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Research article

Temporal topological operators with picture fuzzy multifunctions

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Abstract: This paper introduces the notion of a temporal picture fuzzy modal topological structure (TPFMTS). These structures are based on novel temporal picture fuzzy topological operators as common modal topological operators standing for the closure and interior operators. The paper discusses several fundamental properties of these new TPFMTSs. The constructed temporal picture fuzzy modal topological operators establish that some properties that were considered satisfactory in the temporal intuitionistic fuzzy modal topological structures, as defined by Atanassov in 2023, are not fulfilled. Also, we introduce the concepts of TPF^u and TPF^l-continuous and almost continuous multifunctions. Several properties and characterizations of the presented continuous multifunctions and their types of continuity are established. Some examples are given to explain the correct implications between these notions.

Keywords: temporal picture fuzzy set; picture fuzzy modal operator; picture fuzzy topological structure; topology; ideal; multifunction

1. Introduction

Fuzzification is a crucial tool for addressing humanistic systems in our real-life problems. The first paper on fuzzy set theory was authored by Zadeh in 1965 ([23]). This approach of fuzzy sets (FS) has been widely applied by many scholars. Fuzzy sets described the positivism of an element ϱ of a universal set $\aleph$ to a subset $\mathbb{G} \subseteq \aleph$ by the membership value $\omega_{\mathbb{G}}(\varrho)$, and posited that the negativism of that element $\varrho \in \aleph$ to the set $\mathbb{G}$ is $1 - \omega_{\mathbb{G}}(\varrho)$. Atanassov in [4] based his theory of intuitionistic fuzzy sets (IFS) on the notion of the negativism $\varpi_{\mathbb{G}}(\varrho)$ of an element $\varrho \in \aleph$ to a subset $\mathbb{G} \subseteq \aleph$ that may range from $[0, 1]$ and need not be the complement of the positivism of that element $\varrho \in \aleph$ to $\mathbb{G}$. The values $\omega_{\mathbb{G}}(\varrho)$ and $\varpi_{\mathbb{G}}(\varrho)$ represent the positivism and negativism of each $\varrho \in \aleph$ to $\mathbb{G}$, respectively, with the condition that $0 \leq \omega_{\mathbb{G}}(\varrho) + \varpi_{\mathbb{G}}(\varrho) \leq 1$. In this way, Atanassov encompassed all the fuzzy sets (FS) as a special case of his theory whenever $\omega_{\mathbb{G}}(\varrho) + \varpi_{\mathbb{G}}(\varrho) = 1$. Intuitionistic fuzzy sets (IFS) are more meaningful and applicable to our real-life cases. Cuong in [11] initiated the theory of picture fuzzy sets (PFS) by adding the neutralism of an element $\varrho \in \aleph$ to the subset $\mathbb{G}$, represented by $\sigma_{\mathbb{G}}(\varrho)$. This definition is conditioned with $0 \leq \omega_{\mathbb{G}}(\varrho) + \varpi_{\mathbb{G}}(\varrho) + \sigma_{\mathbb{G}}(\varrho) \leq 1$. In case where $\sigma_{\mathbb{G}}(\varrho) = 0$ for all $\varrho \in \aleph$, then we revert to intuitionistic sets $\mathbb{G}$ in (IFS). Moreover, if $\varpi_{\mathbb{G}}(\varrho) = 1 - \omega_{\mathbb{G}}(\varrho)$ for all $\varrho \in \aleph$, then we revert to a fuzzy set $\mathbb{G}$ in (FS). There are several simple modifications for the intuitionistic fuzzy sets [7], which we shall not discuss here. These modifications include Pythagorean fuzzy sets [16, 21], Fermatean fuzzy sets [18], spherical fuzzy sets [3], q-rung orthopair fuzzy sets [2, 13, 22] and q-rung orthopair picture fuzzy sets [14]. Picture fuzzy modal operators are discussed in [1, 19]. A wide study on the algebraic properties of picture fuzzy sets is found in [20].

The motivation of this paper is to compile the definitions and notions from regions of general topology, of standard modal logic [5, 10, 12] and of picture fuzziness, and to represent the concept of a temporal picture fuzzy modal topological structure, briefly TPFMTS. Picture fuzzy topological spaces are initiated in [17]. There are some examples of TPFMT structures (TPFMTS), based upon the classical picture fuzzy operations "union" ($\cup$) and "intersection" ($\cap$). Also, we presented the general form of temporal picture fuzzy continuous multifunctions. In this paper, we merge the classical definitions of multifunctions in general topology and the standard modal logic [8, 15] with the notion of picture fuzzy sets, further expanding into the realm of TPFMTS. This exploration includes the creation of TPFMTSs facilitated by the standard picture fuzzy operations of "union" ($\cup$) and "intersection" ($\cap$).

The layout of this paper is as follows: Section 1 is an introduction. Section 2 is given for the required basics. Section 3 presents temporal picture fuzzy modal topological structures related to the picture fuzzy sets and studies some important results, including several modal operators. Section 4 investigates the notions of temporal picture fuzzy upper and lower almost continuous multifunctions and introduces many characteristic properties of these defined multifunctions. Section 5 presents the conclusion.

The research on TPFMTS has several important applications in various domains: decision making, pattern recognition, artificial intelligence, information retrieval and data mining. TPFMTS address critical gaps in handling uncertainty, imprecision, and neutrality, which are inherent in real-life problems across diverse domains. To bridge these gaps, picture fuzzy sets (PFS) were introduced, adding a neutrality component to the membership and non-membership values, thereby enabling a more nuanced representation of uncertainty. TPFMTS expands upon these concepts by the integration in modal logic and general topology using the picture fuzzy sets. This integration introduces global operators, such as closure and interior, which modify classical topological and modal relationships. These global operators facilitate a robust analysis of fuzzy sets under modal and topological constraints, providing a suitable tools for theoretical exploration and practical application. The study of TPFMTS not only extends the theory of fuzzy sets but also establishes a wide platform for addressing modern computational challenges. Its ability to integrate neutrality, positivity, and negativity within a unified framework lays the foundation for further exploration and application of TPFMTS in dynamic systems, hybrid models, and emerging technologies, positioning it as a cornerstone of modern mathematical and computational innovation.

2. Preliminaries

Through the paper, denote $I = [0, 1], I_0 = (0, 1]$, and $I_1 = [0, 1)$. Let a universe set $\aleph$, be fixed. An PFS $\mathbb{G}$ in $\aleph$ is an object of the following form: $\mathbb{G} = \{\langle \varrho, \omega_{\mathbb{G}}(\varrho), \varpi_{\mathbb{G}}(\varrho), \sigma_{\mathbb{G}}(\varrho)\rangle \,|\varrho \in \aleph\}$, where $\omega_{\mathbb{G}}(\varrho) \in I$ is called the degree of positive membership of ϱ in $\mathbb{G}$, $\varpi_{\mathbb{G}}(\varrho) \in I$ is called the degree of negative membership of ϱ in $\mathbb{G}$, and $\sigma_{\mathbb{G}}(\varrho) \in I$ is called the degree of neutral membership of ϱ in $\mathbb{G}$, and where $\omega_{\mathbb{G}}(\varrho)$, $\varpi_{\mathbb{G}}(\varrho)$, and $\sigma_{\mathbb{G}}(\varrho)$ satisfy the following condition: $0 \le \omega_{\mathbb{G}}(\varrho) + \varpi_{\mathbb{G}}(\varrho) + \sigma_{\mathbb{G}}(\varrho) \le 1$ for all $\varrho \in \aleph$. Let $\mathfrak{G}$ be a non-empty set (finite or infinite), called the temporal scale with upper and lower boundaries. The elements of $\mathfrak{G}$ are called “time-moments”. We define the temporal PFS (TPFS) as follows:

$\mathbb{G}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\rangle \,|\, \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$, where

(1) $\mathbb{G} \subseteq \aleph$ is a fixed set,

(2) $\omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) + \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) + \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \le 1$ for every $\langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$,

(3) $\omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)$, and $\sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)$ are the degree of positive membership, negative membership and neutral membership, respectively, of the element $\varrho \in \aleph$ at the time-moment $\mathfrak{g} \in \mathfrak{G}$.

Note: each ordinary PFS can be regarded as a TPFS for which $\mathfrak{G}$ is a singleton set.

