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Research article

Parameter estimation for the transmuted inverse Rayleigh distribution using ranked set sampling: Applications and analysis

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Abstract: This paper examines various estimation methods for the parameters of the transmuted inverse Rayleigh distribution (TIRD) using both ranked set sampling (RSS) and simple random sampling (SRS) designs. The parameters are estimated using maximum likelihood estimation, ordinary and weighted least squares, and the maximum product of spacings. Additionally, five goodness-of-fit estimators are evaluated: Anderson-Darling (AD), right-tail AD, left-tail AD, left-tail second-order, and the Cramér-von Mises estimator. A comprehensive simulation study is conducted to assess the performance of these estimators while ensuring an equal number of observations across both sampling designs. Furthermore, an analysis of a real COVID-19 dataset belonging to the Netherlands of 30 days, which is fitted both numerically and graphically to the TIRD, demonstrates the practical applicability of the proposed estimation methods. The results show that RSS-based estimators consistently outperform their SRS counterparts in terms of mean squared error, bias, and mean absolute relative error across all methods. The findings highlight the advantages of RSS for parameter estimation in the TIRD, demonstrating its superiority over SRS for statistical inference. In particular, RSS proves to be more effective when dealing with small sample sizes.

Keywords: ranked set sampling; estimation theory; transmuted inverse Rayleigh distribution; Anderson-Darling; ordinary least squares; maximum likelihood estimation
Mathematics Subject Classification: 60E05, 62F10

Abbreviations

MLE	Maximum likelihood estimation
OLS	Ordinary least squares
WLS	Weighted least squares
CV	Cramér-von Mises estimation
AD	Anderson-Darling estimation
AD_t	Anderson-Darling test
CV_t	Cramér-von Mises test
KS_t	Kolmogorov-Smirnov test
MPS	Maximum product of spacings
RAD	Right Anderson-Darling
LAD	Left Anderson-Darling
LTS	Left Anderson-Darling of second order
AE	Average estimator
Bias	Absolute bias
MSE	Mean squared error
MRE	Mean relative error

1. Introduction

The ranked set sampling was first suggested by [1] for estimating the mean of pasture and forage yield as an efficient method as compared with the commonly used simple random sampling (SRS) method. Many authors considered the RSS method either for estimating the population parameters of suggested new modifications of the method with different applications.

Ranked set sampling is an effective statistical technique that can enhance the efficiency of estimations, particularly when exact measurements are difficult or costly to obtain. By leveraging auxiliary variables to rank observations, RSS improves the accuracy of sampling, making it especially useful for skewed distributions or when direct measurements are challenging. However, its success depends on careful design, as well as the availability of a reliable auxiliary variable that correlates well with the target variable, to ensure its effectiveness.

Let $Z_1, Z_2, ..., Z_m$ be a random sample of size m from a population with a probability density function (PDF) $f(z)$, a cumulative distribution function (CDF) $F(z)$, a mean μ, and a variance σ^2.

The RSS scheme can be described as follows:

(1) Select m SRS samples each of size m (m^2 units) from the target population as:

$$\begin{array}{cccc} Z_{1(1m)}, & Z_{1(2m)}, & \ldots, & Z_{1(mm)}. \\ Z_{2(1m)}, & Z_{2(2m)}, & \ldots, & Z_{2(mm)}. \\ \vdots & \vdots & \vdots & \vdots \\ Z_{m(1m)}, & Z_{m(2m)}, & \ldots, & Z_{m(mm)}. \end{array}$$

(2) Rank the units in each set of size k from lowest to the largest visually or based on any cost-free

method as:

$$\begin{array}{cccc} Z_{1(1:m)}, & Z_{1(2:m)}, & \ldots, & Z_{1(m:m)}. \\ Z_{2(1:m)}, & Z_{2(2:m)}, & \ldots, & Z_{2(m:m)}. \\ \vdots & \vdots & \vdots & \vdots \\ Z_{m(1:m)}, & Z_{m(2:m)}, & \ldots, & Z_{m(m:m)}. \end{array}$$

(3) From the first set of m units, the smallest ranked unit is selected and measured. From the second set, the second smallest ranked unit is selected. This process continues until, from the mth set, the largest ranked unit is measured as we see in the following the underlined units:

$$\begin{array}{cccc} \underline{Z_{1(1:m)}}, & Z_{1(2:m)}, & \ldots, & Z_{1(m:m)}. \\ Z_{2(1:m)}, & \underline{Z_{2(2:m)}}, & \ldots, & Z_{2(m:m)}. \\ \vdots & \vdots & \vdots & \vdots \\ Z_{m(1:m)}, & Z_{m(2:m)}, & \ldots, & \underline{Z_{m(m:m)}}. \end{array}$$

(4) Repeat the Steps 1 through 3 r cycles to obtain a sample of size $n = mr$.

Then, the RSS units are

$$Z_{1(1:m)}, Z_{2(2:m)}, ..., Z_{m(m:m)}.$$

Let $Z_{(k:m)j}$ denote the kth-order statistic from the kth sample of size m, at the jth cycle, where $k = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

The PDF and cumulative distribution function (CDF) [2] of $Z_{(k:m)j}$, respectively, are given by:

$$f(z_{(k:m)j}; \Theta) = \frac{m!}{(k-1)!(m-k)!} f(z_{(k:m)j}; \Theta)[F(z_{(k:m)j}; \Theta)]^{k-1}[1 - F(z_{(k:m)j}; \Theta)]^{m-k}, \tag{1.1}$$

and

$$F(z_{(k:m)j}; \Theta) = \sum_{h=k}^{m} \frac{m!}{h!(m-h)!}[F(z_{(k:m)j}; \Theta)]^{h}[1 - F(z_{(k:m)j}; \Theta)]^{m-h}. \tag{1.2}$$

Various modifications and applications of the RSS are suggested. For example, [3] proposed the partial RSS design as a modification to the usual RSS. The authors of [4] considered the problem of estimating the two-parameter of Xgamma distribution parameters using RSS, whereas [5] used the RSS to estimate the parameters of the inverted Kumaraswamy distribution. In addition, [6] studied median RSS in modeling the stress–strength. The authors [7] used the RSS for estimating the Farlie-Gumbel-Morgenstern bivariate Weibull distribution parameters with a medical data application, whereas [8] studied the distribution function estimation using moving extreme and MiniMax RSS methods. Taconeli [9] suggested dual-rank RSS method. Aldrabseh et al. [10] considered the except-extremes RSS for estimating the population variance, whereas [11] investigated the CDF estimation based on dual-rank RSS, with an application to body mass index data. Alsadat et al. [12] employed the RSS for parameters estimation of the power logarithmic distribution, and [13] investigated the RSS in statistical inference for a novel distribution. Kumar et al. [14] proposed a new modification of the RSS for mean estimation, namely; neutrosophic median RSS. Shafiq et al. [15]

used both the RSS and SRS for a statistical framework for a new Kavya-Manoharan Bilal distribution. Woodall et al. [16] used the RSS to evaluate the performance of control charts. [17] proposed the RSS in estimating multi-stress strength reliability for an inverted Kumaraswamy distribution. Moreover, [18] considered the problem of estimating the Gumbel distribution in terms of an ordered maximum RSS with unequal samples, whereas [19] considered the problem of large sample properties of modified MLE of the location parameter using moving extremes RSS. Yang et al. [20] investigated SRS and moving extremes RSS for estimating the parameters of the log-extended exponential-geometric distribution.

Shala and Merovci [21] suggested a new three-parameter distribution called the transmuted inverse Rayleigh distribution, with the PDF and CDF defined as follows:

$$f(z;\Theta) = \frac{2\alpha(1+\lambda)\sigma^2}{z^3} e^{-\alpha\left(\frac{\sigma}{z}\right)^2} - \frac{4\alpha\lambda\sigma^2}{z^3} e^{-2\alpha\left(\frac{\sigma}{z}\right)^2}, \tag{1.3}$$

for $z \geq 0$, $\sigma > 0$, $\alpha > 0$, $\Theta = \lambda, \alpha, \sigma$, and $|\lambda| \leq 1$,

$$F(z;\Theta) = (1+\lambda)e^{-\alpha\left(\frac{\sigma}{z}\right)^2} - \lambda e^{-2\alpha\left(\frac{\sigma}{z}\right)^2}. \tag{1.4}$$

The hazard rate function (HRF) of the generalized transmuted inverse Rayleigh distribution (TIRD) is given by

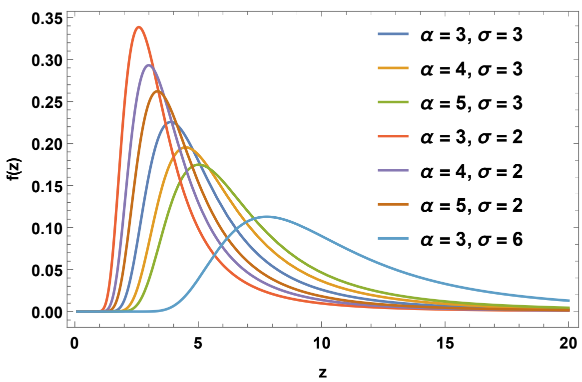
$$h(z;\Theta) = \frac{f(z;\Theta)}{1-F(z;\Theta)} = \frac{2\alpha\sigma^2 B\,(1+\lambda-2\lambda B)}{z^3\left[1-(1+\lambda)B-\lambda B^2\right]}, \quad \text{where } B = \exp\left(-\alpha\left(\frac{\sigma}{z}\right)^2\right), \tag{1.5}$$

with the rth moments given by

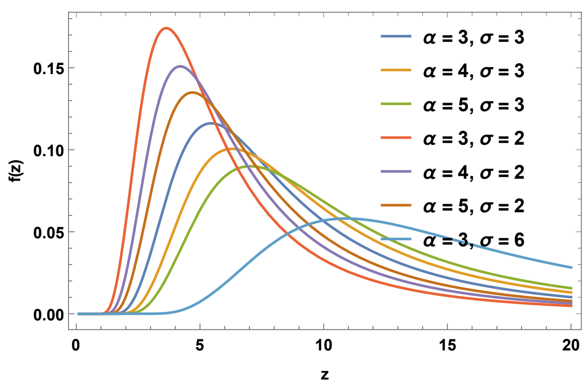
$$E(Z^r) = \alpha^{r/2}\sigma^2\Gamma\left(1-\frac{r}{2}\right)\left[1+\lambda-2^{r/2}\lambda\right], \quad r < 2.$$

The quantile function of the TIRD, with the parameters λ, α, and σ, is given by

$$x_p = Q(p) = \begin{cases} \frac{-\alpha\sigma^2 \ln\left[1+\lambda-\sqrt{(1+\lambda)^2-4\lambda p}\right]}{2\lambda}, & \text{if } \lambda \neq 0, \\ \frac{\sigma}{\sqrt{-\frac{1}{\alpha}\ln(p)}}, & \text{if } \lambda = 0. \end{cases}$$



(a) $\lambda = 0.7$

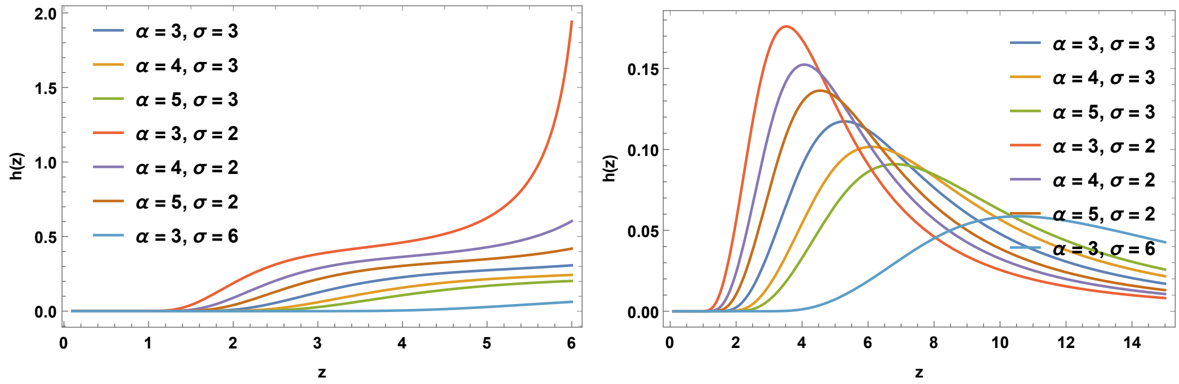


(b) $\lambda = -0.7$

Figure 1. PDF plots of the TIRD for some values of λ and α.

Figure 1 shows the PDF of the TIRD for certain parameter values. It is evident from the figure that the distribution exhibits various shapes of the PDF, demonstrating its flexibility in fitting different types of data.

Figure 2 presents the TIRD's HRF plots for some parameters values. It is clear that the hazard function of the generalized TIRD exhibits various shapes, including decreasing, increasing, and unimodal patterns.



(a) $\lambda = 0.2$ (b) $\lambda = -0.7$

Figure 2. HRF plots of the TIRD for some values of λ and α.

