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Research article

Inference of $P(X < Y)$ for two-parameter exponential-Rayleigh distribution with applications

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Abstract: Let X and Y follow independent exponential-Rayleigh distributions when the shape and scale parameters are different. In this paper, the maximum likelihood and the Bayes estimates of the stress-strength parameter $\delta = P(X < Y)$ are derived. Based on the sampling technique, we use the asymptotic distribution and the Bayes estimate of δ to construct the corresponding confidence and credible intervals. Analyses of two data sets, one simulated data and the other real-life data, are given for illustrative purposes. Finally, Monte Carlo simulations are used to compare the different methods discussed here.

Keywords: asymptotic distributions; Bayesian estimation; confidence and credible intervals; maximum likelihood estimator; stress-strength model
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1. Introduction

The problem of estimating the stress-strength parameter has received continuous interest from researchers. This is because it has many applications, especially in the medical sciences, social sciences, military, engineering, mechanical systems, and many other fields. More applications can be found in Bamber [1], Hall [2], and Kotz *et al*. [3]. If $X < Y$ and Y is the strength of a system under stress X, then a measure of system performance, denoted by $\delta = P(X < Y)$, arises naturally in mechanical reliability systems. When the applied stress exceeds the system's strength, the system fails (or is out of control). For some of the recent studies about the stress-strength model, the readers are referred to Chiang *et al*. [4], Kotb and Raqab [5], Lio and Tsai [6], and Saraoglua *et al*. [7].

Here, we introduce the two-parameter exponential-Rayleigh distribution (denoted ER(θ, β)), which has the following cumulative distribution and density functions:

$$F(x;\theta,\beta) = 1 - R_{E}(x;\theta)R_{R}(x;\beta) = 1 - \exp\left(-\theta x - \frac{x^2}{2\beta^2}\right), \quad x \geq 0,$$

and

$$f(x;\theta,\beta) = \left(\theta + \frac{x}{\beta^2}\right)\exp\left(-\theta x - \frac{x^2}{2\beta^2}\right), \quad x \geq 0,$$

respectively, where $R_{E}(x;\theta)$ and $R_{R}(x;\beta)$ are survival functions of the exponential and Rayleigh distributions, and $\theta > 0$ and $\beta > 0$ are scale parameters. The hazard (instantaneous failure rate) function $h(t)$ at mission time t for the ER(θ,β) is $h(t) = \theta + t/\beta^2$. The mean and variance of ER(θ,β) can be written as follows:

$$\mu = E(X) = \sqrt{\frac{\pi}{2}}\beta\Lambda(\theta,\beta) \quad \text{and} \quad \mathrm{Var}(X) = 2\beta^2 - \sqrt{2\pi}\theta\beta^3\Lambda(\theta,\beta) - \frac{\pi}{2}\beta^2\Lambda^2(\theta,\beta),$$

where Erfc $[t]$ is the complementary error function,

$$\Lambda(\theta,\beta) = \mathrm{Erfc}\left[\frac{\theta\beta}{\sqrt{2}}\right]\exp\left(\frac{1}{2}\theta^2\beta^2\right) \quad \text{and} \quad \mathrm{Erfc}\,[t] = 1 - \frac{2}{\sqrt{\pi}}\int_0^t \exp\left(-u^2\right)\mathrm{d}u. \tag{1.1}$$

It can be verified that

$$\lim_{x\to 0} f(x) = \theta, \quad \lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to 0} h(x) = \theta \quad \text{and} \quad \lim_{x\to\infty} h(x) = \infty.$$

The ER(θ,β) model's convenient structure makes it an excellent tool for studying many types of lifetime data. More studies on the ER model can be found in Kotb and Al Omari [8]. We can obtain several well-known models from the ER(θ,β) model. Clearly, the exponential (E(θ)), Rayleigh (R(β)), and linear exponential (LE(θ,β)) distributions are special cases. The LE$(\theta,\beta)\left(\equiv ER(\theta, 1/\sqrt{\beta})\right)$ model has many applications in reliability analysis and applied statistics. Several aspects of the two-parameter LE model have been studied by Al-khedhairi [9]. For some general references to the LE model, the readers are referred to Mahmoud *et al.* [10] and Mohie El-Din *et al.* [11].

The main aim of our paper is to consider the estimation problems of $\delta = P(X < Y)$, where $X \sim ER(\theta_1,\beta_1)$ and $Y \sim ER(\theta_2,\beta_2)$, and they are independently distributed. One of the very common problems in the statistical literature is the estimation of δ. To the best of our knowledge, no literature has yet been written about this problem. Mainly, this paper aims to develop some estimates for δ based on the maximum likelihood (ML) and Bayesian approaches. We compare the performances of the different methods using a Monte Carlo simulation study.

The article unfolds as follows: In Section 2, we derive the ML estimators of δ. In this section, the asymptotic distribution, approximate and bootstrap confidence intervals (CIs) of δ are presented. In Section 3, the Bayes estimator (BE) and highest posterior density (HPD) credible interval (CrI) of δ are presented. In Section 4, a real-life data set and a simulation data set are analyzed to illustrate our procedures. Section 5 includes a Monte Carlo simulation to illustrate and compare the different proposed methods. Finally, we conclude the paper in Section 6.

2. ML estimator of δ and different CIs

Here, under the assumption that $X \sim \mathrm{ER}(\theta_1,\beta_1)$, $Y \sim \mathrm{ER}(\theta_2,\beta_2)$, and X and Y are independently distributed, the stress-strength reliability of this system is obtained as

$$\begin{aligned}\delta &= \frac{1}{1+(\beta_2/\beta_1)^2} - \sqrt{\frac{\pi}{2}}\frac{\beta_1\beta_2\left(\theta_1\beta_1^2-\theta_2\beta_2^2\right)}{\left(\beta_1^2+\beta_2^2\right)^{3/2}} \\ &\times\ Erfc\left[\frac{(\theta_1+\theta_2)\beta_1\beta_2}{\sqrt{2\left(\beta_1^2+\beta_2^2\right)}}\right]\exp\left[\frac{(\theta_1+\theta_2)^2\beta_1^2\beta_2^2}{2\left(\beta_1^2+\beta_2^2\right)}\right] = \lambda\left(\theta_1,\beta_1,\theta_2,\beta_2\right),\ \text{say},\end{aligned} \tag{2.1}$$

where Erfc$[z]$ gives the complementary error function. To compute the estimate of the stress-strength reliability parameter $\delta = P(X < Y)$, it is required to obtain the estimators of θ_ν and $\beta_\nu (\nu = 1,2)$. Suppose $\mathbf{x} = (X_1, X_2, \ldots, X_n)$ is a random sample from $\mathrm{ER}(\theta_1,\beta_1)$ and $\mathbf{y} = (Y_1, Y_2, \ldots, Y_m)$ is another independent random sample from $\mathrm{ER}(\theta_2,\beta_2)$. Based on $\mathbf{x}$ and $\mathbf{y}$, the log −likelihood function (LF) of $\vartheta = (\theta_1,\beta_1,\theta_2,\beta_2)$ is given by

$$\begin{aligned}\ell(\vartheta|\mathbf{x},\mathbf{y}) &= \sum_{i=1}^{n}\ln\left(\theta_1+\frac{x_i}{\beta_1^2}\right)+\sum_{j=1}^{m}\ln\left(\theta_2+\frac{y_j}{\beta_2^2}\right) \\ &- \theta_1\phi_1-\theta_2\phi_1^*-\frac{1}{2\beta_1^2}\phi_2-\frac{1}{2\beta_2^2}\phi_2^*,\end{aligned}$$

where

$$\phi_r = \sum_{i=1}^{n} x_i^r \quad \text{and} \quad \phi_r^* = \sum_{j=1}^{m} y_j^r. \tag{2.2}$$

The MLEs of θ_ν and β_ν can be obtained by maximizing the log −LF with respect to θ_ν and β_ν. By differentiating the log −LF with respect to θ_ν and β_ν, we immediately have the following non-linear equations:

$$\sum_{i=1}^{n}\frac{1}{\theta_1+x_i/\beta_1^2}-\phi_1=0, \quad \sum_{i=1}^{n}\frac{2x_i}{\theta_1+x_i/\beta_1^2}-\phi_2=0, \tag{2.3}$$

$$\sum_{j=1}^{m}\frac{1}{\theta_2+y_j/\beta_2^2}-\phi_1^*=0, \text{ and } \sum_{j=1}^{m}\frac{2y_j}{\theta_2+y_j/\beta_2^2}-\phi_2^*=0. \tag{2.4}$$

By setting $\rho_\nu = \theta_\nu\beta_\nu^2$, $\rho_\nu > 0$ and then eliminating (θ_1,β_1) from (2.3) and (θ_2,β_2) from (2.4), we obtain the following two non-linear equations, which are free of θ_ν and β_ν $(\nu = 1,2)$:

$$\left(\sum_{i=1}^{n}\frac{2x_i}{\rho_1+x_i}\right)\Big/\left(\sum_{i=1}^{n}\frac{1}{\rho_1+x_i}\right)=\frac{\phi_2}{\phi_1},$$

and

$$\left(\sum_{j=1}^{m}\frac{2y_j}{\rho_2+y_j}\right)\Big/\left(\sum_{j=1}^{m}\frac{1}{\rho_2+y_j}\right)=\frac{\phi_2^*}{\phi_1^*}.$$

After some algebra, the maximum of $\ell(\vartheta|\mathbf{x},\mathbf{y})$ can be obtained as a fixed point solution of:

$$\rho_1 = n/\left(\sum_{i=1}^{n}\frac{1}{\rho_1 + x_i}\right) - \frac{\phi_2}{2\phi_1} = \varphi_1\left(\rho_1;\mathbf{x},\mathbf{y}\right), \quad \text{say}, \tag{2.5}$$

and

$$\rho_2 = m/\left(\sum_{j=1}^{m}\frac{1}{\rho_2 + y_j}\right) - \frac{\phi_2^*}{2\phi_1^*} = \varphi_2\left(\rho_2;\mathbf{x},\mathbf{y}\right), \quad \text{say}. \tag{2.6}$$

It can be easily shown that $\varphi_1\left(\rho_1;\mathbf{x},\mathbf{y}\right)$ and $\varphi_2\left(\rho_2;\mathbf{x},\mathbf{y}\right)$ are unimodal functions of ρ_1 and ρ_2, respectively. The solutions of Eqs (2.5) and (2.6) can be obtained by using a simple iterative procedure as: $\varphi_\nu\left(\rho_\nu^{(\ell)};\mathbf{x},\mathbf{y}\right) = \rho_\nu^{(\ell+1)}$, where $\rho_\nu^{(\ell)}$ is the value of ρ_ν, $\nu = 1,2$ in the ℓth iterate of $\tilde{\rho}_\nu$. For obtaining the roots, the iteration procedure should be stopped when $|\rho_\nu^{(\ell+1)} - \rho_\nu^{(\ell)}|$ is sufficiently small. Once the MLE $\tilde{\rho}_{1ML}$ and $\tilde{\rho}_{2ML}$ are obtained, θ_ν and β_ν can be obtained from

$$\tilde{\beta}_{1ML}^2 = \phi_1/\sum_{i=1}^{n}\frac{1}{\tilde{\rho}_{1ML} + x_i},\ \tilde{\beta}_{2ML}^2 = \phi_1^*/\sum_{j=1}^{m}\frac{1}{\tilde{\rho}_{2ML} + y_j} \quad \text{and} \quad \tilde{\theta}_{\nu ML} = \frac{\tilde{\rho}_{\nu ML}}{\tilde{\beta}_{\nu ML}^2},\ \nu = 1,2. \tag{2.7}$$

Therefore, we compute the MLE of δ as $\tilde{\delta}_{ML} = \lambda\left(\tilde{\theta}_{1ML},\tilde{\beta}_{1ML},\tilde{\theta}_{2ML},\tilde{\beta}_{2ML}\right)$.