Additionally, we mentioned that all operations and operators on the PFSs can be defined for the TPFSs. However, the opposite is also true: each TPFS $\mathbb{G}(\mathfrak{G})$is a standard PFS, but over universe $\aleph \times \mathfrak{G}$. For this reason, we can re-define all operations, relations, and operators defined over standard PFSs, now over TPFSs.

Definition 2.1. *[6, 17]*

Let $\aleph$ be a nonempty set, $\mathfrak{G}$ time-scale, and $\mathbb{G}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\rangle$ $|\, \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$ and $\mathbb{H}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)\rangle \,|\, \langle \varrho, \mathfrak{g}\rangle$ $\in \aleph \times \mathfrak{G}\}$. Then,

(1) $\mathbb{G}(\mathfrak{G}) \subseteq \mathbb{H}(\mathfrak{G})$iff $\forall \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$, $\omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \le \omega_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)$, $\varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \ge \varpi_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)$, and $\sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \le \sigma_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)$ or $\sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \ge \sigma_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)$.

(2) $\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \vee \omega_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \wedge \varpi_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)\rangle \,|$ $\langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}\}$.

(3) $\mathbb{G}(\mathfrak{G}) \cap \mathbb{H}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \wedge \omega_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \vee \varpi_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle \varrho, \mathfrak{g}\rangle)\rangle \,|$ $\langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}\}$.

(4) $\daleth\, \mathbb{G}(\mathfrak{G}) = \{\langle\langle \varrho, \mathfrak{g}\rangle, \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \omega_{\mathbb{G}}(\langle \varrho\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\rangle \,|\, \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$.

(5) $\mathbb{G}(\mathfrak{G}) \barwedge \mathbb{H}(\mathfrak{G}) = \{\langle\langle\varrho, \mathfrak{g}\rangle, 0, 1, 0\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$ *if* $\mathbb{G}(\mathfrak{G}) \subseteq \mathbb{H}(\mathfrak{G})$, *and* $\mathbb{G}(\mathfrak{G}) \barwedge \mathbb{H}(\mathfrak{G}) = \mathbb{G}(\mathfrak{G}) \cap (\exists\, \mathbb{H}(\mathfrak{G}))$ *otherwise.*

(6) $\sharp(\mathfrak{G}) = \{\langle\langle\varrho, \mathfrak{g}\rangle, 1, 0, 0\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}, \flat(\mathfrak{G}) = \{\langle\langle\varrho, \mathfrak{g}\rangle, 0, 1, 0\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$.

(7) $\natural(\mathfrak{G}) = \{\langle\langle\varrho, \mathfrak{g}\rangle, 0, 0, 1\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}, \mho(\mathfrak{G}) = \{\langle\langle\varrho, \mathfrak{g}\rangle, 0, 0, 0\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$.

The map $\mathfrak{F} : \aleph \times \mathfrak{G} \looparrowright \Upsilon \times \mathfrak{G}$ is called a temporal picture fuzzy multifunction ($TPFM$, for short) for any $\langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$, $\mathfrak{F}(\langle\varrho, \mathfrak{g}\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$. The degree of membership of $\langle\zeta, \mathfrak{g}\rangle \in \Upsilon \times \mathfrak{G}$ at the time moment $\mathfrak{g} \in \mathfrak{G}$ is denoted by: $\mathfrak{F}(\langle\varrho, \mathfrak{g}\rangle)(\langle\zeta, \mathfrak{g}\rangle) = \Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$. The domain of $\mathfrak{F}$, denoted by $D(\mathfrak{F})$, and the range of $\mathfrak{F}$, denoted by $R(\mathfrak{F})$, are defined by: for any $\langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$ and $\langle\zeta, \mathfrak{g}\rangle \in (\Upsilon \times \mathfrak{G})$, $D(\mathfrak{F})(\langle\varrho, \mathfrak{g}\rangle) = \bigcup_{\langle\zeta,\mathfrak{g}\rangle\in(\Upsilon\times\mathfrak{G})} \Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$ and $R(\mathfrak{F})(\langle\zeta, \mathfrak{g}\rangle) = \bigcup_{\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}} \Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$. $\mathfrak{F}$ is called crisp iff $\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle) = \sharp(\mathfrak{G})\ \forall\ \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$ and $\langle\zeta, \mathfrak{g}\rangle \in (\Upsilon \times \mathfrak{G})$. $\mathfrak{F}$ is called Normalized ($NTPFM$, for short) iff $\forall\ \langle\varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}$, there exists $\langle\zeta_0, \mathfrak{g}\rangle \in (\Upsilon \times \mathfrak{G})$ such that $\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta_0, \mathfrak{g}\rangle) = \sharp(\mathfrak{G})$. $\mathfrak{F}$ is called surjective iff $R(\mathfrak{F})\langle\zeta, \mathfrak{g}\rangle = \sharp(\mathfrak{G})\ \forall\ \langle\zeta, \mathfrak{g}\rangle \in (\Upsilon \times \mathfrak{G})$. The inverse of $\mathfrak{F}$ denoted by $\mathfrak{F}^- : \Upsilon \looparrowright \aleph$ is a $TPFM$ defined by: $\mathfrak{F}^-(\langle\zeta, \mathfrak{g}\rangle)(\langle\varrho, \mathfrak{g}\rangle) = \mathfrak{F}(\langle\varrho, \mathfrak{g}\rangle)((\langle\zeta, \mathfrak{g}\rangle)) = \Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$. One easily verifies that $D(\mathfrak{F}^-) = R(\mathfrak{F})$ and $D(\mathfrak{F}) = R(\mathfrak{F}^-)$. The image $\mathfrak{F}\mathbb{G}(\mathfrak{G}))$ of $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, the lower inverse $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$ and the upper inverse $\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ of $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$ are defined respectively as follows:

$$\mathfrak{F}(\mathbb{G}(\mathfrak{G}))\langle\zeta, \mathfrak{g}\rangle = \bigcup_{\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}} \left[\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle) \cap \mathbb{G}(\mathfrak{G})(\langle\varrho, \mathfrak{g}\rangle)\right],$$

$$\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))\langle\varrho, \mathfrak{g}\rangle = \bigcup_{\langle\zeta,\mathfrak{g}\rangle\in(\Upsilon\times\mathfrak{G})} \left[\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle) \cap \mathbb{U}(\mathfrak{G})\langle\zeta, \mathfrak{g}\rangle\right],$$

$$\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))\langle\varrho, \mathfrak{g}\rangle = \bigcap_{\langle\zeta,\mathfrak{g}\rangle\in(\Upsilon\times\mathfrak{G})} \left[\exists\, \Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle) \cup \mathbb{U}(\mathfrak{G})\langle\zeta, \mathfrak{g}\rangle\right].$$

Definition 2.2. *[1] A temporal picture fuzzy topology on* $\aleph$ *is a map* τ*:* $\left(I^3\right)^{\aleph\times\mathfrak{G}} \to I^3$ *defined by* $\tau(\mathbb{G}(\mathfrak{G})) = \langle\omega_\tau(\mathbb{G}(\mathfrak{G})), \varpi_\tau(\mathbb{G}(\mathfrak{G})), \sigma_\tau(\mathbb{G}(\mathfrak{G}))\rangle$ *which satisfies the following properties:*

(1) $\tau(\flat(\mathfrak{G})) = \tau(\sharp(\mathfrak{G})) = \langle 1, 0, 0\rangle$.

(2) $\tau(\mathbb{G} \cap \mathbb{H}) \geq \tau(\mathbb{G}) \wedge \tau(\mathbb{H})$*, for each* $\mathbb{G}, \mathbb{H} \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$.

(3) $\tau(\bigcup_{i\in\Gamma} \mathbb{G}_i) \geq \bigwedge_{i\in\Gamma} \tau(\mathbb{G}_i)$*, for each* $\mathbb{G}_i \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$*,* $i \in \Gamma$.

