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Research article

Caputo-Hadamard fractional Wirtinger-type inequalities via Taylor expansion with applications to classical means

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Abstract: In this paper, we explored Caputo-Hadamard fractional Wirtinger-type inequalities using Taylor's formula. The main findings were derived by utilizing Hölder's inequality to derive results for Caputo-Hadamard fractional derivatives in terms of $\mathbf{L}_q$ norms for $q > 1$. Through graphical interpretation, we confirmed the validity of the results. A flowchart summarizing the logical progression from lemma to theorem was added for clarity. Furthermore, inequalities were also derived for Hadamard fractional derivatives. Finally, we discussed the applications of Wirtinger-type inequalitie which incorporates arithmetic mean and geometric mean-type inequalities.

Keywords: Taylor's formula; Wirtinger inequality; Caputo-Hadamard fractional derivative; Hölder's inequality
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1. Introduction

Fractional calculus (FC) is a branch of mathematics that generalizes the concepts of integration and differentiation to fractional orders. Around the same time as the classical operators were developed, the idea of fractional operators was proposed. FC enhances our understanding of complex processes and improves our ability to construct and evaluate intricate phenomena. Over the past few decades, several types of fractional derivatives and integrals have been developed and applied across diverse fields, including machine learning [1], robotics [2], control theory [3], bioengineering [4], atmospheric and statistical physics [5], optical and mechanical engineering [6], viscoelasticity [7], optimization, and the study of the effects of global warming [8]. The most commonly used fractional derivatives in the literature are the Riemann-Liouville fractional derivative in [9] and Hadamard fractional derivative, originally defined by Hadamard in [10]. Another significant type of fractional derivative, the Caputo fractional derivative, was defined in [9,11]. Later, the Caputo-type modification of Hadamard fractional derivative, referred to as Caputo-Hadamard fractional derivative, was defined in [12].

Inequalities also play a vital role and find applications in different fields such as physics, economics, engineering, and the social sciences. In the recent years, numerous researchers have contributed to the advancement of fractional integral inequalities, which are frequently used in optimization problems. Furthermore, these inequalities have been used to determine the lower and upper bounds of solutions for systems of fractional DEs. Additionally, fractional inequalities find applications in quantitative quadrature, probability [13], and plenty of other fields. Over time, several generalizations of the different classical inequalities to FC have been developed in the mathematical literature by numerous authors. One of the most significant inequalities in the literature is Wirtinger's inequality, named after Wilhelm Wirtinger. It was initially applied in 1904 in the development of the isoperimetric inequality. Now the reason why this result is so important is that the function and its derivatives are compared in integrals. The integral of the square of a function and the square of the derivative of a function are compared in classical Wirtinger's inequality. In other words, the inequality in [14] expresses that if $\chi \in \mathbf{C}^1[\nu, \mu]$ satisfies $\chi(\nu) = \chi(\mu) = 0$, then

$$\int_{\nu}^{\mu} \chi^2(z)dz \leq \int_{\nu}^{\mu} (\chi'(z))^2 dz. \tag{1.1}$$

Because of its significance, many authors have focused on Wirtinger-type inequalities, including those by Bessel, Blaschke, Beesack, Poincare, and Sobolev. For example, Beesack expanded inequality (1.1) in the following manner.

Theorem 1.1. *[15] If $\chi \in \boldsymbol{C}^2[\nu, \mu]$ with $\chi(\nu) = \chi(\mu) = 0$, then the inequality stated below holds:*

$$\int_{\nu}^{\mu} \chi^4(z)dz \leq \frac{4}{3} \int_{\nu}^{\mu} (\chi'(z))^4 dz. \tag{1.2}$$

Several issues, involving integral estimation, series convergence, and the determination of the least eigenvalues of differential operators, rely on Beesack's results and similar conclusions from other researchers. For instance, the most effective constant in the Poincare inequality is the first eigenvalue of the Laplace operator in [16]. This inequality is the more extended version of the Wirtinger inequality. In [17] Polya et al. explored fundamental isoperimetric inequalities and their applications in spectral theory, partial differential equations, and mathematical physics. In [18], Samet Erden established Wirtinger-type inequalities for higher-order differential functions. In [19], Robert Osserman reviewed refined versions, extensions, and recent applications of the isoperimetric inequality. C. A. Swanson proved general forms of the Wirtinger inequality in n-dimensions in [20]. Further, Bottcher and Widom, in [21], investigated a series of constants that appear in particular cases, focusing on the determination of the optimal constant in the Wirtinger-Sobolev inequality. This inequality provides a precise relationship between the integral of the square of a function and the square of its higher-order derivatives. Horst Alzer presented modified forms of continuous and discrete versions of the Wirtinger inequality in [22]. Mohammad W. Alomari proved Beesack Wirtinger-type inequalities for absolutely continuous functions with derivatives in L_p spaces ($p > 1$) in [23]. Paul R. Beesack in [24] examined the connection between a function and its rate of change in the framework of inequalities. Mehmet Zeki Sarikaya demonstrated a few expanded Wirtinger-type inequalities in [25].

Theorem 1.2. *Let $\chi \in \boldsymbol{C}^1[\nu,\mu]$ with the conditions $\chi(\nu) = \chi(\mu) = 0$ and $\chi' \in \boldsymbol{L}_2[\nu,\mu]$, and then we have the following inequality:*

$$\int_{\nu}^{\mu} [\chi(z)]^2 dz \leq \frac{(\mu-\nu)^2}{6} \int_{\nu}^{\mu} [\chi'(z)]^2 dz. \tag{1.3}$$

We will use following elementary inequalities in our results:

$$\left(\prod_{h=1}^{m} \Upsilon_h\right)^{\frac{1}{m}} \leq \frac{1}{m} \sum_{h=1}^{m} \Upsilon_h, \tag{1.4}$$

for $\Upsilon_1, \Upsilon_2, ..., \Upsilon_m$ reals and $m \geq 1$, and

$$\left(\sum_{h=1}^{m} \Upsilon_h\right)^2 \leq m \sum_{h=1}^{m} \Upsilon_h^2, \tag{1.5}$$

for $\Upsilon_1, \Upsilon_2, ..., \Upsilon_m$ reals given in [26].

For our study to proceed, we need to keep in mind the following definitions and basic findings. We indicate by $\mathbf{AC}^e[\nu,\mu]$ the space of real-valued functions $\chi(\xi)$ defined on the interval $[\nu,\mu]$, where these functions have derivatives up to order $e-1$ ($e \in \mathbf{N}$). Furthermore, the space $\mathbf{AC}\ [\nu,\mu]$ represents the set of absolutely continuous functions on $[\nu,\mu]$, and the $(e-1)$th derivative of these functions $\chi^{(e-1)}(\xi)$, exists and is well-defined. It can also be stated as

$$\mathbf{AC}^e[\nu,\mu] = \left\{\chi : [\nu,\mu] \to R : \chi^{(e-1)}(\xi) \in \mathbf{AC}[\nu,\mu]\right\}.$$

According to [9], the definitions of Caputo fractional derivatives of order ψ are given below.

Definition 1.1. *Suppose the function $\chi \in \boldsymbol{AC}^e[\nu,\mu]$ and $e = \lceil\psi\rceil$ with $e \in \boldsymbol{N}$ and $\psi > 0$. Then the left-sided and right-sided Caputo fractional derivatives of order ψ are defined as*

$$
\begin{aligned}
{}^{C}D^{\psi}_{\nu+}\chi(v) &= {}^{RL}I^{e-\psi}_{\nu+}\chi^{(e)}(v) = \frac{1}{\Gamma(e-\psi)} \int_{\nu}^{v} \frac{\chi^{(e)}(\xi)}{(v-\xi)^{\psi-e+1}} d\xi, \ v > \nu, \\
{}^{C}D^{\psi}_{\mu-}\chi(v) &= (-1)^{e\,RL}I^{e-\psi}_{\mu-}\chi^{(e)}(v) = \frac{(-1)^e}{\Gamma(e-\psi)} \int_{\nu}^{\mu} \frac{\chi^{(e)}(\xi)}{(\xi-v)^{\psi-e+1}} d\xi, \ v < \mu.
\end{aligned}
$$

If $\psi = e$ and the usual derivatives of f of order e exist, then ${}^{C}D^{\psi}_{\nu+}\chi(v) = \chi^{(e)}(v)$ and ${}^{C}D^{\psi}_{\mu-}\chi(v) = (-1)^e\chi^{(e)}(v)$. In particular, we have the following case for $e = 1$ and $\psi = 0$:

$${}^{C}D^{0}_{\nu+}\chi(v) = {}^{C}D^{0}_{\mu-}\chi(v) = \chi(v).$$

According to [10], Hadamard fractional integrals are stated as follows.