2.1. Asymptotic distribution

Now the asymptotic distribution of $\tilde{\vartheta}_{ML}$ and of $\tilde{\delta}_{ML}$ can be obtained. Asymptotic CIs of δ can also be constructed using the asymptotic distribution. Based on the asymptotic normality results of $\tilde{\delta}_{ML}$, we have the following result:

Theorem 1. As $n,m \to \infty$, then

$$\left[\sqrt{n}\left(\tilde{\theta}_1 - \theta_1\right),\ \sqrt{n}\left(\tilde{\beta}_1 - \beta_1\right),\ \sqrt{m}\left(\tilde{\theta}_2 - \theta_2\right),\ \sqrt{m}\left(\tilde{\beta}_2 - \beta_2\right)\right] \xrightarrow{d} N_4\left(0,\mathbf{S}^{-1}\left(\vartheta\right)\right),$$

where

$$\mathbf{S}\left(\vartheta\right) = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix},$$

and

$$s_{11} = \frac{\beta_1^2}{2}\eta\left(\theta_1,\beta_1\right), \quad s_{12} = s_{21} = \theta_1\beta_1\eta\left(\theta_1,\beta_1\right) - \sqrt{2\pi}\Lambda\left(\theta_1,\beta_1\right),$$
$$s_{22} = \frac{4}{\beta_1^2} + 2\theta_1^2\eta\left(\theta_1,\beta_1\right) - 4\sqrt{2\pi}\frac{\theta_1}{\beta_1}\Lambda\left(\theta_1,\beta_1\right), \quad s_{33} = \frac{\beta_2^2}{2}\eta\left(\theta_2,\beta_2\right),$$
$$s_{34} = s_{43} = \theta_2\beta_2\eta\left(\theta_2,\beta_2\right) - \sqrt{2\pi}\Lambda\left(\theta_2,\beta_2\right),$$

$$s_{44} = \frac{4}{\beta_2^2} + 2\theta_2^2 \eta(\theta_2, \beta_2) - 4\sqrt{2\pi}\frac{\theta_2}{\beta_2}\Lambda(\theta_2, \beta_2).$$

Proof. See the Appendix.

We provide the following theorem based on the fact that the two-parameter ER model satisfies all the regularity conditions and the asymptotic properties of the MLEs (cf. Bain [12]).

Theorem 2. As $n, m \to \infty$ and $\frac{n}{m} \to \epsilon$, then

$$\sqrt{n}\left(\tilde{\delta} - \delta\right) \xrightarrow{d} N(0, \mathbf{C}_S),$$

where

$$\begin{aligned} \mathbf{C}_S &= \frac{1}{s_{11}s_{22} - s_{12}^2}\left(s_{11}\left(\frac{\partial\delta}{\partial\beta_1}\right)^2 - 2s_{12}\left(\frac{\partial\delta}{\partial\theta_1}\right)\left(\frac{\partial\delta}{\partial\beta_1}\right) + s_{22}\left(\frac{\partial\delta}{\partial\theta_1}\right)^2\right) \\ &+ \frac{1}{s_{33}s_{44} - s_{34}^2}\left(s_{33}\left(\frac{\partial\delta}{\partial\beta_2}\right)^2 - 2s_{34}\left(\frac{\partial\delta}{\partial\theta_2}\right)\left(\frac{\partial\delta}{\partial\beta_2}\right) + s_{44}\left(\frac{\partial\delta}{\partial\theta_2}\right)^2\right). \end{aligned}$$

Proof. By applying the delta method and using Theorem 1, we immediately conclude that the asymptotic distribution of $\tilde{\delta} = \lambda\left(\tilde{\theta}_1, \tilde{\beta}_1, \tilde{\theta}_2, \tilde{\beta}_2\right)$ is the following:

$$\sqrt{n}\left(\tilde{\delta} - \delta\right) \xrightarrow{d} N(0, \mathbf{C}_S),$$

where $\mathbf{C}_S = \mathbf{M}_S^t \mathbf{S}^{-1} \mathbf{M}_S$,

$$\mathbf{M}_S = \begin{pmatrix} \frac{\partial\delta}{\partial\theta_1} \\ \frac{\partial\delta}{\partial\beta_1} \\ \frac{\partial\delta}{\partial\theta_2} \\ \frac{\partial\delta}{\partial\beta_2} \end{pmatrix} \quad \text{and} \quad \mathbf{S}^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 & 0 \\ \Sigma_{21} & \Sigma_{22} & 0 & 0 \\ 0 & 0 & \Sigma_{33} & \Sigma_{34} \\ 0 & 0 & \Sigma_{43} & \Sigma_{44} \end{pmatrix},$$

with the elements of $\mathbf{S}^{-1}$ being

$$\Sigma_{11} = \frac{s_{22}}{s_{11}s_{22} - s_{12}^2}, \quad \Sigma_{12} = \frac{-s_{12}}{s_{11}s_{22} - s_{12}^2} = \Sigma_{21}, \quad \Sigma_{22} = \frac{s_{11}}{s_{11}s_{22} - s_{12}^2},$$
$$\Sigma_{33} = \frac{s_{44}}{s_{33}s_{44} - s_{34}^2}, \quad \Sigma_{34} = \frac{-s_{34}}{s_{33}s_{44} - s_{34}^2} = \Sigma_{43}, \text{ and } \Sigma_{44} = \frac{s_{33}}{s_{33}s_{44} - s_{34}^2}.$$

Therefore

$$\begin{aligned} \mathbf{C}_S &= \frac{1}{s_{11}s_{22} - s_{12}^2}\left(s_{11}\left(\frac{\partial\delta}{\partial\beta_1}\right)^2 - 2s_{12}\left(\frac{\partial\delta}{\partial\theta_1}\right)\left(\frac{\partial\delta}{\partial\beta_1}\right) + s_{22}\left(\frac{\partial\delta}{\partial\theta_1}\right)^2\right) \\ &+ \frac{1}{s_{33}s_{44} - s_{34}^2}\left(s_{33}\left(\frac{\partial\delta}{\partial\beta_2}\right)^2 - 2s_{34}\left(\frac{\partial\delta}{\partial\theta_2}\right)\left(\frac{\partial\delta}{\partial\beta_2}\right) + s_{44}\left(\frac{\partial\delta}{\partial\theta_2}\right)^2\right). \end{aligned}$$

The proof is thus completed.

Using Theorem 2, we can construct the asymptotic CI of δ. To compute the CI of δ, the variance $\mathbf{C}_S$ has to be estimated. Consequently, the $100(1-\alpha)\%$ $(0<\alpha<1)$ approximate CI for δ is

$$\left(\tilde{\delta} - Z_{1-\alpha/2}\frac{\sqrt{\tilde{\mathbf{C}}_S}}{\sqrt{n}}, \tilde{\delta} + Z_{1-\alpha/2}\frac{\sqrt{\tilde{\mathbf{C}}_S}}{\sqrt{n}}\right),$$

where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)$-th percentile of $N(0,1)$.

2.2. Bootstrap confidence intervals

Due to the difficulty in constructing the CI based on the asymptotic results, it is also expected not to perform very well for a small sample size(s). So, here, we propose that the construction of CIs be done using the bootstrap methodology. The following algorithm is used to estimate the CIs of δ using percentile bootstrap (we refer to it as Boot-p) methods (see Efron [13]):

Algorithm (1) for obtaining Boot-p CIs:

(1) Generate $\mathbf{x}_0 = (x_1, x_2, \ldots, x_n)$ of size n from ER(θ_1,β_1) and $\mathbf{y}_0 = (y_1, y_2, \ldots, y_m)$ of size m from ER(θ_2,β_2). From the sample $\mathbf{x}_0$ and $\mathbf{y}_0$, compute $\tilde{\theta}_1^{[0]}$, $\tilde{\beta}_1^{[0]}$, $\tilde{\theta}_2^{[0]}$ and $\tilde{\beta}_2^{[0]}$, using the maximum likelihood method.

(2) Generate a bootstrap sample $\mathbf{x}_1 = \left(x_1^{[1]}, x_2^{[1]}, \ldots, x_n^{[1]}\right)$ of size n from ER($\tilde{\theta}_1^{[0]}$,$\tilde{\beta}_1^{[0]}$) and $\mathbf{y}_1 = \left(y_1^{[1]}, y_2^{[1]}, \ldots, y_m^{[1]}\right)$ of size m from ER($\tilde{\theta}_2^{[0]}$,$\tilde{\beta}_2^{[0]}$). Based on $\mathbf{x}_1$ and $\mathbf{y}_1$, compute the Boot-p estimate of δ say $\tilde{\delta}^{[1]} = \lambda\left(\tilde{\theta}_1^{[1]}, \tilde{\beta}_1^{[1]}, \tilde{\theta}_2^{[1]}, \tilde{\beta}_2^{[1]}\right)$.