To analyze the shape of the HRD, we employed the method proposed by [22] that relies solely on the PDF of the distribution, which is defined as follows:

$$\tau(z;\Theta) = -\frac{f'(z;\Theta)}{f(z;\Theta)},$$

where $h(z;\Theta)$ and $\tau(z;\Theta)$ have the same properties. Hence, if $\tau'(z;\Theta) > 0$ for $z \in (0, z_0)$, $\tau'(z_0;\Theta) = 0$, and $\tau'(z;\Theta) < 0$ for $z \in (z_0, 0)$, then the distribution hasan upside-down bathtub hazard rate (UBT). The first derivative of the TIRD's PDF is

$$f'(z;\Theta) = \frac{1}{z^6} 2\alpha\sigma^2 e^{-\frac{2\alpha\sigma^2}{z^2}} \left(6\lambda z^2 - 8\alpha\lambda\sigma^2 + (\lambda+1) e^{\frac{\alpha\sigma^2}{z^2}} \left(2\alpha\sigma^2 - 3z^2\right)\right),$$

$$\tau(z;\Theta) = -\frac{\left(\frac{4\alpha^2(\lambda+1)\sigma^4}{z^6} - \frac{6\alpha(\lambda+1)\sigma^2}{z^4}\right) e^{-\frac{\alpha\sigma^2}{z^2}} + \left(-\frac{16\alpha^2\lambda\sigma^4}{z^6} + \frac{12\alpha\lambda\sigma^2}{z^4}\right) e^{-\frac{2\alpha\sigma^2}{z^2}}}{\left(\frac{2\alpha(\lambda+1)\sigma^2}{z^3} - \frac{4\alpha\lambda\sigma^2}{z^3} e^{-\frac{\alpha\sigma^2}{z^2}}\right) e^{-\frac{\alpha\sigma^2}{z^2}}},$$

and when $\tau'(z_0) = 0$ implies

$$\begin{aligned} z_0^4 \Bigg(& (3\lambda^2 - 12\alpha\lambda^2 - 6\alpha\lambda^2 e^{\frac{2\alpha}{z_0^2}} - 8\alpha^2\lambda^2 e^{\frac{\alpha}{z_0^2}}) \\ & + (6\lambda - 12\lambda + 36\alpha\lambda - 12\alpha\lambda e^{\frac{2\alpha}{z_0^2}} + 8\alpha^2\lambda e^{\frac{\alpha}{z_0^2}}) + 3 - 6\alpha e^{\frac{2\alpha}{z_0^2}} \Bigg) = 0. \end{aligned}$$

Solving this equation analytically is quite challenging because it combines polynomial and exponential terms in z_0. Such equations rarely have straightforward closed-form solutions and are generally tackled using numerical methods for given values of λ, α, and σ. For more details, see [21].

The integration of RSS with the TIRD offers significant improvements over traditional methods. RSS increases the estimation precision, especially in small-sample scenarios, by incorporating prior ranking information. The TIRD is known for its flexibility in modeling positively skewed and heavy-tailed data, and it addresses limitations of classical distributions. When combined, the RSS and TIRD provide a robust framework that enhances the parameter estimation accuracy, reduces the influence of outliers, and supports more efficient statistical inference in real-world applications.

To the best of our knowledge, this is the first study in the literature to consider RSS for estimating the parameters of the TIRD. This study aims to:

- Compare some estimation methods for the TIRD parameters under RSS and SRS designs. Then, assess performance of the estimators using methods such as using maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares (WLS), maximum product of spacing (MPS), Anderson–Darling estimation (AD), right AD (RAD), left AD (LAD), second-order LAD (LTS), and Cramér–von Mises estimation (CvM).
- Conduct simulations to evaluate the accuracy of RSS with different estimation methods based on mean square error (MSE), bias, and mean relative error (MRE).
- Apply the estimation methods to a COVID-19 dataset to demonstrate its practical use. Additionally, to compare the efficiency of RSS and SRS designs, highlighting the advantages of RSS, particularly for small sample sizes.

The remainder of this paper is structured as follows: Section 2 presents the estimation methods considered in this study along with the goodness-of-fit tests. Section 3 provides numerical simulations for various sample sizes. In Section 4, an application to real data is discussed. Finally, the conclusions are drawn in Section 5.

2. Methods of estimation

This section provides a detailed explanation of RSS and derives the estimators of the TIRD using various estimation methods under both RSS and SRS.

2.1. Maximum likelihood estimation

Let $z_{k(k:m)l}, k = 1, 2, \ldots, m, l = 1, 2, \ldots, r$ be the RSS of size $n = mr$ where m is the set size and r is the number of cycles, gathered form the TIRD. For abbreviation, we will consider $\hat{\Theta} = \hat{\alpha}, \hat{\sigma}, \hat{\lambda}$. The likelihood function of the TIRD, is obtained by inserting Eqs 1.3 and 1.4 in Eq 1.1 as shown below:

$$\begin{aligned} L(\Theta) &\propto \prod_{k=1}^{m}\prod_{l=1}^{r}\left[F(z_{k(k:m)l}, \Theta)\right]^{k-1}\left[1 - F(z_{k(k:m)l}, \Theta)\right]^{m-k} f(z_{k(k:m)l}, \Theta) \\ &= \prod_{k=1}^{m}\prod_{l=1}^{r}\left((1+\lambda)e^{-\alpha\left(\frac{\sigma}{z}\right)^2} - \lambda e^{-2\alpha\left(\frac{\sigma}{z}\right)^2}\right)^{k-1} \\ &\times\left(1 - (1+\lambda)e^{-\alpha\left(\frac{\sigma}{z}\right)^2} - \lambda e^{-2\alpha\left(\frac{\sigma}{z}\right)^2}\right)^{m-k} \\ &\times\left(\frac{2\alpha(1+\lambda)\sigma^2}{z^3}e^{-\alpha\left(\frac{\sigma}{z}\right)^2} - \frac{4\alpha\lambda\sigma^2}{z^3}e^{-2\alpha\left(\frac{\sigma}{z}\right)^2}\right). \end{aligned} \tag{2.1}$$

The logarithmic of Eq 2.1, indicated by $l^{\bullet}(\Theta)$, is expressed by

$$\begin{aligned}
l^{\bullet}(\Theta) \quad \propto \quad & mr\log(\alpha) + 2mr\log(\sigma) + \sum_{k=1}^{m}\sum_{l=1}^{r}(k-1)\log\left((\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right. \\
& \left. -\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right) + \sum_{k=1}^{m}\sum_{l=1}^{r}(m-k)\log\left\{1-\left((\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right.\right. \\
& \left.\left. -\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right)\right\} + \sum_{k=1}^{m}\sum_{l=1}^{r}\log\left((\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\right) \\
- \quad & 2\alpha\sigma^2\sum_{k=1}^{m}\sum_{l=1}^{r}\frac{1}{z^2_{k(k:m)l}} - \sum_{k=1}^{m}\sum_{l=1}^{r}\log(z^3_{k(k:m)l}). \qquad (2.2)
\end{aligned}$$

Differentiate Eq 2.2 with respect to Θ and equate to zero to obtain

$$\begin{aligned}
\frac{\partial l(\Theta)}{\partial\alpha} \quad = \quad & \frac{mr}{\alpha} + \sum_{k=1}^{m}\sum_{l=1}^{r}(k-1)\frac{\sigma^2}{z^2_{k(k:m)l}}\frac{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)} \\
- \quad & \sum_{k=1}^{m}\sum_{l=1}^{r}(m-k)\frac{\sigma^2}{z^2_{k(k:m)l}}\frac{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{1-\left((\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right)} \\
+ \quad & \sum_{k=1}^{m}\sum_{l=1}^{r}\frac{(\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\sigma^2}{z^2_{k(k:m)l}\left((\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\right)} - 2\sigma^2\sum_{k=1}^{m}\sum_{l=1}^{r}\frac{1}{z^2_{k(k:m)l}} = 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l(\Theta)}{\partial\lambda} \quad = \quad & \sum_{k=1}^{m}\sum_{l=1}^{r}(k-1)\frac{\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)} \\
+ \quad & \sum_{k=1}^{m}\sum_{l=1}^{r}(m-k)\frac{-\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)+\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{1-\left((\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right)} \\
+ \quad & \sum_{k=1}^{m}\sum_{l=1}^{r}\frac{\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2}{(\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda} = 0,
\end{aligned}$$

$$\frac{\partial l(\Theta)}{\partial\sigma} \quad = \quad \frac{2mr}{\sigma} + \sum_{k=1}^{m}\sum_{l=1}^{r}(k-1)\frac{2\alpha\sigma}{z^2_{k(k:m)l}}\frac{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}$$

$$- \sum_{k=1}^{m}\sum_{l=1}^{r}(m-k)\frac{2\alpha\sigma}{z^2_{k(k:m)l}}\frac{(\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{1-\left((\lambda+1)\exp\left(-\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-\lambda\exp\left(-\frac{2\alpha\sigma^2}{z^2_{k(k:m)l}}\right)\right)}$$

$$+ \sum_{k=1}^{m}\sum_{l=1}^{r}\frac{2\alpha\sigma(\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)}{z^2_{k(k:m)l}\left((\lambda+1)\exp\left(\frac{\alpha\sigma^2}{z^2_{k(k:m)l}}\right)-2\lambda\right)} - 4\alpha\sigma\sum_{k=1}^{m}\sum_{l=1}^{r}\frac{1}{z^2_{k(k:m)l}} = 0. \tag{2.3}$$

The MLE $\hat{\Theta}$ of Θ is the solution of the non-linear Eq 2.3. Since Eq 2.3 do not have an analytical solution, a numerical method can be used to obtain $\hat{\Theta}$.

2.2. Least square estimators

In this section, the OLS and WLS are derived using the RSS method. Swain et al. [23] introduced the OLS and WLS for estimating the parameters of the beta distribution. Consider an ordered sample $Z_{1(1:m)}, Z_{2(2:m)}, \ldots, Z_{M(m:m)}$ which constitutes an RSS of size $n = mr$, selected from the TIRD. By minimizing the respective objective functions with respect to Θ , the OLSE $\hat{\Theta}_{OLS}$ and WLSE $\hat{\Theta}_{WLS}$ are obtained. The objective functions to be minimized are given by:

$$\Lambda(\Theta) = \sum_{k=1}^{m}\left[F\left(z_{k(k:m)}\,|\Theta\right) - \frac{k}{m+1}\right]^2,$$

$$\Phi(\Theta) = \sum_{k=1}^{m}\frac{(n+1)^2(m+2)}{k(m-k+1)}\left[F\left(z_{k(k:m)}\,|\Theta\right) - \frac{k}{m+1}\right]^2.$$

The nonlinear equations below can be solved to get $\hat{\Theta}_{OLS}$ and $\hat{\Theta}_{WLS}$, respectively as follows:

$$\frac{\partial\Lambda(\Theta)}{\partial\Theta} = \sum_{k=1}^{m}\left[F\left(z_{k(k:m)}\,|\Theta\right) - \frac{k}{n+1}\right]\Delta\left(z_{k(k:m)}\,|\Theta\right) = 0,$$

$$\frac{\partial\Phi(\Theta)}{\partial\Theta} = \sum_{k=1}^{m}\frac{(m+1)^2(m+2)}{k(m-k+1)}\left[F\left(z_{k(k:m)}\,|\Theta\right) - \frac{k}{m+1}\right]\Delta\left(z_{k(k:m)}\,|\Theta\right) = 0,$$

where

$$\Delta\left(z_{k(k:m)}\,|\Theta\right) = \frac{\partial}{\partial\Theta}F\left(z_{k(k:m)}\,|\Theta\right) = \frac{z_{k(k:m)}\Theta\left(\Theta - z_{k(k:m)} + 2\right)e^{-\frac{z_{k(k:m)}\Theta}{1-z_{k(k:m)}}}}{\left(z_{k(k:m)}-1\right)^2(\Theta+1)^2}. \tag{2.4}$$

2.3. Maximum product of spacings estimation

The MPS method has been recognized as a strong alternative to traditional MLE, as highlighted by the pioneering work of [24, 25]. Their findings indicate that MPS provides notable benefits in parameter estimation for continuous univariate distributions. Specifically, MPS ensures greater consistency compared with MLE methods while achieving the same level of efficiency as MLE.

Consider an ordered sample $Z_{1(1:m)}, Z_{2(2:m)}, \ldots, Z_{m(m:m)}$ that forms an RSS of size m, collected from the TIRD. The corresponding uniform spacings are defined as:

$$\Pi_k(\Theta) = F\left(z_{k(k:m)}\,|\Theta\right) - F\left(z_{k(k-1:m)}\,|\Theta\right), \quad i = 1, 2, ..., m+1,$$

where the following boundary conditions hold: $F\left(z_{k(0:m)}\,|\Theta\right) = 0, \quad F\left(z_{k(m+1:m)}\,|\Theta\right) = 1$, ensuring that the sum of spacings satisfies $\sum_{k=1}^{m+1} \Pi_k(\Theta) = 1$.