Definition 1.2. *The left-sided Hadamard fractional integral of order $\psi > 0$ has the form*

$${}^{H}I^{\psi}_{\nu+}\chi(v) = \frac{1}{\Gamma(\psi)} \int_{\nu}^{v} (\log v - \log u)^{\psi-1} \frac{\chi(u)}{u} du.$$

The right-sided Hadamard fractional integral of order $\psi > 0$ is defined by

$${}^{H}I^{\psi}_{\mu-}\chi(v) = \frac{1}{\Gamma(\psi)} \int_{\nu}^{\mu} (\log u - \log v)^{\psi-1} \frac{\chi(u)}{u} du.$$

In [10], the definitions of Hadamard fractional derivatives are as follows.

Definition 1.3. *The left-sided Hadamard fractional derivative of order $\psi > 0$ is provided below:*

$$^{H}D^{\psi}_{\nu+}\chi(v) = \left(v\frac{d}{dv}\right)^{e} {}^{H}I^{e-\psi}_{\nu+}\chi(v),\ e = [\psi] + 1,$$

and the right-sided Hadamard fractional derivative of order $\psi > 0$ has the form

$$^{H}D^{\psi}_{\mu-}\chi(v) = \left(-v\frac{d}{dv}\right)^{e} {}^{H}I^{e-\psi}_{\mu-}\chi(v).$$

The definitions of Caputo-Hadamard fractional derivatives of order $\psi > 0$ are presented here as in [12, 17].

Definition 1.4. *Suppose $R(\psi) \geq 0$ and $e = [R(\psi)] + 1$. Consider the function $\chi(v) \in \boldsymbol{AC}^{e}_{\delta}[\nu, \mu]$, where $0 < \nu < \mu < \infty$ and $\boldsymbol{AC}^{e}_{\delta}[\nu, \mu]=\left\{l : [\nu, \mu] \to \boldsymbol{C} : \delta^{e-1}[l(v)] \in \boldsymbol{AC}[\nu, \mu], \delta = v\frac{d}{dv}\right\}$. The definitions of left-sided and right-sided Caputo-Hadamard fractional derivatives are given below:*

$$^{CH}D^{\psi}_{\nu+}\chi(v) = {}^{H}D^{\psi}_{\nu+}\left[\chi(u) - \sum_{k=0}^{e-1}\frac{\delta^{k}\chi(\nu)}{k!}\left(log\frac{u}{\nu}\right)^{k}\right](v), \tag{1.6}$$

$$^{CH}D^{\psi}_{\mu-}\chi(v) = {}^{H}D^{\psi}_{\mu-}\left[\chi(u) - \sum_{k=0}^{e-1}\frac{\delta^{k}\chi(\mu)}{k!}\left(log\frac{\mu}{u}\right)^{k}\right](v). \tag{1.7}$$

From the above-mentioned definition, one can generalize to the well-known fractional derivative.

$$\begin{cases} {}^{CH}D^{\psi}_{\nu+}\chi(z) = {}^{H}D^{\psi}_{\nu+}\chi(z), & \text{if we place } \delta^{k}\chi(\nu) = 0, \\ {}^{CH}D^{\psi}_{\mu-}\chi(z) = {}^{H}D^{\psi}_{\mu-}\chi(z), & \text{if we place } \delta^{k}\chi(\mu) = 0, \end{cases} \tag{1.8}$$

where $k = 0, 1, ..., e - 1$.

Here, $\delta = v\frac{d}{dv}$ is the logarithmic differential operator, which is important in formulating Hadamard-type fractional derivatives due to the presence of a logarithmic kernel.

Theorem 1.3. *Let $R(\psi) \geq 0$ and $e = [R(\psi)] + 1$. Consider the function $\chi(v) \in \boldsymbol{AC}^{e}_{\delta}[\nu, \mu]$, where $0 < \nu < \mu < \infty$. Then ${}^{CH}D^{\psi}_{\nu+}\chi(v)$ and ${}^{CH}D^{\psi}_{\mu-}\chi(v)$ exist everywhere on $[\nu, \mu]$ with the following representations:*

(i) If $\psi \notin N \cup \{0\}$, then ${}^{CH}D^{\psi}_{\nu+}\chi(v)$ and ${}^{CH}D^{\psi}_{\mu-}\chi(v)$ can be represented as

$$^{CH}D^{\psi}_{\nu+}\chi(v) = {}^{H}I^{e-\psi}_{\nu+}\delta^{e}\chi(v) = \frac{1}{\Gamma(e-\psi)}\int_{\nu}^{v}(\log v - \log u)^{e-\psi-1}\delta^{e}\chi(u)\frac{du}{u},$$
$$^{CH}D^{\psi}_{\mu-}\chi(v) = (-1)^{e}{}^{H}I^{e-\psi}_{\mu-}\delta^{e}\chi(v) = \frac{(-1)^{e}}{\Gamma(e-\psi)}\int_{v}^{\mu}(\log u - \log v)^{e-\psi-1}\delta^{e}\chi(u)\frac{du}{u}.$$

(ii) If $\psi \in N$, then

$$^{CH}D^{\psi}_{\nu+}\chi(v) = \delta^{e}\chi(v),\ {}^{CH}D^{\psi}_{\mu-}\chi(v) = (-1)^{e}\delta^{e}\chi(v).$$

(iii) Also

$$ {}^{CH}D^{0}_{\nu+}\chi(v) = {}^{CH}D^{0}_{\mu-}\chi(v) = \chi(v). $$

This theorem corresponds to Theorem 2.1 presented by Jarad et al. in [12].

In contrast to the classical Hadamard derivative which is formed by using a fractional operator with a logarithmic kernel directly, the Caputo-Hadamard is established by incorporating an integer-order logarithmic differential operator first and then a fractional integral. This approach provides greater flexibility in describing memory-dependent systems and makes it more compatible with problems where initial conditions are provided in the form of classical derivatives.

The beneficial aspects of both Caputo and Hadamard operators are merged in the Caputo-Hadamard operator. It is appropriate for modeling multiplicative or scale invariant procedures since it uses a logarithmic kernel similar to the Hadamard derivative. At the same time, it allows the classical initial condition like the Caputo derivative which is crucial in physical and engineering applications. This combination provides greater versatility for analyzing systems with memory effect and logarithmic-type behavior. The Caputo-Hadamard fractional derivative has proven to be a useful tool for modeling intricate memory-dependent systems and in capturing non-locality effects. Thus it finds applications in a wide range of fields including biology, finance, physics, engineering, signal processing, and control theory. As stated in the work of He et al. in [27], stability results for systems involving Hadamard and Caputo-Hadamard fractional derivatives have been constructed by applying Lyapunov methods and derivative inequalities. For instance, Wang et al. in [28] offered a mathematical modeling framework for solving the fractional Allen-Cahn equation based on the Caputo-Hadamard derivative. Zhao et al. introduced an effective spectral collocation approach for evaluating differential equations including the Caputo-Hadamard derivative in [29], showing its applicability to nonlinear initial and boundary value problems. Our study primarily focuses on applications of this derivative in establishing the Wirtinger-type ineqalities. These inequalities have been used to establish new large spaces between the zeros of the Riemann zeta function in [30] by Samir H. Saker and for time-delay systems by Seuret et al. in [31]. In [32], Bo Du explored the existence of anti-periodic solutions for an inertial competitive neutral-type network using the Wirtinger inequality. The Wirtinger inequality has demonstrated its importance to improve system stability and computing efficiency by its effective application in control design for large-scale connected systems with sampled data input by Lynnyk et al. in [33].