(3) Repeat Step 2 for NBOOT times and label all Boot-p estimates of δ by

$$\tilde{\delta}^{[k]} = \lambda\left(\tilde{\theta}_1^{[k]}, \tilde{\beta}_1^{[k]}, \tilde{\theta}_2^{[k]}, \tilde{\beta}_2^{[k]}\right), \quad k = 1, 2, \ldots, \text{NBOOT}.$$

(4) Let $\hat{S}(x)$ be the empirical distribution of $\tilde{\delta}^{[k]}$. Define $\tilde{\delta}_{Bp}(x) = \hat{S}^{-1}(x)$ for given x. The approximate $100(1-\alpha)\%$ CI of δ is given by

$$\left(\tilde{\delta}_{Bp}\left(\frac{\alpha}{2}\right), \tilde{\delta}_{Bp}\left(1-\frac{\alpha}{2}\right)\right).$$

3. Bayes estimation and HPD CrI of δ

In this section, under the assumption that the parameters $\theta_\nu (\nu = 1,2)$ and β_ν are random variables, we compute the BE of the stress-strength parameter and then obtain the corresponding CrI of this parameter. It is known that the posterior distribution is computationally efficient and analytically tractable when the prior and posterior density functions are of similar families. So, here, we assume that θ_ν and β_ν have independent Gamma(a_ν, b_ν) and square root inverted gamma (denoted SRIG(σ_ν, μ_ν)) priors, respectively, with the following pdfs:

$$\pi_\nu(\theta_\nu | a_\nu, b_\nu) \propto \theta_\nu^{a_\nu - 1} \exp(-b_\nu \theta_\nu), \; \theta_\nu > 0, \; \nu = 1, 2, \tag{3.1}$$

and

$$\pi_{\nu+2}(\beta_\nu | \sigma_\nu, \mu_\nu) \propto \beta_\nu^{-2\sigma_\nu - 1} \exp\left(-\frac{\mu_\nu}{2\beta_\nu^2}\right), \; \beta_\nu > 0, \; \nu = 1, 2, \tag{3.2}$$

where $a_v,\ b_v > 0$ and $\sigma_v,\ \mu_v > 0$. By making use of the identity

$$\prod_{\ell=1}^{p}\left(\theta+\frac{z_\ell}{\beta^2}\right)=\sum_{\ell=0}^{p}W_\ell(\mathbf{z})\theta^{p-\ell}\beta^{-2\ell},$$

see Kotb and Raqab [14], where $\mathbf{z}=\left(z_1,z_2,\ldots,z_p\right)$, $W_0(\mathbf{z})=1$, and

$$W_\ell(\mathbf{z})=\sum_{k_1=1}^{p-\ell+1}z_{k_1}\sum_{k_2=k_1+1}^{p-\ell+2}z_{k_2}\times\cdots\times\sum_{k_\ell=k_{\ell-1}+1}^{p}z_{k_\ell},$$

we have the LF of the observed data as

$$L\left(\mathbf{x},\mathbf{y}|\vartheta\right)=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{m}W_{\ell_1,\ell_2}\left(\mathbf{x},\mathbf{y}\right)\frac{\theta_1^{n-\ell_1}\theta_2^{m-\ell_2}}{\beta_1^{2\ell_1}\beta_2^{2\ell_2}}\exp\left(-\theta_1\phi_1-\theta_2\phi_1^*-\frac{\phi_2}{2\beta_1^2}-\frac{\phi_2^*}{2\beta_2^2}\right),\tag{3.3}$$

where $W_{\ell_1,\ell_2}\left(\mathbf{x},\mathbf{y}\right)=W_{\ell_1}(\mathbf{x})W_{\ell_2}(\mathbf{y})$, ϕ_r and ϕ_r^* are defined in (2.2).

Combining (3.1) and (3.2) with (3.3), the joint posterior density of θ_v and β_v given $\mathbf{x}$ and $\mathbf{y}$ is obtained as

$$\begin{aligned}\pi\left(\vartheta|\mathbf{x},\mathbf{y}\right) &= £^{-1}\left(a_1,a_2,\sigma_1,\sigma_2\right)\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{m}W_{\ell_1,\ell_2}\left(\mathbf{x},\mathbf{y}\right)\frac{\theta_1^{n+a_1-\ell_1-1}\theta_2^{m+a_2-\ell_2-1}}{\beta_1^{2(\sigma_1+\ell_1)+1}\beta_2^{2(\sigma_2+\ell_2)+1}}\\ &\times\ \exp\left(-\theta_1C_x-\theta_2C_y-\frac{C_x^*}{2\beta_1^2}-\frac{C_y^*}{2\beta_2^2}\right),\end{aligned}\tag{3.4}$$

where $C_x=b_1+\phi_1$, $C_y=b_2+\phi_1^*$, $C_x^*=\mu_1+\phi_2$, $C_y^*=\mu_2+\phi_2^*$ and

$$\begin{aligned}£\left(a_1,a_2,\sigma_1,\sigma_2\right) &= \sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{m}2^{(\sigma_1+\ell_1-1)+(\sigma_2+\ell_2-1)}W_{\ell_1,\ell_2}\left(\mathbf{x},\mathbf{y}\right)\\ &\times\ \frac{\Gamma\left(n+a_1-\ell_1\right)\Gamma\left(m+a_2-\ell_2\right)}{C_x^{n+a_1-\ell_1}C_y^{m+a_2-\ell_2}}\times\frac{\Gamma\left(\sigma_1+\ell_1\right)\Gamma\left(\sigma_2+\ell_2\right)}{\left(C_x^*\right)^{\sigma_1+\ell_1}\left(C_y^*\right)^{\sigma_2+\ell_2}}.\end{aligned}$$

Since the posterior densities of (θ_1,β_1) and (θ_2,β_2) are independent, we can rewrite the posterior density (3.4) in the following form

$$\pi\left(\vartheta|\mathbf{x},\mathbf{y}\right)=\pi_1\left(\theta_1,\beta_1|\mathbf{x}\right)\pi_2\left(\theta_2,\beta_2|\mathbf{y}\right),$$

where

$$\pi_1\left(\theta_1,\beta_1|\mathbf{x}\right)=£_1^{-1}\left(a_1,\sigma_1\right)\sum_{\ell_1=0}^{n}W_{\ell_1}\left(\mathbf{x}\right)\frac{\theta_1^{n+a_1-\ell_1-1}}{\beta_1^{2(\sigma_1+\ell_1)+1}}\exp\left(-\theta_1C_x-\frac{C_x^*}{2\beta_1^2}\right),\tag{3.5}$$

and

$$\pi_2\left(\theta_2,\beta_2|\mathbf{y}\right)=£_2^{-1}\left(a_2,\sigma_2\right)\sum_{\ell_2=0}^{m}W_{\ell_2}\left(\mathbf{y}\right)\frac{\theta_2^{m+a_2-\ell_2-1}}{\beta_2^{2(\sigma_2+\ell_2)+1}}\exp\left(-\theta_2C_y-\frac{C_y^*}{2\beta_2^2}\right),\tag{3.6}$$

with $£_1(a_1, \sigma_1)$ and $£_2(a_2, \sigma_2)$ being normalizing constants. Subsequently, the BE of any function $G(\vartheta)$ under the squared error loss (SEL) function, is given by

$$\tilde{G}_{BE} = \int_0^\infty G(\vartheta)\pi(\vartheta|\mathbf{x},\mathbf{y})\,\mathrm{d}\vartheta.$$

From Eq (3.4), the estimates of θ_ν and β_ν under the SEL function can be obtained as follows:

$$\tilde{\theta}_{\nu BE} = E(\theta_\nu|\mathbf{x},\mathbf{y}) = \frac{£_\nu(a_\nu+1,\sigma_\nu)}{£_\nu(a_\nu,\sigma_\nu)} \quad \text{and} \quad \tilde{\beta}_{\nu BE} = \frac{£_\nu(a_\nu,\sigma_\nu-1/2)}{£_\nu(a_\nu,\sigma_\nu)},\ \nu = 1,2,$$

respectively. Once the BEs of θ_1, β_1, θ_2, and β_2 are obtained, the approximate BE of δ is computed from (2.1). We can also obtain the BE $\tilde{\delta}_{BE}$ under the SEL function as the posterior mean of $\delta = \lambda(\theta_1,\beta_1,\theta_2,\beta_2)$, i.e.,

$$\tilde{\delta}_{BE} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \lambda(\theta_1,\beta_1,\theta_2,\beta_2)\,\pi(\vartheta|\mathbf{x},\mathbf{y})\,\mathrm{d}\theta_1\mathrm{d}\beta_1\theta_2\mathrm{d}\beta_2.$$

It is clear that the BE computation of δ cannot be analytically obtained and must be calculated numerically. Now, we adopt the Metropolis-Hastings (M-H) technique with a normal proposal distribution to compute the desired BE of δ and the corresponding CrI. Also, we use the M-H algorithm to obtain the highest posterior density (HPD) CrI of δ.

3.1. MCMC method

The Gibbs sampling technique requires being able to generate Markov chain Monte Carlo (MCMC) samples and then compute the BE $\tilde{\delta}$ of δ as well as the corresponding CrI. From Eqs (3.5) and (3.6), the marginal posterior pdfs of θ_1, β_1, θ_2, and β_2 are obtained to be

$$\begin{aligned}
&\theta_1|\mathbf{x} \sim \mathrm{Gamma}(n+a_1, C_x),\\
&\pi(\beta_1|\theta_1,\mathbf{x}) \propto \sum_{\ell_1=0}^{n} W_{\ell_1}(\mathbf{x})\,\alpha_1^{-\ell_1}\ \mathrm{InvGamma}\left(\sigma_1+\ell_1,\ \sqrt{\frac{C_x^*}{2}}, 2\right),\\
&\theta_2|\mathbf{y} \sim \mathrm{Gamma}\left(m+a_2, C_y\right),\\
&\pi(\beta_2|\theta_2,\mathbf{y}) \propto \sum_{\ell_2=0}^{m} W_{\ell_2}(\mathbf{y})\,\alpha_2^{-\ell_2}\ \mathrm{InvGamma}\left(\sigma_2+\ell_2,\ \sqrt{\frac{C_y^*}{2}}, 2\right),
\end{aligned}$$

respectively, where the pdf of $\mathrm{InvGamma}(u, v, \jmath)$ is given by

$$h(z;u,v,\jmath) = \frac{\jmath}{v\Gamma(u)}\left(\frac{v}{z}\right)^{u\jmath+1}\exp\left(-\left(\frac{v}{z}\right)^{\jmath}\right),\quad u,v,\jmath > 0.$$

Here u and $\jmath$ are the shape parameters and v is the scale parameter. Since the posterior pdfs of β_1 and β_2 are mixtures of inverse gamma densities, we use the M-H method with normal proposal distribution to generate random samples from their densities. The following algorithm summarizes the M-H method for BEs of δ as well as the corresponding CrIs.

Algorithm (2): M-H algorithm for estimation:

(1) Start with the initial values $\left(\theta_1^{(0)}, \beta_1^{(0)}, \theta_2^{(0)}, \beta_2^{(0)}\right) = \left(\tilde{\theta}_{1ML}, \tilde{\beta}_{1ML}, \tilde{\theta}_{2ML}, \tilde{\beta}_{2ML}\right)$.
(2) Set $u = 1$.
(3) Generate $\theta_1^{(u)}$ from Gamma $(n + a_1, C_x)$ and $\theta_2^{(u)}$ from Gamma $\left(m + a_2, C_y\right)$.
(4) Using M-H method, generate $\beta_1^{(u)}$ from $\pi\left(\beta_1^{(u)}|\theta_1^{(u)}, \mathbf{x}\right)$ and $\beta_2^{(u)}$ from $\pi\left(\beta_2^{(u)}|\theta_2^{(u)}, \mathbf{y}\right)$, respectively, with $N\left(\beta_\nu^{(u-1)}, \mathrm{Var}\left(\beta_\nu^{(u-1)}\right)\right)$, $\nu = 1, 2$, as proposal models.
(5) Calculate $\delta^{(u)} = \lambda\left(\alpha_1^{(u)}, \alpha_2^{(u)}, \sigma_1^{(u)}, \sigma_2^{(u)}\right)$.
(6) Set $u = u + 1$.
(7) Repeat Steps 2–6 , n^* times and obtain $\delta = \left(\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n^*)}\right)$.