The MPSE $\hat{\Theta}_{MPS}$ of Θ is produced by maximizing the following function with respect to Θ:

$$\Upsilon = \frac{1}{m+1} \sum_{k=1}^{m+1} \log\left[\Pi_s(\Theta)\right].$$

The MPSE $\hat{\Theta}_{MPS}$ is determined by solving the following equation numerically:

$$\frac{\partial \Upsilon}{\partial \Theta} = \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{1}{\Pi_k(\Theta)} \left[\Delta\left(z_{k(k:m)}\,|\Theta\right) - \Delta\left(z_{k(k-1:m)}\,|\Theta\right)\right] = 0,$$

where $\Delta\left(.\,|\Theta\right)$ is given in Eq (2.4).

2.4. Goodness-of-fit estimators

Next, we explore techniques for estimating the parameter Θ by optimizing the goodness-of-fit measures. These approaches aim to minimize the discrepancy between the empirical CDF and the fitted CDF.

2.4.1. Anderson-Darling methods

The Anderson-Darling test, proposed by [26] as an alternative to traditional normality tests, is recognized for its rapid convergence to the asymptotic distribution. This characteristic improves its reliability in statistical inference [27, 28].

Consider an ordered sample, denoted as $\{Z_{1(1:m)}, Z_{2(2:m)}, \ldots, Z_{m(m:m)}\}$, forming an RSS of size m drawn from the TIRD. The estimators $\hat{\Theta}_{AD}$ (AD), $\hat{\Theta}_{RAD}$ (RAD), $\hat{\Theta}_{LAD}$ (LAD), and $\hat{\Theta}_{LTS}$ (LTS) for Θ are obtained by minimizing their respective objective functions as follows:

$$\begin{aligned}
\mathrm{AD}(\Theta) &= -m - \frac{1}{m} \sum_{k=1}^{m} (2k-1) \left\{\log F\left(z_{k(k:m)}\,|\Theta\right) + \log \bar{F}\left(z_{k(m-k+1:m)}\,|\Theta\right)\right\}, \\
\mathrm{RAD}(\Theta) &= \frac{m}{2} - 2 \sum_{k=1}^{m} F\left(z_{k(k:m)}\,|\Theta\right) - \frac{1}{m} \sum_{k=1}^{m} (2k-1) \log \bar{F}\left(z_{k(m+1-k:m)}\,|\Theta\right), \\
\mathrm{LAD}(\Theta) &= \frac{-3m}{2} + 2 \sum_{k=1}^{m} F\left(z_{k(k:m)}\,|\Theta\right) - \frac{1}{m} \sum_{k=1}^{m} (2k-1) \log H\left(z_{k(k:m)}\,|\Theta\right), \\
\mathrm{LTS}(\Theta) &= 2 \sum_{k=1}^{m} \log\left[F\left(z_{k(k:m)}\,|\Theta\right)\right] + \frac{1}{n} \sum_{k=1}^{m} \frac{(2k-1)}{F\left(z_{k(k:m)}\,|\Theta\right)},
\end{aligned} \tag{2.5}$$

where $\bar{F}\left(.\,|\Theta\right)$ is the survival function.

2.4.2. Cramér-von Mises estimators

MacDonald [29] demonstrated that the Cramér-von Mises (CV) statistic exhibits lower bias compared with other minimum distance estimators, making it a valuable alternative for parameter estimation. Consider an ordered sample $Z_{1(1:m)}, Z_{2(2:m)}, ..., Z_{m(m:m)}$ forming a RSS of size m, collected from the TIRD. The CVM estimator (CV) $\hat{\Theta}_{CV}$ of Θ is obtained by minimizing the following function:

$$\eta(\Theta) = \frac{1}{12m} + \sum_{k=1}^{m} \left\{ F\left(z_{k(k:m)} | \Theta\right) - \frac{2k-1}{2m} \right\}^2 . \tag{2.6}$$

Alternatively, instead of directly minimizing Eq (2.6), the CVME $\hat{\Theta}_{CV}$ can be obtained by solving the following nonlinear equation:

$$\frac{\partial \eta(\Theta)}{\partial \Theta} = \sum_{k=1}^{m} \left\{ F\left(z_{k(k:m)} | \Theta\right) - \frac{2k-1}{2m} \right\} \Delta\left(z_{k(k:m)} | \Theta\right) = 0,$$

where $\Delta\left(. | \Theta\right)$ is defined in Eq (2.4), and

$$\chi(\Theta) = \underset{1 \leqslant k \leqslant m}{Max} \left[\frac{k}{m} - F\left(z_{k(k:m)} | \Theta\right), F\left(z_{k(k:m)} | \Theta\right) - \frac{k-1}{n} \right].$$

3. Numerical simulation

This section presents a simulation study to assess the performance of the proposed estimators for the TIRD using both SRS and RSS. Here, we considered three sample sizes: $n = m \times r$, $n = 5 \times 3 = 15$ (small); $n = 7 \times 3 = 21$, $n = 5 \times 6 = 10 \times 3 = 30$ (moderate); and $n = 7 \times 6 = 42, n = 10 \times 6 = 60$ (large). The choices $m = 7, 10$ align with those used by [30] in quartile RSS and by [31] in double percentile RSS, assuming perfect ranking for 10,000 iterations. For each generated sample, the estimation is conducted for the parameter values $(\alpha, \sigma, \lambda) = (2, 0.8, 0.75)$ using various estimation methods, including MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS. We primarily utilized the Nelder–Mead algorithm implemented through the Optim function in R Software. To evaluate the accuracy of each estimation method and sampling design, four statistical metrics are computed: average estimator (AE), absolute bias (Bias), MSE, and mean absolute relative error (MRE), which are defined as follows:

$$AE(\hat{\Theta}) = \frac{1}{10000} \sum_{i=1}^{10000} \hat{\Theta}_i, Bias(\hat{\Theta}) = \frac{1}{10000} \sum_{i=1}^{10000} |\hat{\Theta}_i - \Theta|,$$

and

$$MSE(\hat{\Theta}) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\Theta}_i - \Theta)^2, MRE(\hat{\Theta}) = \frac{1}{10000} \sum_{i=1}^{10000} \frac{|\hat{\Theta}_i - \Theta|}{\Theta}, \text{where } \Theta = \alpha, \sigma, \lambda.$$

The results obtained using SRS are presented in Tables 1 and 2 for the sample sizes of (5×3), (5×6), (7×3), (7×6), (10×3), and (10×6), respectively. Similarly, the results based on RSS are shown in Tables 3 and 4 for the same sample sizes. In addition, a summary of the results is illustrated in Figures 3–6 for α, σ, and λ, respectively.

Table 1. Simulation results of MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS under SRS design for sample sizes of (5×3), (5×6), and (7×3) with $(\alpha, \sigma, \lambda) = (2, 0.8, 0.75)$.

n	Par.	Metric	MLE	OLS	WLS	CV	MPS	AD	RAD	LAD	LTS
5×3	$\hat{\alpha}$	AE	1.9332	2.1079	2.1149	2.0279	2.2121	2.0888	2.0643	2.0743	2.0766
	$\hat{\alpha}$	Bias	0.0831	0.3308	0.3134	0.2543	0.3327	0.2602	0.2838	0.2989	0.3777
	$\hat{\alpha}$	MSE	0.0249	0.2514	0.2988	0.1573	0.2716	0.1668	0.2273	0.2175	0.3064
	$\hat{\alpha}$	MRE	0.0416	0.1654	0.1567	0.1271	0.1663	0.1301	0.1419	0.1494	0.1888
	$\hat{\sigma}$	AE	0.8165	0.7614	0.7658	0.8313	0.7158	0.7841	0.8048	0.8008	0.816
	$\hat{\sigma}$	Bias	0.065	0.1351	0.1302	0.1164	0.1448	0.1157	0.1316	0.133	0.1777
	$\hat{\sigma}$	MSE	0.0076	0.0369	0.0339	0.0305	0.0357	0.0312	0.0532	0.042	0.0726
	$\hat{\sigma}$	MRE	0.0813	0.1689	0.1628	0.1455	0.181	0.1446	0.1645	0.1663	0.2221
	$\hat{\lambda}$	AE	0.7186	0.5767	0.5949	0.8635	0.4449	0.6801	0.7189	0.77	0.8335
	$\hat{\lambda}$	Bias	0.294	0.4928	0.4562	0.4267	0.5076	0.4018	0.4142	0.5182	0.6843
	$\hat{\lambda}$	MSE	0.1503	0.4038	0.3632	0.3017	0.4546	0.2923	0.3083	0.4421	0.7658
	$\hat{\lambda}$	MRE	0.3921	0.6571	0.6083	0.569	0.6768	0.5358	0.5523	0.6909	0.9124
5×6	$\hat{\alpha}$	AE	1.9412	2.0569	2.0627	2.0363	2.183	2.0586	2.0523	2.0616	2.1097
	$\hat{\alpha}$	Bias	0.0702	0.2314	0.2071	0.1741	0.2524	0.1814	0.2116	0.2073	0.3036
	$\hat{\alpha}$	MSE	0.021	0.1328	0.1192	0.0769	0.1746	0.1047	0.1238	0.1048	0.2316
	$\hat{\alpha}$	MRE	0.0351	0.1157	0.1036	0.0871	0.1262	0.0907	0.1058	0.1037	0.1518
	$\hat{\sigma}$	AE	0.8126	0.78	0.7814	0.8145	0.732	0.7861	0.7951	0.7921	0.7846
	$\hat{\sigma}$	Bias	0.0476	0.0926	0.0823	0.0759	0.105	0.0756	0.0884	0.0876	0.1321
	$\hat{\sigma}$	MSE	0.0039	0.0342	0.014	0.0175	0.021	0.012	0.0179	0.0156	0.0363
	$\hat{\sigma}$	MRE	0.0596	0.1157	0.1029	0.0948	0.1313	0.0945	0.1104	0.1095	0.1651
	$\hat{\lambda}$	AE	0.727	0.6475	0.6701	0.814	0.5233	0.6963	0.7098	0.7473	0.7186
	$\hat{\lambda}$	Bias	0.239	0.3378	0.3011	0.2853	0.3562	0.2741	0.2745	0.3635	0.5455
	$\hat{\lambda}$	MSE	0.1022	0.2016	0.165	0.1324	0.271	0.1365	0.135	0.2265	0.49
	$\hat{\lambda}$	MRE	0.3186	0.4504	0.4014	0.3804	0.475	0.3655	0.366	0.4847	0.7274
7×3	$\hat{\alpha}$	AE	1.9369	2.0804	2.0803	2.0341	2.2005	2.0766	2.0483	2.0567	2.0829
	$\hat{\alpha}$	Bias	0.0771	0.2782	0.2449	0.2079	0.2864	0.2183	0.2391	0.2583	0.3338
	$\hat{\alpha}$	MSE	0.0237	0.1826	0.1532	0.0993	0.2182	0.1285	0.1456	0.1621	0.2531
	$\hat{\alpha}$	MRE	0.0386	0.1391	0.1225	0.1039	0.1432	0.1091	0.1196	0.1291	0.1669
	$\hat{\sigma}$	AE	0.8152	0.7722	0.7761	0.8222	0.7244	0.785	0.8016	0.8005	0.8009
	$\hat{\sigma}$	Bias	0.0569	0.1103	0.102	0.0934	0.1243	0.0923	0.1046	0.1119	0.1539
	$\hat{\sigma}$	MSE	0.0058	0.0266	0.0211	0.0176	0.0302	0.0173	0.0249	0.0286	0.058
	$\hat{\sigma}$	MRE	0.0711	0.1379	0.1275	0.1167	0.1554	0.1154	0.1308	0.1399	0.1924
	$\hat{\lambda}$	AE	0.7202	0.6216	0.643	0.842	0.4807	0.6896	0.7199	0.7557	0.7613
	$\hat{\lambda}$	Bias	0.2706	0.4152	0.3749	0.3514	0.4306	0.333	0.3302	0.4442	0.6176
	$\hat{\lambda}$	MSE	0.1345	0.2953	0.2495	0.2037	0.3603	0.2027	0.1906	0.3385	0.6161
	$\hat{\lambda}$	MRE	0.3608	0.5536	0.4999	0.4685	0.5742	0.444	0.4403	0.5923	0.8235

Table 2. Simulation results of MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS under the SRS design for sample sizes of (7×6), (10×3), and (10×6) with $(\alpha, \sigma, \lambda) = (2, 0.8, 0.75)$.

