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Research article

Explicit solutions and non-solutions for the Diophantine equation $p^x + q^{2y} = z^{2n}$ involving primes $p \not\equiv q \pmod 4$

Kittipong Laipaporn[1], **Saeree Wananiyakul**[2] **and Prathomjit Khachorncharoenkul**[1,*]

[1] Center of Excellence for Ecoinformatics, School of Science, Walailak University, Nakhon Si Thammarat 80160, Thailand

[2] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

* **Correspondence:** Email: prathomjit@gmail.com.

Abstract: Over the past decade, significant research has been conducted on the equation $a^x + b^y = z^2$ under various conditions imposed on a and b or on x and y. Most studies focus on conditions where the equation has no solution, while some explore cases with infinitely many solutions, often considering scenarios where x or y is even. Motivated by this line of inquiry, we have been inspired to investigate and analyze equations of the form $p^x + q^{2y} = z^{2n}$ for two distinct primes p and q, and to present explicit forms of their solutions (p, x, q, y, z, n). Recent studies on the exponential Diophantine equation $p^x + q^y = z^2$, where p and q are primes, have addressed cases where $p = 2$ or $p \equiv q \pmod 4$. In this paper, we address the case where $p \not\equiv q \pmod 4$ and y is even. In addition, we explore special cases where z is the prime and provide the complete set of solutions for $p^x + q^{2y} = z^{2n}$. We also show that the equation has no solution when $\{2, 3\} \not\subseteq \{p, q, z\}$. In other words, we provide almost explicit solutions to $p^x + q^y = z^{2n}$ except for the case where both x and y are odd.

Keywords: Diophantine equation; Catalan's conjecture; Legendre symbol
Mathematics Subject Classification: 11D61, 11D72

1. Introduction

The solutions of the exponential Diophantine equation

$$a^x \pm b^y = z^n,$$

where a, b, x, y, z are non-negative integers and n is a positive integer, have been explored for a long time. A significant number of articles have previously suggested the exact values of the parameters

a and b, such as $2^x + 5^y = z^2$ [1]. This equation served the role as the model of the other equations from then until now. However, publications in the past five years have inspired us to realize that the development of this field has been directed to the generalization of the equation $a^x \pm b^y = z^n$. We have provided some of them in Table 1.

Table 1. Examples of exponential Diophantine equations with prime bases and solution techniques.

Year/Ref.	Equations	Solutions	Techniques
2013/ [2]	$p^x + (p+1)^y = z^2$	$(p, x, y, z) = (3, 2, 2, 5)$ and $(7, 0, 1, 3)$ are the only two solutions, where p is a Mersenne prime.	Applying Catalan's conjecture.
2017/ [3]	$2^x + q = z^4$	– Exactly one solution in which x is even and q is a prime. – Infinitely many solutions in which x is even and q is composite, or x is odd and $q \geq 2$.	Using the principles of division and the prime number property.
	$p^x + q = z^4$	– Exactly one solution in which x is even, $p = 3$, and q is a prime. – Infinitely many solutions in which x is even and q is composite, or x is odd and $q \geq 2$.	
2022/ [4]	$7^x + 32^y = z^2$	$(x, y, z) = (2, 1, 9)$ is the unique solution.	Applying Catalan's conjecture.
	$2^x + 7^y = z^2$	$(x, y, z) = (3, 0, 3), (5, 2, 9)$ are the only two solutions, where $x \neq 1$.	
2022/ [5]	$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$	$(x, y, z) = (1, 1, 2)$ is the unique solution, provided that a, b, and c satisfy the stipulated conditions and $m > \max\{10^8, c^2\}$.	Using the Baker method.
2022/ [6]	$M^x + (M-1)^y = z^2$	$(M, x, y, z) = (3, 1, 0, 2), (3, 0, 3, 3)$, and $(3, 2, 4, 5)$ are all solutions, where M is a Mersenne prime.	Applying Catalan's conjecture and Euler's criterion.
2024/ [7]	$a^x + a^y = z^n$	All solutions are $(a, n, x, y, z) \in A \cup B$ where A is the set of all elements of the form $\left(2, 2, 2k, 2k+3, 3 \cdot 2^k\right)$ or $\left(2, 2, 2k+3, 2k, 3 \cdot 2^k\right)$ for all $k \in \mathbb{N}_0$ and B is the set of all elements of the form $\left(a, n, nk, nk+1, a^k \sqrt[n]{a+1}\right)$ or $\left(a, n, nk+1, nk, a^k \sqrt[n]{a+1}\right)$ for all $k \in \mathbb{N}_0$ under the conditions $a, n \geq 2$ and $x \neq y$.	Applying Catalan's conjecture.
	$a^x - a^y = z^n$	All solutions are $(a, n, x, y, z) \in \left\{\left(3, 3, 3k+2, 3k, 2 \cdot 3^k\right), \left(a, n, nk+1, nk, a^k \sqrt[n]{a-1}\right) : k \in \mathbb{N}_0\right\}$, where $a, n \geq 2$ and $x > y$.	

