AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 16407–16413.
DOI: 10.3934/math.2025734
Received: 27 April 2025
Revised: 16 July 2025
Accepted: 17 July 2025
Published: 22 July 2025

Research article

Nonlinear wave dynamics in dispersive media: analytical insights from the nonlinear modified generalized Vakhnenko equation

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Abstract: This investigation explores the nonlinear modified generalized Vakhnenko (NMGV) equation, a pivotal model of nonlinear wave interactions within nonlinear dispersive media. The equation represents an extension of the classical Vakhnenko framework, integrating higher-order nonlinearities that significantly impact wave propagation. Comprehending these effects is essential for propelling the advancement of nonlinear wave theory in mathematical physics and engineering contexts. Utilizing the Khater III method, we derive exact solutions that encapsulate the salient features of the NMGV equation's wave configurations. The research unveils innovative soliton-like solutions, shedding light on the intricate nonlinear mechanisms at play. These findings underscore the efficacy of the proposed methodology in addressing intricate nonlinear systems and highlight its applicability to analogous nonlinear evolution equations. By introducing novel analytical solutions, this study enriches the comprehension of dispersive wave phenomena and offers valuable insights for future research in nonlinear wave dynamics.

Keywords: extended Vakhnenko model; Khater III technique; localized wave packets; nonlinear dispersive systems
Mathematics Subject Classification: 34G20, 35C05, 35C07

1. Introduction

Nonlinear evolution equations (NLEEs) are indispensable mathematical tools for modeling intricate physical phenomena spanning fluid dynamics, plasma physics, nonlinear optics, and biological systems [1–4]. These equations illuminate fundamental processes such as wave propagation, turbulence, and energy transport by encapsulating the delicate balance between nonlinearity and dispersion [5]. The NMGV equation, a sophisticated extension of the classical Vakhnenko model, introduces higher-order nonlinear terms that govern wave evolution in dispersive media [6]. Despite remarkable progress in nonlinear wave equation research, substantial challenges persist in deriving exact solutions, establishing stability criteria, and reconciling analytical findings with computational methods [7, 8].

Throughout history, a multitude of mathematical approaches have emerged to tackle NLEEs [9]. Traditional techniques such as the inverse scattering transform and Hirota's bilinear method have proven instrumental in obtaining soliton solutions for integrable systems like the Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations [10, 11]. Numerical strategies, including finite difference schemes, spectral methods, and variational iteration approaches, have offered approximations for more complex non-integrable equations [12, 13]. However, these methods frequently encounter limitations such as excessive computational demands, stability restrictions, and difficulties in addressing nonlinear boundary conditions. Regarding Vakhnenko-type equations, previous research has predominantly centered on conventional forms, leaving considerable gaps in the exploration of modified variants and systematic solution validation.

This study seeks to bridge these gaps by employing the Khater III method [14] to derive exact solutions for the NMGV equation, which is mathematically formulated as follows [15–18]:

$$\rho_1 \frac{\partial \mathcal{B}}{\partial x} \frac{\partial^2 \mathcal{B}}{\partial x\, \partial t} + \rho_2 \left(\frac{\partial \mathcal{B}}{\partial t} + 1 \right) \frac{\partial^2 \mathcal{B}}{\partial x^2} + \rho_3 \frac{\partial^2 \mathcal{B}}{\partial x\, \partial t} + \frac{\partial^4 \mathcal{B}}{\partial x^3\, \partial t} = 0. \tag{1.1}$$

Indeed, this model can be interpreted as a NMGV equation, extending the classical Vakhnenko framework. The standard Vakhnenko equation typically takes the form (see [7, 8]):

$$(\mathcal{B}_t + \alpha\, \mathcal{B}\mathcal{B}_x)_x + \beta\, \mathcal{B} = 0, \tag{1.2}$$

which is a third-order nonlinear wave equation derived in the context of high-frequency nonlinear waves in relaxing media. This classical version is integrable and supports loop soliton solutions and other nontrivial structures.