n	Est.	Metric	MLE	OLS	WLS	CV	MPS	AD	RAD	LAD	LTS
7×6	$\hat{\alpha}$	AE	1.9447	2.0528	2.0618	2.0451	2.1640	2.0646	2.0480	2.0607	2.1168
	$\hat{\alpha}$	Bias	0.0649	0.1911	0.1676	0.1479	0.2119	0.1573	0.1824	0.1752	0.2627
	$\hat{\alpha}$	MSE	0.0180	0.0960	0.0893	0.0527	0.1229	0.0722	0.1003	0.0757	0.1822
	$\hat{\alpha}$	MRE	0.0324	0.0956	0.0838	0.0739	0.1055	0.0787	0.0912	0.0876	0.1313
	$\hat{\sigma}$	AE	0.8115	0.7816	0.7830	0.8058	0.7406	0.7849	0.7949	0.7899	0.7718
	$\hat{\sigma}$	Bias	0.0395	0.0724	0.0645	0.0588	0.0854	0.0613	0.0716	0.0706	0.1127
	$\hat{\sigma}$	MSE	0.0027	0.0131	0.0086	0.0071	0.0146	0.0084	0.0128	0.0103	0.0256
	$\hat{\sigma}$	MRE	0.0493	0.0905	0.0806	0.0735	0.1067	0.0766	0.0895	0.0882	0.1408
	$\hat{\lambda}$	AE	0.7341	0.6746	0.6948	0.7983	0.5697	0.7064	0.7183	0.7412	0.6671
	$\hat{\lambda}$	Bias	0.2099	0.2775	0.2440	0.2355	0.2860	0.2257	0.2254	0.2998	0.4838
	$\hat{\lambda}$	MSE	0.0781	0.1357	0.1068	0.0900	0.1857	0.0913	0.0886	0.1547	0.4000
	$\hat{\lambda}$	MRE	0.2799	0.3700	0.3253	0.3140	0.3813	0.3010	0.3005	0.3997	0.6451
10×3	$\hat{\alpha}$	AE	1.9404	2.0575	2.0609	2.0400	2.1787	2.0685	2.0540	2.0659	2.1165
	$\hat{\alpha}$	Bias	0.0708	0.2261	0.1933	0.1710	0.2442	0.1819	0.2063	0.2116	0.3011
	$\hat{\alpha}$	MSE	0.0203	0.1272	0.1045	0.0865	0.1640	0.0977	0.1152	0.1284	0.2263
	$\hat{\alpha}$	MRE	0.0354	0.1130	0.0966	0.0855	0.1221	0.0909	0.1032	0.1058	0.1505
	$\hat{\sigma}$	AE	0.8120	0.7763	0.7795	0.8113	0.7310	0.7830	0.7949	0.7903	0.7762
	$\hat{\sigma}$	Bias	0.0469	0.0891	0.0793	0.0731	0.1032	0.0741	0.0869	0.0865	0.1290
	$\hat{\sigma}$	MSE	0.0039	0.0165	0.0130	0.0110	0.0225	0.0116	0.0207	0.0155	0.0341
	$\hat{\sigma}$	MRE	0.0586	0.1114	0.0991	0.0913	0.1290	0.0927	0.1086	0.1082	0.1612
	$\hat{\lambda}$	AE	0.7286	0.6432	0.6688	0.8111	0.5203	0.6924	0.7113	0.7383	0.6827
	$\hat{\lambda}$	Bias	0.2327	0.3322	0.2927	0.2782	0.3474	0.265	0.2663	0.3503	0.5359
	$\hat{\lambda}$	MSE	0.0979	0.1942	0.1551	0.1269	0.2548	0.1282	0.1264	0.2081	0.4751
	$\hat{\lambda}$	MRE	0.3102	0.4429	0.3902	0.3709	0.4632	0.3533	0.3551	0.4670	0.7146
10×6	$\hat{\alpha}$	AE	1.9553	2.0565	2.0476	2.0479	2.1328	2.0544	2.0435	2.0641	2.1005
	$\hat{\alpha}$	Bias	0.0528	0.1593	0.1362	0.1267	0.1639	0.1288	0.1582	0.1447	0.2236
	$\hat{\alpha}$	MSE	0.0109	0.0668	0.0488	0.0418	0.0680	0.0428	0.0723	0.0485	0.1427
	$\hat{\alpha}$	MRE	0.0264	0.0797	0.0681	0.0634	0.0819	0.0644	0.0791	0.0723	0.1118
	$\hat{\sigma}$	AE	0.8109	0.7837	0.7896	0.8026	0.7545	0.7876	0.7962	0.7892	0.7736
	$\hat{\sigma}$	Bias	0.0332	0.0576	0.0512	0.0466	0.0654	0.0483	0.0601	0.0549	0.0943
	$\hat{\sigma}$	MSE	0.0019	0.0093	0.0082	0.0047	0.0086	0.0049	0.0132	0.0061	0.0199
	$\hat{\sigma}$	MRE	0.0415	0.0720	0.0640	0.0583	0.0817	0.0604	0.0751	0.0686	0.1179
	$\hat{\lambda}$	AE	0.7502	0.7029	0.7224	0.7943	0.6327	0.7263	0.7318	0.7526	0.6653
	$\hat{\lambda}$	Bias	0.1723	0.2193	0.1911	0.1898	0.2100	0.1816	0.1838	0.2384	0.4146
	$\hat{\lambda}$	MSE	0.0507	0.0831	0.0633	0.0576	0.0993	0.0560	0.0585	0.0943	0.3052
	$\hat{\lambda}$	MRE	0.2297	0.2924	0.2548	0.2530	0.2800	0.2421	0.2451	0.3178	0.5528

Table 3. Simulation results of MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS under the RSS design for sample sizes of (5×3), (5×6), and (7×3) with $(\alpha, \sigma, \lambda) = (2, 0.8, 0.75)$.

n	Par.	Metric	MLE	OLS	WLS	CV	MPS	AD	RAD	LAD	LTS
5×3	$\hat{\alpha}$	AE	1.943	2.0896	2.1033	2.0482	2.2753	2.0893	2.0667	2.0955	2.1361
	$\hat{\alpha}$	Bias	0.0673	0.2983	0.2875	0.2016	0.3414	0.2243	0.2702	0.2513	0.3459
	$\hat{\alpha}$	MSE	0.0194	0.2253	0.2223	0.1064	0.2482	0.1344	0.1934	0.1808	0.3086
	$\hat{\alpha}$	MRE	0.0337	0.1492	0.1438	0.1008	0.1707	0.1122	0.1351	0.1257	0.173
	$\hat{\sigma}$	AE	0.8096	0.7432	0.7471	0.8096	0.6841	0.7663	0.7939	0.7719	0.7757
	$\hat{\sigma}$	Bias	0.0481	0.1172	0.1129	0.0865	0.1475	0.093	0.1118	0.1105	0.1572
	$\hat{\sigma}$	MSE	0.004	0.031	0.0266	0.0168	0.038	0.0184	0.0391	0.0241	0.0498
	$\hat{\sigma}$	MRE	0.0602	0.1466	0.1412	0.1081	0.1843	0.1163	0.1397	0.1382	0.1965
	$\hat{\lambda}$	AE	0.7294	0.4829	0.5107	0.8035	0.3521	0.6198	0.6752	0.6655	0.6998
	$\hat{\lambda}$	Bias	0.2469	0.4544	0.4196	0.3364	0.5207	0.3497	0.363	0.4365	0.6082
	$\hat{\lambda}$	MSE	0.1076	0.3521	0.3142	0.191	0.4733	0.2272	0.2321	0.3334	0.5996
	$\hat{\lambda}$	MRE	0.3291	0.6059	0.5595	0.4485	0.6942	0.4662	0.484	0.582	0.8109
5×6	$\hat{\alpha}$	AE	1.955	2.0575	2.0668	2.0514	2.2222	2.0722	2.0516	2.0696	2.1342
	$\hat{\alpha}$	Bias	0.052	0.1928	0.1724	0.1419	0.2478	0.1525	0.1898	0.1734	0.2823
	$\hat{\alpha}$	MSE	0.0112	0.0984	0.0819	0.0507	0.1569	0.0615	0.1053	0.0791	0.2195
	$\hat{\alpha}$	MRE	0.026	0.0964	0.0862	0.0709	0.1239	0.0762	0.0949	0.0867	0.1412
	$\hat{\sigma}$	AE	0.8069	0.7657	0.7698	0.8002	0.7148	0.774	0.7887	0.7776	0.7642
	$\hat{\sigma}$	Bias	0.0348	0.0735	0.067	0.0556	0.1013	0.0597	0.0707	0.0714	0.1225
	$\hat{\sigma}$	MSE	0.002	0.0125	0.0103	0.0069	0.0197	0.008	0.0132	0.0108	0.0326
	$\hat{\sigma}$	MRE	0.0435	0.0918	0.0838	0.0695	0.1266	0.0746	0.0884	0.0892	0.1531
	$\hat{\lambda}$	AE	0.7424	0.6149	0.6447	0.7899	0.4987	0.6775	0.6979	0.7001	0.643
	$\hat{\lambda}$	Bias	0.1871	0.2936	0.2583	0.2309	0.3397	0.2301	0.2375	0.3056	0.5002
	$\hat{\lambda}$	MSE	0.0592	0.1524	0.1204	0.0869	0.2461	0.0951	0.0991	0.1641	0.4214
	$\hat{\lambda}$	MRE	0.2495	0.3914	0.3444	0.3079	0.4529	0.3067	0.3166	0.4074	0.667
7×3	$\hat{\alpha}$	AE	1.9573	2.0674	2.07	2.052	2.2594	2.0802	2.0422	2.0678	2.1286
	$\hat{\alpha}$	Bias	0.0508	0.2186	0.195	0.1476	0.2925	0.1668	0.212	0.1881	0.2951
	$\hat{\alpha}$	MSE	0.0115	0.1274	0.1067	0.0506	0.2025	0.0865	0.1266	0.0949	0.2324
	$\hat{\alpha}$	MRE	0.0254	0.1093	0.0975	0.0738	0.1462	0.0834	0.106	0.094	0.1475
	$\hat{\sigma}$	AE	0.8091	0.7578	0.7628	0.8041	0.6973	0.7724	0.7948	0.778	0.767
	$\hat{\sigma}$	Bias	0.0382	0.0825	0.0753	0.0575	0.1201	0.0638	0.0807	0.0782	0.1309
	$\hat{\sigma}$	MSE	0.0024	0.0154	0.0126	0.007	0.0266	0.0088	0.0191	0.0171	0.0353
	$\hat{\sigma}$	MRE	0.0478	0.1032	0.0942	0.0719	0.1501	0.0798	0.1009	0.0977	0.1637
	$\hat{\lambda}$	AE	0.7485	0.5705	0.5991	0.8046	0.4219	0.6678	0.6948	0.6847	0.6474
	$\hat{\lambda}$	Bias	0.2004	0.3294	0.2938	0.2484	0.4165	0.2516	0.2625	0.3346	0.5312
	$\hat{\lambda}$	MSE	0.0672	0.1941	0.16	0.1006	0.3439	0.1172	0.1212	0.2052	0.4672
	$\hat{\lambda}$	MRE	0.2672	0.4392	0.3917	0.3312	0.5553	0.3355	0.35	0.4461	0.7083

Table 4. Simulation results of MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS under the RSS design for sample sizes of (7×6), (10×3), and (10×6) with $(\alpha, \sigma, \lambda) = (2, 0.8, 0.75)$.