Many researchers have investigated Wirtinger-type inequalities in fractional and classical context. Samet Erden established results for classical derivatives in [18]. Later, Erden et al. in [34] constructed Wirtinger-type inequalities for Caputo and Riemann-Liouville fractional derivatives. Inspired by aforementioned work and their applications in diverse fields, the primary objective of this study is to explore and establish a new family of inequalities involving the Caputo-Hadamard fractional differentiable operator with a particular emphasis on Wirtinger-type inequalities. By utilizing Hölder's inequality, we obtain significant results applicable across various fields, including mathematical analysis, statistical modeling, engineering, and computational sciences. Moreover, the findings of this study may inspire further exploration of inequalities associated with Caputo-Hadamard fractional derivatives.

This study contributes to the growing subject of fractional calculus by proposing Wirtinger-type inequalities for Caputo-Hadamard and Hadamard fractional derivatives. These findings provide approaches for analyzing non-local and memory-dependent systems represented by fractional differential equations along with expanding classical inequalities. We broaden our approach to the Hadamard derivative which has remained relatively less studied in the literature while the Caputo-Hadamard derivative offers the main focus of our work due to its practical significance and analytical quality. The key objective of this paper is to improve the theoretical understanding of Wirtinegr-type inequalities in the Caputo-Hadamard context with further extensions offered for the Hadamard derivatives.

The first section presents the introduction that incorporates the background, purpose, and importance of the study. In the second section, Wirtinger-type inequalities are established for Caputo-Hadamard and Hadamard fractional derivatives. The third section explores the applications in the context of arithmetic and geometric means. The final section offers concluding ideas based on our findings.

2. Inequalities for Caputo-Hadamard and Hadamard fractional derivatives

Within this section, we examine Wirtinger-type inequalities related to Caputo-Hadamard fractional derivatives. By using the relation of fractional operators defined including those involving Hadamard fractional derivatives, we explore the connection between Caputo-Hadamard and Hadamard fractional derivatives as presented in [12]. The analysis begins with Taylor's formula which serves as the foundation for deriving the following two key identities utilized throughout the article.

Lemma 2.1. *Let χ belong to $\boldsymbol{AC}^e_\delta[\nu,\mu]$ or $\boldsymbol{C}^e_\delta[\nu,\mu]$ and $\psi \in C$. Then,*

$$\chi(v) = \frac{1}{\Gamma(\psi)} \int_\nu^v (\log v - \log\tau)^{\psi-1\,CH}D^\psi_{\nu+}\chi(\tau)\frac{d\tau}{\tau} + \sum_{k=0}^{e-1} \frac{\delta^k\chi(\nu)}{k!}\left(log\frac{v}{\nu}\right)^k, \tag{2.1}$$

$$\chi(v) = \frac{1}{\Gamma(\psi)} \int_v^\mu (\log v - \log\tau)^{\psi-1\,CH}D^\psi_{\mu-}\chi(\tau)\frac{d\tau}{\tau} + \sum_{k=0}^{e-1} \frac{\delta^k\chi(\mu)}{k!}\left(log\frac{\mu}{v}\right)^k. \tag{2.2}$$

Theorem 2.1. *Consider that $\chi \in \boldsymbol{C}^e_\delta[\nu,\mu]$ with $e \in \boldsymbol{N}\setminus\{0\}$, ${}^{CH}D^\psi_{\nu+}\chi(\omega), {}^{CH}D^\psi_{\mu-}\chi(\omega) \in \boldsymbol{L}_2[\nu,\mu]$ with $\delta^k\chi(\nu) = \delta^k\chi(\mu) = 0$, $k = 0,1,...,e$, $e = \lceil\psi\rceil$, and $\psi \geq 1$. Then the following result holds:*

$$\int_\nu^\mu |\chi(v)|^2 dv \leq \frac{1}{6\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{2\psi-2}(\mu-\nu)^2 \times \int_\nu^\mu \left[|{}^{CH}D^\psi_{\nu+}\chi(\omega)|^2 + \left|{}^{CH}D^\psi_{\mu-}\chi(\omega)\right|^2\right] d\omega. \tag{2.3}$$

Proof. Under the condition $\delta^k\chi(\nu) = \delta^k\chi(\mu) = 0$, utilizing (2.1) and (2.2), applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\chi(v)|^2 &= \left|\frac{1}{\Gamma(\psi)} \int_\nu^v (\log v - \log\eta)^{(\psi-1)\,CH}D^\psi_{\nu+}\chi(\eta)\frac{d\eta}{\eta}\right|^2 \\
&\leq \frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\left(\int_\nu^v \left|{}^{CH}D^\psi_{\nu+}\chi(\eta)\right| d\eta\right)^2 \\
&\leq \frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}(v-\nu)\int_\nu^v \left|{}^{CH}D^\psi_{\nu+}\chi(\eta)\right|^2 d\eta,
\end{aligned} \tag{2.4}$$

and

$$|\chi(v)|^2 = \left|\frac{1}{\Gamma(\psi)} \int_v^\mu (\log\eta - \log v)^{(\psi-1)\,CH}D^\psi_{\mu-}\chi(\eta)|\frac{d\eta}{\eta}\right|^2$$

$$\leq \frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\left(\int_{\nu}^{\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\eta)\right|d\eta\right)^2$$
$$\leq \frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}(\mu-\nu)\int_{\nu}^{\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\eta)\right|^2 d\eta. \tag{2.5}$$

Integrating (2.4) with respect to v from ν to $\lambda\nu + (1-\lambda)\mu$ for $\lambda \in [0, 1]$, and then applying Dirichlet's integral formula on the resulting expression, we obtain the following:

$$\int_{\nu}^{\lambda\nu+(1-\lambda)\mu}|\chi(v)|^2 dv$$
$$\leq\frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\nu}^{\lambda\nu+(1-\lambda)\mu}(v-\nu)\int_{\nu}^{v}\left|{}^{CH}D^{\psi}_{\nu+}\chi(\eta)\right|^2 d\eta dv$$
$$=\frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\nu}^{\lambda\nu+(1-\lambda)\mu}\left|{}^{CH}D^{\psi}_{\nu+}\chi(\eta)\right|^2\int_{\eta}^{\lambda\nu+(1-\lambda)\mu}(v-\nu)dvd\eta$$
$$=\frac{1}{2\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\nu}^{\lambda\nu+(1-\lambda)\mu}\left|{}^{CH}D^{\psi}_{\nu+}\chi(\eta)\right|^2\times\left((\mu-\nu)^2(1-\lambda)^2-(\eta-\nu)^2\right)d\eta. \tag{2.6}$$

Integrating (2.5) from $\lambda\nu + (1-\lambda)\mu$ to μ and following a similar procedure, we get

$$\int_{\lambda\nu+(1-\lambda)\mu}^{\mu}|\chi(v)|^2 dv$$
$$\leq\frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\lambda\nu+(1-\lambda)\mu}^{\mu}(\mu-v)\int_{v}^{\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\eta)\right|^2 d\eta dv$$
$$=\frac{1}{\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\mu}^{\lambda\nu+(1-\lambda)\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\eta)\right|^2\int_{\eta}^{\lambda\mu+(1-\lambda)\mu}(\mu-v)dvd\eta$$
$$=\frac{1}{2\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\int_{\lambda\nu+(1-\lambda)\mu}^{\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\eta)\right|^2\times\left((\mu-\nu)^2\lambda^2-(\mu-\eta)^2\right)d\eta. \tag{2.7}$$