The pair $(\aleph, \tau)$ is called a temporal picture fuzzy topological space in Šostak's sense. For any $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ the number $\omega_\tau(\mathbb{G}(\mathfrak{G}))$ is called the openness degree at a certain times , $\varpi_\tau(\mathbb{G}(\mathfrak{G}))$ is called the non openness degree at a certain times, while $\sigma_\tau(\mathbb{G}(\mathfrak{G}))$ is called the neutral degree at a certain times. For $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$

$$cl_\tau(\mathbb{G}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle) = \bigcap\{\mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}} : \mathbb{G}(\mathfrak{G}) \subseteq \mathbb{H}(\mathfrak{G}), \tau(\exists\, \mathbb{H}(\mathfrak{G})) \geq \langle\varsigma, \varkappa, \vartheta\rangle\},$$
$$int_\tau(\mathbb{G}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle) = \bigcup\{\mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}} : \mathbb{G}(\mathfrak{G}) \supseteq \mathbb{H}(\mathfrak{G}), \tau(\mathbb{H}(\mathfrak{G})) \geq \langle\varsigma, \varkappa, \vartheta\rangle\}.$$

Definition 2.3. *[1] Let* $\mathfrak{F} : (\aleph, \tau) \looparrowright (\Upsilon, \sigma)$ *be a* $TPFM$*,* $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$*. Then,* $\mathfrak{F}$ *is called:*

(1) TPF^uS-continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$ and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$, such that $\mathbb{G}(\mathfrak{G}) \cap D(\mathfrak{F}) \subseteq \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$.

(2) TPF^lS-continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$, such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$.

Remark 2.4. (1) *If $\mathfrak{F}$ is $NTPFM$, then $\mathfrak{F}$ is TPF^uS -continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma,\varkappa,\vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$, such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$.*

Definition 2.5. *[1] The map $\mathfrak{L}^P$: $\left(I^3\right)^{\aleph\times\mathfrak{G}} \to I^3$ is called temporal picture fuzzy ideal on $\aleph$ if it satisfies the following conditions for $\mathbb{G}(\mathfrak{G}), \mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$:*

(1) $\mathfrak{L}^P(\flat(\mathfrak{G})) = \langle 1,0,0\rangle$, $\mathfrak{L}^P(\sharp(\mathfrak{G})) = \langle 0,1,0\rangle$.

(2) $\mathbb{G}(\mathfrak{G}) \subseteq \mathbb{H}(\mathfrak{G}) \Rightarrow \mathfrak{L}^P(\mathbb{G}(\mathfrak{G})) \geq \mathfrak{L}^P(\mathbb{H}(\mathfrak{G}))$.

(3) $\mathfrak{L}^P(\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G})) \geq \mathfrak{L}^P(\mathbb{G}(\mathfrak{G})) \wedge \mathfrak{L}^P(\mathbb{H}(\mathfrak{G}))$.

Also, $\mathfrak{L}^P$ is called proper if $\mathfrak{L}^P(\sharp(\mathfrak{G})) = \langle 0,1,0\rangle$ and there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ such that $\mathfrak{L}^P(\mathbb{G}(\mathfrak{G})) > \langle 0,1,0\rangle$.

If $\mathfrak{L}^P_1$ and $\mathfrak{L}^P_2$ are temporal picture fuzzy ideals on $\aleph\times\mathfrak{G}$, we say that $\mathfrak{L}^P_1$ is finer than $\mathfrak{L}^P_2$ ($\mathfrak{L}^P_2$ is coarser than $\mathfrak{L}^P_1$), denoted by $\mathfrak{L}^P_2 \subseteq \mathfrak{L}^P_1$, iff $\mathfrak{L}^P_2(\mathbb{G}(\mathfrak{G})) \leq \mathfrak{L}^P_1(\mathbb{G}(\mathfrak{G}))\ \forall\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$. Let us define the special picture fuzzy ideals $\mathfrak{L}^{P0}$, $\mathfrak{L}^{P1}$ by

$$\mathfrak{L}^{P0}(\mathbb{G}(\mathfrak{G})) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{G}(\mathfrak{G}) = \flat(\mathfrak{G}), \\ \langle 0,1,0\rangle & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathfrak{L}^{P1}(\mathbb{G}(\mathfrak{G})) = \begin{cases} \langle 0,1,0\rangle & \text{if } \mathbb{G}(\mathfrak{G}) = \sharp(\mathfrak{G}), \\ \langle 1,0,0\rangle & \text{otherwise.} \end{cases}$$

3. Temporal picture fuzzy modal topological structures

In a trial to extend the conditions (C1–C9) presented in [6, 9], we will present the coming four kinds of TPFTSs.

cl-cl-TPFTS, *in-in*-TPFTS, *cl-in*-TPFTS, and *in-cl*-TPFTS. The *cl-cl*-TPFTS is the object $\left\langle\left(I^3\right)^{\aleph\times\mathfrak{G}}, O, \mathfrak{M}, \mathfrak{N}, \circ\right\rangle$ where $\mathfrak{G}$ is a fixed time-scale, $\aleph$ is the universe set, $O : \left(I^3\right)^{\aleph\times\mathfrak{G}} \to \left(I^3\right)^{\aleph\times\mathfrak{G}}$ is an operator stands for the closure type, $\mathfrak{M}, \mathfrak{N} : \left(I^3\right)^{\aleph\times\mathfrak{G}} \times \left(I^3\right)^{\aleph\times\mathfrak{G}} \to \left(I^3\right)^{\aleph\times\mathfrak{G}}$ are operations over TPFS, $\circ : \left(I^3\right)^{\aleph\times\mathfrak{G}} \to \left(I^3\right)^{\aleph\times\mathfrak{G}}$ is a temporal operator referring to the closure type.

Now, from Definition 2.1, we can see that those four conditions (C1–C4) in [6, 9] refer to a topological operator "closure", and those conditions (C5–C8) in [6, 9] refer again to a topological operator but as a temporal topological operator "closure", and that condition (CC9) in [6, 9] determines the relation between the two types of operators. On the other hand, if we have a set of

PFTSs for which we know that they are related to fixed time-moments $\mathfrak{g}_1 < \mathfrak{g}_2 < ...$, respectively, then on their basis we can construct a TPFTS. Few conditions are invalid for some objects that resemble TPFTS. In certain cases, some conditions are nonexistent, whilst in other instances the relations in the conditions are exchanged with weak (feeble) relations. Denoted for these structures the word "feeble", then we can name these structures "Temporal Picture Fuzzy Feeble Topological Structure" (TPFFTS).

Theorem 3.1. *For any universal set* $\aleph$ *and for any time-scale* $\mathfrak{G}$ *, then* $\left\langle \left(I^3\right)^{\aleph\times\mathfrak{G}}, cl_\cap, \cup, \cap, cl^\circ_\cap \right\rangle$ *is a* cl-cl-*TPFFTS for which, in condition CC5, the relation " = " is changed to the relation " $\subseteq$ ".*

Proof. Let $\mathbb{G}(\mathfrak{G}), \mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ be given. Then, we check in on the validity of the nine (CC1–CC9) conditions.

$(CC1)$

$$
\begin{aligned}
& cl_\cap\left(\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G})\right) \\
&= cl_\cap\left(\left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \cup \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \\ \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right) \\
&= cl_\cap\left(\left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \vee \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right) \\
&= \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigvee_{\varrho\in\aleph} \left(\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \vee \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \\ \bigwedge_{\varrho\in\aleph} \left(\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \bigwedge_{\varrho\in\aleph} \left(\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \\
&= \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \left(\bigvee_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \vee \left(\bigvee_{\varrho\in\aleph} \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \\ \left(\bigwedge_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \wedge \left(\bigwedge_{\varrho\in\aleph} \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \left(\bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \wedge \left(\bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \\
&= cl_\cap\left(\mathbb{G}(\mathfrak{G})\right) \cup cl_\cap\left(\mathbb{H}(\mathfrak{G})\right);
\end{aligned}
$$

$(CC2)$

$$
\mathbb{G}(\mathfrak{G}) = \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \subseteq \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigvee_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \\ \bigwedge_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}
$$

$$
= cl_\cap\left(\mathbb{G}(\mathfrak{G})\right);
$$

$(CC3)$

$$
cl_\cap\left(\flat(\mathfrak{G})\right) = cl_\cap\left(\left\{\left\langle \langle\varrho,\mathfrak{g}\rangle, 0, 1, 0\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right) = \left\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigvee_{\varrho\in\aleph} 0, \\ \bigwedge_{\varrho\in\aleph} 1, \bigwedge_{\varrho\in\aleph} 0 \end{array}\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}
$$

$$
= \left\{\left\langle \langle\varrho,\mathfrak{g}\rangle, 0, 1, 0\right\rangle \mid \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} = \flat(\mathfrak{G});
$$

$(CC4)$

$$cl_{\cap}\left(cl_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\right)\right)=cl_{\cap}\left(\left\{\left\langle\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}\right)$$

$$=\left\{\left\langle\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{G}\rangle),\bigwedge_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$=cl_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\right);$$