n	Par.	Metric	MLE	OLS	WLS	CV	MPS	AD	RAD	LAD	LTS
7×6	$\hat{\alpha}$	AE	1.9682	2.0505	2.0575	2.0565	2.1689	2.0644	2.051	2.0601	2.1125
	$\hat{\alpha}$	Bias	0.0376	0.1421	0.1232	0.108	0.1843	0.1137	0.1578	0.1289	0.2269
	$\hat{\alpha}$	MSE	0.0058	0.0511	0.038	0.0263	0.0801	0.0322	0.0772	0.0429	0.16
	$\hat{\alpha}$	MRE	0.0188	0.071	0.0616	0.054	0.0921	0.0569	0.0789	0.0644	0.1135
	$\hat{\sigma}$	AE	0.8073	0.7745	0.7771	0.7974	0.7366	0.7804	0.7919	0.783	0.7651
	$\hat{\sigma}$	Bias	0.0274	0.0507	0.0443	0.0375	0.0731	0.0418	0.0543	0.0487	0.0973
	$\hat{\sigma}$	MSE	0.0012	0.0055	0.004	0.0029	0.0107	0.0038	0.0154	0.0054	0.0207
	$\hat{\sigma}$	MRE	0.0343	0.0634	0.0553	0.0469	0.0914	0.0523	0.0678	0.0608	0.1216
	$\hat{\lambda}$	AE	0.7534	0.6612	0.6837	0.7848	0.5775	0.7062	0.7156	0.7188	0.6306
	$\hat{\lambda}$	Bias	0.1485	0.2105	0.1843	0.1728	0.2387	0.1698	0.1784	0.2213	0.4185
	$\hat{\lambda}$	MSE	0.0353	0.075	0.0579	0.0469	0.1258	0.048	0.0527	0.0843	0.3139
	$\hat{\lambda}$	MRE	0.198	0.2807	0.2458	0.2303	0.3182	0.2264	0.2379	0.2951	0.558
10×3	$\hat{\alpha}$	AE	1.9685	2.0587	2.0613	2.0577	2.2116	2.0675	2.0485	2.0602	2.1187
	$\hat{\alpha}$	Bias	0.0375	0.1595	0.1408	0.1135	0.2289	0.1203	0.164	0.1378	0.2436
	$\hat{\alpha}$	MSE	0.0057	0.0645	0.0555	0.0291	0.1195	0.0351	0.0784	0.0532	0.18
	$\hat{\alpha}$	MRE	0.0187	0.0798	0.0704	0.0567	0.1144	0.0602	0.082	0.0689	0.1218
	$\hat{\sigma}$	AE	0.8071	0.765	0.7699	0.7976	0.7172	0.7767	0.7906	0.7797	0.7619
	$\hat{\sigma}$	Bias	0.029	0.0577	0.052	0.0401	0.0913	0.0448	0.0581	0.0531	0.1066
	$\hat{\sigma}$	MSE	0.0013	0.0068	0.0059	0.0035	0.0163	0.0042	0.0123	0.0063	0.025
	$\hat{\sigma}$	MRE	0.0362	0.0721	0.065	0.0501	0.1142	0.056	0.0726	0.0664	0.1332
	$\hat{\lambda}$	AE	0.7512	0.6186	0.6436	0.7842	0.505	0.6896	0.7039	0.6997	0.6142
	$\hat{\lambda}$	Bias	0.1539	0.2361	0.2054	0.1792	0.3022	0.1789	0.1918	0.2367	0.4482
	$\hat{\lambda}$	MSE	0.0366	0.0958	0.0753	0.0519	0.1992	0.0561	0.062	0.1004	0.3601
	$\hat{\lambda}$	MRE	0.2052	0.3148	0.2739	0.2389	0.4029	0.2386	0.2558	0.3156	0.5976
10×6	$\hat{\alpha}$	AE	1.9751	2.0509	2.0528	2.0567	2.1309	2.0627	2.0601	2.0405	2.0916
	$\hat{\alpha}$	Bias	0.0291	0.1112	0.0948	0.0888	0.141	0.09	0.1215	0.0957	0.1896
	$\hat{\alpha}$	MSE	0.0034	0.0316	0.0197	0.0162	0.0398	0.0174	0.0394	0.0195	0.1219
	$\hat{\alpha}$	MRE	0.0146	0.0556	0.0474	0.0444	0.0705	0.045	0.0608	0.0478	0.0948
	$\hat{\sigma}$	AE	0.8057	0.7784	0.7814	0.795	0.752	0.7825	0.7864	0.7879	0.7676
	$\hat{\sigma}$	Bias	0.0208	0.0354	0.0308	0.0262	0.054	0.0303	0.036	0.032	0.0825
	$\hat{\sigma}$	MSE	7e-04	0.0027	0.002	0.0015	0.0055	0.0019	0.0036	0.0022	0.0163
	$\hat{\sigma}$	MRE	0.026	0.0443	0.0385	0.0328	0.0675	0.0379	0.0451	0.0399	0.1032
	$\hat{\lambda}$	AE	0.7537	0.6879	0.7056	0.7763	0.6337	0.7211	0.7257	0.7304	0.6265
	$\hat{\lambda}$	Bias	0.1152	0.1531	0.1332	0.1274	0.1691	0.1263	0.1352	0.1626	0.3666
	$\hat{\lambda}$	MSE	0.0205	0.0381	0.0287	0.0254	0.0557	0.0254	0.0292	0.0428	0.2586
	$\hat{\lambda}$	MRE	0.1537	0.2042	0.1776	0.1699	0.2254	0.1684	0.1802	0.2168	0.4888

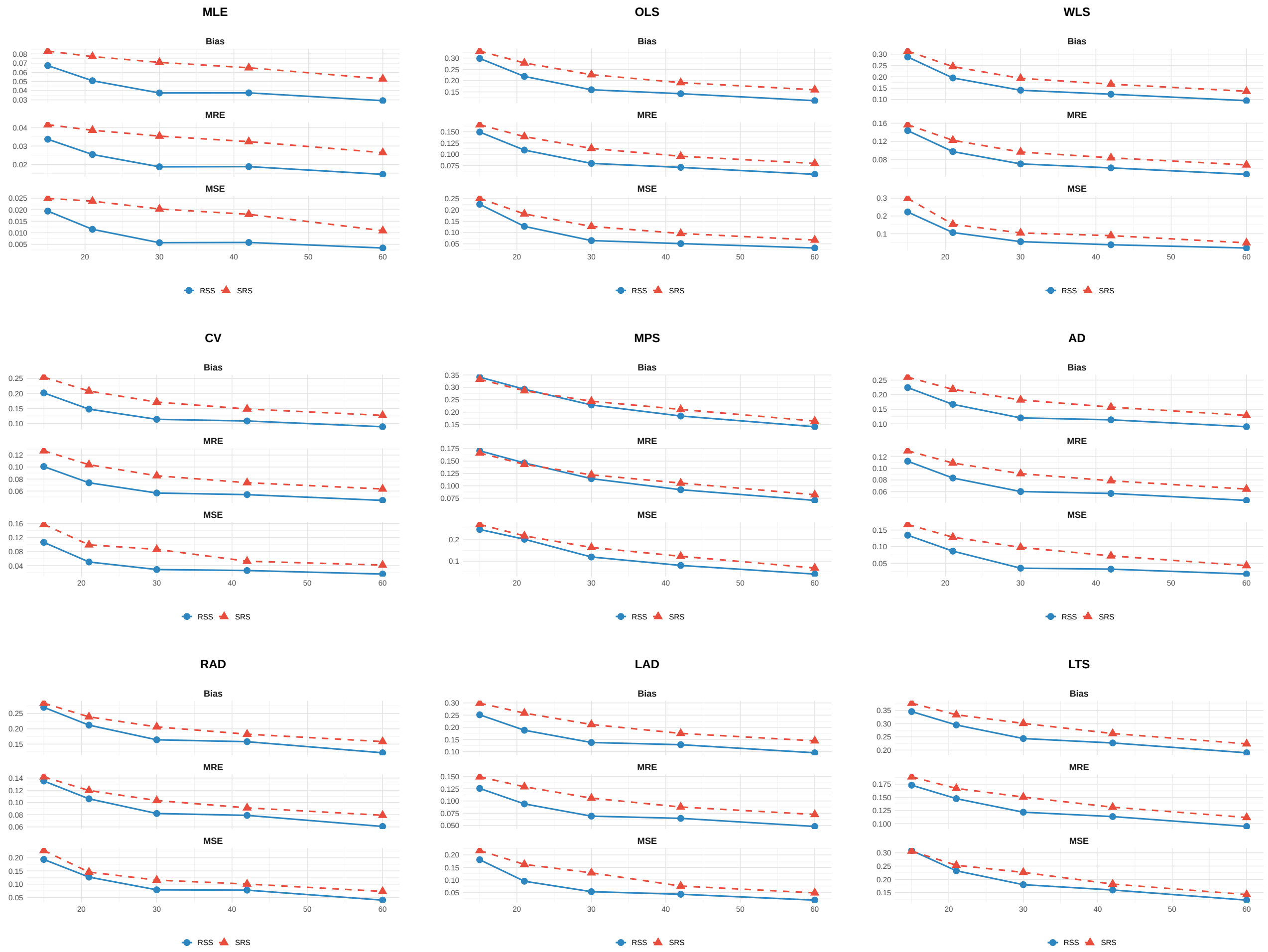


Figure 3. Comparison between SRS and RSS for estimating α using the MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS methods.

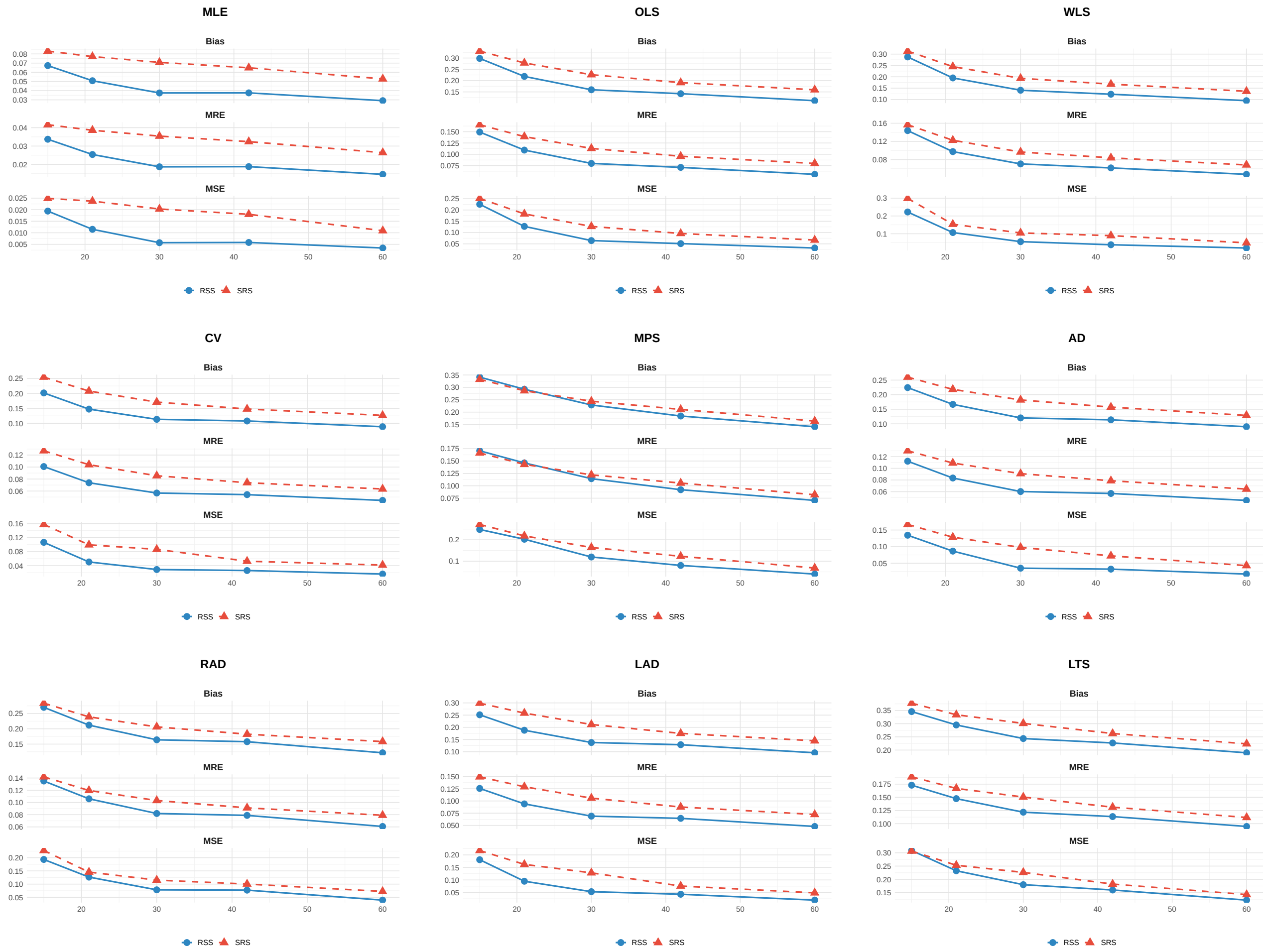


Figure 4. Comparison between SRS and RSS for estimating σ using the MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS methods.