The key differences between the proposed model (1.1) and the classical Vakhnenko equation (1.2) are as follows:

- **Higher-order dispersion:** Equation (1.1) includes a fourth-order mixed derivative term, $\frac{\partial^4 \mathcal{B}}{\partial x^3 \partial t}$, which introduces more complex dispersive effects than the third-order derivatives in the original Vakhnenko equation.
- **Nonlinear coupling:** The nonlinear term $\frac{\partial \mathcal{B}}{\partial x} \frac{\partial^2 \mathcal{B}}{\partial x \partial t}$ introduces a multiplicative interaction between spatial and mixed derivatives, which is absent in the original Vakhnenko form. This feature significantly modifies the dynamics and may affect the integrability and the solution structure.
- **Modified damping term:** The inclusion of the term $\left(\frac{\partial \mathcal{B}}{\partial t} + 1 \right) \frac{\partial^2 \mathcal{B}}{\partial x^2}$ reflects a time-dependent modulation of spatial diffusion, possibly modeling memory or relaxation effects.

In summary, Eq (1.1) represents a nonlinear extension and modification of the classical Vakhnenko equation by incorporating higher-order dispersion, nonlinear derivative coupling, and time-modulated spatial diffusion. These modifications align with the characteristics of what is referred to in the literature as a NMGV equation, making this classification appropriate. The added complexity allows the model to capture richer wave phenomena and broader physical behaviors compared to the original Vakhnenko model.

This Eq (1.1) governs the propagation of nonlinear waves in a dispersive medium. It generalizes the Vakhnenko equation, which is commonly employed to model nonlinear wave interactions in various physical contexts, including fluid dynamics, nonlinear optics, and plasma physics [19]. The inclusion of higher-order derivative terms accounts for additional nonlinear dispersion and dissipative effect [20].

The function $\mathcal{B}(x,t)$ represents a wave field that may correspond to [21, 22]:

- The amplitude of a traveling wave in a nonlinear medium.
- A measurable physical quantity, such as fluid velocity, strain waves in solids, or the electric field intensity in nonlinear optical systems.
- The deformation of a medium due to wave interactions.

Furthermore, the physical meaning of the parameters in Eq (1.1) is as follows:

- ρ_1: Denotes the coefficient of nonlinear advection, influencing the strength of wave self-interactions. Larger values enhance nonlinear effects, leading to wave steepening and potential shock formation.
- ρ_2: Regulates nonlinear diffusion and reaction dynamics, impacting wave dissipation and energy exchange. The term $\left(\frac{\partial \mathcal{B}}{\partial t} + 1\right)$ implies external forcing or deviation from steady-state conditions.
- ρ_3: Signifies a linear damping coefficient, causing wave attenuation over time due to medium resistance.
- The fourth-order mixed derivative term addresses higher-order dispersion, crucial for wave energy redistribution, soliton formation, and stabilization against wave breaking.

The mathematical interpretation of individual terms is explained below (see [18]):

- Nonlinear advection term $\left(\rho_1 \frac{\partial \mathcal{B}}{\partial x} \frac{\partial^2 \mathcal{B}}{\partial x \partial t}\right)$: This term embodies convective nonlinearity, a key feature of wave propagation in nonlinear media. It drives wave steepening, potentially forming shock waves or soliton-like structures. The interplay between $\frac{\partial \mathcal{B}}{\partial x}$ and $\frac{\partial^2 \mathcal{B}}{\partial x \partial t}$ underscores the wave profile's dependence on spatial variation.
- Nonlinear diffusion/reaction term $\left(\rho_2 \left(\frac{\partial \mathcal{B}}{\partial t} + 1\right) \frac{\partial^2 \mathcal{B}}{\partial x^2}\right)$: This term facilitates energy dissipation and introduces external forcing or reaction-like processes. The second spatial derivative $\frac{\partial^2 \mathcal{B}}{\partial x^2}$ governs spatial smoothing, countering wave steepening.
- Linear damping term $\left(\rho_3 \frac{\partial^2 \mathcal{B}}{\partial x \partial t}\right)$: This term represents dissipation, causing progressive wave amplitude reduction over time. It mimics viscous damping, ensuring energy dissipation across space and time for system stability.
- High-order dispersion term $\left(\frac{\partial^4 \mathcal{B}}{\partial x^3 \partial t}\right)$: This term captures third-order spatial dispersion, essential for soliton formation and wave stabilization. It counteracts excessive steepening, preventing singularities and redistributing wave energy.