Here and in the sequel, $\mathbb{N}_0$ denotes the set of all non-negative integers.

Crucially, N. Burshtein [3], in 2017, demonstrated that the equation $p^x + q = z^4$ has infinitely many solutions in which x is odd and $q \geq 2$. In the following year, the equation $2^x + q^y = z^2$, where q is an odd prime, was further examined [8]. In 2019, R. J. S. Mina and J. B. Bacani [9] focused on the non-existence of solutions for Diophantine equations of the form $p^x + q^y = z^{2n}$, where at least one of the exponents is required to be odd. In 2021, Mina and Bacani [10] also studied the equation $p^x + q^y = z^2$, where p and q are odd primes with $p \equiv q \pmod 4$. Together, these studies addressed almost all cases of the equation $p^x + q^y = z^2$ for the arbitrary prime numbers p and q.

In this article, we address the remaining case where $p \not\equiv q \pmod 4$ and y is restricted to even values. To achieve this, we rewrite the equation in the form

$$p^x + q^{2y} = z^{2n},$$

where p and q are distinct primes, n is a positive integer, and x, y, and z are non-negative integers. We organize our work into four sections as follows. In Section 2, we present the essential results needed to prove the main results. Section 3 states all the results, divided into two parts. The first part presents our main result: Theorem 8. Additionally, the equations from [8, 11, 12] are applied to the sets A through E of Theorem 8, as discussed in Remarks 1 and 2. For the special case where z is a prime, we conclude that the equation $p^x + q^{2y} = z^{2n}$ has no solutions if $\{2, 3\} \not\subseteq \{p, q, z\}$ (see Corollary 12). The second part presents some consequences of Theorem 8, including an alternative proof of the equation in [8], where we provide all explicit solution sets for $2^x + q^y = z^{2n}$ (see Proposition 13). Moreover, we provide additional information in Propositions 13 and 16 and Corollaries 14 and 15. Finally, we discuss and summarize all the results in Section 4.

2. Preliminaries

In this paper, we assume that all variables are non-negative integers. Here, we list some well-known properties for the Legendre symbol.

Theorem 1. *Let a, b be integers and p be an odd prime. Then*

(1) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod p$,

(2) $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$,

(3) $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$,

(4) $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ *for any odd prime $q \neq p$.*

The rest of this section focuses on the essential theorems for proving the main results.

Theorem 2. *[13] The Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive integer solutions, namely $(x, n) = (1, 1), (2, 2)$, and $(11, 5)$.*

Theorem 3. *[7] Let p be a prime and $n \geq 2$ be a positive integer. All solutions of the equation $p^x + 1 = z^n$ are*

$$(p, x, z, n) \in \{(2, 3, 3, 2), (2^n - 1, 1, 2, n)\}.$$

In particular, the equation has at the most one solution when p and n are fixed. Furthermore, if n is a composite number, the equation has no solution.

Theorem 4. *[7] Let p be a prime and $n \geq 2$ be a positive integer. All solutions of the equation $p^x - 1 = z^n$ are*

$$(p, x, z, n) \in \left\{(3, 2, 2, 3), (p, 0, 0, n), (2, 1, 1, n), \left((2v)^{2^\ell} + 1, 1, 2v, 2^\ell\right) : v, \ell \in \mathbb{N}\right\}.$$

In particular, this equation has at most two solutions when p and n are fixed.

Finally, we restate the results relevant to this paper from [8, 10] as follows: In 2018, Burshtein studied the equation

$$2^x + q^y = z^2, \tag{2.1}$$

where q is an odd prime and x, y, z are positive integers.