Subsequently, introducing the change of variable $\eta = \rho\nu + (1-\rho)\mu$ in (2.6) and (2.7), and after simplification, we get the following relations:

$$\int_{\nu}^{\lambda\nu+(1-\lambda)\mu}|\chi(v)|^2 dv \leq \frac{(\mu-\nu)^3}{2\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\times\int_{\lambda}^{1}\left((1-\lambda)^2-(1-\rho)^2\right)\left|{}^{CH}D^{\psi}_{\nu+}\chi(\rho\nu+(1-\rho)\mu)\right|^2 d\rho,$$

and

$$\int_{\lambda\nu+(1-\lambda)\mu}^{\mu}|\chi(v)|^2 dv \leq \frac{(\mu-\nu)^3}{2\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{(2\psi-2)}\times\int_{0}^{\lambda}\left(\lambda^2-\rho^2\right)\left|{}^{CH}D^{\psi}_{\mu-}\chi(\rho\nu+(1-\rho)\mu)\right|^2 d\rho.$$

By aligning the resultant inequalities and integrating both sides of the resultant expression with respect to λ over the interval $[0, 1]$, one can observe that

$$\int_{\nu}^{\mu}|\chi(v)|^2 dv \leq \frac{(\mu-\nu)^3}{2\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{(2\psi-2)}\times\left[\int_{0}^{1}\int_{\lambda}^{1}\left((1-\lambda)^2-(1-\rho)^2\right)\left|{}^{CH}D^{\psi}_{\nu+}\chi(\rho\nu+(1-\rho)\mu)\right|^2 d\rho d\lambda\right.$$
$$\left.+\int_{0}^{1}\int_{0}^{\lambda}\left(\lambda^2-\rho^2\right)\left|{}^{CH}D^{\psi}_{\mu-}\chi(\rho\nu+(1-\rho)\mu)\right|^2 d\rho d\lambda\right]. \tag{2.8}$$

Now by changing the order of integration, we can write

$$\int_{\nu}^{\mu}|\chi(v)|^2dv\leq\frac{(\mu-\nu)^3}{2\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{(2\psi-2)}\times\Bigg[\int_0^1\left|{}^{CH}D^{\psi}_{\nu+}\chi(\rho\nu+(1-\rho)\mu)\right|^2\int_0^{\rho}\left((1-\lambda)^2-(1-\rho)^2\right)d\lambda d\rho$$
$$+\int_0^1\left|{}^{CH}D^{\psi}_{\mu-}\chi(\rho\nu+(1-\rho)\mu)\right|^2\int_{\rho}^1\left(\lambda^2-\rho^2\right)d\lambda d\rho\Bigg].$$

Furthermore

$$\int_{\nu}^{\mu}|\chi(v)|^2dv\leq\frac{(\mu-\nu)^3}{2\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{(2\psi-2)}\times\Bigg[\int_0^1 m(\rho)\left|{}^{CH}D^{\psi}_{\nu+}\chi(\rho\nu+(1-\rho)\mu)\right|^2d\rho$$
$$+\int_0^1 s(\rho)\left|{}^{CH}D^{\psi}_{\mu-}\chi(\rho\nu+(1-\rho)\mu)\right|^2d\rho\Bigg], \tag{2.9}$$

where

$$m(\rho)=\frac{1}{3}-\frac{(1-\rho)^3}{3}-\rho(1-\rho)^2,$$

and

$$s(\rho)=\frac{1}{3}-\frac{\rho^3}{3}-\rho^2(1-\rho).$$

Finally, by applying the change of variable $\omega=\rho\nu+(1-\rho)\mu$ and $d\omega=(\nu-\mu)d\rho$ in (2.9) and observing that the maximum value of the functions $m(\rho)$ and $s(\rho)$ for $\rho\in[0,1]$ is $\frac{1}{3}$, we arrive at the result. □

Example 2.1. *Consider the function* $\chi(v)=(\log v-\log 1)^e(\log 2-\log v)^e$ *on* $[1,2]$*, by selecting* $e=2$ *and* $\psi=1.5$ *on the left-hand side of (2.3), and under the condition of the theorem, which is* $\chi(1)=\chi(2)=\delta\chi(1)=\delta\chi(2)=0$*, we obtain that*

$$\int_1^2|\chi(v)|^2dv=0.0000833631.$$

Thus, the left side of (2.3) evaluates to an integral value of 0.0000833631*. To solve the right side of (2.3), we proceed as follows:*

$${}^{CH}D^{1.5}_{1+}\chi(v)=0.56419\Big[1.92181(\log v)^{0.5}-13.6528(\log v)^{1.5}+22.1848(\log v)^{2.5}$$
$$-8.58175(\log v)^{3.5}+0.812698(\log v)^{4.5}\Big],$$

and

$${}^{CH}D^{1.5}_{2-}\chi(v)=\frac{0.56419}{(\log v)^{0.5}}\Big[1.92181((0.693147-\log v)\log v)^{0.5}+(\log v-1.4427(\log v)^2)^{0.5}$$
$$\times(-1.3125+\log v(-6.18231+\log v(16.1157+(-6.91028+0.676616\log v)\log v)))\Big].$$

By result of the theorem, it follows that

$$\int_1^2 \left[|{}^{CH}D^{1.5}_{1+}\chi(v)|^2 + |{}^{CH}D^{1.5}_{2-}\chi(v)|^2\right] dv = 0.0346331.$$

We also obtain a result:

$$\frac{(\log(\mu) - \log(\nu))^{2\psi-2}(\mu - \nu)^2}{6\nu^2[\Gamma(\psi)]^2} = 0.14709,$$

for $\psi = 1.5$, $\nu = 1$, *and* $\mu = 2$, *so the right side of (2.3) becomes*

$$\frac{(\log(\mu)-\log(\nu))^{2\psi-2}(\mu-\nu)^2}{6\nu^2[\Gamma(\psi)]^2}\Gamma(1.5)]^2\int_1^2 \left[|{}^{CH}D^{\psi}_{\nu+}\chi(v)|^2+|{}^{CH}D^{\psi}_{\mu-}\chi(v)|^2\right]dv = 0.14709*0.0346331 = 0.0050942.$$

So, the numerical result of (2.3) gives us

$$0.0000833631 \leq 0.0050942,$$

which tells us that the inequality is valid.

To validate the inequality the corresponding graphs are given in Figure 1 and a table of numerical values is presented in Table 1.

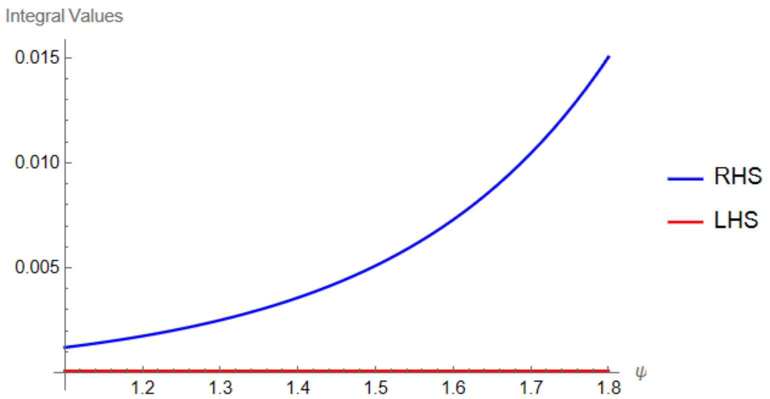

Figure 1. The graph of inequality (2.3) for $1.1 \leq \psi \leq 1.8$.

Table 1. Some left-hand and right-hand-side values of (2.3) corresponding to $\psi \in [1.1, 1.9]$.