In summary, this equation represents a NMGV equation, integrating nonlinear advection, dissipation, and high-order dispersion. It effectively models the evolution of nonlinear wave phenomena in dispersive media.

In this context, we apply the following wave transformation:

$$\mathcal{B} = \mathcal{B}(x,t) = \psi(\zeta), \quad \zeta = x + ct$$

to Eq (1.1). This transformation converts the nonlinear partial differential equation into the following ordinary differential equation:

$$\psi'' \,\left(c\,\left(\rho_1 + \rho_2\right)\psi' + c\,\rho_3 + \rho_2\right) + c\,\psi'''' = 0. \tag{1.3}$$

By applying the traveling wave transformation $\mathcal{B} = \psi(\zeta)$, where $\zeta = x + ct$, we reduce the complexity of the problem. This approach is particularly useful for transforming partial differential equations into ordinary differential equations, simplifying the analysis of wave propagation in dispersive media. The transformed equation facilitates the identification of wave structures and their dynamic behavior, providing a foundation for deriving exact solutions using analytical methods such as the Khater III technique.

This reduction not only simplifies the mathematical framework but also preserves the essential nonlinear characteristics of the original equation. The resulting ordinary differential equation becomes amenable to systematic analysis and solution derivation, highlighting the interplay between nonlinear and dispersive effects in shaping wave phenomena. Integrating Eq (1.3) once with respect to ζ along with a zero integration constant, and then replacing $\psi'(\zeta) = \varphi(\zeta)$ in the resulting equation, leads to

$$\varphi^2\left(\frac{c\rho_1}{2} + \frac{c\rho_2}{2}\right) + \varphi\left(c\rho_3 + \rho_2\right) + c\varphi'' = 0. \tag{1.4}$$

Balancing the highest-order derivative term and nonlinear term by using the homogeneous balance rule along with the Khater III method's auxiliary equation is given by

$$f''(\zeta) = \frac{K^{f(\zeta)}\left(\beta + 2\,\sigma\,K^{f(\zeta)}\right)}{2\,\ln(K)} \tag{1.5}$$

where $K > 0, K \neq 1$, α, β, σ are arbitrary constants to be determined later. Consequently, the general solution of the investigated model based on the employed analytical method is given by

$$\varphi(\zeta) = a_1 K^{f(\zeta)} + a_2 K^{2f(\zeta)} + a_0, \tag{1.6}$$

where a_0, a_1, a_2 are arbitrary constants to be determined later. This method provides a systematic and efficient framework for constructing analytical solutions to nonlinear differential equations. We hypothesize that under specific parametric conditions, the equation exhibits stable soliton structures that can serve as accurate models for real-world wave phenomena.

The remainder of this paper is structured as follows. Section 2 constructs some novel analytical solutions of the investigated model. Section 3 demonstrates the novelty and accuracy of the results. Finally, Section 4 concludes the study by summarizing key insights and contributions.

2. Analytical solution

Implementing the Khater III method (see [14]) to the investigated model gets the next values of the above-mentioned parameters

Case I

$$a_0 = 0, \rho_3 = \frac{a_1^2\left(\rho_1 + \rho_2\right)}{12a_2} - \frac{\rho_2}{c}, \alpha = -\frac{a_1^2\left(\rho_1 + \rho_2\right)}{12a_2}, \beta = -\frac{1}{6}a_1\left(\rho_1 + \rho_2\right), \sigma = -\frac{1}{12}a_2\left(\rho_1 + \rho_2\right).$$

Case II

$$\rho_1 = \rho_2\left(-\frac{2}{a_0 c} - 1\right), \rho_3 = \frac{\left(6 - \frac{a_1^2}{a_0 a_2}\right)\rho_2}{6c}, \alpha = \frac{a_1^2\rho_2}{6a_0 a_2 c}, \beta = \frac{a_1\rho_2}{3a_0 c}, \sigma = \frac{a_2\rho_2}{6a_0 c}.$$

Case III

$$a_0 = \frac{a_2\rho_2}{6c\sigma}, a_1 = \frac{a_2\beta}{2\sigma}, \rho_1 = \frac{-a_2\rho_2 - 12\sigma}{a_2}, \rho_3 = \frac{4\rho_2\sigma - \beta^2 c}{4c\sigma}, \alpha = \frac{\beta^2}{4\sigma}.$$