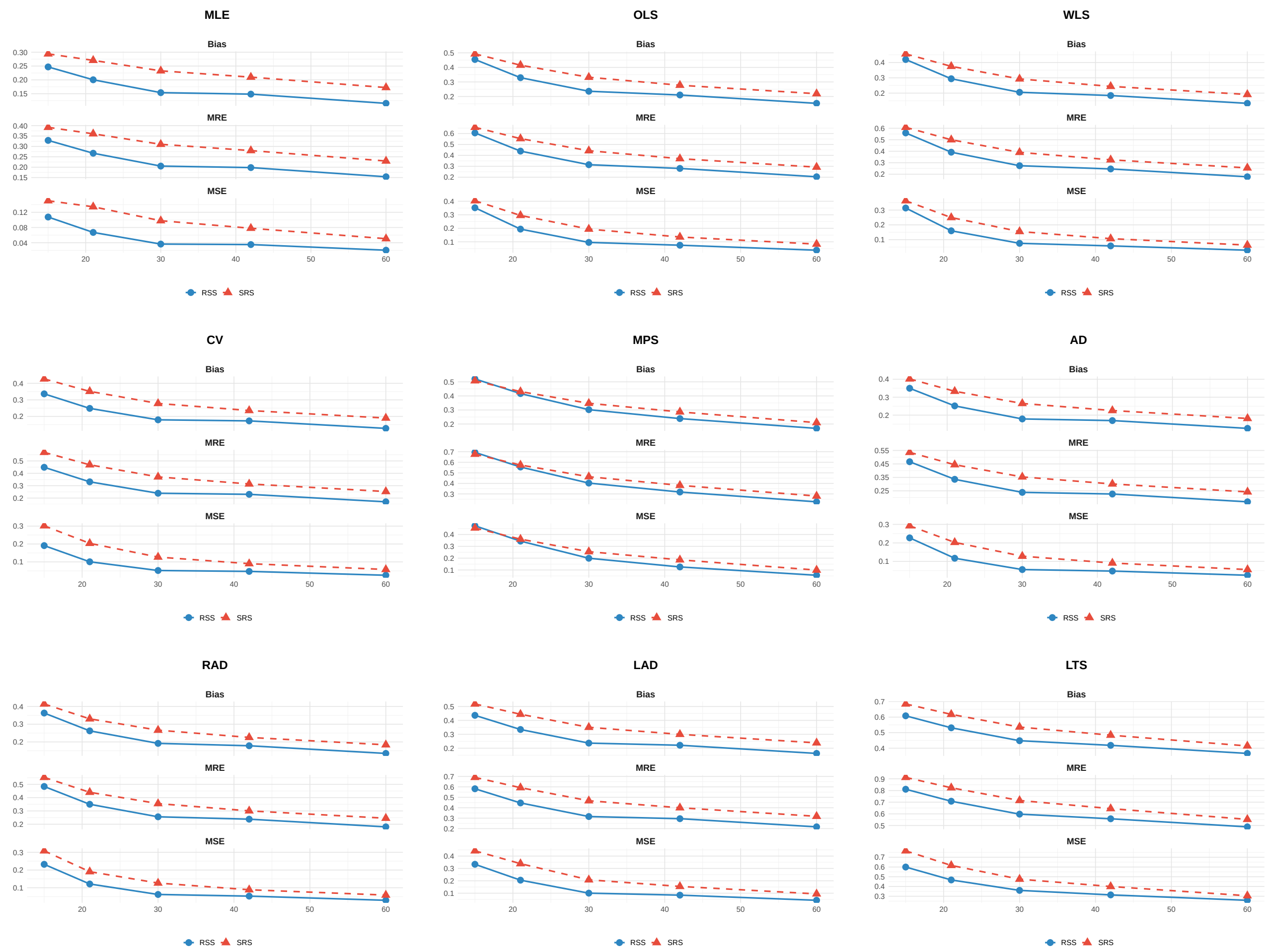


Figure 5. Comparison between SRS and RSS for estimating λ using the MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS methods.

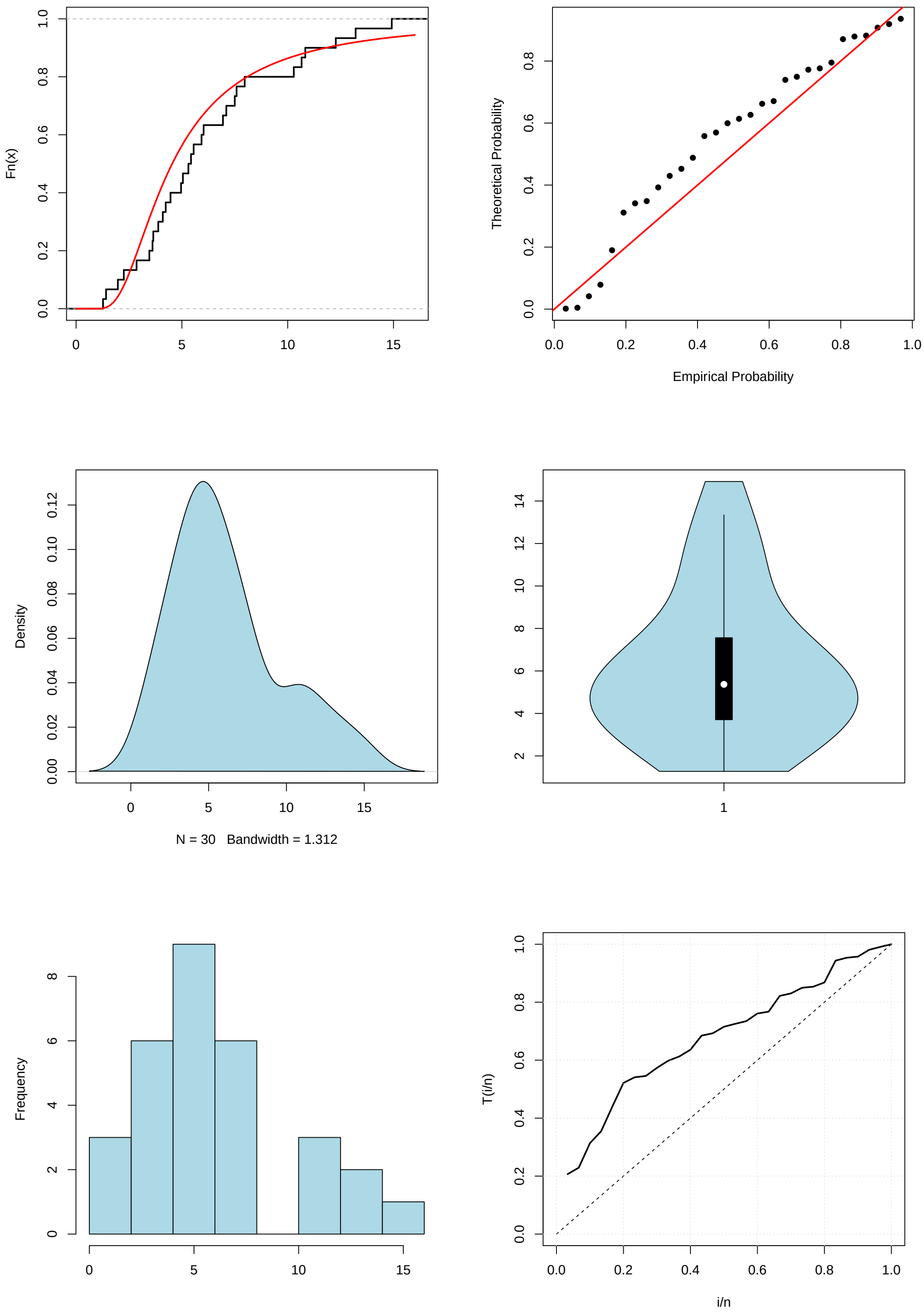


Figure 6. The plots of estimated CDF, P-P, density, violin, histogram, and TTT plot for the dataset.

Based on Tables 1 to 4, the following comments can be made:

- The MLE consistently shows lower AE, bias, and MSE compared with the other estimators, indicating its robustness under SRS.
- In terms of the MRE, one can note that the MRE based on the SRS is greater than of the RSS for fixed sample size. For illustration, when the sample size is 60 for estimating λ using the MLE, the MRE based on the SRS and RSS are 0.2297 and using 0.1537, respectively. However, the MRE is changing from one sampling method to other one, as an example, for a sample size of 42 for estimating α, the MRE values are 0.0324, 0.0956, 0.0838, and 0.0739 using MLE, OLS, WLS, and CV, respectively.
- The OLS and WLS methods generally exhibit higher AE and bias, making them less efficient compared with other methods. Among the methods considered, the MPS estimator shows the highest bias and MSE, indicating poor accuracy in estimating the parameters.
- As the sample size increases (e.g., from 5×3 to 10×6), the AE and Bias decrease, which aligns with expectations that larger samples yield more reliable estimates. The MSE also decreases with larger sample sizes, confirming the efficiency gain.
- Regarding the parameters, for $\hat{\alpha}$, the CV and AD estimators tend to perform better in terms of lower bias and MSE in most cases. For $\hat{\sigma}$, the MLE provides the smallest bias and MSE values, reinforcing its reliability. But for $\hat{\lambda}$, WLS and LAD often show lower AE and MRE, suggesting these estimators might be preferable when estimating this parameter.
- The AD, RAD, and LAD estimators yield competitive results, often achieving lower MSE than OLS and WLS, making them viable alternatives. However, the LTS estimator shows the highest bias and MSE, indicating it is the least effective among the methods considered.
- Across different sample sizes, the estimators based on RSS tend to have lower AE, MSE, bias, and MRE compared with their SRS counterparts. This confirms the efficiency of RSS in parameter estimation. The RSS approach results in smaller variances (MSE values) for most estimators, particularly for the shape parameter ($\hat{\alpha}$) and scale parameter ($\hat{\sigma}$). This suggests that RSS provides more reliable parameter estimations compared with SRS.
- The MLE generally provides the most stable and accurate estimates across all parameters, showing lower MSE and bias compared with other methods. This behavior is consistent with its known property of asymptotic efficiency.
- As the sample size increases, all estimators improve in performance, with reductions in bias and MSE. However, the efficiency gains are more pronounced under RSS, highlighting its advantage over SRS in small-sample scenarios.
- The WLS and OLS exhibit higher biases compared with MLE but remain competitive in terms of MSE. The MPS and AD estimators perform well for certain parameters but exhibit variability across different scenarios.
- The right-tail AD and left-tail AD estimators demonstrate stable results, particularly in capturing distributional tail behavior, but may not always outperform MLE. The AD-based methods (AD, RAD, LAD, and LTS) perform well under RSS, particularly for extreme values. This suggests that RSS enhances tail estimation accuracy, which is crucial for reliability analysis and risk assessment.

4. Application to COVID-19 data

The data considered here represent COVID-19 data from the Netherlands, covering a 30-day period from 31 March to 30 April 2020. The data give the rough mortality rate. The data are considered by [32] and regiven in Table 5.

Table 5. COVID-19 rough mortality rate data-set in the Netherlands.

14.918	10.656	12.274	10.289	10.832	7.099	5.928	13.211	7.968	7.584
5.555	6.027	4.097	3.611	4.960	7.498	6.940	5.307	5.048	2.857
2.254	5.431	4.462	3.883	3.461	3.647	1.974	1.273	1.416	4.235

Table 6 presents the descriptive statistics for the COVID-19 rough mortality rate dataset in the Netherlands. First we fit the TIRD to the dataset in terms of goodness-of-fit tests such as the Anderson–Darling test (AD_t), Cramér–von Mises test (CV_t) statistic, and Kolmogorov–Smirnov test (KS_t) along with their associated p-values. Also, the estimated CDF, Probability-Probability (P-P) plot, density plot, Violin plot, histogram plot, and a total time on test (TTT) plot for the dataset are given in Figure 6.

Table 6. Descriptive statistics for the COVID-19 rough mortality rate in Netherlands.

n	$Mean$	$Median$	sd	$Skewness$	$Kurtosis$	$Range$
30	6.157	5.369	3.533	0.834	2.953	13.645
Min	Max	$Q1$	$Q3$	IQR	CV	
1.273	14.918	3.706	7.562	3.857	57.392	

The estimation methods described above, including MLE, OLS, WLS, CV, MPS, AD, RAD, LAD, and LTS are applied using both the SRS and RSS methods. Two cases of the sample size are considered, ($m = 5, r = 2$) and ($m = 5, r = 3$), utilizing sampling without replacement, assuming the MLE estimates as the true values of the parameters in each case. We summarize the process in Table 7 for $m = 5$, and $r = 2$, where the last 5 units are considered for actual measurement.

Table 7. Selection process in RSS for $m = 5$ and $r = 2$.

Set	Cycle 1					Cycle 2				
	Step 1: 5 SRS each of 5 units									
	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Set 1	14.918	10.656	12.274	10.289	10.832	3.647	1.974	1.273	1.416	10.289
Set 2	7.099	5.928	13.211	7.968	7.584	4.235	4.097	3.611	4.960	7.498
Set 3	5.555	6.027	4.097	3.611	4.960	5.431	4.462	3.883	3.461	3.647
Set 4	7.498	6.940	5.307	5.048	2.857	10.656	12.274	10.289	5.431	4.462
Set 5	2.254	5.431	4.462	3.883	3.461	10.832	7.099	4.235	4.097	7.584
	Step 2: Rank the units in each set									
	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Set 1	10.289	10.656	10.832	12.274	14.918	1.273	1.416	1.974	3.647	10.289
Set 2	5.928	7.099	7.584	7.968	13.211	3.611	4.097	4.235	4.960	7.498
Set 3	3.611	4.097	4.960	5.555	6.027	3.461	3.647	3.883	4.462	5.431
Set 4	2.857	5.048	5.307	6.940	7.498	4.462	5.431	10.289	10.656	12.274
Set 5	2.254	3.461	3.883	4.462	5.431	4.097	4.235	7.099	7.584	10.832
	Step 3: Select the *k*th unit in the *k*th set									
	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Set 1	<u>10.289</u>	10.656	10.832	12.274	14.918	<u>1.273</u>	1.416	1.974	3.647	10.289
Set 2	5.928	<u>7.099</u>	7.584	7.968	13.211	3.611	<u>4.097</u>	4.235	4.960	7.498
Set 3	3.611	4.097	<u>4.960</u>	5.555	6.027	3.461	3.647	<u>3.883</u>	4.462	5.431
Set 4	2.857	5.048	5.307	<u>6.940</u>	7.498	4.462	5.431	10.289	<u>10.656</u>	12.274
Set 5	2.254	3.461	3.883	4.462	<u>5.431</u>	4.097	4.235	7.099	7.584	<u>10.832</u>
	Final selected RSS for measurement									
	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Set 1	10.289					1.273				
Set 2		7.099					4.097			
Set 3			4.960					3.883		
Set 4				6.940					10.656	
Set 5					5.431					10.832

The results are presented in Tables 8 and 9 for ($m = 5, r = 2$) and ($m = 5, r = 3$), respectively. The plots of the estimated PDFs and CDFs of the TIRD for $m = 5$ and $r = 2$ are given in Figures 7 and 8 based on different estimation methods using SRS and RSS. The same plots are given in Figures 9 and 10 for $m = 5$ and $r = 3$.