ψ	1.1	1.3	1.5	1.7	1.9
Left side	0.0000833631	0.0000833631	0.0000833631	0.0000833631	0.0000833631
Right side	0.00120259	0.00250153	0.0050942	0.0104482	0.0215159

Discussion. The above example is constructed to demonstrate the validity of Theorem 2.1. The selected function satisfies all the conditions of the theorem namely, it belongs to the space of e times the absolutely continuous function on its domain. The detailed calculations and graphs have been obtained using Wolfram Mathematica 13.2. It can be clearly observed that, for increasing values of ψ, the right-hand-side values also increase.

Remark 2.1. *If we substitute $\psi = e = 1$ in (2.4), then we get the following:*

$$\int_{\nu}^{\mu} |\chi(v)|^2 \, dv \leq \frac{(\mu-\nu)^2}{3\nu^2} \int_{\nu}^{\mu} \left[|\omega\chi(\omega)|^2 \right] d\omega.$$

Also, we have the Wirtinger-type inequality for the Hadamard fractional derivative.

Theorem 2.2. *Under the conditions of Theorem 2.1 and by utilizing (1.8), we have the following result:*

$$\int_{\nu}^{\mu} |\chi(v)|^2 dv \leq \frac{(\mu-\nu)^2}{6\nu^2[\Gamma(\psi)]^2} (\log\mu - \log\nu)^{(2\psi-2)} \times \int_{\nu}^{\mu} \left[\left| {}^{H}D_{\nu+}^{\psi}\chi(\omega) \right|^2 + \left| {}^{H}D_{\mu-}^{\psi}\chi(\omega)|^2 \right| \right] d\omega. \tag{2.10}$$

Theorem 2.3. *Suppose that $\chi \in \boldsymbol{C}_{\delta}^{e}[\nu,\mu]$ with $e \in \boldsymbol{N} \setminus \{0\}$ and ${}^{CH}D_{\nu+}^{\psi}\chi(\xi), {}^{CH}D_{\mu-}^{\psi}\chi(\xi) \in \boldsymbol{L}_q[\nu,\mu]$ with $q > 1$, $\psi \geq 1$. If $\delta^k\chi(\nu) = \delta^k\chi(\mu) = 0$, $k = 0, 1, ..., e$ and $e = \lceil \psi \rceil$, then the following inequality is true:*

$$\int_{\nu}^{\mu} |\chi(v)|^q dv \leq \frac{(\mu-\nu)^q}{q(q+1)\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{\psi q - q} \times \int_{\nu}^{\mu} \left[\left| {}^{CH}D_{\nu+}^{\psi}\chi(\xi) \right|^q + \left| {}^{CH}D_{\mu-}^{\psi}\chi(\xi) \right|^q \right] d\xi. \tag{2.11}$$

Proof. First by taking the absolute values of (2.1) and (2.2), and then applying Hölder's inequality with the exponents q and $\frac{q}{q-1}$, under the condition $\delta^k\chi(\nu) = \delta^k\chi(\mu) = 0$, $k = 0, 1, ..., e$, we find that

$$\begin{aligned} |\chi(v)|^q &= \left| \frac{1}{\Gamma(\psi)} \int_{\nu}^{v} (\log v - \log\eta)^{\psi-1\, CH}D_{\nu+}^{\psi}\chi(\eta) \frac{d\eta}{\eta} \right|^q \\ &\leq \frac{1}{\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} \left[\int_{\nu}^{v} \left| {}^{CH}D_{\nu+}^{\psi}\chi(\eta) \right| d\eta \right]^q \\ &\leq \frac{1}{\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} (v-\nu)^{q-1} \int_{\nu}^{v} \left| {}^{CH}D_{\nu+}^{\psi}\chi(\eta) \right|^q d\eta, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |\chi(v)|^q &= \left| \frac{1}{\Gamma(\psi)} \int_{\nu}^{\mu} (\log\eta - \log\nu)^{\psi-1\, CH}D_{\mu-}^{\psi}\chi(\eta) \frac{d\eta}{\eta} \right|^q \\ &\leq \frac{1}{\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} \left[\int_{\nu}^{\mu} \left| {}^{CH}D_{\mu-}^{\psi}\chi(\eta) \right| d\eta \right]^q \\ &\leq \frac{1}{\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} (\mu-\nu)^{q-1} \int_{\nu}^{\mu} \left| {}^{CH}D_{\mu-}^{\psi}\chi(\eta) \right|^q d\eta. \end{aligned} \tag{2.13}$$

Integrating (2.12) with respect to v from ν to $\lambda\nu + (1-\lambda)\mu$ for $\lambda \in [0,1]$ and by applying Dirichlet's formula, we obtain

$$\begin{aligned} &\int_{\nu}^{\lambda\nu+(1-\lambda)\mu} |\chi(v)|^q dv \\ \leq& \frac{1}{\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} \times \int_{\nu}^{\lambda\nu+(1-\lambda)\mu} (v-\nu) \int_{\nu}^{v} \left| {}^{CH}D_{\nu+}^{\psi}\chi(\eta) \right|^q d\eta dv \\ =& \frac{1}{q\nu^q[\Gamma(\psi)]^q} (\log\mu - \log\nu)^{q\psi - q} \int_{\nu}^{\lambda\nu+(1-\lambda)\mu} \Big((1-\lambda)^q \times (\mu-\nu)^q - (\eta-\nu)^q \Big) \left| {}^{CH}D_{\nu+}^{\psi}\chi(\eta) \right|^q d\eta. \end{aligned} \tag{2.14}$$

Now integrating (2.13) with respect to v from $\lambda\nu + (1-\lambda)\mu$ to μ for $\lambda \in [0,1]$, and applying similar steps as in the process for (2.12), we obtain

$$\begin{aligned}
&\int_{\lambda\nu+(1-\lambda)\mu}^{\mu} |\chi(v)|^q dv \\
\leq &\frac{1}{\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \int_{\lambda\nu+(1-\lambda)\mu}^{\mu} (\mu - v) \int_{v}^{\mu} \left|{}^{CH}D_{\mu-}^{\psi}\chi(\eta)\right|^q d\eta dv \\
= &\frac{1}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \int_{\lambda\nu+(1-\lambda)\mu}^{\mu} \Big(\lambda^q(\mu-\nu)^q - (\mu-\eta)^q\Big)\left|{}^{CH}D_{\mu-}^{\psi}\chi(\eta)\right|^q d\eta. \qquad (2.15)
\end{aligned}$$

By applying the change of variable $\eta = \rho\nu + (1-\rho)\mu$ in (2.14) and (2.15), we obtain the following integrals:

$$\int_{\nu}^{\lambda\nu+(1-\lambda)\mu} |\chi(v)|^q dv \leq \frac{(\mu-\nu)^{q+1}}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \int_{\lambda}^{1} \Big((1-\lambda)^q - (1-\rho)^q\Big)\left|{}^{CH}D_{\nu+}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho,$$

and

$$\int_{\lambda\nu+(1-\lambda)\mu}^{\mu} |\chi(v)|^q dv \leq \frac{(\mu-\nu)^{q+1}}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \int_{0}^{\lambda} \Big(\lambda^q - \rho^q\Big)\left|{}^{CH}D_{\mu-}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho.$$

Integrating the resulting inequalities with respect to λ over the interval $[0,1]$ and arranging the results next to one another, we observe that

$$\begin{aligned}
\int_{\nu}^{\mu} |\chi(v)|^q dv \leq &\frac{(\mu-\nu)^{q+1}}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \\
&\times \Big[\int_0^1 \int_{\lambda}^{1} \Big((1-\lambda)^q - (1-\rho)^q\Big)\left|{}^{CH}D_{\nu+}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho d\lambda \\
&+ \int_0^1 \int_0^{\lambda} \Big(\lambda^q - \rho^q\Big)\left|{}^{CH}D_{\mu-}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho d\lambda\Big].
\end{aligned}$$

