correspondence between the graphical relations, and topological relations such as the isomorphism properties. For our applications, we noticed that the human circulatory network $(\vartheta(\delta), T_{\Gamma_\delta})$ in Figures 3(a),(b) does not have a monophonic discrete property and has a monophonic connectedness. For the nervous network in Figures 4(a),(b), which are taken from [11], we got that $(\vartheta(\delta), T_{\Gamma_\delta})$ has the monophonic discrete property and does not have the monophonic connectedness. For future research, we suggest generating a rough approximation system via the closeness property and via the concept of monophonic eccentric vertices to study the monophonic connectedness and monophonic discrete properties.

Author contributions

Faten H. Damag: Conceptualization, methodology, validation, investigation, resources; Amin Saif: Conceptualization, methodology, validation, investigation, writing-original draft; Adem Kiliçman: Methodology, validation, investigation, resources, writing-original draft, writing-review and editing; Fozaiyah Alhubairah: Resources, writing-original draft, writing-review and editing; Khaled M. Saad: Validation, resources, writing-original draft; Ekram E. Ali: Validation, resources; Mouataz Billah Mesmouli: Validation, resources, writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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