Table 8. Estimates and test statistics for each method under the SRS and RSS sampling methods, with $m = 5$ and $r = 2$.

Method	SRS estimates			RSS estimates			AD_t		CV_t		KS_t		p-value	
	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\lambda}$	SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
MLE	1.4729	2.2386	-0.5647	1.5909	2.6391	-0.4044	2.6328	1.0944	0.5517	0.1520	0.2330	0.1439	0.0647	0.5172
OLS	4.3886	1.5032	-0.4930	1.6589	2.6436	-0.4954	1.2184	0.8648	0.1984	0.0734	0.1590	0.1080	0.3932	0.8386
WLS	0.7526	3.5528	-0.5108	0.8048	3.6237	-0.5030	1.3350	0.9995	0.2322	0.1316	0.1685	0.1365	0.3248	0.5841
CV	0.7519	3.5452	-0.5197	0.6876	3.9396	-0.4999	1.3289	0.9829	0.2310	0.1255	0.1682	0.1340	0.3269	0.6065
MPS	0.7980	2.7844	-0.5896	4.1573	1.4606	-0.4693	4.0711	1.8412	0.8682	0.3614	0.2881	0.1983	0.0107	0.1649
AD	1.1869	2.6333	-0.6095	7.1427	1.1999	-0.4993	1.6918	1.0809	0.3281	0.1578	0.1917	0.1460	0.1936	0.4986
RAD	1.6076	2.5105	-0.5053	2.8591	2.0491	-0.5154	1.1117	0.8446	0.1678	0.0528	0.1494	0.0917	0.4696	0.9424
LAD	1.3053	2.3770	-0.6186	1.5407	2.4601	-0.4825	2.3591	1.5176	0.4888	0.2796	0.2222	0.1803	0.0879	0.2515
LTS	0.5818	2.9838	-0.5851	2.9302	1.5087	-0.8074	6.1300	2.2132	1.2898	0.4497	0.3490	0.2164	9e-04	0.1033

Table 9. Estimates and test statistics for each method under the SRS and RSS sampling methods, with $m = 5$ and $r = 3$.

Method	SRS estimates			RSS estimates			AD_t		CV_t		KS_t		p-value	
	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\lambda}$	SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
MLE	1.5001	2.2927	-0.5564	1.6688	2.5612	-0.4822	2.1903	0.9611	0.4486	0.1156	0.2150	0.1298	0.1070	0.6454
OLS	1.1037	3.0126	-0.5231	0.9653	3.4615	-0.5026	1.1038	0.8601	0.1666	0.0720	0.1491	0.1071	0.4724	0.8456
WLS	0.5208	4.2569	-0.5404	6.6196	1.2910	-0.5228	1.2706	0.8861	0.2159	0.0903	0.1641	0.1179	0.3551	0.7548
CV	1.0954	2.9105	-0.5601	1.7721	2.5055	-0.5284	1.2799	0.8695	0.2194	0.0830	0.1651	0.1139	0.3480	0.7897
MPS	0.7744	2.9671	-0.5950	2.1870	2.1254	-0.4809	3.1176	1.2685	0.6612	0.2119	0.2532	0.1628	0.0350	0.3644
AD	0.7711	3.3447	-0.5858	4.2336	1.5995	-0.5252	1.5292	0.9107	0.2865	0.1012	0.1822	0.1234	0.2410	0.7048
RAD	0.7127	3.5817	-0.6338	2.3334	2.3166	-0.4802	1.1347	0.8680	0.1805	0.0479	0.1539	0.0859	0.4326	0.9659
LAD	1.2271	2.5598	-0.5852	3.3880	1.7229	-0.5251	1.9303	1.0867	0.3866	0.1615	0.2036	0.1474	0.1444	0.4869
LTS	0.6088	3.1097	-0.6108	2.5626	1.6223	-0.8777	4.4392	1.8831	0.9458	0.3630	0.3000	0.2002	0.0069	0.1571

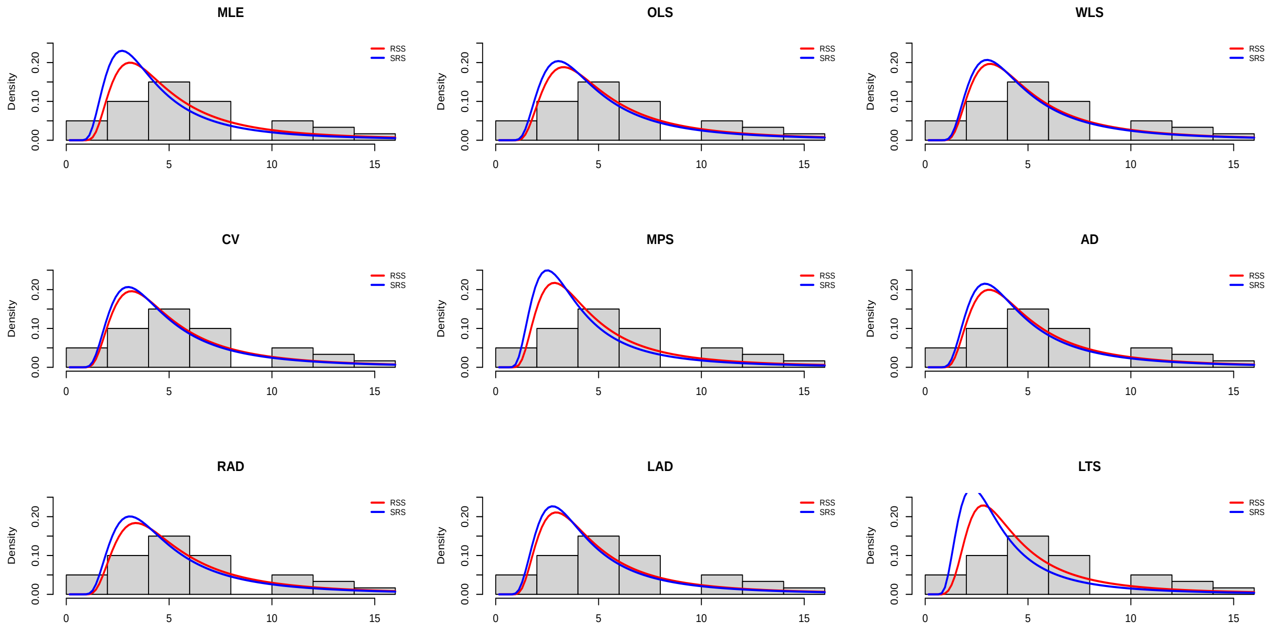


Figure 7. Plots of the estimated PDFs of the TIRD for $m = 5$ and $r = 2$.

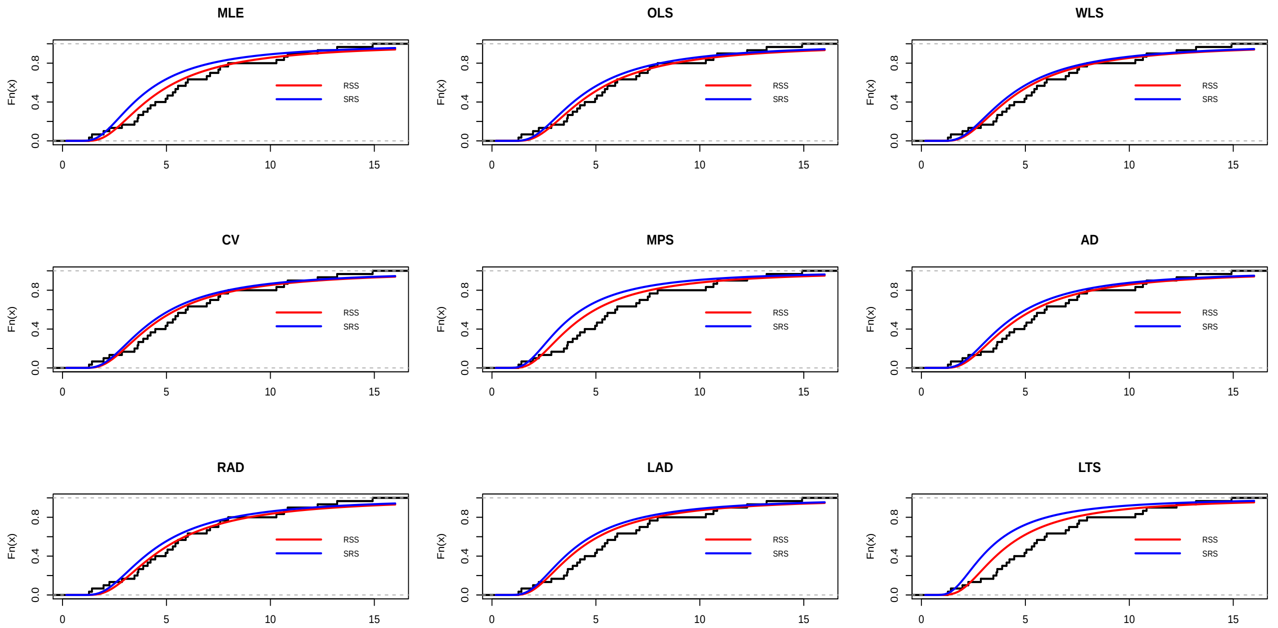


Figure 8. Plots of the estimated CDFs of the TIRD for $m = 5$ and $r = 2$.

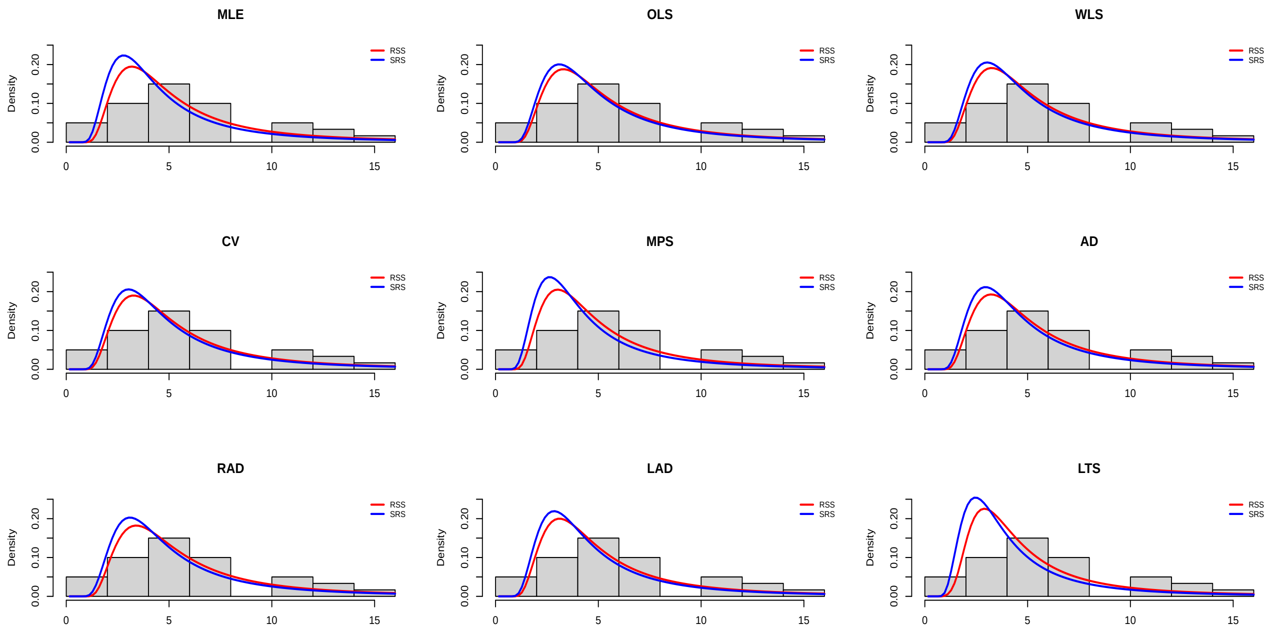


Figure 9. Plots of the estimated PDFs of the TIRD for $m = 5$ and $r = 3$.