By changing the integration order,

$$\begin{aligned}
\int_{\nu}^{\mu} |\chi(v)|^q dx \leq &\frac{(\mu-\nu)^{q+1}}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q}(\log\mu - \log\nu)^{q\psi-q} \\
&\times \Big[\int_0^1 \left|{}^{CH}D_{\nu+}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q \int_0^{\rho} \Big((1-\lambda)^q - (1-\rho)^q\Big) d\lambda d\rho \\
&+ \int_0^1 \left|{}^{CH}D_{\mu-}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q \int_{\rho}^{1} \Big(\lambda^q - \rho^q\Big) d\rho d\lambda\Big],
\end{aligned}$$

and we get

$$\begin{aligned}
\int_{\nu}^{\mu} |\chi(v)|^q dv \leq &\frac{(\mu-\nu)^{q}}{q\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \Big[\int_0^1 h(\rho)\left|{}^{CH}D_{\nu+}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho \\
&+ \int_0^1 t(\rho)\left|{}^{CH}D_{\mu-}^{\psi}\chi(\rho\nu + (1-\rho)\mu)\right|^q d\rho\Big],
\end{aligned}$$

where

$$h(\rho) = \frac{1}{q+1} - \frac{(1-\rho)^{q+1}}{q+1} - \rho(1-\rho)^q,$$

and

$$t(\rho) = \frac{1}{q+1} - \frac{\rho^{q+1}}{q+1} - \rho^q(1-\rho).$$

It is observed that $\max[h(\rho), t(\rho)] = \frac{1}{q+1}$ and

$$\int_\nu^\mu |\chi(v)|^q dv$$
$$\leq \frac{(\mu-\nu)^{q+1}}{q(q+1)\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q\psi-q} \times \int_0^1 \left[\left|{}^{CH}D^{\psi}_{\nu+}\chi(\rho\nu+(1-\rho)\mu)\right|^q + \left|{}^{CH}D^{\psi}_{\mu-}\chi(\rho\nu+(1-\rho)\mu)\right|^q\right] d\rho.$$

Finally by applying the change of variable $\xi = \rho\nu + (1-\rho)\mu$, we can derive (2.11). □

Figure 2 is represents logical form from Lemma 2.1 to Theorem 2.3.

```
                    Lemma 2.1
             (Foundational identities)
                        ↓
                   Theorem 2.1
       (Caputo-Hadamard inequality in L₂[a, b])
            /           |           \
            ↓           ↓            ↓
     Example 2.1    Remark 2.1    Theorem 2.2
                                   (Hadamard inequality in L₂[a, b])
                       ↓
                 Theorem 2.3
 (Caputo-Hadamard inequality in L_q[a, b], q > 1)
```

Figure 2. The logical path from Lemma 2.1 to Theorem 2.3.

Example 2.2. *Consider the function* $\chi(v) = (\log v - \log 1)^e(\log 2 - \log v)^e$ *on the interval* $[1, 2]$. *By picking* $q = 4$, $e = 2$, *and* $\psi = 1.5$ *in (2.11), with* $\chi(1) = \chi(2) = \delta\chi(1) = \delta\chi(2) = 0$, *we obtain that*

$$\int_1^2 |\chi(v)|^4 dv = 0.0000000127612.$$

Therefore, the result of the integral for the left side of the (2.11) is found to be 0.0000000127612. *For the right side, we find*

$$\begin{aligned} {}^{CH}D^{1.5}_{1+}\chi(v) =& 0.56419\Big[1.92181(\log v)^{0.5} - 13.6528(\log v)^{1.5} + 22.1848(\log v)^{2.5} \\ & - 8.58175(\log v)^{3.5} + 0.812698(\log v)^{4.5}\Big], \end{aligned}$$

and

$$ {}^{CH}D^{1.5}_{2-}\chi(v) = \frac{0.56419}{(\log v)^{0.5}}\Big[1.92181((0.693147-\log v)\log v)^{0.5} + (\log v - 1.4427(\log v)^2)^{0.5} \times(-1.3125+\log v(-6.18231+\log v(16.1157+(-6.91028+0.676616\log v)\log v)))\Big]. $$

In this case, it follows that

$$ \int_1^2 \left[|{}^{CH}D^{1.5}_{1+}\chi(v)|^4 + |{}^{CH}D^{1.5}_{2-}\chi(v)|^4\right]dv = 0.000916457. $$

We also have the result:

$$ \frac{(\mu-v)^4(\log\mu-\log v)^{4\psi-4}}{20v^4[\Gamma(\psi)]^4} = 0.0389441, $$

for the case $\psi = 1.5, v = 1$, *and* $\mu = 2$. *Thus, the right-hand-side of (2.10) becomes*

$$ \frac{(\mu-v)^4(\log\mu-\log v)^{4\psi-4}}{20v^4[\Gamma(\psi)]^4}\int_1^2\left[|{}^{CH}D^{1.5}_{v+}\chi(\xi)|^4+|{}^{CH}D^{1.5}_{\mu-}\chi(\xi)|^4\right]dv = 0.0389441*0.000916457 = 0.0000356906. $$

Thus the (2.11) yields the mathematical result

$$ 0.0000000127612 \le 0.0000356906, $$

and this demonstrates that the inequality holds true.

The table of numerical values is shown in Table 2 and the corresponding graph is given in Figure 3 to verify the inequality.

Table 2. Some left-hand and right-hand-side values of (2.11) corresponding to $\psi \in [1.1, 1.9]$.

ψ	1.1	1.3	1.5	1.7	1.9
Left side	$1.27612 * 10^{-8}$	$1.27612 * 10^{-8}$	$1.27612 * 10^{-8}$	$1.27612 * 10^{-8}$	$1.27612 * 10^{-8}$
Right side	$2.11158 * 10^{-6}$	$8.84039 * 10^{-6}$	0.0000356906	0.000153715	0.000812788

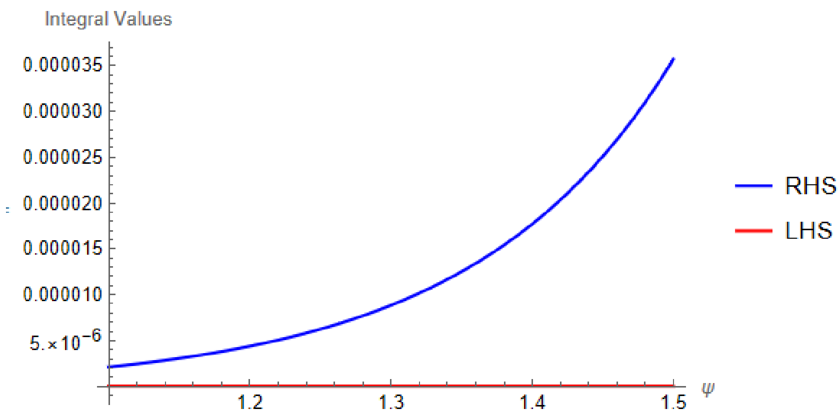

Figure 3. The graph of (2.11) for $1.1 \le \psi \le 1.5$.

Discussion. The above example is provided to justify (2.11) and the function is chosen according to the conditions of the theorem. The derivation and graph are obtained by making use of Wolfram Mathematica 13.2, right-hand-side values also increase with the increasing value of ψ.

Remark 2.2. *If we substitute $q = 4$ in (2.11), we get the following inequality:*

$$\int_{\nu}^{\mu} |\chi(\nu)|^4 d\nu \leq \frac{(\mu-\nu)^4}{20\nu^4[\Gamma(\psi)]^4}(\log\mu - \log\nu)^{4(\psi-1)} \times \int_{\nu}^{\mu} \left[\left|{}^{CH}D^{\psi}_{\nu+}\chi(\xi)\right|^4 + \left|{}^{CH}D^{\psi}_{\mu-}\chi(\xi)\right|^4\right] d\xi.$$

Remark 2.3. *If we substitute $q = 2$ in (2.11), we get (2.3).*

Remark 2.4. *If we replace $q = 4$ and $e = \psi = 1$ in (2.11), then we arrive at the following outcome:*

$$\int_{\nu}^{\mu} |\chi(\nu)|^4 \, d\nu \leq \frac{(\mu-\nu)^4}{10\nu^4} \int_{\nu}^{\mu} |\xi\chi'(\xi)|^4 \, d\xi.$$

Furthermore, here are some results for the Hadamard fractional derivative.