Immediately, using the above procedures, the average BE based on the SEL function and MSEs of δ can be obtained as

$$\tilde{\delta}_{BE} = \frac{1}{n^*}\sum_{u=1}^{n^*}\delta^{(u)} \quad \text{and} \quad \mathrm{MSE}(\tilde{\delta}) = \frac{1}{n^*}\sum_{u=1}^{n^*}\left(\delta^{(u)} - \delta_t\right)^2,$$

where δ_t is the true value of δ. Next, based on the M-H algorithm, we use the method proposed by Chen and Shao [15] (see Theorem 2, p. 73) to establish the HPD CrI of δ. Based on n^* ordered values $\tilde{\delta}_{(1)} \leq \tilde{\delta}_{(2)} \leq \ldots \leq \tilde{\delta}_{(n^*)}$ from generated values δ, the $100(1-\alpha)\%$ HPD CrI of δ can be constructed as $(\delta_{(\epsilon)}, \delta_{(\epsilon+[(1-\alpha)n^*])})$, where ϵ is chosen such that

$$\delta_{(\epsilon+[(1-\alpha)n^*])} - \delta_{(\epsilon)} = \min_{1\leq i\leq n^*-[(1-\alpha)n^*]}\left(\delta_{(i+[(1-\alpha)n^*])} - \delta_{(i)}\right),$$

$[(1-\alpha)n^*]$ denotes the integer part of $(1-\alpha)n^*$.

4. Data analysis

Here, we present a complete analysis for both a simulated and real-life data set to illustrate the techniques. The computations are performed using Wolfram Mathematica 13.3 codes.

4.1. Example 1 (Simulated data)

It is assumed that the prior distributions of θ_ν and β_ν ($\nu = 1, 2$) are gamma and SRIG distributions given by Eqs (3.1) and (3.2), respectively. In this example, the computations are done using the following priors:

$$\pi_1 : \ a_1 = 3, b_1 = 0.665,$$
$$\pi_2 : \ a_2 = 2, b_2 = 1.5,$$
$$\pi_3 : \ \sigma_1 = 2, \ \mu_1 = 0.64,$$
$$\pi_4 : \ \sigma_2 = 2.5, \ \mu_2 = 2.3.$$

In this case the true value of $\delta = 0.55$. Now, we generate the following two independent samples from the ER distribution:

Sample 1: $\mathbf{x}$ =(0.02286, 0.02457, 0.02673, 0.04646, 0.06218, 0.12491, 0.19077, 0.19221, 0.20679, 0.21335, 0.23212, 0.23840, 0.27462, 0.28671, 0.28940, 0.30800, 0.33557, 0.34647, 0.35868, 0.44168, 0.44299, 0.48523, 0.54205, 0.76352, 0.99740), where $X \sim \mathrm{ER}(2, 0.5)$ and $n = 25$.

Sample 2: $\mathbf{y}$ =(0.00794, 0.00982, 0.03656, 0.08241, 0.08658, 0.09059, 0.09473, 0.10978, 0.12002, 0.12154, 0.15144, 0.16146, 0.17330, 0.19688, 0.23391, 0.25392, 0.27027, 0.33117, 0.378003, 0.38103, 0.43225, 0.51243, 0.51498, 0.68838, 0.87013), where $Y \sim \text{ER}(3, 0.8)$ and $m = 25$.

From Figure 1, the profile log-likelihood functions of ρ_1 and ρ_2 are unimodal functions. So, we can apply the proposed iterative procedure with starting values (an initial guess) of $\rho_1 = 0.3$ and $\rho_2 = 0.8$, where the iterative procedure converges after eight iterations within an approximate relative error of less than 10^{-8}. We obtained $\tilde{\rho}_1 = 0.34918$ and $\tilde{\rho}_2 = 0.79236$. Once $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are obtained, from Eq (2.7), the MLEs for $(\theta_1, \beta_1, \theta_2, \beta_2)$ are found to be $(\tilde{\theta}_1, \tilde{\beta}_1, \tilde{\theta}_2, \tilde{\beta}_2)$ =(2.00747, 0.41706, 3.11302, 0.50451). Therefore, we compute the MLE of δ as $\tilde{\delta} = 0.56707$, and the corresponding 95% CIs based on asymptotic distribution and boot-p are computed as (0.43067,0.66902) and (0.42164,0.67862), respectively. The resulting BEs of different parameters based on the SEL function are found to be $(\tilde{\theta}_1, \tilde{\beta}_1, \tilde{\theta}_2, \tilde{\beta}_2)$ =(2.43897, 0.51299, 2.90556, 0.70434) and $\tilde{\delta} = 0.51679$. Moreover, the 95% HPD CrI of $\tilde{\delta}$ is (0.45003,0.66183).

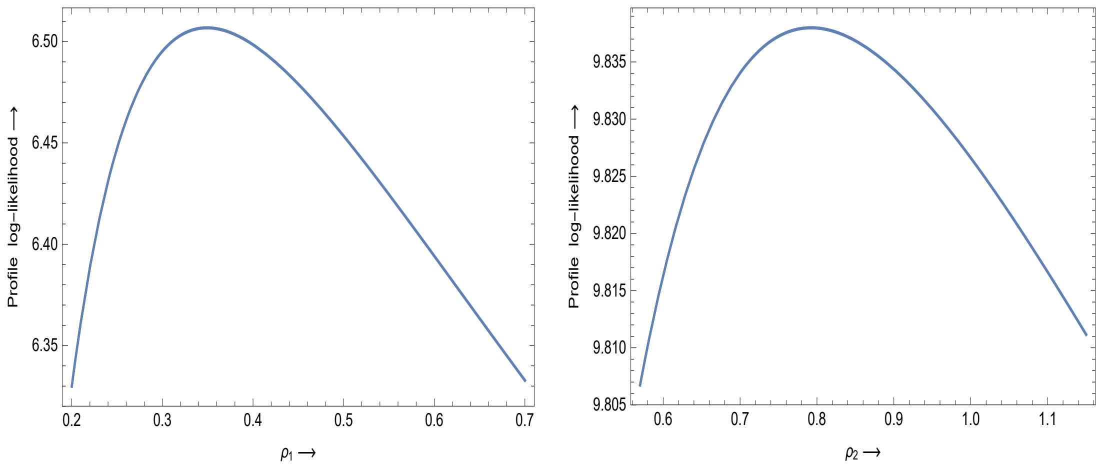

Figure 1. Profile log-likelihood function of the simulated data sets.

To determine which of the estimates (MLEs or BEs) provides a better fit to the two data sets 1 and 2, we compute the goodness-of-fit tests and the corresponding p-values based on MLEs and BEs. We use the Anderson-Darling (A-D), Cramér-von Mises (CvM), Kolmogorov-Smirnov (K-S) distance, and Pearson χ^2 goodness-of-fit tests. Based on the results of goodness-of-fit tests given in Table 1, BEs provide a better fit than the MLEs to data set 1 and the other way to data set 2.

Table 1. A-D, CvM, K-S, and Pearson χ^2 statistics and corresponding p-values of data sets 1 and 2.

	Data Set		A-D	CvM	K-S	Pearson χ^2
MLE	**x**	statistic	0.42212	0.06689	0.14590	6.68000
		p-value	0.82499	0.77077	0.61030	0.46294
	y	statistic	0.21206	0.03123	0.11654	2.84000
		p-value	0.98683	0.97197	0.84791	0.89940
BE	**x**	statistic	0.49337	0.09168	0.17400	6.68000
		p-value	0.75167	0.62723	0.39037	0.46294
	y	statistic	0.42286	0.07388	0.10638	0.92000
		p-value	0.82424	0.72799	0.91154	0.99602

4.2. *Example 2 (Real life data)*

To verify the goodness of fit of the ER distribution, two real data sets representing the wait times (in minutes) before the service of the customers at two different banks, I and II, are provided; see Lindley [16].

First sample: The first set of data represents the waiting times for clients of Bank I. The data set is listed as follows: **x** =(0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5).

Second sample: The second set of data represents the waiting times for clients of Bank II. The data set is listed as follows: **y** =(0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0).

First, the ER model is fitted to the two data sets separately. Now, we want to check by using the Kaplan and Meier [17] estimator (KME), A-D, CvM, K-S distance, and Pearson χ^2 goodness-of-fit tests whether the ER model is suitable for this data. We wish to test the null hypothesis

$$H_0 : F(x) = \text{ER distribution} \quad \text{against} \quad H_1 : F(x) \neq \text{ER distribution}.$$

If p-value $< \alpha$, we reject H_0 (and accept H_1) at a significance level of $\alpha = 0.05$. Table 2 presents the A-D, CvM, K-S distance, and Pearson χ^2 goodness-of-fit tests and the corresponding p-values based on MLEs and BEs. Based on the p-values, one cannot reject the hypothesis that data sets I and II come from the ER model. Figures 2 and 3 show, respectively, the empirical survival function (SF) versus the empirical function and the P-P plot of the KME versus the fitted SF, as well as box plots for the given data sets. Visually, Figure 2 indicates a reasonable match between the fitted SF and the empirical SF. Also, Figure 3 shows the depicted points for the fitted SF are very near the 45^o line. It can be easily seen that the ER works quite well for these strength data.

Table 2. A-D, CvM, K-S, and Pearson χ^2 statistics and corresponding p-values of data sets I and II.

Data Set			A-D	CvM	K-S	Pearson χ^2
MLE	**x**	statistic	0.92427	0.13319	0.07300	13.1000
		p-value	0.39908	0.44511	0.63395	0.36181
	y	statistic	0.30008	0.04301	0.07568	13.3333
		p-value	0.93785	0.91707	0.85617	0.20563
BE	**x**	statistic	0.96185	0.12079	0.07721	16.7400
		p-value	0.37747	0.49222	0.56356	0.15964
	y	statistic	0.28600	0.04249	0.06880	11.8667
		p-value	0.94809	0.91990	0.92026	0.29408

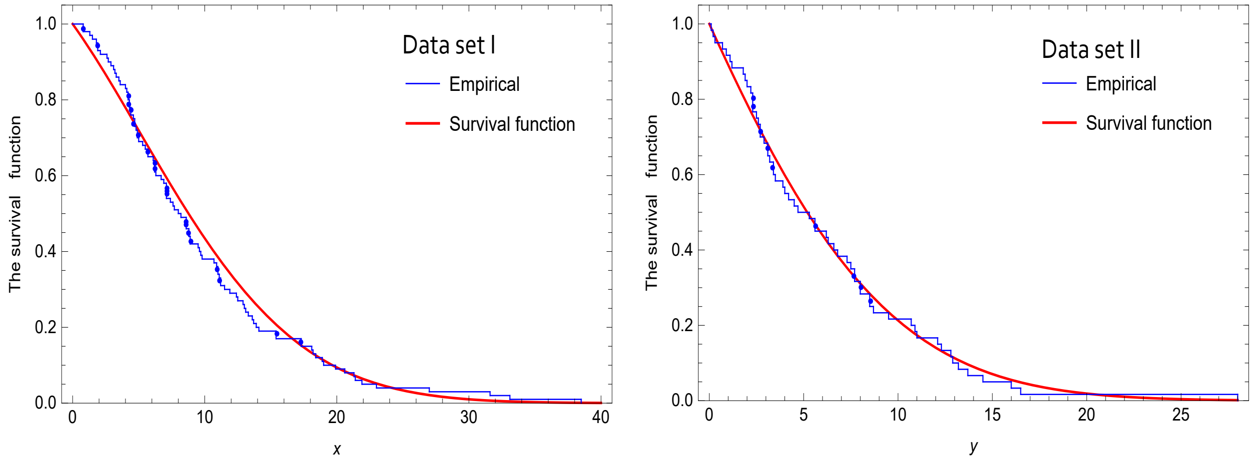

Figure 2. The empirical and fitted survival function.