$(CC5)$

$$cl^{\circ}_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\cap\mathbb{H}\left(\mathfrak{G}\right)\right)$$

$$=cl^{\circ}_{\cap}\left(\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\wedge\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle),\\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\vee\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle),\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\wedge\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}\right)$$

$$=\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\left(\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\wedge\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\left(\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\vee\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\left(\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\wedge\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$\subseteq\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\left(\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right)\wedge\left(\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right),\\ \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right)\vee\left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right),\left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right)\wedge\left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$=\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}\cap\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$=cl^{\circ}_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\right)\cap cl^{\circ}_{\cap}\left(\mathbb{H}\left(\mathfrak{G}\right)\right);$$

$(CC6)$

$$\mathbb{G}\left(\mathfrak{G}\right)=\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$\subseteq\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{G}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$$=cl^{\circ}_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\right);$$

$(CC7)$

$$cl^{\circ}_{\cap}\left(\sharp\left(\mathfrak{G}\right)\right)=cl^{\circ}_{\cap}(\{\langle\langle\varrho,\mathfrak{g}\rangle,1,0,0\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\})=\{\langle\langle\varrho,\mathfrak{g}\rangle,1,0,0\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\}=\sharp\left(\mathfrak{G}\right);$$

$(CC8)$

$$cl^{\circ}_{\cap}\left(cl^{\circ}_{\cap}\left(\mathbb{G}\left(\mathfrak{G}\right)\right)\right)=cl^{\circ}_{\cap}\left(\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{G}\rangle),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}\right)$$

$$=\left\{\left\langle\begin{array}{c}\langle\varrho,\mathfrak{g}\rangle,\bigvee_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle),\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\end{array}\right\rangle\mid\langle\varrho,\mathfrak{g}\rangle\in\aleph\times\mathfrak{G}\right\}$$

$= cl_{\cap}^{\circ}(\mathbb{G}(\mathfrak{G}))$;

$(CC9)$

$cl_{\cap}^{\circ}(cl_{\cap}(\mathbb{G}(\mathfrak{G})))$

$$= cl_{\cap}^{\circ}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \bigvee\limits_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \\ \bigwedge\limits_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \bigwedge\limits_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right)$$

$$= \left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \bigvee\limits_{\mathfrak{g}\in\mathfrak{G}}\left(\bigvee\limits_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right), \\ \bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\left(\bigwedge\limits_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right), \bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\left(\bigwedge\limits_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right) \end{array}\right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}$$

$$= \left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \bigvee\limits_{\varrho\in\aleph}\left(\bigvee\limits_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right), \\ \bigwedge\limits_{\varrho\in\aleph}\left(\bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right), \bigwedge\limits_{\varrho\in\aleph}\left(\bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right) \end{array}\right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}$$

$$= cl_{\cap}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \bigvee\limits_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \\ \bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \bigwedge\limits_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \end{array}\right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right)$$

$= cl_{\cap}(cl_{\cap}^{\circ}(\mathbb{G}(\mathfrak{G})))$. □

Example 3.2. *Let* $\aleph = \{\varrho_1, \varrho_2, \varrho_3\}$, $\mathfrak{G} = \{4, 5\}$, *and* $\mathbb{G}(\mathfrak{G})$, $\mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ *are defined by:*

$$\mathbb{G}(\mathfrak{G}) = \left\{\begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.2, 0.4\rangle, & \langle\langle\varrho_1, 5\rangle, 0.5, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.1, 0.5\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.0, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.1, 0.2, 0.2\rangle. \end{array}\right\},$$

$$\mathbb{H}(\mathfrak{G}) = \left\{\begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.2, 0.5, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.1, 0.1, 0.1\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.4, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.6, 0.2, 0.2\rangle, & \langle\langle\varrho_3, 5\rangle, 0.3, 0.2, 0.1\rangle. \end{array}\right\}.$$

Then,

$$cl_{\cap}^{\circ}(\mathbb{G}(\mathfrak{G})) = \left\{\begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.5, 0.1, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.5, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.1, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.1, 0.2, 0.1\rangle. \end{array}\right\},$$

$$cl_{\cap}^{\circ}(\mathbb{H}(\mathfrak{G})) = \left\{\begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.2, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.2, 0.1, 0.1\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.2, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.6, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.6, 0.2, 0.1\rangle. \end{array}\right\},$$

$$cl_{\cap}^{\circ}(\mathbb{G}(\mathfrak{G})) \cap cl_{\cap}^{\circ}(\mathbb{H}(\mathfrak{G})) = \left\{\begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.2, 0.1, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.2, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.2, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.1, 0.2, 0.1\rangle. \end{array}\right\},$$

$$\mathbb{G}(\mathfrak{G}) \cap \mathbb{H}(\mathfrak{G}) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.2, 0.5, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.1, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.4, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.0, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.1, 0.2, 0.1\rangle. \end{array} \right\},$$

$$cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G}) \cap \mathbb{H}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.2, 0.1, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.2, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.2, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.1, 0.2, 0.1\rangle. \end{array} \right\},$$

$cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G})) \cap cl^{\circ}_{\cap}(\mathbb{H}(\mathfrak{G})) \neq cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G}) \cap \mathbb{H}(\mathfrak{G}))$, *and*

$$cl_{\cap}(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.5, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.1, 0.1\rangle, & \langle\langle\varrho_2, 5\rangle, 0.5, 0.1, 0.0\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.3, 0.1, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.5, 0.1, 0.0\rangle. \end{array} \right\},$$

and then $cl_{\cap}(\mathbb{G}(\mathfrak{G})) \nsubseteq cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G}))$ *and* $cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G})) \nsubseteq cl_{\cap}(\mathbb{G}(\mathfrak{G}))$,

but $cl^{\circ}_{\cap}(cl_{\cap}(\mathbb{G}(\mathfrak{G}))) = cl_{\cap}(cl^{\circ}_{\cap}(\mathbb{G}(\mathfrak{G})))$

$$= \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.5, 0.1, 0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.5, 0.1, 0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.5, 0.1, 0\rangle, & \langle\langle\varrho_2, 5\rangle, 0.5, 0.1, 0\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.5, 0.1, 0\rangle, & \langle\langle\varrho_3, 5\rangle, 0.5, 0.1, 0\rangle. \end{array} \right\}.$$

Also, for the conditions (II1–II9) in [6, 9], we will deduce these coming results in TPFSs.

Theorem 3.3. *For any universal set* $\aleph$ *and for any time-scale set* $\mathfrak{G}$*, then* $\left\langle \left(I^3\right)^{\aleph\times\mathfrak{G}}, int_{\cup}, \cap, \cup, int^{\circ}_{\cup} \right\rangle$ *is an* in-in-*TPFFTS for which, in condition (II5), the relation " = " is changed to the relation " $\supseteq$ ".*

Proof. Let $\mathbb{G}(\mathfrak{G})$, $\mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ be given. Then, we check in the validity of the nine (II1–II9) conditions.

$(II1)$

$$\begin{aligned}
& int_{\cup}(\mathbb{G}(\mathfrak{G}) \cap \mathbb{H}(\mathfrak{G})) \\
&= int_{\cup}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \cap \left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \omega_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle), \\ \varpi_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right) \\
&= int_{\cup}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \wedge \omega_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \vee \varpi_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right) \\
&= \left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph} (\omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \wedge \omega_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle)), \\ \bigvee_{\varrho\in\aleph} (\varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \vee \varpi_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle)), \bigwedge_{\varrho\in\aleph} (\sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle)) \end{array} \right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \\
&= \left\{\left\langle \begin{array}{c} \langle\varrho, \mathfrak{g}\rangle, \left(\bigwedge_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right) \wedge \left(\bigwedge_{\varrho\in\aleph} \omega_{\mathbb{H}}(\langle\varrho, \mathfrak{G}\rangle)\right), \\ \left(\bigvee_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right) \vee \left(\bigvee_{\varrho\in\aleph} \varpi_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle)\right), \left(\bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle\varrho, \mathfrak{g}\rangle)\right) \wedge \left(\bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{H}}(\langle\varrho, \mathfrak{g}\rangle)\right) \end{array} \right\rangle \mid \langle\varrho, \mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\right\} \\
&= int_{\cup}(\mathbb{G}(\mathfrak{G})) \cap int_{\cup}(\mathbb{H}(\mathfrak{G}));
\end{aligned}$$

$(II2)$

$$int_{\cup}(\mathbb{G}(\mathfrak{G})) = \{\left\langle \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigvee_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigwedge_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}$$

$$\subseteq \{\langle\langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\} = \mathbb{G}(\mathfrak{G});$$