Theorem 5. *[8]*

(1) For $q \equiv 1 \pmod 4$,

- *If y is even, then Eq (2.1) has no solution;*
- *If y is odd and x is even, then Eq (2.1) has no solution except when $y = 1$;*
- *If y and x are odd, then Eq (2.1) has no solution except when $x \neq 1$.*

(2) For $q \equiv 3 \pmod 4$,

- *If y is even, then Eq (2.1) has no solution except when $y = 2$;*
- *If y is odd, then Eq (2.1) has no solution except when $x = y = 1$.*

In 2021, Mina and Bacani studied the equation

$$p^x + q^y = z^2, \tag{2.2}$$

where p and q are odd primes with $p \equiv q \pmod 4$ and x, y, z are non-negative integers.

Theorem 6. *[10]*

(1) If x and y are of the same parity, then Eq (2.2) has no solution.

(2) If x and y are of different parity, all solutions of Eq (2.2) are $(p, x, q, y, z) \in A \cup B \cup C$, where

- *$A = \{(3, 1, q, 0, 2) : q \equiv 3 \pmod 4\}$;*
- *$B = \{(3, 5, 11, 4, 122)\}$;*
- *$C = \{(p, 2m, 2p^m + 1, 1, p^m + 1) : p \equiv 3 \pmod 4, m \in \mathbb{N}_0\}$.*

Additionally, Burshtein provided a remark identifying all solutions to $2^x + q^y = z^{2n}$, where q is an odd prime, as stated in Theorem 5. Similarly, Mina and Bacani provided a remark on all solutions to $p^x + q^y = z^{2n}$, where p and q are odd primes with $p \equiv q \pmod 4$, as stated in Theorem 6. However, the explicit solutions for Theorem 5 were not provided. Therefore, we present Proposition 13 to determine the explicit solutions in the cases where either x or y is even. We further extend the primes 2 and q to the distinct primes p and q in Theorem 8, where all explicit solutions are given, including the case where $p \not\equiv q \pmod 4$.

3. Results

In this section, we deliver our main result–Theorem 8–with Proposition 7 as the prelude.

3.1. Main results

From this section on, we concentrate on identifying all solutions to the equation

$$p^x + q^{2y} = z^{2n}, \tag{3.1}$$

where p and q are primes with $p \not\equiv q \pmod 4$.

First, we provide the crucial instrument to support the main result, Theorem 8.

Proposition 7. *Let n be a positive integer, and p and q be two distinct primes. If $\alpha \geq 2$ such that $\gcd(p, \alpha) = 1$, then all solutions of the equation $p^x + q^{\alpha y} = z^{\alpha n}$ are of the form $(p, x, q, y, z, n) \in A \cup B \cup C$, where*

- $A = \left\{\left(p, \log_p((q^y + 1)^\alpha - q^{\alpha y}), q, y, q^y + 1, 1\right) : \log_p((q^y + 1)^\alpha - q^{\alpha y}) \text{ is an integer}\right\}$;
- $B = \left\{\left(p, \log_p\left(3^{2\alpha} - 2^{3\alpha}\right), 2, 3, 3, 2\right) : \log_p\left(3^{2\alpha} - 2^{3\alpha}\right) \text{ is an integer}\right\}$;
- $C = \left\{\left(p, \log_p(2^{n\alpha} - (2^n - 1)^\alpha), 2^n - 1, 1, 2, n\right) : n \text{ is prime and } \log_p(2^{n\alpha} - (2^n - 1)^\alpha) \text{ is an integer}\right\}$.

In particular, the equation has at most one solution for fixed p, q and $n \neq 1$.

Proof. If $x = 0$, then we apply Theorem 3 to the equation $1 + q^{\alpha y} = z^{\alpha n}$ and its solutions are

$$(q, \alpha y, z, \alpha n) = (2, 3, 3, 2), (2^{\alpha n} - 1, 1, 2, \alpha n).$$

However, both of the solutions lead us to contradict the assumptions that $\alpha \geq 2$ and integer y. Hence, there is no solution in the case where $x = 0$. Now, we assume that $x \neq 0$ and we consider the equation $p^x = z^{\alpha n} - q^{\alpha y}$ by separating into two cases.