Theorem 2.4. *Under the conditions of Theorem 2.3 and from (1.8), we arrive at the inequality:*

$$\int_{\nu}^{\mu} |\chi(\nu)|^q \, d\nu \leq \frac{(\mu-\nu)^q}{q(q+1)\nu^q[\Gamma(\psi)]^q}(\log\mu - \log\nu)^{q(\psi-1)} \times \int_{\nu}^{\mu} \left[\left|{}^{H}D^{\psi}_{\nu+}\chi(\xi)\right|^q + \left|{}^{H}D^{\psi}_{\mu-}\chi(\xi)\right|^q\right] d\xi. \quad (2.16)$$

Remark 2.5. *By substituting $q = 4$ in* (2.16)*, we will obtain following inequality:*

$$\int_{\nu}^{\mu} |\chi(\nu)|^q \, d\nu \leq \frac{(\mu-\nu)^4}{20\nu^4[\Gamma(\psi)]^4}(\log\mu - \log\nu)^{4(\psi-1)} \times \int_{\nu}^{\mu} \left[\left|{}^{H}D^{\psi}_{\nu+}\chi(\xi)\right|^4 + \left|{}^{H}D^{\psi}_{\mu-}\chi(\xi)\right|^4\right] d\xi.$$

Remark 2.6. *By setting $q = 2$ in* (2.16)*, we will get* (2.10).

3. Some applications in terms of AM-GM-type inequalities

This section involves some Wirtinger-type inequalities that align with the arithmetic and geometric mean involving Caputo-Hadamard fractional derivatives. The arithmetic-mean and geometric-mean inequalities are extremely important in mathematics since they provides a basic relationship between the average of a set of non-negative numbers and their geometric mean. These inequalities are also extensively applied in the estimation, proof, and solving of problems in a number of areas such as algebra, calculus, and even economics, where they are used to analyze situations involving growth rates and returns. Moreover, these inequalities serve as a powerful tools for establishing bounds and inequalities in various mathematical contexts. These inequalities find practical application in fields such as signal processing and control theory. In control theory, they are commonly used to determine stability conditions, energy estimates for dynamic systems, and performance bounds. In signal processing, they find application in designing optimal filters, noise reduction, signal reconstruction by bounding errors, and ensuring stable algorithm behavior. Presenting the Wirtinger-type inequality in this form facilitates its effective use in analyzing real-world engineering problems particularly when dealing with fractional-order systems. In the following theorems we have geometric-type expressions on the left-hand-side and arithmetic-type relations on the right-hand-side.

Theorem 3.1. *Consider a real-valued function $\Phi(\varkappa)$ which is a non-negative continuous function defined on the interval $\boldsymbol{I} = [\nu, \mu]$. Consider χ to be such that ${}^{CH}D^{\psi}_{\nu+}\chi(\varkappa), {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa) \in \boldsymbol{C}^{m-1}(\boldsymbol{I})$ and $D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa), D^{\kappa-1}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)$ are absolutely continuous for $\varkappa \in \boldsymbol{I}$ with $\chi(\nu) = \chi(\mu) = 0$, for $\kappa = 1, 2, ...m$. Then the following inequality holds:*

$$\int_{\nu}^{\mu} |\Phi(\varkappa)|^2 \Bigg[\left(\prod_{\kappa=1}^{m} \left|D^{\kappa-1CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|\right)^{\frac{2}{m}} + \left(\prod_{\kappa=1}^{m} \left|D^{\kappa-1CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|\right)^{\frac{2}{m}}\Bigg]d\varkappa$$
$$\leq \frac{(\mu-\nu)^3}{4m}\frac{1}{6\nu^2[\Gamma(\psi)]^2}(\log\mu - \log\nu)^{2\psi-2} \times \int_{\nu}^{\mu}\left[\left|{}^{CH}D^{\psi}_{\nu+}\chi(\omega)\right|^2 + \left|{}^{CH}D^{\psi}_{\mu-}\chi(\omega)\right|^2\right]d\omega$$
$$\times \int_{\nu}^{\mu}\left[\sum_{\kappa=1}^{m}\left(\left|D^{\kappa CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 + \left|D^{\kappa CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2\right)\right]d\varkappa. \tag{3.1}$$

Proof. One can easily observe the following identities:

$$D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)d\varkappa = \int_{\nu}^{\varkappa} D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(s)ds, \tag{3.2}$$

$$D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)d\varkappa = -\int_{\varkappa}^{\mu} D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(s)ds, \tag{3.3}$$

$$D^{\kappa-1}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)d\varkappa = \int_{\nu}^{\varkappa} D^{\kappa}\ {}^{CH}D^{\psi}_{\mu-}\chi(s)ds, \tag{3.4}$$

$$D^{\kappa-1}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)d\varkappa = -\int_{\varkappa}^{\mu} D^{\kappa}\ {}^{CH}D^{\psi}_{\mu-}\chi(s)ds, \tag{3.5}$$

for $\varkappa \in \mathbf{I}$ and $\kappa = 1, 2, ...m$. From (3.2)–(3.5), we get the following two inequalities:

$$\left|D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right| \leq \frac{1}{2}\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa, \tag{3.6}$$

$$\left|D^{\kappa-1}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right| \leq \frac{1}{2}\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|d\varkappa, \tag{3.7}$$

for $\varkappa \in \mathbf{I}$ and $\kappa = 1, 2, ...m$. Now taking (3.6) into consideration, using Cauchy-Schwarz, (1.4), and (1.5), we get the following:

$$\left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|\right)^{\frac{2}{m}}$$
$$\leq \left[\prod_{\kappa=1}^{m}\left(\frac{1}{2}\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)\right]^{\frac{2}{m}}$$
$$= \left(\frac{1}{2}\right)^2\left[\prod_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)\right]^{\frac{2}{m}}$$
$$\leq \frac{1}{4}\left[\frac{1}{m}\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)\right]^2$$

$$\leq \frac{1}{4m^2}\left[m\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|d\varkappa\right)^2\right]$$
$$\leq \frac{\mu-\nu}{4m}\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 d\varkappa\right)$$
$$= \frac{\mu-\nu}{4m}\int_{\nu}^{\mu}\left(\sum_{\kappa=1}^{m}\left|D^{\kappa\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2\right)d\varkappa. \tag{3.8}$$

By applying similar steps on (3.7), we get:

$$\left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|\right)^{\frac{2}{m}} \leq \frac{\mu-\nu}{4m}\int_{\nu}^{\mu}\left(\sum_{\kappa=1}^{m}\left|D^{\kappa\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2\right)d\varkappa. \tag{3.9}$$

Adding (3.8) and (3.9), we obtain:

$$\left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|\right)^{\frac{2}{m}} + \left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|\right)^{\frac{2}{m}}$$
$$\leq \frac{\mu-\nu}{4m}\int_{\nu}^{\mu}\left[\sum_{\kappa=1}^{m}\left(\left|D^{\kappa\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 + \left|D^{\kappa\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2\right)\right]d\varkappa. \tag{3.10}$$

Multiplying by $|\Phi(\varkappa)|^2$ on both sides of (3.10), integrating from ν to μ, and by utilizing Theorem 2.1, we obtain the required result. □

Remark 3.1. *If we substitute $m = 1$ in* (3.1)*, then we get the following:*

$$\int_{\nu}^{\mu}\Phi(\varkappa)\left[\left|{}^{CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 + \left|{}^{CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2\right]d\varkappa$$
$$\leq \frac{(\mu-\nu)^3}{4}\frac{1}{6\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{2\psi-2}\times\int_{\nu}^{\mu}\left[\left|{}^{CH}D_{\nu+}^{\psi}\chi(\omega)\right|^2 + \left|{}^{CH}D_{\mu-}^{\psi}\chi(\omega)\right|^2\right]d\omega$$
$$\times\int_{\nu}^{\mu}\left[\left(\left|D\ {}^{CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 + \left|D\ {}^{CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2\right)\right]d\varkappa, \tag{3.11}$$

which represents the Wirtinger-type inequality.