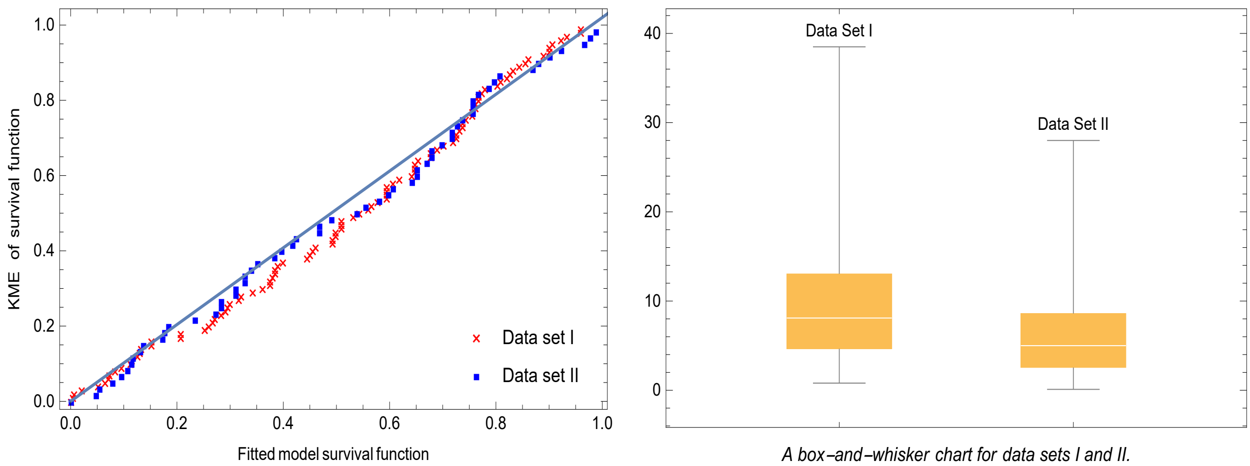

Figure 3. *P-P* plot of KME versus fitted SF (left); box-plots (right) for data sets I and II.

The computations are done using the same values of hyper-parameters in Example 1. In Figure 4, we have presented the profile log-likelihood functions of ρ_1 and ρ_2. So, we shall consider the initial guess of ρ_1 and ρ_2 as 6.5 and 12, respectively. Based on these initial guesses, the iterative procedure converges after six and ten iterations, respectively, within an approximate relative error of less than 10^{-8}. We obtained $\tilde{\rho}_1 = 6.90882$ and $\tilde{\rho}_2 = 12.59690$. Once $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are obtained, from Eq (2.5), the MLEs for $(\theta_1, \beta_1, \theta_2, \beta_2)$ are found to be $(\tilde{\theta}_1, \tilde{\beta}_1, \tilde{\theta}_2, \tilde{\beta}_2)$ =(0.04833, 11.95580, 0.11077, 10.66300). Therefore, we compute the MLE of δ as $\tilde{\delta} = 0.65816$, and the corresponding 95% CIs based on asymptotic distribution and boot-p are computed as (0.63635,0.68816) and (0.60989,0.69525), respectively. The resulting BEs of different parameters based on the SEL function are found to be $(\tilde{\theta}_1, \tilde{\beta}_1, \tilde{\theta}_2, \tilde{\beta}_2)$ =(0.05503, 12.56520, 0.10432, 9.65935) and $\tilde{\delta} = 0.64726$. Moreover, the 95% HPD CrI of $\tilde{\delta}$ is (0.63194, 0.67005). Overall, based on the goodness-of-fit tests given in Table 2, BEs provide a better fit than the MLEs to data sets I and II.

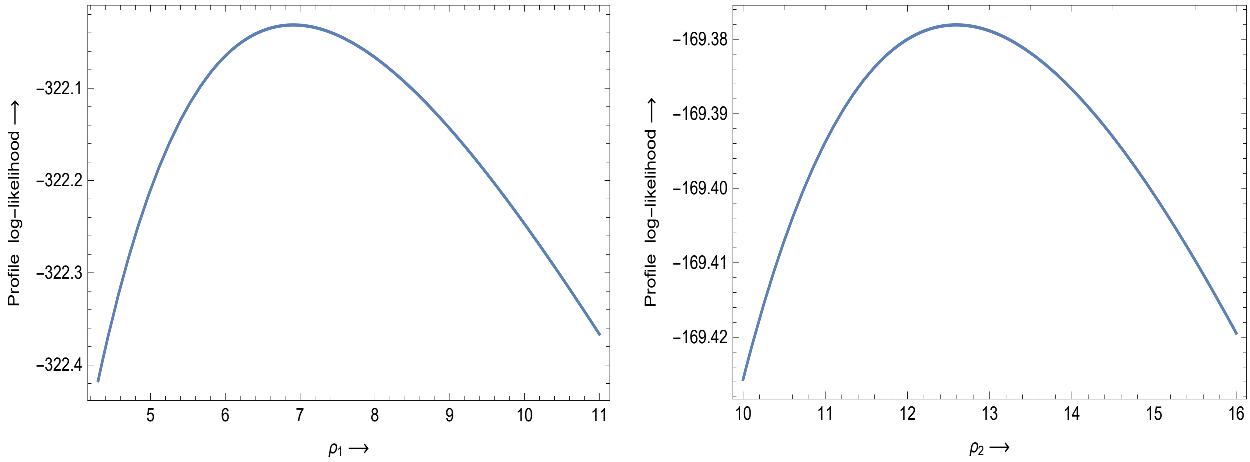

Figure 4. Profile log-likelihood function of the data sets I and II.

5. Numerical comparison

To assess and compare the so-developed estimators of δ in terms of biases and mean square errors (MSEs), we carry out a Monte Carlo (MC) simulation study. Also, we compare different CIs obtained by using the asymptotic distribution of the MLEs, the Bootstrap CI, and the HPD CrIs in terms of the average lengths (ALs). We have considered different sample sizes and different choices of hyper-parameters that allow us to generate the parameter values $\theta_\nu(\nu = 1, 2)$ and β_ν, namely, $(a_\nu, b_\nu)\theta_\nu$ = (1.25, 0.4)0.5, (1.43,0.7)1, (1.66,1.5)2.5 and $(\sigma_\nu, \mu_\nu)\beta_\nu$ = (1.5, 0.4)0.5, (1.4,1.38)1, (0.8,1.9)2.5. Based on that generated value of parameter values, we generate $n(n = 15, 45)$ random samples from the ER(θ_1,β_1) and $m(m = 15, 45)$ random samples from the ER(θ_2,β_2), using the transformation $X_i = -\theta_1\beta_1^2 + \beta_1\left(\theta_1^2\beta_1^2 - 2\log[1 - U_i]\right)^{1/2}$, $i = 1, 2, \ldots, n$; and $Y_j = -\theta_2\beta_2^2 + \beta_2\left(\theta_2^2\beta_2^2 - 2\log\left[1 - U_j\right]\right)^{1/2}$, $j = 1, 2, \ldots, m$, where $U_i \sim \mathrm{U}(0, 1)$. The MLE and BE of δ are computed by using the results in Sections 2 and 3. We report the average biases, MSEs, and ALs over 5000 replications. The simulated bias and MSEs are, respectively,

$$\tilde{\delta}_{\mathrm{Bias}} = \frac{1}{n^*}\sum_{i=1}^{n^*}\left(\tilde{\delta}_i - \delta\right) \quad \text{and} \quad MSE(\tilde{\delta}) = \frac{1}{n^*}\sum_{i=1}^{n^*}\left(\tilde{\delta}_i - \delta\right)^2.$$

The results of the simulation are summarized in Tables 3 and 4 for the different parameter values and different n and m. From the numerical results, some points could be observed as follows:

(1) The performances of BEs of δ obtained from the SEL function are better than those of MLEs and MCMC. We also observe that the MLE method is the second-best estimation method.
(2) We note that the MSEs increase when both β_1 and β_2 increase together, while this is not always true for parameters θ_1 and θ_2.
(3) It is observed that overall, the MSEs decrease when the difference between β_1 and β_2 increases.
(4) We observe that Bootstrap CIs provide the smallest average CI lengths for different parameter values and for different sample sizes.
(5) It is evident from Table 4 that HPD CrIs are not very sensitive to the hyper-parameter values that are assumed.

(6) As expected, when n and m increase, then the MSEs of all the estimators and ALs decrease.

Table 3. Biases and MSEs of the MLE and 95% CI. of δ.

			$\tilde{\delta}_{ML}$		AL	
(n,m)	β_1	β_2	Bias	MSE	95% CI.	Boot-p
			$\theta_1 = 0.5, \theta_2 = 0.5$			
(15,15)	0.5	0.5	-0.00457	0.00853	0.28462	0.19681
		1	0.00550	0.00841	0.26457	0.19304
		2.5	-0.00849	0.00644	0.22969	0.12697
	1	0.5	-0.00631	0.00779	0.26513	0.21092
		1	0.00095	0.01065	0.28441	0.21078
		2.5	-0.00917	0.00962	0.27272	0.06700
	2.5	0.5	0.00815	0.00638	0.22991	0.15215
		1	0.00630	0.00988	0.27312	0.18923
		2.5	-0.00066	0.01187	0.28657	0.18258
(15,45)	0.5	0.5	-0.00421	0.00613	0.20200	0.16467
		1	0.00267	0.00476	0.18839	0.12456
		2.5	-0.00386	0.00324	0.16338	0.10247
	1	0.5	-0.00417	0.00647	0.18813	0.17878
		1	-0.00047	0.00715	0.20246	0.15122
		2.5	-0.00551	0.00545	0.19531	0.11059
	2.5	0.5	0.00944	0.00530	0.16354	0.16101
		1	0.00995	0.00785	0.19437	0.16282
		2.5	0.00047	0.00765	0.20434	0.16961
(45,15)	0.5	0.5	-0.00081	0.00610	0.20201	0.14413
		1	0.00161	0.00649	0.18763	0.16766
		2.5	-0.00765	0.00547	0.16381	0.10207
	1	0.5	-0.00310	0.00486	0.18856	0.15128
		1	-0.00149	0.00740	0.20232	0.14819
		2.5	-0.01017	0.00791	0.19412	0.16518
	2.5	0.5	0.00156	0.00333	0.16387	0.09769
		1	0.00279	0.00552	0.19543	0.14565
		2.5	-0.00282	0.00767	0.20429	0.16178
(45,45)	0.5	0.5	-0.00222	0.00299	0.16509	0.11162
		1	0.00096	0.00307	0.15398	0.10375
		2.5	-0.00377	0.00250	0.14620	0.08317
	1	0.5	-0.00104	0.00311	0.15416	0.11517
		1	-0.00052	0.00360	0.16577	0.12255
		2.5	-0.00600	0.00325	0.15970	0.08814
	2.5	0.5	0.00458	0.00236	0.13392	0.08741
		1	0.00537	0.00336	0.15991	0.12147
		2.5	-0.00066	0.00377	0.16742	0.11401
			$\theta_1 = 1, \theta_2 = 1$			
(15,15)	0.5	0.5	-0.00277	0.00942	0.28609	0.18640
		1	-0.00254	0.00939	0.27703	0.15603
		2.5	-0.01278	0.00829	0.26252	0.06063
	1	0.5	0.00062	0.00972	0.27727	0.20315
		1	-0.00199	0.01089	0.28667	0.11270
		2.5	-0.01003	0.01086	0.28303	0.20343
	2.5	0.5	0.01161	0.00860	0.26267	0.18460
		1	0.00689	0.01126	0.28276	0.20176
		2.5	-0.00271	0.01445	0.28642	0.19729
(15,45)	0.5	0.5	-0.00522	0.00646	0.20338	0.14450
		1	-0.00330	0.00554	0.19798	0.14763
		2.5	-0.00614	0.00431	0.18649	0.14195

Table 3. Continued.