$(II3)$

$$int_{\cup}(\sharp(\mathfrak{G})) = int_{\cup}(\{\langle\langle\varrho,\mathfrak{g}\rangle, 1, 0, 0\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\})$$

$$=\{\left\langle \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph}1, \bigvee_{\varrho\in\aleph}0, \bigwedge_{\varrho\in\aleph}0\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\} = \{\langle\langle\varrho,\mathfrak{g}\rangle, 1, 0, 0\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\} = \sharp(\mathfrak{G});$$

$(II4)$

$$int_{\cup}(int_{\cup}(\mathbb{G}(\mathfrak{G}))) = int_{\cup}\left(\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \\ \bigvee_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigwedge_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}\right)$$

$$= \{\left\langle \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{G}\rangle), \bigvee_{\varrho\in\aleph}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigwedge_{\varrho\in\aleph}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}$$

$$= int_{\cup}(\mathbb{G}(\mathfrak{G}));$$

$(II5)$

$$int_{\cup}^{\circ}(\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G}))$$

$$= int_{\cup}^{\circ}\left(\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \vee \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \\ \varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}\right)$$

$$= \{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}}(\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \vee \omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}}(\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)), \bigwedge_{\mathfrak{g}\in\mathfrak{G}}(\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle) \wedge \sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}$$

$$\supseteq \{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \vee \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \\ \left(\bigvee_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \wedge \left(\bigvee_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right), \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle)\right) \wedge \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle)\right) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}$$

$$= \{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{G}}(\langle\varrho,\mathfrak{g}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{G}}(\langle\varrho,\mathfrak{G}\rangle) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\} \cup$$

$$\{\left\langle \begin{array}{c} \langle\varrho,\mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\omega_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}}\varpi_{\mathbb{H}}(\langle\varrho,\mathfrak{G}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}}\sigma_{\mathbb{H}}(\langle\varrho,\mathfrak{g}\rangle) \end{array}\right\rangle | \langle\varrho,\mathfrak{g}\rangle \in \aleph\times\mathfrak{G}\}$$

$$= int_{\cup}^{\circ}(\mathbb{G}(\mathfrak{G})) \cup int_{\cup}^{\circ}(\mathbb{H}(\mathfrak{G}));$$

$(II6)$

$$int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G})) = \{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$$

$$\subseteq \{\langle \langle \varrho, \mathfrak{g}\rangle, \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\} = \mathbb{G}(\mathfrak{G});$$

($II7$)

$$int^{\circ}_{\cup}(\flat(\mathfrak{G})) = int^{\circ}_{\cup}(\{\langle \langle \varrho, \mathfrak{g}\rangle, 0, 1, 0\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\})$$

$$= \{\langle \langle \varrho, \mathfrak{g}\rangle, 0, 1, 0\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\} = \flat(\mathfrak{G});$$

($II8$)

$$int^{\circ}_{\cup}(int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}))) = int^{\circ}_{\cup}\left(\{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{G}\rangle) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}\right)$$

$$= \{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}} \left(\bigvee_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$$

$$= \{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\} = int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}));$$

($II9$)

$$int^{\circ}_{\cup}(int_{\cup}(\mathbb{G}(\mathfrak{G}))) = int^{\circ}_{\cup}\left(\{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \\ \bigvee_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}\right)$$

$$= \{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \left(\bigwedge_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \\ \bigvee_{\mathfrak{g}\in\mathfrak{G}} \left(\bigvee_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \left(\bigwedge_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$$

$$= \{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigwedge_{\varrho\in\aleph} \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \\ \bigvee_{\varrho\in\aleph} \left(\bigvee_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right), \bigwedge_{\varrho\in\aleph} \left(\bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle)\right) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}$$

$$= int_{\cup}\left(\{\left\langle \begin{array}{c} \langle \varrho, \mathfrak{g}\rangle, \bigvee_{\mathfrak{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \\ \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle \varrho, \mathfrak{g}\rangle), \bigwedge_{\mathfrak{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle \varrho, \mathfrak{G}\rangle) \end{array} \right\rangle | \langle \varrho, \mathfrak{g}\rangle \in \aleph \times \mathfrak{G}\}\right)$$

$$= int_{\cup}(int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}))).$$ □

Example 3.4. *From Example 3.2, we have*

$$\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G}) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.2, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.5, 0.1, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.1, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.6, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.3, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.2, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.3, 0.2, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.3, 0.1, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.3, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.3, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.3, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.2, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.3, 0.2, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.1, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.0, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.0, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^{\circ}_{\cup}(\mathbb{H}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.1, 0.5, 0.1\rangle, & \langle\langle\varrho_1, 5\rangle, 0.1, 0.5, 0.1\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.4, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.4, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.3, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.3, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G})) \cup int^{\circ}_{\cup}(\mathbb{H}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 4\rangle, 0.3, 0.2, 0.0\rangle, & \langle\langle\varrho_1, 5\rangle, 0.3, 0.2, 0.0\rangle, \\ \langle\langle\varrho_2, 4\rangle, 0.2, 0.1, 0.2\rangle, & \langle\langle\varrho_2, 5\rangle, 0.2, 0.1, 0.2\rangle, \\ \langle\langle\varrho_3, 4\rangle, 0.3, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 5\rangle, 0.3, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}) \cup \mathbb{H}(\mathfrak{G})) \neq int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G})) \cup int^{\circ}_{\cup}(\mathbb{H}(\mathfrak{G})).$$

Also, for the conditions (CI1–CI9) in [6,9], we will deduce these coming results in TPFSs.

Theorem 3.5. *For any universal set* $\aleph$ *and for any time-scale set* $\mathfrak{G}$*, then* $\left\langle \left(I^3\right)^{\aleph\times\mathfrak{G}}, cl_{\cap}, \cup, \cap, int^{\circ}_{\cup} \right\rangle$ *is a* cl -in-*TPFFTS for which, in condition (CI9), the relation " = " is changed to the relation " ⊇ ".*

Proof. Let $\mathbb{G}(\mathfrak{G}), \mathbb{H}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, and we will only check the condition (CI9).

$(CI9)$

$$int^{\circ}_{\cup}(cl_{\cap}(\mathbb{G}(\mathfrak{G}))) = int^{\circ}_{\cup}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathrm{g}\rangle, \bigvee\limits_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \\ \bigwedge\limits_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \bigwedge\limits_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathrm{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right)$$

$$= \left\{\left\langle \begin{array}{c} \langle\varrho, \mathrm{g}\rangle, \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \bigvee\limits_{\varrho\in\aleph} \omega_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \\ \bigvee\limits_{\mathrm{g}\in\mathfrak{G}} \bigwedge\limits_{\varrho\in\aleph} \varpi_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \bigwedge\limits_{\varrho\in\aleph} \sigma_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathrm{g}\rangle \in \aleph\times\mathfrak{G}\right\}$$

$$\supseteq \left\{\left\langle \begin{array}{c} \langle\varrho, \mathrm{g}\rangle, \bigvee\limits_{\varrho\in\aleph} \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \\ \bigwedge\limits_{\varrho\in\aleph} \bigvee\limits_{\mathrm{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \bigwedge\limits_{\varrho\in\aleph} \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathrm{g}\rangle \in \aleph\times\mathfrak{G}\right\}$$

$$= cl_{\cap}\left(\left\{\left\langle \begin{array}{c} \langle\varrho, \mathrm{g}\rangle, \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \omega_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \\ \bigvee\limits_{\mathrm{g}\in\mathfrak{G}} \varpi_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle), \bigwedge\limits_{\mathrm{g}\in\mathfrak{G}} \sigma_{\mathbb{G}}(\langle\varrho, \mathrm{g}\rangle) \end{array} \right\rangle \mid \langle\varrho, \mathrm{g}\rangle \in \aleph\times\mathfrak{G}\right\}\right)$$

$$= cl_{\cap}\left(int^{\circ}_{\cup}(\mathbb{G}(\mathfrak{G}))\right).$$ □

Similarly, for the conditions (IC1–IC9) in [6, 9], we ensure these results come in TPFSs.