Theorem 3.2. *Under the assumptions of Theorem 3.1, we have the following inequality.*

$$\int_{\nu}^{\mu}|\Phi(\varkappa)|^2\left[\left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2\right)^{\frac{2}{m}} + \left(\prod_{\kappa=1}^{m}\left|D^{\kappa-1 CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2\right)^{\frac{2}{m}}\right]d\varkappa$$
$$\leq \frac{(\mu-\nu)^2}{m}\frac{1}{6\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{2\psi-2}\times\int_{\nu}^{\mu}\left[\left|{}^{CH}D_{\nu+}^{\psi}\chi(\omega)\right|^2 + \left|{}^{CH}D_{\mu-}^{\psi}\chi(\omega)\right|^2\right]d\omega$$
$$\times\left[\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa-1CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 d\varkappa\int_{\nu}^{\mu}\left|D^{\kappa\ CH}D_{\nu+}^{\psi}\chi(\varkappa)\right|^2 d\varkappa\right.\right.$$
$$\left.\left.+\int_{\nu}^{\mu}\left|D^{\kappa-1\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2 d\varkappa\int_{\nu}^{\mu}\left|D^{\kappa\ CH}D_{\mu-}^{\psi}\chi(\varkappa)\right|^2 d\varkappa\right)\right]. \tag{3.12}$$

Proof. To start the proof, the following identities are easily observable:

$$\left[D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right]^2 = 2\int_{\nu}^{\varkappa} D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(s)D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(s)ds, \tag{3.13}$$

$$\left[D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right]^2 = -2\int_{\varkappa}^{\mu} D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(s)D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(s)ds, \tag{3.14}$$

$$\left[D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right]^2 = 2\int_{\nu}^{\varkappa} D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(s)D^{\kappa}\,{}^{CH}D^{\psi}_{\mu-}\chi(s)ds, \tag{3.15}$$

$$\left[D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right]^2 = -2\int_{\varkappa}^{\mu} D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(s)D^{\kappa}\,{}^{CH}D^{\psi}_{\mu-}\chi(s)ds, \tag{3.16}$$

for $\varkappa \in \mathbf{I}$ and $\kappa = 1, 2, ..., m$. From (3.13) and (3.14), we get the following:

$$\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 \leq \int_{\nu}^{\mu} \left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 \left|D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa. \tag{3.17}$$

Similarly, from (3.15) and (3.16), we get the following inequality:

$$\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 \leq \int_{\nu}^{\mu} \left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 \left|D^{\kappa}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa. \tag{3.18}$$

Consider (3.17), by utilizing the Cauchy-Schwarz and elementary inequalities given in (1.4) and (1.5), we obtain:

$$\begin{aligned}
&\left[\prod_{\kappa=1}^{m}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2\right]^{\frac{2}{m}}\\
\leq&\left[\prod_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|\left|D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)\right]^{\frac{2}{m}}\\
\leq&\left[\frac{1}{m}\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|\left|D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)\right]^{2}\\
\leq&\frac{1}{m^2}\left[m\sum_{\kappa=1}^{m}\left(\int_{\nu}^{\mu}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|\left|D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|d\varkappa\right)^{2}\right]\\
\leq&\frac{1}{m}\left[\sum_{\kappa=1}^{m}\left(\left(\int_{\nu}^{\mu}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\right)\times\left(\int_{\nu}^{\mu}\left|D^{\kappa}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\right)\right)\right].
\end{aligned} \tag{3.19}$$

Similarly, from (3.18), we obtain following result:

$$\left[\prod_{\kappa=1}^{m}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2\right]^{\frac{2}{m}} \leq \frac{1}{m}\left[\sum_{\kappa=1}^{m}\left(\left(\int_{\nu}^{\mu}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\right)\times\left(\int_{\nu}^{\mu}\left|D^{\kappa}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\right)\right)\right]. \tag{3.20}$$

By adding (3.19) and (3.20),

$$\left[\prod_{\kappa=1}^{m}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2\right]^{\frac{2}{m}} + \left[\prod_{\kappa=1}^{m}\left|D^{\kappa-1}\,{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2\right]^{\frac{2}{m}}$$

$$\leq \frac{1}{m}\Bigg[\sum_{\kappa=1}^{m}\Bigg(\bigg(\int_{\nu}^{\mu}\left|D^{\kappa-1}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\bigg)\bigg(\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\bigg)$$

$$+\bigg(\int_{\nu}^{\mu}\left|D^{\kappa-1}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\bigg)\bigg(\int_{\nu}^{\mu}\left|D^{\kappa}\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\bigg)\Bigg)\Bigg]. \tag{3.21}$$

Multiplying by $|\Phi(\varkappa)|^2$ on both sides of (3.21), integrating from ν to μ, with respect to $\varkappa$, and by using Theorem 2.1, we obtain (3.12). □

Remark 3.2. *If we substitute $m = 1$ in (3.12), we obtain:*

$$\int_{\nu}^{\mu}|\Phi(\varkappa)|^2\left[\left|{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^4+\left|{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^4\right]d\varkappa$$

$$\leq \frac{(\mu-\nu)^2}{6\nu^2[\Gamma(\psi)]^2}(\log\mu-\log\nu)^{2\psi-2}\times\int_{\nu}^{\mu}\left[\left|{}^{CH}D^{\psi}_{\nu+}\chi(\omega)\right|^2+\left|{}^{CH}D^{\psi}_{\mu-}\chi(\omega)\right|^2\right]d\omega \tag{3.22}$$

$$\times\Bigg[\bigg(\int_{\nu}^{\mu}\left|{}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\int_{\nu}^{\mu}\left|D\ {}^{CH}D^{\psi}_{\nu+}\chi(\varkappa)\right|^2 d\varkappa\bigg)+\bigg(\int_{\nu}^{\mu}\left|{}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\int_{\nu}^{\mu}\left|D\ {}^{CH}D^{\psi}_{\mu-}\chi(\varkappa)\right|^2 d\varkappa\bigg)\Bigg],$$

which is a Wirtinger-type inequality.

4. Conclusions

A valuable tool for describing memory and inherited features of various components and procedures is the fractional derivative. It is a more appropriate way to illustrate real-world problems than the usual derivative of integer order. Whenever analyzing systems with changes of non-integer order, the Caputo-Hadamard fractional derivative works particularly well because it offers an explanation for problems with memory effect, and scaling. The study of inequalities is important for determining boundaries, stability, and distinction of strategies for fractional DEs. The presence, uniqueness, and consistency of a system of fractional DEs have been evaluated using fractional inequalities, also referred to as inequalities that include a derivative and integral of arbitrary order. In this paper, we derived Wirtinger-type inequalities for the Caputo-Hadamard fractional derivatives and our results apply to functions whose left- and right-sided Caputo-Hadamard fractional derivatives belong to $\mathbf{L}_2$ and $\mathbf{L}_q$ space for $q>1$. Results for the Hadamard fractional derivative are presented and remarks are also concluded from our results. Furthermore, essential examples are provided to validate the accuracy of our main findings. Applications incorporating arithmetic-mean and geometric-mean-type inequalities are also presented. The current results are achieved for a limited class of functions under certain boundary conditions. We expect that the concept and methodology considered in this work will provide researchers with new ideas.

Author contributions

Muhammad Samraiz: Methodology, Validation, Investigation, Writing-original draft preparation, Writing-review and editing; Humaira Javaid: Conceptualization, Formal analysis, Writing-original draft preparation; Muath Awadalla: Conceptualization, Validation, Writing-original draft preparation, Writing-review and editing, Supervision; Hajer Zaway: Software, Formal analysis, Investigation, Project administration. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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