(n,m)	β_1	β_2	$\tilde{\delta}_{ML}$ Bias	MSE	AL 95% CI.	Boot-p
	1	0.5	0.00221	0.00751	0.19729	0.16599
		1	0.00068	0.00698	0.20425	0.17104
		2.5	-0.00357	0.00677	0.20154	0.17122
	2.5	0.5	0.01483	0.00706	0.18663	0.17676
		1	0.00939	0.00825	0.20112	0.13777
		2.5	0.00524	0.00972	0.20394	0.11471
(45,15)	0.5	0.5	-0.00338	0.00648	0.20317	0.13122
		1	-0.00345	0.00726	0.19700	0.13956
		2.5	-0.01495	0.00700	0.18667	0.16717
	1	0.5	0.00177	0.00554	0.19779	0.15224
		1	0.00061	0.00758	0.20431	0.11343
		2.5	-0.01064	0.00771	0.20105	0.15010
	2.5	0.5	0.00627	0.00467	0.18642	0.12138
		1	0.00359	0.00675	0.20163	0.14043
		2.5	-0.00087	0.00962	0.20386	0.17136
(45,45)	0.5	0.5	-0.00235	0.00340	0.16633	0.12088
		1	-0.00087	0.00327	0.16206	0.08526
		2.5	-0.00635	0.00292	0.15238	0.10624
	1	0.5	0.00138	0.00356	0.16197	0.11572
		1	0.00166	0.00368	0.16742	0.10032
		2.5	-0.00463	0.00345	0.16486	0.11361
	2.5	0.5	0.00760	0.00298	0.15224	0.10413
		1	0.00455	0.00336	0.16486	0.11058
		2.5	-0.00015	0.00653	0.16665	0.12017
			$\theta_1 = 2.5, \theta_2 = 2.5$			
(15,15)	0.5	0.5	-0.00224	0.01065	0.28742	0.17487
		1	-0.00968	0.01145	0.28518	0.12457
		2.5	-0.01738	0.01269	0.28337	0.15860
	1	0.5	0.00811	0.01121	0.28529	0.20454
		1	-0.00010	0.01373	0.28663	0.08602
		2.5	-0.00364	0.01586	0.28626	0.18562
	2.5	0.5	0.01314	0.01241	0.28336	0.19420
		1	0.00653	0.01492	0.28655	0.18330
		2.5	0.00277	0.01776	0.28627	0.18061
(15,45)	0.5	0.5	-0.00085	0.00700	0.20446	0.14051
		1	-0.00410	0.00723	0.20310	0.15228
		2.5	-0.01295	0.00798	0.20185	0.16197
	1	0.5	0.00618	0.00816	0.20297	0.18190
		1	0.00120	0.00979	0.20387	0.15338
		2.5	0.00020	0.01224	0.20347	0.15698
	2.5	0.5	0.01098	0.00783	0.20147	0.16313
		1	0.00774	0.01097	0.20352	0.16099
		2.5	0.00343	0.01621	0.20332	0.13081
(45,15)	0.5	0.5	0.00078	0.00732	0.20446	0.14640
		1	-0.00939	0.00792	0.20277	0.13469
		2.5	-0.01750	0.00828	0.20169	0.13852
	1	0.5	0.00438	0.00709	0.20308	0.13916
		1	-0.00327	0.00988	0.20392	0.13820
		2.5	-0.00907	0.01096	0.20342	0.15146

Table 3. Continued.

(n,m)	θ_1	θ_2	$\tilde{\delta}_{ML}$ Bias	MSE	AL 95% CI.	Boot-p
	2.5	0.5	0.00787	0.00800	0.20138	0.16719
		1	-0.00266	0.01241	0.20354	0.14256
		2.5	-0.00513	0.01612	0.20342	0.18329
(45,45)	0.5	0.5	0.00059	0.00373	0.16746	0.09664
		1	-0.0319	0.00378	0.16600	0.11624
		2.5	-0.01151	0.00392	0.16483	0.14603
	1	0.5	0.00370	0.00382	0.16606	0.10661
		1	-0.00125	0.00565	0.16672	0.11742
		2.5	-0.00366	0.00855	0.16624	0.12028
	2.5	0.5	0.00741	0.00389	0.16463	0.10918
		1	0.00124	0.00880	0.16624	0.13118
		2.5	0.00282	0.01568	0.16620	0.28298
			$\beta_1 = 0.5, \beta_2 = 0.5$			
(15,15)	0.5	1	-0.01122	0.00854	0.28207	0.16763
		2.5	-0.01731	0.00653	0.25395	0.17887
	1	0.5	0.00752	0.00904	0.28142	0.17200
		2.5	-0.01549	0.00880	0.27103	0.18172
	2.5	0.5	0.00172	0.00682	0.25409	0.14067
		1	0.00316	0.00867	0.27052	0.10600
(15,45)	0.5	1	-0.01476	0.00585	0.20044	0.16139
		2.5	-0.02033	0.00456	0.18009	0.16434
	1	0.5	0.00520	0.00639	0.19942	0.13704
		2.5	-0.01636	0.00581	0.19236	0.16743
	2.5	0.5	-0.00482	0.00483	0.17958	0.14828
		1	0.00220	0.00611	0.19248	0.12482
(45,15)	0.5	1	-0.00830	0.00618	0.20038	0.13690
		2.5	-0.01194	0.00471	0.17997	0.14996
	1	0.5	0.00383	0.00614	0.19962	0.16430
		2.5	-0.00552	0.00610	0.19271	0.15242
	2.5	0.5	0.01168	0.00472	0.17836	0.12612
		1	0.00371	0.00571	0.19216	0.11249
		2.5	-0.00059	0.00706	0.20441	0.13312
(45,45)	0.5	1	-0.00680	0.00298	0.16388	0.11931
		2.5	-0.01391	0.00238	0.14709	0.09247
	1	0.5	0.00126	0.00312	0.16317	0.09829
		2.5	-0.00500	0.00292	0.15771	0.09868
	2.5	0.5	0.00606	0.00236	0.14580	0.09996
		1	0.00020	0.00303	0.15723	0.09800
		2.5	-0.00116	0.00367	0.16743	0.11197
			$\beta_1 = 1, \beta_2 = 1$			
(15,15)	0.5	1	-0.00595	0.01022	0.27801	0.21092
		2.5	-0.00480	0.00628	0.22571	0.18878
	1	0.5	-0.00245	0.01007	0.27697	0.17116
		2.5	0.00117	0.00942	0.26002	0.17020
	2.5	0.5	0.00216	0.00669	0.22538	0.08400
		1	-0.00112	0.00963	0.26074	0.14605

Table 3. Continued.

(n,m)	θ_1	θ_2	$\tilde{\delta}_{ML}$ Bias	MSE	AL 95% CI.	Boot-p
(15,45)	0.5	1	-0.00300	0.00668	0.19731	0.28221
		2.5	-0.00510	0.00418	0.16014	0.14031
	1	0.5	0.00021	0.00679	0.19757	0.12195
		2.5	-0.00063	0.00069	0.18464	0.23256
	2.5	0.5	-0.00110	0.00398	0.16068	0.07432
		1	-0.00266	0.00549	0.18519	0.16049
(45,15)	0.5	1	-0.00071	0.00734	0.19765	0.13714
		2.5	0.00125	0.00382	0.16024	0.10641
	1	0.5	-0.00106	0.00637	0.19691	0.17264
		2.5	0.00098	0.00540	0.18551	0.13463
	2.5	0.5	0.00016	0.00446	0.15911	0.13185
		1	-0.00315	0.00673	0.18451	0.15646
(45,45)	0.5	1	0.00063	0.00348	0.16161	0.12003
		2.5	0.00129	0.00218	0.13079	0.10608
	1	0.5	0.00090	0.00335	0.16158	0.10739
		2.5	0.00001	0.00309	0.15157	0.08808
	2.5	0.5	-0.00108	0.00218	0.13104	0.09394
		1	0.00054	0.00313	0.15145	0.06872
			$\beta_1 = 2.5, \beta_2 = 2.5$			
(15,15)	0.5	1	0.00706	0.01086	0.26957	0.19495
		2.5	0.00570	0.00785	0.19402	0.09770
	1	0.5	-0.00194	0.01105	0.27042	0.13638
		2.5	0.00873	0.01281	0.24789	0.11903
	2.5	0.5	-0.00566	0.00805	0.19555	0.05100
		1	-0.00970	0.01130	0.24804	0.16011
		2.5	-0.00292	0.01754	0.28635	0.17047
(15,45)	0.5	1	0.00431	0.00802	0.19167	0.12564
		2.5	0.00612	0.00695	0.13758	0.10422
	1	0.5	0.00115	0.00683	0.19261	0.08212
		2.5	0.01115	0.01118	0.17562	0.12113
	2.5	0.5	-0.00198	0.00365	0.13755	0.12663
		1	0.00184	0.01065	0.17595	0.18319
		2.5	0.00231	0.01493	0.20348	0.17266
(45,15)	0.5	1	0.00102	0.00637	0.19249	0.19967
		2.5	-0.00089	0.00478	0.13851	0.37964
	1	0.5	-0.00491	0.00736	0.19181	0.16185
		2.5	0.00263	0.00864	0.17534	0.07764
	2.5	0.5	-0.00561	0.00674	0.13904	0.04772
		1	-0.01212	0.01003	0.17566	0.10443
		2.5	-0.00307	0.01492	0.20355	0.11709
(45,45)	0.5	1	0.00235	0.00364	0.15710	0.10082
		2.5	0.00204	0.00261	0.11235	0.08526
	1	0.5	-0.00128	0.00369	0.15710	0.22441
		2.5	-0.00066	0.00964	0.14271	0.10667
	2.5	0.5	-0.00169	0.00351	0.11230	0.07386
		1	-0.00182	0.00712	0.14292	0.08446
		2.5	0.00093	0.01389	0.16615	0.11787

Table 4. Biases and MSEs of the BE and 95% HPD CrI of ψ_1.