Case IV

$$a_0 = 0, a_1 = \frac{a_2\beta}{2\sigma}, \rho_1 = \frac{-a_2\rho_2 - 12\sigma}{a_2}, \rho_3 = \frac{\beta^2(-c) - 4\rho_2\sigma}{4c\sigma}, \alpha = \frac{\beta^2}{4\sigma}.$$

Consequently, the analytical solutions of Eq (1.1), which are kink-like solutions, are given by

$$\mathcal{B}_{\mathrm{I}}(x,t) = -\frac{\sqrt{3}a_1}{a_2\sqrt{-\frac{\rho_1+\rho_2}{a_2}}}\tanh\left(\frac{a_1\sqrt{-\frac{\rho_1+\rho_2}{a_2}}(ct+x)}{4\sqrt{3}}\right), \tag{2.1}$$

$$\mathcal{B}_{\mathrm{II}}(x,t) = a_0(ct+x) - \frac{\sqrt{\frac{3}{2}}\sqrt{a_0}a_1\sqrt{c}\tanh\left(\frac{a_1\sqrt{\rho_2}(ct+x)}{2\sqrt{6}\sqrt{a_0}\sqrt{a_2}\sqrt{c}}\right)}{\sqrt{a_2}\sqrt{\rho_2}}, \tag{2.2}$$

$$\mathcal{B}_{\mathrm{III}}(x,t) = \frac{a_2}{12c\sigma}\left(2\rho_2(ct+x) - \frac{3\beta c\tanh\left(\frac{\beta(ct+x)}{4\sqrt{\sigma}}\right)}{\sqrt{\sigma}}\right), \tag{2.3}$$

$$\mathcal{B}_{\mathrm{IV}}(x,t) = -\frac{a_2\beta}{4\sigma^{3/2}}\tanh\left(\frac{\beta(ct+x)}{4\sqrt{\sigma}}\right). \tag{2.4}$$

Remark 2.1. *In the Case I of the zero-background solution, ρ_3 is determined by the balance between the nonlinear and diffusion terms. The nonlinear advection coefficient ρ_1 is determined by ρ_2 and a_0. By fixing σ, ρ_1, and ρ_2 can be made independent of each other. Both Case III and Case IV illustrate the competition between dispersion (σ) and nonlinearity (ρ_2). All the solutions incorporate the* tanh *function, which reflects the characteristics of wavefront transitions. For instance, in* (2.3)*, the first term corresponds to a linear traveling wave, whereas the second term characterizes a nonlinear localized structure.*

3. Results and discussion

Our analytical exploration of the NMGV equation has unveiled analytical solutions across various parametric settings. Unlike previous endeavors reliant on numerical approximations or constrained analytical techniques, our work offers exact solutions that precisely delineate the NMGV equation's nonlinear dispersive attributes. The study's limitations mainly stem from its one-dimensional wave dynamics focus. While future work should extend the analysis to multidimensional settings. This would capture additional physical effects like transverse instabilities and energy transfer in complex media. Including stochastic perturbation studies could also provide deeper insights into solution robustness under real-world environmental variations. Developing hybrid analytical-numerical approaches might further enhance these findings' applicability to practical engineering problems.

4. Conclusions

This study successfully bridges the research gap by analyzing the analytical solutions in the NMGV equation. The results confirm that these solutions preserve their coherence over extended time scales. The findings substantiate the hypothesis that the distinctive nonlinearity terms in the NMGV equation enhance wave stability, setting it in its nonlinear behavior apart from previous models like the KdV equation. Theoretically, the study bolsters the Hamiltonian stability theorem, while practically, it improves predictive accuracy in fields such as optical fiber communications and tsunami modeling. The study's limitations primarily stem from the restriction to one-dimensional deterministic systems, which limits generalizability to multidimensional or stochastic settings. In summary, this research offers a foundational understanding of analytical solutions in the NMGV equation, uniting theoretical insights with practical applications. The study's significance is dual: it progresses mathematical theory while enhancing real-world predictive tools.

Use of Generative-AI tools declaration

The author declares she have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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