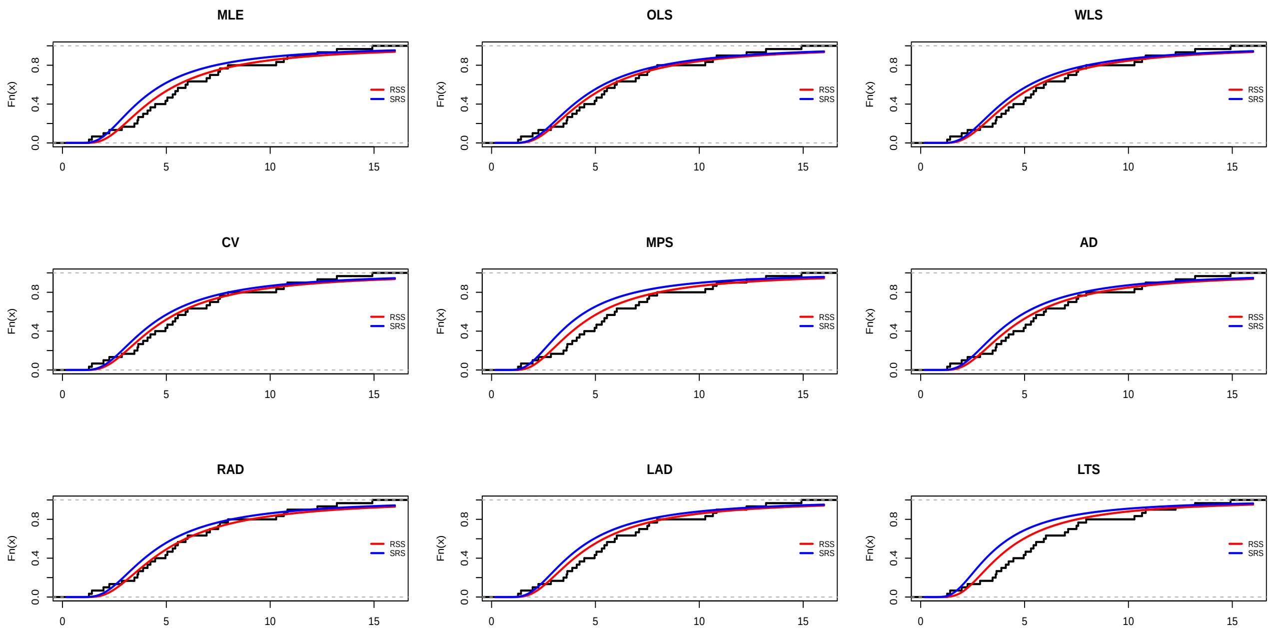


Figure 10. Plots of the estimated CDFs of the TIRD for the two sampling method for $m = 5$ and $r = 3$

Figures 9 and 10 showed that the RSS has a better fit than the SRS, as it closer to the data. Based on Tables 7 and 8, the following results can be concluded:

- RSS consistently outperforms SRS across all estimation methods, demonstrating better model fit (lower AD_t, CV_t, and KS_t values; higher p-values). This aligns with the theoretical advantage of ranked data, leading to more precise parameter estimates.

- Under the RSS, RAD and OLS emerge as the most effective estimation methods, while MLE and CV demonstrate consistent performance across both the SRS and RSS frameworks. In contrast, LTS and MPS exhibit relatively poor performance, suggesting their limited suitability for this dataset or model.
- Increasing r from 2 to 3 enhances the efficiency of RSS, particularly improving the performance of methods like WLS and AD. However, this increase also introduces a slight rise in the variability of the estimates.

5. Conclusions

This study investigates various estimation methods for the parameters of the transmuted inverse Rayleigh distribution under ranked set sampling and simple random sampling frameworks. By employing maximum likelihood estimation, ordinary and weighted least squares, maximum product of spacings, and goodness-of-fit-based estimators (Anderson-Darling, right-tail AD, left-tail AD, left-tail second-order AD, and Cramér-von Mises), we provide a comprehensive assessment of parameter estimation performance. The simulation study demonstrates that RSS consistently enhances estimation accuracy, reducing the mean squared error, bias, and mean absolute relative error compared to SRS. Furthermore, the analysis of a COVID-19 dataset reaffirms the practical advantages of RSS in real-world applications. These findings highlight the efficiency of RSS in parameter estimation for the TIRD, making it a preferred choice for statistical inference, particularly in scenarios where precision and reliability are crucial. However, this study advances the literature by integrating RSS with the TIRD, offering improved parameter estimation, particularly in small-sample contexts. While prior research has explored either advanced distributional modeling or the efficiency of RSS independently, this work uniquely combines both, offering a more robust framework for parameter estimation. It addresses the limitations of earlier methods, enhances robustness to outliers, and introduces greater flexibility in modeling diverse hazard rate shapes. For future works, the TIRD model can be considered for acceptance sampling plans [33] using other RSS modifications, as suggested by [34, 35].

Author contributions

Amer Al-Omari: Conceptualization, methodology, software, investigation, validation, writing original draft; Sid Ahmed Benchiha: Conceptualization, methodology, investigation, validation, software, writing review and editing; Ghadah Alomani: Funding acquisition, conceptualization, validation, data curation, writing original draft. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflict of interest.

References

1. G. A. McIntyre, A method of unbiased selective sampling, using ranked sets, *Aust. J. Agr. Res.*, **3** (1952), 385–390.
2. H. A. David, H. N. Nagaraja, *Order statistics*, John Wiley & Sons, 2004.
3. A. Haq, J. Brown, E. Moltchanova, A. I. Al-Omari, Partial ranked set sampling design, *Environmetrics*, **24** (2013), 201–207. https://doi.org/10.1002/env.2203
4. A. I. Al-Omari, S. Benchiha, I. M. Almanjahie, Efficient estimation of two-parameter xgamma distribution parameters using ranked set sampling design, *Mathematics*, **10** (2022), 3170. https://doi.org/10.3390/math10173170
5. H. F. Nagy, A. I. Al-Omari, A. S. Hassan, G. A. Alomani, Improved estimation of the inverted Kumaraswamy distribution parameters based on ranked set sampling with an application to real data, *Mathematics*, **10** (2022), 4102. https://doi.org/10.3390/math10214102
6. A. S. Hassan, I. M. Almanjahie, A. I. Al-Omari, L. Alzoubi, H. F. Nagy, Stress-strength modeling using median-ranked set sampling: Estimation, simulation, and application, *Mathematics*, **11** (2023), 318. https://doi.org/10.3390/math11020318
7. A. Hanandeh, A. Al-Omari, Estimation based on ranked set sampling for Farlie-Gumbel-Morgenstern bivariate Weibull distribution parameters with an application to medical data, *Pak. J. Stat. Oper. Res.*, **19** (2023), 671–689. https://doi.org/10.18187/pjsor.v19i4.4435
8. A. I. Al-Omari, M. S. Abdallah, Estimation of the distribution function using moving extreme and MiniMax ranked set sampling, *Commun. Stat. Simul. C.*, **52** (2023), 1909–1925. https://doi.org/10.1080/03610918.2021.1891433
9. C. A. Taconeli, Dual-rank ranked set sampling, *J. Stat. Comput. Sim.*, **94** (2024), 29–49. https://doi.org/10.1080/00949655.2023.2229472

10. M. Aldrabseh, M. T. Ismail, A. I. Al-Omari, Except-extremes ranked set sampling for estimating the population variance with two applications of real data sets, *Aust. J. Stat.*, **53** (2024), 99–113. https://doi.org/10.17713/ajs.v53i4.1872

11. M. S. Abdallah, A. I. Al-Omari, An efficient CDF estimator based on dual-rank ranked set sampling with an application to body mass index data, *J. Indian Soc. Prob. Stat.*, **25** (2024), 67–84. https://doi.org/10.1007/s41096-023-00171-8

12. N. Alsadat, A. S. Hassan, M. Elgarhy, A. Johannssen, A. M. Gemeay, Estimation methods based on ranked set sampling for the power logarithmic distribution, *Sci. Rep.*, **14** (2024), 17652. https://doi.org/10.1038/s41598-024-67693-4

13. H. M. Aljohani, Statistical inference for a novel distribution using ranked set sampling with applications, *Heliyon*, **10** (2024), e26893. https://doi.org/10.1016/j.heliyon.2024.e26893

14. A. Kumari, R. Singh, F. Smarandache, New modification of ranked set sampling for estimating population mean: Neutrosophic median ranked set sampling with an application to demographic data, *Int. J. Comput. Intell. Syst.*, **17** (2024), 210. https://doi.org/10.1007/s44196-024-00548-y

15. A. Shafiq, T. N. Sindhu, M. B. Riaz, M. K. Hassan, T. A. Abushal, A statistical framework for a new Kavya-Manoharan Bilal distribution using ranked set sampling and simple random sampling, *Heliyon*, **10** (2024), e30762. https://doi.org/10.1016/j.heliyon.2024.e30762

16. W. H. Woodall, A. Haq, M. A. Mahmoud, N. A. Saleh, Reevaluating the performance of control charts based on ranked-set sampling, *Qual. Eng.*, **36** (2024), 365–370. https://doi.org/10.1080/08982112.2023.2212751

17. A. S. Hassan, N. Alsadat, M. Elgarhy, H. Ahmad, H. F. Nagy, On estimating multi-stress strength reliability for inverted Kumaraswamy under ranked set sampling with application in engineering, *J. Nonlinear Math. Phys.*, **31** (2024), 30. https://doi.org/10.1007/s44198-024-00196-y

18. N. M. Hassan, O. A. Alamri, Estimation of Gumbel distribution based on ordered maximum ranked set sampling with unequal samples, *Axioms*, **13** (2024), 279. https://doi.org/10.3390/axioms13040279

19. H. Wang, W. Chen, Large sample properties of modified maximum likelihood estimator of the location parameter using moving extremes ranked set sampling, *Stat. Prob. Lett.*, **223** (2025), 110430. https://doi.org/10.1016/j.spl.2025.110430

20. R. Yang, W. Chen, Y. Dong, Log-extended exponential-geometric parameters estimation using simple random sampling and moving extremes ranked set sampling, *Commun. Stat. Simul. C.*, **52** (2023), 267–277. https://doi.org/10.1080/03610918.2020.1853167

21. M. Shala, F. Merovci, A new three-parameter inverse Rayleigh distribution: Simulation and application to real data, *Symmetry*, **16** (2024), 634. https://doi.org/10.3390/sym16050634

22. R. E. Glaser, Bathtub and related failure rate characterizations, *J. Amer. Stat. Assoc.*, **75** (1980), 667–672. https://doi.org/10.2307/2287666

23. J. J. Swain, S. Venkatraman, J. R. Wilson, Least-squares estimation of distribution functions in Johnson's translation system, *J. Stat. Comput. Sim.*, **29** (1988), 271–297. https://doi.org/10.1080/00949658808811068

24. R. C. H. Cheng, N. A. K. Amin, Maximum product of spacings estimation with application to the lognormal distribution, Math. Rep., 1979.
25. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. R. Stat. Soc. B*, **45** (1983), 394–403. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x
26. T. W. Anderson, D. A. Darling, Asymptotic theory of certain goodness of fit criteria based on stochastic processes, *Ann. Math. Stat.*, **23** (1952), 193–212. https://doi.org/10.1214/aoms/1177729437
27. T. W. Anderson, D. A. Darling, A test of goodness of fit, *J. Amer. Stat. Assoc.*, **49** (1954), 765–769. https://doi.org/10.2307/2281537
28. A. N. Pettitt, A two-sample Anderson-Darling rank statistic, *Biometrika*, **63** (1976), 161–168. https://doi.org/10.2307/2335097
29. P. D. M. MacDonald, Comment on "an estimation procedure for mixtures of distributions" by Choi and Bulgren, *J. R. Stat. Soc. B*, **33** (1971), 326–329.
30. H. A. Muttlak, Investigating the use of quartile ranked set samples for estimating the population mean, *Appl. Math. Comput.*, **146** (2003), 437–443. https://doi.org/10.1016/S0096-3003(02)00595-7
31. A. A. Jemain, A. I. Al-Omari, Double percentile ranked set samples for estimating the population mean, *Adv. Appl. Stat.*, **6** (2006), 261–276.
32. H. M. Almongy, E. M. Almetwally, H. M. Aljohani, A. S. Alghamdi, E. H. Hafez, A new extended Rayleigh distribution with applications of COVID-19 data, *Result. Phys.*, **23** (2021), 104012. https://doi.org/10.1016/j.rinp.2021.104012
33. A. I. Al-Omari, The transmuted generalized inverse Weibull distribution in acceptance sampling plans based on life tests, *Trans. Inst. Meas. Control.*, **40** (2018), 4432-4443. https://doi.org/10.1177/0142331217749695
34. A. A. Hanandeh, A. D. Al-Nasser, A. I. Al-Omari, New mixed ranked set sampling variations, *Stat*, **11** (2022), e417. https://doi.org/10.1002/sta4.417
35. E. Zamanzade, A. I. Al-Omari, New ranked set sampling for estimating the population mean and variance, *Hacet. J. Math. Stat.*, **45** (2016), 1891–1905.