			$\tilde{\delta}_{BE}$		$\tilde{\delta}_{MCMC}$		AL
(n,m)	β_1	β_2	Bias	MSE	Bias	MSE	HPD
			$\theta_1 = 0.5, \theta_2 = 0.5$				
(15,15)	0.5	0.5	0.00091	0.00584	-0.01032	0.01195	0.27150
		1	0.00943	0.00553	-0.00012	0.01126	0.24722
		2.5	-0.00720	0.00393	-0.01409	0.00754	0.21588
	1	0.5	-0.00684	0.00505	-0.00001	0.01013	0.25129
		1	-0.00009	0.00717	-0.01924	0.01304	0.27975
		2.5	-0.01475	0.00661	-0.00021	0.01352	0.26757
	2.5	0.5	0.00822	0.00395	0.00849	0.00826	0.21492
		1	0.01303	0.00674	0.00007	0.01267	0.26957
		2.5	0.00099	0.00834	0.00099	0.01503	0.28989
(15,45)	0.5	0.5	-0.01383	0.00446	-0.00695	0.01028	0.22646
		1	-0.00744	0.00336	0.00010	0.00732	0.20465
		2.5	-0.01181	0.00216	-0.01032	0.00453	0.16117
	1	0.5	-0.01510	0.00471	-0.00017	0.00793	0.22503
		1	-0.01200	0.00508	-0.00710	0.01087	0.23643
		2.5	-0.01918	0.00411	-0.00025	0.00751	0.20942
	2.5	0.5	0.00344	0.00373	0.02603	0.00676	0.19397
		1	0.00787	0.00564	0.00022	0.00994	0.23828
		2.5	-0.00224	0.00570	-0.01084	0.01088	0.24174
(45,15)	0.5	0.5	0.01196	0.00438	0.00098	0.00877	0.22925
		1	0.01512	0.00425	-0.00011	0.00795	0.22008
		2.5	-0.00450	0.00358	-0.01135	0.00693	0.19792
	1	0.5	0.00787	0.00337	0.00019	0.00681	0.19888
		1	0.01121	0.00486	-0.01583	0.01087	0.23568
		2.5	-0.00777	0.00564	-0.00013	0.00951	0.24062
	2.5	0.5	0.01159	0.00229	0.00563	0.00501	0.16132
		1	0.01729	0.00407	0.00021	0.00710	0.20861
		2.5	0.00657	0.00583	0.00183	0.01090	0.24173
(45,45)	0.5	0.5	-0.00125	0.00259	-0.00284	0.00581	0.17461
		1	-0.00010	0.00237	0.00004	0.00497	0.16517
		2.5	-0.00744	0.00172	-0.00929	0.00331	0.14494
	1	0.5	0.00025	0.00241	0.00015	0.00474	0.16356
		1	-0.00184	0.00271	-0.00442	0.00586	0.18011
		2.5	-0.01157	0.00260	-0.00027	0.00572	0.17317
	2.5	0.5	0.00743	0.00177	0.00600	0.00351	0.14185
		1	0.01102	0.00267	0.00023	0.00552	0.17403
		2.5	-0.00113	0.00285	-0.00971	0.00660	0.18479
			$\theta_1 = 1, \theta_2 = 1$				
(15,15)	0.5	0.5	0.00017	0.00627	0.00006	0.01381	0.27590
		1	-0.00191	0.00619	-0.00050	0.01210	0.26733
		2.5	-0.01513	0.00577	-0.00060	0.01256	0.25001
	1	0.5	0.00170	0.00637	0.00029	0.01179	0.26972
		1	-0.00081	0.00743	0.00033	0.01422	0.28579
		2.5	-0.01474	0.00765	-0.00015	0.01461	0.28061
	2.5	0.5	0.01510	0.00604	0.00069	0.01104	0.25021
		1	0.01343	0.00780	0.00020	0.01439	0.28214
		2.5	-0.00212	0.00886	0.00021	0.01433	0.29051
(15,45)	0.5	0.5	-0.00679	0.00454	0.00012	0.00944	0.23320
		1	-0.00795	0.00393	-0.00021	0.00868	0.21317
		2.5	-0.01506	0.00330	-0.00040	0.00648	0.19177

Table 4. Continued.

(n,m)	β_1	β_2	$\tilde{\delta}_{BE}$ Bias	$\tilde{\delta}_{BE}$ MSE	$\tilde{\delta}_{MCMC}$ Bias	$\tilde{\delta}_{MCMC}$ MSE	AL HPD
	1	0.5	-0.00071	0.00515	0.00044	0.01029	0.23511
		1	-0.00306	0.00499	0.00014	0.01039	0.23994
		2.5	-0.01281	0.00464	-0.00036	0.00913	0.22481
	2.5	0.5	0.01358	0.00525	0.00025	0.00982	0.23216
		1	0.01138	0.00631	-0.00016	0.00990	0.24823
		2.5	0.00220	0.00575	0.00003	0.00959	0.24118
(45,15)	0.5	0.5	0.00351	0.00441	0.00016	0.00876	0.23365
		1	0.00086	0.00498	-0.00020	0.01049	0.23803
		2.5	-0.01341	0.00522	-0.00009	0.00888	0.23212
	1	0.5	0.00745	0.00385	0.00001	0.00794	0.21719
		1	0.00488	0.00540	-0.00016	0.01092	0.23921
		2.5	-0.01212	0.00609	-0.00035	0.01137	0.24578
	2.5	0.5	0.01618	0.00328	0.00051	0.00721	0.18916
		1	0.01282	0.00463	0.00061	0.00878	0.22430
		2.5	0.00079	0.00595	0.00011	0.00946	0.24050
(45,45)	0.5	0.5	-0.00109	0.00272	0.00005	0.00565	0.18013
		1	-0.00257	0.00252	-0.00017	0.00612	0.17614
		2.5	-0.01217	0.00245	-0.00067	0.00495	0.16001
	1	0.5	0.00341	0.00273	0.00015	0.00568	0.17623
		1	0.00153	0.00291	0.00016	0.00620	0.18392
		2.5	-0.01032	0.00291	-0.00044	0.00575	0.17968
	2.5	0.5	0.01330	0.00247	0.00049	0.00425	0.15992
		1	0.01011	0.00280	0.00024	0.00563	0.17922
		2.5	0.00156	0.00318	-0.00027	0.00585	0.18210
			$\theta_1 = 2.5, \theta_2 = 2.5$				
(15,15)	0.5	0.5	0.00040	0.00579	-0.00011	0.01526	0.28406
		1	-0.01288	0.00607	-0.00029	0.01378	0.28355
		2.5	-0.01485	0.00638	-0.00015	0.01415	0.28016
	1	0.5	0.01261	0.00607	-0.00026	0.01438	0.28583
		1	0.00009	0.00626	0.00025	0.01462	0.28678
		2.5	-0.00242	0.00691	-0.00016	0.01478	0.28314
	2.5	0.5	0.01269	0.00643	0.00027	0.01414	0.27722
		1	0.00389	0.00652	0.00007	0.01436	0.28739
		2.5	0.00205	0.00679	0.00017	0.01484	0.28409
(15,45)	0.5	0.5	0.02341	0.00476	0.00016	0.01130	0.23810
		1	0.01626	0.00416	-0.00006	0.01075	0.22649
		2.5	0.00882	0.00381	-0.00031	0.00874	0.22081
	1	0.5	0.03083	0.00580	-0.00002	0.01207	0.24587
		1	0.02296	0.00495	-0.00005	0.01097	0.23996
		2.5	0.02187	0.00495	0.00026	0.01109	0.23511
	2.5	0.5	0.02962	0.00553	0.00032	0.01124	0.24398
		1	0.02614	0.00539	-0.00018	0.01106	0.24168
		2.5	0.02202	0.00537	-0.00013	0.01104	0.23886
(45,15)	0.5	0.5	-0.02216	0.00468	0.00021	0.00996	0.23986
		1	-0.03196	0.00560	-0.00015	0.01133	0.24466
		2.5	-0.03469	0.00601	-0.00015	0.01098	0.24424
	1	0.5	-0.01567	0.00408	0.00004	0.00954	0.22711
		1	-0.02461	0.00519	-0.00010	0.01057	0.24006
		2.5	-0.02591	0.00554	0.00005	0.01209	0.24099

Table 4. Continued.

(n,m)	θ_1	θ_2	$\tilde{\delta}_{BE}$ Bias	$\tilde{\delta}_{BE}$ MSE	$\tilde{\delta}_{MCMC}$ Bias	$\tilde{\delta}_{MCMC}$ MSE	AL HPD
	2.5	0.5	-0.01205	0.00401	0.00013	0.00847	0.21979
		1	-0.02212	0.00498	-0.00001	0.00997	0.23418
		2.5	-0.02323	0.00543	-0.00037	0.01101	0.23821
(45,45)	0.5	0.5	0.00078	0.00267	-0.00011	0.00608	0.18275
		1	-0.00396	0.00274	-0.00012	0.00532	0.18062
		2.5	-0.00976	0.00271	-0.00026	0.00548	0.17362
	1	0.5	0.00459	0.00271	0.00012	0.00617	0.17990
		1	-0.00092	0.00274	0.00005	0.00681	0.17984
		2.5	-0.00293	0.00284	0.00004	0.00523	0.17786
	2.5	0.5	0.00630	0.00270	0.00018	0.00524	0.17588
		1	0.00285	0.00283	-0.00011	0.00624	0.17893
		2.5	0.00125	0.00293	-0.00011	0.00514	0.17662
	θ_1	θ_2			$\beta_1 = 0.5, \beta_2 = 0.5$		
(15,15)	0.5	1	-0.02349	0.00645	0.00011	0.01117	0.26842
		2.5	-0.06964	0.00976	-0.00002	0.00900	0.23287
	1	0.5	0.02643	0.00688	0.00016	0.01338	0.26716
		2.5	-0.07028	0.01101	-0.00021	0.01200	0.25969
	2.5	0.5	0.06120	0.00884	0.00002	0.00871	0.23368
		1	0.05290	0.00857	0.00009	0.01250	0.25881
(15,45)	0.5	1	-0.03165	0.00521	-0.00015	0.00904	0.22225
		2.5	-0.05356	0.00638	-0.00008	0.00634	0.18842
	1	0.5	0.01048	0.00477	-0.00006	0.00850	0.23145
		2.5	-0.04578	0.00624	-0.00012	0.00874	0.21583
	2.5	0.5	0.04186	0.00550	-0.00016	0.00822	0.20445
		1	0.04796	0.00646	-0.00011	0.00828	0.22373
(45,15)	0.5	1	-0.00793	0.00456	0.00015	0.00935	0.22927
		2.5	-0.05198	0.00641	0.00004	0.00745	0.20306
	1	0.5	0.02686	0.00511	0.00012	0.00802	0.22198
		2.5	-0.04761	0.00644	-0.00003	0.00818	0.22353
	2.5	0.5	0.05274	0.00636	0.00032	0.00630	0.19044
		1	0.03252	0.00506	0.00048	0.00811	0.21923
		2.5	-0.02279	0.00468	-0.00037	0.01051	0.23968
(45,45)	0.5	1	-0.01180	0.00270	-0.00014	0.00521	0.17455
		2.5	-0.03562	0.00344	0.00013	0.00469	0.15062
	1	0.5	0.01100	0.00270	-0.00024	0.00505	0.17373
		2.5	-0.02700	0.00320	-0.00031	0.00582	0.17146
	2.5	0.5	0.03465	0.00334	0.00015	0.00401	0.15367
		1	0.02561	0.00310	-0.00025	0.00490	0.16935
		2.5	-0.00086	0.00270	-0.00018	0.00568	0.18196
					$\beta_1 = 1, \beta_2 = 1$		
(15,15)	0.5	1	-0.02325	0.00762	-0.00022	0.01152	0.27443
		2.5	-0.05993	0.00841	-0.00063	0.00872	0.22214
	1	0.5	0.01846	0.00716	-0.00019	0.01140	0.27157
		2.5	-0.04577	0.00822	-0.00024	0.01205	0.25810
	2.5	0.5	0.06010	0.00843	0.00041	0.00956	0.21719
		1	0.04753	0.00832	0.00065	0.01262	0.26340
(15,45)	0.5	1	-0.02487	0.00536	0.00017	0.00820	0.22588
		2.5	-0.04093	0.00474	-0.00036	0.00577	0.18250
	1	0.5	0.00872	0.00506	-0.00013	0.00960	0.23105
		2.5	-0.02586	0.00509	-0.00005	0.00835	0.21590

Table 4. Continued.