Corollary 3.6. *For any universal set $\aleph$ and a time-scale set $\mathfrak{G}$, then $\left\langle \left(I^3\right)^{\aleph\times\mathfrak{G}}, int_\cup, \cap, \cup, cl^\circ_\cap \right\rangle$ is an* in -cl-*TPFFTS for which, in condition (IC9), the relation" = " is changed to the relation " $\subseteq$ ".*

Example 3.7. *Let $\aleph = \{\varrho_1, \varrho_2, \varrho_3\}$, $\mathfrak{G} = \{8, 9\}$, and $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$ be defined by:*

$$\mathbb{G}(\mathfrak{G}) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.5, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.3, 0.3, 0.3\rangle, & \langle\langle\varrho_2, 9\rangle, 0.2, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.7, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.8, 0.1, 0.1\rangle. \end{array} \right\}. \text{ Then,}$$

$$cl_\cap(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.8, 0.1, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.8, 0.1, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.8, 0.1, 0.1\rangle. \end{array} \right\},$$

$$int^\circ_\cup(cl_\cap(\mathbb{G}(\mathfrak{G}))) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.7, 0.1, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.7, 0.1, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.7, 0.1, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.7, 0.1, 0.1\rangle. \end{array} \right\},$$

$$int^\circ_\cup(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.5, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.5, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.2, 0.3, 0.2\rangle, & \langle\langle\varrho_2, 9\rangle, 0.2, 0.3, 0.2\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.7, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.7, 0.2, 0.1\rangle. \end{array} \right\},$$

$$cl_\cap\left(int^\circ_\cup(\mathbb{G}(\mathfrak{G}))\right) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.7, 0.2, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.7, 0.2, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.7, 0.2, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.7, 0.2, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.7, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.7, 0.2, 0.1\rangle. \end{array} \right\},$$

$$int^\circ_\cup(cl_\cap(\mathbb{G}(\mathfrak{G}))) \neq cl_\cap\left(int^\circ_\cup(\mathbb{G}(\mathfrak{G}))\right).$$

$$int_\cup(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.5, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.1, 0.5, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.1, 0.5, 0.1\rangle. \end{array} \right\},$$

$$cl^\circ_\cap(int_\cup(\mathbb{G}(\mathfrak{G}))) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.3, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.1, 0.3, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.1, 0.3, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.1, 0.3, 0.1\rangle. \end{array} \right\},$$

$$cl^\circ_\cap(\mathbb{G}(\mathfrak{G})) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.1, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.1, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.3, 0.2, 0.2\rangle, & \langle\langle\varrho_2, 9\rangle, 0.3, 0.2, 0.2\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.8, 0.1, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.8, 0.1, 0.1\rangle. \end{array} \right\},$$

$$int_\cup(cl^\circ_\cap(\mathbb{G}(\mathfrak{G}))) = \left\{ \begin{array}{ll} \langle\langle\varrho_1, 8\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_1, 9\rangle, 0.1, 0.2, 0.1\rangle, \\ \langle\langle\varrho_2, 8\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_2, 9\rangle, 0.1, 0.2, 0.1\rangle, \\ \langle\langle\varrho_3, 8\rangle, 0.1, 0.2, 0.1\rangle, & \langle\langle\varrho_3, 9\rangle, 0.1, 0.2, 0.1\rangle. \end{array} \right\},$$

$$cl^\circ_\cap(int_\cup(\mathbb{G}(\mathfrak{G}))) \neq int_\cup\left(cl^\circ_\cap(\mathbb{G}(\mathfrak{G}))\right).$$

4. Temporal picture fuzzy almost continuous multifunctions

In this section, we use the notion of picture fuzzy multifunctions and study some of its properties and implications, and so we combine the picture fuzzy multifunctions with the temporal picture fuzzy modal topological operators defined in the previous section. Also, we introduced various types of continuity of these picture fuzzy multifunctions. Several properties of the presented multifunctions between temporal picture fuzzy structures will be discussed.

Definition 4.1. *Let $\mathfrak{F} : (\aleph, \tau) \looparrowright (\Upsilon, \sigma)$ be a TPFM, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Then, $\mathfrak{F}$ is called:*

(1) TPF^uS-continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \cap D(\mathfrak{F}) \subseteq \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$.

(2) TPF^lS-continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$.

(3) TPF^u almost continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \cap D(\mathfrak{F}) \subseteq \mathfrak{F}^u(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$.

(4) TPF^l almost continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$.

(5) TPF^uS (resp. TPF^lS)-continuous iff it is TPF^uS (resp. TPF^lS)-continuous at every fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$.

(6) TPF^u (resp. TPF^l) almost continuous iff it is TPF^u (resp. TPF^l) almost continuous at every fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$.

Remark 4.2. (1) *If $\mathfrak{F}$ is NTPFM, then $\mathfrak{F}$ is TPF^uS -continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$.*

(2) *If $\mathfrak{F}$ is NTPFM, then $\mathfrak{F}$ is TPF^u almost continuous at a fuzzy point $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$ iff $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$ for each $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle \varsigma, \varkappa, \vartheta\rangle$, and $\langle \varrho, \mathfrak{g}\rangle_{\langle \varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^u(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle \varsigma, \varkappa, \vartheta\rangle), \langle \varsigma, \varkappa, \vartheta\rangle))$.*

(3) *TPF^uS (resp. TPF^lS)-continuity$\Longrightarrow$ TPF^u (resp. TPF^l) almost continuity.*

Theorem 4.3. *Let $\mathfrak{F} : (\aleph, \tau) \looparrowright (\Upsilon, \sigma)$ be a TPFM. Then, the following are equivalent: for $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$.*

(1) $\mathfrak{F}$ is TPF^l almost continuous.

(2) $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle)$ if $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(3) $\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ if $\mathbb{U}(\mathfrak{G}) = int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$.

(4) $\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle))) \geq \sigma(\mathbb{U}(\mathfrak{G}))$.

Proof. (1) $\Longrightarrow$ (2) Let $\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$, $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, and $\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$. Then, there exists $\mathbb{G}(\mathfrak{G}) \in \left(I^3\right)^{\aleph\times\mathfrak{G}}$, $\tau(\mathbb{G}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G})$ such that $\mathbb{G}(\mathfrak{G}) \subseteq \mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle))$. Thus,

$$\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{G}(\mathfrak{G}) \subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle),$$

and hence $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle)$.

(2) $\Longrightarrow$ (3) Let $(\mathbb{U}(\mathfrak{G}) = int_\sigma(cl_\sigma((\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$. Then by (2),

$$\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle) = int_\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle).$$

Thus, $\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(3) $\Longrightarrow$ (4) Suppose that there exist $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$ such that $\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle))) < \langle\varsigma,\varkappa,\vartheta\rangle \leq \sigma(\mathbb{U}(\mathfrak{G}))$. Since $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and

$$int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle) = int_\sigma(cl_\sigma(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle).$$

Then by (3), we have $\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$. It is a contradiction. Thus, $\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle))) \geq \sigma(\mathbb{U}(\mathfrak{G}))$.

(4) $\Longrightarrow$ (1) Let $\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathfrak{F})$, $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))$. Then by (4), we have $\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)) = \mathbb{G}(\mathfrak{G})$ with $\tau(\mathbb{G}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$. Since $\mathbb{U}(\mathfrak{G}) \subseteq int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$,

$$\langle\varrho,\mathfrak{g}\rangle_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \subseteq \mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)) = \mathbb{G}(\mathfrak{G}), \text{ and (1) follows.}$$

□

Theorem 4.4. *Let $\mathfrak{F} : (\aleph,\tau) \leftrightarrow (\Upsilon,\sigma)$ be a $TPFM$. Then, we get these equivalent properties for $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\partial \in I_0, \eth \in I_1$ and $\wp \in I_1$.*

(1) $\mathfrak{F}$ is TPF^l almost continuous.

(2) $\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})) \supseteq cl_\tau(\mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle)$ if $\sigma(\dashv \mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(3) $\tau(\dashv(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ if $\mathbb{U}(\mathfrak{G}){=}cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$.

Proof. (1) $\Longrightarrow$ (2) Let $\mathfrak{F}$ be TPF^l almost continuous and $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$ with $\sigma(\dashv \mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

Then by Theorem 4.3(2),

$$\begin{aligned}\mathfrak{F}^l(\dashv \mathbb{U}(\mathfrak{G})) &\subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\dashv \mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= int_\tau(\mathfrak{F}^l(int_\sigma(\dashv(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= int_\tau(\mathfrak{F}^l(\dashv(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle))),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle).\end{aligned}$$

Since

$$\begin{aligned}\dashv(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))) &= \mathfrak{F}^l(\dashv \mathbb{U}(\mathfrak{G}))\\ &\subseteq int_\tau(\mathfrak{F}^l(\dashv(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle))),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= int_\tau(\dashv\mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= \dashv(cl_\tau(\mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)).\end{aligned}$$

Thus, we obtain $\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})) \supseteq cl_\tau(\mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)$.

(2) $\Longrightarrow$ (3) Let $\mathbb{U}(\mathfrak{G}) = cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$. Then by(2),

$$\begin{aligned}\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})) &\supseteq cl_\tau(\mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= cl_\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle)).\end{aligned}$$

Thus, $\tau(\dashv(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(3) $\Longrightarrow$ (1) Let $\mathbb{U}(\mathfrak{G}) = int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$. Then, $\dashv \mathbb{U}(\mathfrak{G}) = cl_\sigma(int_\sigma(\dashv \mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$ and hence by (3), $\tau(\dashv(\mathfrak{F}^u(\dashv\mathbb{U}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, that is, $\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$. Then by Theorem 4.3(3), $\mathfrak{F}$ is TPF^l almost continuous. □

Theorem 4.5. *Let $\mathfrak{F}:(\aleph,\tau)\looparrowright(\Upsilon,\sigma)$ be a $TPFM$. Then, we get these equivalent properties for $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma\in I_0,\varkappa\in I_1$ and $\vartheta\in I_1$.*

(1) $\mathfrak{F}$ is TPF^u almost continuous.

(2) $\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})) \subseteq int_\tau(\mathfrak{F}^u(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)$ if $\sigma(\mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(3) $\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ if $\mathbb{U}(\mathfrak{G}) = int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.

(4) $\tau(\mathfrak{F}^u(int_\sigma(cl_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))) \geq \sigma(\mathbb{U}(\mathfrak{G}))$.

(5) $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \supseteq cl_\tau(\mathfrak{F}^l(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)$ if $\sigma(\dashv \mathbb{U}(\mathfrak{G})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(6) $\tau(\dashv(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ if $\mathbb{U}(\mathfrak{G}) = cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.

Theorem 4.6. *Let $\mathfrak{F}:(\aleph,\tau)\looparrowright(\Upsilon,\sigma)$ be a $TPFM$, $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma\in I_0,\varkappa\in I_1$ and $\vartheta\in I_1$. Then, the coming are fulfilled.*

(1) If $\mathfrak{F}$ is $NTPFM$, then $\mathfrak{F}$ is TPF^uS-continuous. iff $\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))) \geq \sigma(\mathbb{U}(\mathfrak{G}))$.

(2) $\mathfrak{F}$ is TPF^lS-continuous. iff $\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))) \geq \sigma(\mathbb{U}(\mathfrak{G}))$.

The following example shows that generally a TPF^l almost continuous and TPF^u almost continuous need not be either TPF^lS-continuous or TPF^uS-continuous.

Example 4.7. *Let $\aleph = \{\varrho_1, \varrho_2\}$, $\Upsilon = \{\zeta_1, \zeta_2, \}$, $\mathfrak{G} = \{\mathfrak{g}_1, \mathfrak{g}_2\}$ and $\mathfrak{F} : \aleph \looparrowright \Upsilon$ be $TPFMs$.*

Define $\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$ by:

$\Psi_{\mathfrak{F}}(\langle\varrho, \mathfrak{g}\rangle, \langle\zeta, \mathfrak{g}\rangle)$	$\langle\zeta_1, \mathfrak{g}_1\rangle$	$\langle\zeta_1, \mathfrak{g}_2\rangle$	$\langle\zeta_2, \mathfrak{g}_1\rangle$	$\langle\zeta_2, \mathfrak{g}_2\rangle$
$\langle\varrho_1, \mathfrak{g}_1\rangle$	$\langle 0.1, 0.3, 0.1\rangle$	$\langle 0, 1, 0\rangle$	$\langle 0.2, 0.3, 0.5\rangle$	$\langle 1, 0, 0\rangle$
$\langle\varrho_1, \mathfrak{g}_2\rangle$	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 1, 0, 0\rangle$	$\langle 0.3, 0.4, 0.1\rangle$	$\langle 0.6, 0.2, 0.1\rangle$
$\langle\varrho_2, \mathfrak{g}_1\rangle$	$\langle 0.2, 0.4, 0.1\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.1, 0.5, 0.2\rangle$	$\langle 1, 0, 0\rangle$
$\langle\varrho_2, \mathfrak{g}_2\rangle$	$\langle 0.3, 0.1, 0.4\rangle$	$\langle 1, 0, 0\rangle$	$\langle 0.2, 0.5, 0.3\rangle$	$\langle 0.6, 0.3, 0.1\rangle$.

Define Temporal picture fuzzy topologies τ, σ on $\aleph$ and Υ by:

$$\tau(\mathbb{G}(\mathfrak{G})) = \begin{cases} \langle 1, 0, 0\rangle, & \mathbb{G}(\mathfrak{G}) \in \{\flat(\mathfrak{G}), \sharp(\mathfrak{G})\} \\ \langle 0.55, 0.25, 0.20\rangle, & \mathbb{G}(\mathfrak{G}) \in \{\mathbb{G}_1(\mathfrak{G}), \mathbb{G}_2(\mathfrak{G})\} \\ \langle 0, 1, 0\rangle, & o.w, \end{cases}$$

$$\sigma(\mathbb{U}(\mathfrak{G})) = \begin{cases} \langle 1, 0, 0\rangle, & \mathbb{U}(\mathfrak{G}) \in \{\flat(\mathfrak{G}), \sharp(\mathfrak{G})\} \\ \langle 0.35, 0.45, 0.20\rangle, & \mathbb{U}(\mathfrak{G}) = \mathbb{U}_1(\mathfrak{G}) \\ \langle 0, 1, 0\rangle, & o.w, \end{cases}$$

where

$$\mathbb{G}_1(\mathfrak{G}) = \left\{ \begin{array}{l} \langle\langle\varrho, \mathfrak{g}_1\rangle, 0.03, 0.43, 0.39\rangle, \\ \langle\langle\varrho, \mathfrak{g}_2\rangle, 0.02, 0.41, 0.55\rangle, \end{array}, \varrho \in \aleph \right\}, \ \mathbb{G}_2(\mathfrak{G}) = \left\{ \begin{array}{l} \langle\langle\varrho, \mathfrak{g}_1\rangle, 0.04, 0.37, 0.39\rangle, \\ \langle\langle\varrho, \mathfrak{g}_2\rangle, 0.04, 0.37, 0.55\rangle, \end{array}, \varrho \in \aleph \right\}$$

and $\mathbb{U}_1(\mathfrak{G}) = \left\{ \begin{array}{l} \langle\langle\zeta, \mathfrak{g}_1\rangle, 0.03, 0.38, 0.21\rangle, \\ \langle\langle\zeta, \mathfrak{g}_2\rangle, 0.04, 0.37, 0.57\rangle, \end{array}, \zeta \in \Upsilon \right\}.$

Then, $\mathfrak{F}$ is TPF^u (resp. TPF^l) almost continuous, but it is not TPF^uS (resp. TPF^lS)−continuous because

$$\begin{aligned} \mathfrak{F}^u(\mathbb{U}_1(\mathfrak{G})) &\subseteq int_\tau(\mathfrak{F}^u(int_\sigma(cl_\sigma(\mathbb{U}_1(\mathfrak{G}), \langle 0.35, 0.45, 0.2\rangle), \langle 0.35, 0.45, 0.2\rangle)), \langle 0.35, 0.45, 0.2\rangle) \\ \mathfrak{F}^l(\mathbb{U}_1(\mathfrak{G})) &\subseteq int_\tau(\mathfrak{F}^l(int_\sigma(cl_\sigma(\mathbb{U}_1(\mathfrak{G}), \langle 0.35, 0.45, 0.2\rangle), \langle 0.35, 0.45, 0.2\rangle)), \langle 0.35, 0.45, 0.2\rangle) \end{aligned}$$

but

$$\begin{aligned} \tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})_1)) &= \langle 0, 1, 0\rangle \not\geq \sigma(\mathbb{U}(\mathfrak{G})_1) = \langle 0.35, 0.45, 0.2\rangle, \\ \tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})_1)) &= \langle 0, 1, 0\rangle \not\geq \sigma(\mathbb{U}(\mathfrak{G})_1) = \langle 0.35, 0.45, 0.2\rangle. \end{aligned}$$