(n,m)	θ_1	θ_2	$\tilde{\delta}_{BE}$ Bias	$\tilde{\delta}_{BE}$ MSE	$\tilde{\delta}_{MCMC}$ Bias	$\tilde{\delta}_{MCMC}$ MSE	AL HPD
	2.5	0.5	0.04527	0.00522	0.00016	0.00637	0.18499
		1	0.04068	0.00562	-0.00041	0.00834	0.21162
(45,15)	0.5	1	-0.00802	0.00543	-0.00027	0.00995	0.23159
		2.5	-0.04459	0.00510	-0.00050	0.00723	0.18846
	1	0.5	0.02284	0.00493	-0.00006	0.00834	0.22646
		2.5	-0.04249	0.00562	0.00038	0.00815	0.21294
	2.5	0.5	0.03903	0.00489	-0.00001	0.00469	0.17763
		1	0.02433	0.00491	0.00003	0.00878	0.21565
(45,45)	0.5	1	-0.01051	0.00292	-0.00018	0.00511	0.17831
		2.5	-0.02480	0.00257	0.00004	0.00316	0.14081
	1	0.5	0.01251	0.00283	-0.00020	0.00542	0.17660
		2.5	-0.02098	0.00288	0.00012	0.00482	0.16499
	2.5	0.5	0.02616	0.00257	0.00002	0.00338	0.14053
		1	0.02188	0.00301	-0.00023	0.00477	0.16435
			$\beta_1 = 2.5, \beta_2 = 2.5$				
(15,15)	0.5	1	-0.01051	0.00782	-0.00005	0.01498	0.26972
		2.5	-0.03537	0.00520	0.00026	0.00647	0.18465
	1	0.5	0.01433	0.00758	0.00001	0.01303	0.27119
		2.5	-0.03281	0.00747	0.00055	0.01105	0.24155
	2.5	0.5	0.03630	0.00539	-0.00025	0.00632	0.18530
		1	0.03481	0.00729	0.00019	0.01184	0.24972
		2.5	-0.00100	0.00695	0.00045	0.01570	0.28502
(15,45)	0.5	1	-0.01226	0.00529	0.00001	0.00953	0.22829
		2.5	-0.01858	0.00302	0.00019	0.00438	0.16256
	1	0.5	0.00995	0.00419	0.00009	0.00891	0.22543
		2.5	-0.01061	0.00444	0.00017	0.00789	0.21230
	2.5	0.5	0.03247	0.00360	0.00018	0.00443	0.15731
		1	0.03501	0.00517	0.00019	0.00740	0.20497
		2.5	0.02326	0.00545	-0.00013	0.01101	0.23584
(45,15)	0.5	1	-0.00905	0.00500	-0.00007	0.00912	0.22523
		2.5	-0.03367	0.00366	0.00007	0.00443	0.15285
	1	0.5	0.01333	0.00535	-0.00029	0.00894	0.22642
		2.5	-0.03363	0.00525	0.00047	0.00724	0.19670
	2.5	0.5	0.01990	0.00330	-0.00039	0.00443	0.15955
		1	0.01230	0.00432	-0.00013	0.00803	0.20943
		2.5	-0.02176	0.00543	0.00012	0.00977	0.23590
(45,45)	0.5	1	-0.00759	0.00289	-0.00011	0.00549	0.17320
		2.5	-0.01625	0.00173	0.00014	0.00282	0.12160
	1	0.5	0.00820	0.00280	-0.00007	0.00532	0.17339
		2.5	-0.01285	0.00247	0.00018	0.00407	0.15444
	2.5	0.5	0.01518	0.00166	0.00005	0.00311	0.12565
		1	0.01319	0.00243	-0.00026	0.00404	0.15304
		2.5	0.00046	0.00292	-0.00018	0.00641	0.17632

6. Conclusions

In this paper, we compare different methods of estimating the stress-strength parameter δ of a two-parameter ER model with different shape and scale parameters. We have compared the performance of the BEs under the SEL function with the performance of the MLE using Monte Carlo simulation. As well as the respective bootstrap CIs and CrIs using the Gibbs sampling technique. Further, the details have been explained using two data sets, one simulated data and the other real-life data. It is evident that the BEs based on the SEL function outperform the MLEs and the MCMC method. Also, it is clearly observed that the Boot-p CIs compete with all other CIs and CrIs discussed in this article. Since stress-strength measures are important and can be used in many areas beyond just checking reliability, our study can be useful in real-life fields such as reliability testing, medical sciences, biological research, electrical product quality checks, and communication systems. We can conduct studies on additional distributions that are relevant in this field. Further research in this area is required.

Appendix

The Fisher information matrix of $\theta = (\alpha_1, \sigma_1, \alpha_2, \sigma_2)$ is defined as $£(\theta) = E\left(\Im(\theta)\right)$, where $\Im(\theta) = \left(\Im_{ij}(\theta)\right)$

$$\Im(\theta) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \theta_1^2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \beta_2} \\ \frac{\partial^2 \ell}{\partial \beta_1 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \beta_1^2} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \theta_2} & \frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 \ell}{\partial \theta_2 \theta_1} & \frac{\partial^2 \ell}{\partial \theta_2 \partial \beta_1} & \frac{\partial^2 \ell}{\partial \theta_2^2} & \frac{\partial^2 \ell}{\partial \theta_2 \partial \beta_2} \\ \frac{\partial^2 \ell}{\partial \beta_2 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \beta_2 \beta_1} & \frac{\partial^2 \ell}{\partial \beta_2 \partial \theta_2} & \frac{\partial^2 \ell}{\partial \beta_2^2} \end{pmatrix} = \begin{pmatrix} \Im_{11} & \Im_{12} & \Im_{13} & \Im_{14} \\ \Im_{21} & \Im_{22} & \Im_{23} & \Im_{24} \\ \Im_{31} & \Im_{32} & \Im_{33} & \Im_{34} \\ \Im_{41} & \Im_{42} & \Im_{43} & \Im_{44} \end{pmatrix}.$$

It is easy to see that

$$\Im_{11} = \sum_{i=1}^{n} \left(\theta_1 + \frac{x_i}{\beta_1^2}\right)^{-2}, \quad \Im_{12} = -\sum_{i=1}^{n} \frac{2x_i}{\left(\theta_1 + x_i/\beta_1^2\right)^2 \beta_1^3}, \quad \Im_{13} = \Im_{31} = \Im_{14} = \Im_{41} = 0,$$

$$\Im_{21} = \Im_{12},\ \Im_{22} = -\sum_{i=1}^{n} \frac{2x_i\left(x_i + 3\theta_1\beta_1^2\right)}{\left(\theta_1 + x_i/\beta_1^2\right)^2 \beta_1^6} + \frac{3\phi_2}{\beta_1^4},\ \Im_{23} = \Im_{32} = \Im_{24} = \Im_{42} = 0,$$

$$\Im_{33} = \sum_{j=1}^{m} \left(\theta_2 + \frac{y_j}{\beta_2^2}\right)^{-2},\ \Im_{34} = -\sum_{j=1}^{m} \frac{2y_j}{\left(\theta_2 + y_j/\beta_2^2\right)^2 \beta_2^3} = \Im_{43},$$

$$\Im_{44} = -\sum_{j=1}^{m} \frac{2y_j\left(y_j + 3\theta_2\beta_2^2\right)}{\left(\theta_2 + y_j/\beta_2^2\right)^2 \beta_2^6} + \frac{3\phi_2^*}{\beta_2^4}.$$

By using the integrals of the forms

$$\int_0^\infty \frac{t}{t+a} \exp\left(-\frac{1}{b}(t+a)^2\right) \mathrm{d}t = \frac{\sqrt{b\pi}}{2}\mathrm{Erfc}\left[\frac{a}{\sqrt{b}}\right] - \frac{a}{2}\Gamma\left(0, \frac{a^2}{b}\right),$$

$$\int_0^\infty t^2 (t+a) \exp\left(-\frac{1}{b}(t+a)^2\right) \mathrm{d}t = \frac{b^2}{2}\exp\left(-\frac{a^2}{b}\right) - \frac{1}{2}ab\sqrt{\pi b}\mathrm{Erfc}\left[\frac{a}{\sqrt{b}}\right],$$

we obtain the elements of the Fisher information matrix as

$$\begin{aligned}
£_{11} &= \frac{n\beta_1^2}{2}\,\eta(\theta_1,\beta_1), \quad £_{12} = £_{21} = n\theta_1\beta_1\eta(\theta_1,\beta_1) - n\sqrt{2\pi}\Lambda(\theta_1,\beta_1), \\
£_{22} &= \frac{4n}{\beta_1^2} + 2n\theta_1^2\eta(\theta_1,\beta_1) - 4n\sqrt{2\pi}\frac{\theta_1}{\beta_1}\Lambda(\theta_1,\beta_1), \\
£_{33} &= \frac{m\beta_2^2}{2}\,\eta(\theta_2,\beta_2), \quad £_{34} = £_{43} = m\theta_2\beta_2\eta(\theta_2,\beta_2) - m\sqrt{2\pi}\Lambda(\theta_2,\beta_2), \\
£_{44} &= \frac{4m}{\beta_2^2} + 2m\theta_2^2\eta(\theta_2,\beta_2) - 4m\sqrt{2\pi}\frac{\theta_2}{\beta_2}\Lambda(\theta_2,\beta_2),
\end{aligned}$$

where $\Lambda(\theta,\beta)$ is as in Eq (1.1) and

$$\eta(\theta,\beta) = \Gamma\left(0, \frac{1}{2}\theta^2\beta^2\right)\exp\left(\frac{1}{2}\theta^2\beta^2\right).$$

Based on the limiting standard-normal distribution, we obtain the elements of the variance-covariance matrix as

$$s_{ij} = \lim_{n\to\infty}\frac{£_{ij}}{n}, \quad i,j = 1,2, \quad \text{and} \quad s_{ij} = \lim_{m\to\infty}\frac{£_{ij}}{m}, \quad i,j = 3,4.$$

Author contributions

Mohammed S. Kotb: Methodology, writing-original draft, writing-review & editing, software; Ghaithah A. Alzhrani: Writing-original draft, formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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