Theorem 4.8. *Let $\mathfrak{F} : (\aleph, \tau) \looparrowright (\Upsilon, \sigma)$ be a $TPFM$. Then, $\mathfrak{F}$ is a TPF^l almost continuous iff $cl_\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathfrak{F}^u(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle))$ for any $\mathbb{U}(\mathfrak{G}) \subseteq cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\partial \in I_0, \eth \in I_1$ and $\wp \in I_1$.*

Proof. Let $\mathfrak{F}$ be TPF^l almost continuous. Then, for each $\mathbb{U}(\mathfrak{G}) \subseteq cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) = \mathbb{V}(\mathfrak{G})$ (say) where $\mathbb{V}(\mathfrak{G}) = cl_\sigma(int_\sigma(\mathbb{V}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$. By Theorem 4.4(3), $\tau(\dashv(\mathfrak{F}^u(\mathbb{V}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, and thus

$$\begin{aligned} cl_\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle)) &\subseteq cl_\tau(\mathfrak{F}^u(\mathbb{V}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle)) \\ &= \mathfrak{F}^u(cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)) \\ &\subseteq \mathfrak{F}^u(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle))), \forall \mathfrak{g} \in \mathfrak{G}. \end{aligned}$$

Conversely; if $\mathbb{U}(\mathfrak{G}) = cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$, then $\mathbb{U}(\mathfrak{G}) \subseteq cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$, and we have $cl_\tau(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle)) \subseteq \mathfrak{F}^u(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle))) = \mathfrak{F}^u(\mathbb{U}(\mathfrak{G}))$. Thus, $\tau(\dashv(\mathfrak{F}^u(\mathbb{U}(\mathfrak{G})))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$. Hence by Theorem 4.4(3), $\mathfrak{F}$ is TPF^l almost continuous. □

Theorem 4.9. *Let $\mathfrak{F} : (\aleph, \tau) \looparrowright (\Upsilon, \sigma)$ be a $NTPFM$. Then, $\mathfrak{F}$ is TPF^u almost continuous iff $cl_\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle) \subseteq \mathfrak{F}^l(cl_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle)))$ for any $\mathbb{U}(\mathfrak{G}) \subseteq cl_\sigma(int_\sigma(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\partial \in I_0$, $\eth \in I_1$ and $\wp \in I_1$.*

Proof. Similar to the proof of Theorem 4.8. □

Consider $\mathfrak{M}, \mathfrak{N}, id_{\aleph\times\mathfrak{G}} : \left(I^3\right)^{\aleph\times\mathfrak{G}} \times I^3 \to \left(I^3\right)^{\aleph\times\mathfrak{G}}$ are operators on $\aleph$ and $\mathfrak{W}, \mathfrak{V}, id_{\Upsilon\times\mathfrak{G}} : \left(I^3\right)^{\Upsilon\times\mathfrak{G}} \times I^3 \to \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$ are operators on Υ.

Remark 4.10. (1) *Let $\mathfrak{F} : (\aleph, \tau, \mathfrak{L}^P) \looparrowright (\Upsilon, \sigma)$ be a $TPFM$. Then, $\mathfrak{F}$ is TPF^l $(\mathfrak{M}, \mathfrak{N}, \mathfrak{W}, \mathfrak{V}, \mathfrak{L}^P)$-continuous iff for every $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$*

$$\mathfrak{L}^P[\mathfrak{M}(\mathfrak{F}^l(\mathfrak{V}(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle) \barwedge \mathfrak{N}(\mathfrak{F}^l(\mathfrak{W}(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle)] \geq \sigma(\mathbb{U}(\mathfrak{G})).$$

(2) *Let $\mathfrak{F} : (\aleph, \tau, \mathfrak{L}^P) \looparrowright (\Upsilon, \sigma,)$ be a $NTPFM$. Then, $\mathfrak{F}$ is TPF^u $(\mathfrak{M}, \mathfrak{N}, \mathfrak{W}, \mathfrak{V}, \mathfrak{L}^P)$-continuous iff for every $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$*

$$\mathfrak{L}^P[\mathfrak{M}(\mathfrak{F}^u(\mathfrak{V}(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle) \barwedge \mathfrak{N}(\mathfrak{F}^u(\mathfrak{W}(\mathbb{U}(\mathfrak{G}), \langle\varsigma,\varkappa,\vartheta\rangle)), \langle\varsigma,\varkappa,\vartheta\rangle)] \geq \sigma(\mathbb{U}(\mathfrak{G})).$$

We can see that the above remark generalizes the concept of TPF^lS (resp. TPF^uS)-continuous, whenever it has been taken $\mathfrak{M}$ = identity operator on $\aleph$, $\mathfrak{N}$ = interior operator on $\aleph$, while $\mathfrak{W}$ and $\mathfrak{V}$ both are equal to the identity operator on Υ, and moreover $\mathfrak{L}^P = \mathfrak{L}^{P0}$. It is supposed that there exist $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$ such that $\tau\left(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))\right) < \langle\varsigma,\varkappa,\vartheta\rangle \leq \sigma(\mathbb{U}(\mathfrak{G}))$. Since

$$\mathfrak{L}^{P0}[(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \barwedge int_\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle)] \geq \sigma(\mathbb{U}(\mathfrak{G})).$$

For every $\mathbb{U}(\mathfrak{G}) \in \left(I^3\right)^{\Upsilon\times\mathfrak{G}}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Thus, $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \barwedge int_\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle) = \flat(\mathfrak{G})$, and hence $\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})) \leq int_\tau(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G})), \langle\varsigma,\varkappa,\vartheta\rangle)$, it follows that $\tau\left(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))\right) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, it is a contradiction. Then, $\tau\left(\mathfrak{F}^l(\mathbb{U}(\mathfrak{G}))\right) \geq \sigma(\mathbb{U}(\mathfrak{G}))$. So $\mathfrak{F}$ is TPF^lS-continuous. The other case is similarly proved.

It is clear that

(1) TPF^l (resp. $NTPF^u$) almost continuous multifunction$\Leftrightarrow$ TPF^l (resp. $NTPF^u$) ($id_{\aleph\times\mathfrak{G}}$, int_τ, $int_\sigma(cl_\sigma)$, $id_{\Upsilon\times\mathfrak{G}}$, $\mathfrak{L}^{P0}$)-continuous multifunction.

(2) TPF^l (resp. $NTPF^u$) weakly continuous multifunction$\Leftrightarrow$ TPF^l (resp. $NTPF^u$) ($id_{\aleph\times\mathfrak{G}}$, int_τ, cl_σ, $id_{\Upsilon\times\mathfrak{G}}$, $\mathfrak{L}^{P0}$)-continuous multifunction.

(3) TPF^l (resp. $NTPF^u$) almost weakly continuous multifunction$\Leftrightarrow$ TPF^l (resp. $NTPF^u$) ($id_{\aleph\times\mathfrak{G}}$, $int_\tau\,(cl_\tau)$, cl_σ, $id_{\Upsilon\times\mathfrak{G}}$, $\mathfrak{L}^{P0}$)-continuous multifunction.

5. Conclusions

We have introduced the definitions of the two standard modal operators joined to a time-scale set. These new operators build the defined temporal picture fuzzy structures (TPFSs). These topological operators are investigated depending on the degrees of positivism, negativism, and neutralism, which generalize new types of closure and interior operators in the temporal picture fuzzy topological structures (TPFTS). There are four types of temporal picture fuzzy modal operators constructed in the context in detail. These four operators are analogous to the corresponding ones defined in [6, 9], but there are few differences in their properties. We have explained the differences by clear examples. Although there are a variety of operators defined in this paper, all operators defined here did not satisfy some of Kuratowski's conditions of closure or Kuratowski's conditions of interior. The generated structures (TPFFMTSs) are called ”feeble”, standing for not all conditions for a topological (closure or interior) operator are satisfied. Also, this paper submitted the notions of TPF^u and TPF^l as almost continuous multifunctions. Some characterizations of these types of continuous multifunctions are proved, and many examples are submitted to explain the allowed implications between these types of temporal picture fuzzy continuity. In future work, we will generalize these notions to a wider form of temporal picture fuzzy semi-continuity. Also, we will try to study the variety of temporal picture fuzzy continuity in the soft set theory using special operators.

Author contributions

D. L. Shi: Methodology, Funding, Visualization; M. N. AbuShugair: Methodology, Funding, Validation and Formal analysis, Writing-original draft; S. E. Abbas: Validation and Formal analysis, Reviewing and Investigation the final version; Ismail Ibedou: Reviewing and Investigation the final version, Writing-original draft, Visualization. All authors have read and agreed to the published version of the manuscript.

Data availability statement

The data sets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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