Case 1: $z^n - q^y \neq 1$. With the assumptions that p is a prime, $x \neq 0$, and the fact that

$$p^x = z^{\alpha n} - q^{\alpha y} = (z^n - q^y) \sum_{i=0}^{\alpha-1} z^{in} q^{(\alpha-i)y},$$

we have $z^n \equiv q^y \pmod p$. Next, we consider

$$1 < q^{\alpha y} + z^n q^{(\alpha-1)y} \leq \sum_{i=0}^{\alpha-1} z^{in} q^{(\alpha-i)y} = p^\gamma \quad \text{for some } 1 \leq \gamma < x.$$

Then,

$$0 \equiv p^\gamma = \sum_{i=0}^{\alpha-1} z^{in} q^{(\alpha-i)y} \equiv \sum_{i=0}^{\alpha-1} q^{iy} q^{(\alpha-i)y} \equiv \sum_{i=0}^{\alpha-1} q^{\alpha y} \equiv \alpha q^{\alpha y} \pmod p.$$

Since p and q are distinct primes, we obtain $p \mid \alpha$, which contradicts $\gcd(p, \alpha) = 1$. Hence, there is no solution in this case.

From this point onward, all solutions in this proposition are provided for Case 2 only.
Case 2: $z^n - q^y = 1$.
Case 2.1: $n = 1$. We have $z = q^y + 1$, and this implies that

$$p^x = z^\alpha - q^{\alpha y} = (q^y + 1)^\alpha - q^{\alpha y}.$$

Hence, in a straightforward manner, $(p, x, q, y, z, n) = \left(p, \log_p\left((q^y + 1)^\alpha - q^{\alpha y}\right), q, y, q^y + 1, 1\right)$.
Case 2.2: $n \geq 2$. Applying Theorem 3 with the equation $z^n = q^y + 1$, we have $(q, y, z, n) = (2, 3, 3, 2)$, $(2^n - 1, 1, 2, n)$, where n is a prime. Hence, the solutions of $p^x + q^{\alpha y} = z^{\alpha n}$ are

$$(p, x, q, y, z, n) = \left(p, \log_p\left(9^\alpha - 8^\alpha\right), 2, 3, 3, 2\right), \left(p, \log_p\left(2^{n\alpha} - (2^n - 1)^\alpha\right), 2^n - 1, 1, 2, n\right),$$

where n is a prime. □

In this context, we focus on the case where $p \not\equiv q \pmod 4$. For conciseness and comprehensiveness of the references, we state Theorem 8 for two distinct primes p and q. However, from the results of this theorem, the sets E and F clearly contain the solutions for the case where $p \not\equiv q \pmod 4$, though not all solutions are included. The remaining solutions are in $A \cup B \cup C$. Therefore, $A \cup B \cup C$ encompasses the case where either p or q equals 2. This indicates that the work in [8] specifically applies to our case. Additionally, we provide explicit solutions that are not covered in [8]. For the case where $p \equiv q \pmod 4$, the result is in set D of Theorem 8 following from Theorem 6.

Theorem 8. *Let p and q be two distinct primes and n be a positive integer. All solutions to the equation*

$$p^x + q^{2y} = z^{2n}$$

are of the form $(p, x, q, y, z, n) \in A \cup B \cup C \cup D \cup E \cup F$, *where*

- $A = \{(2, 5, 7, 1, 3, 2), (3, 2, 2, 2, 5, 1), (3, 5, 11, 2, 122, 1), (7, 1, 3, 1, 2, 2), (17, 1, 2, 3, 3, 2)\}$;
- $B = \{(3, 1, q, 0, 2, 1)\} \cup \left\{\left(2^{2^\ell} + 1, 1, 2, 2^\ell - 1, 2^{2^{\ell-1}} + 1, 1\right) : \ell \in \mathbb{N}\right\}$;
- $C = \{(2, 3, q, 0, 3, 1), (2, 2 + \log_2(q + 1), q, 1, q + 2, 1) : q \text{ is an odd prime}\}$;
- $D = \{(2q^y + 1, 1, q, y, q^y + 1, 1) : q \equiv 3 \pmod 4 \text{ and } y \geq 1\}$;
- $E = \{(3, \log_3(2q^y + 1), q, y, q^y + 1, 1) : q \equiv 1 \pmod{12} \text{ and } y \text{ is odd}\}$;
- $F = \left\{\left(p, \log_p(2q^y + 1), q, y, q^y + 1, 1\right) : \begin{array}{l} p \equiv 3 \pmod 4 \text{ with } p \neq 3, q \equiv 1 \pmod 4 \\ \text{and } p \equiv 1 \pmod q, y \geq 1 \end{array}\right\}$,

where all logarithmic values in each set must be integers. In particular, if $n \neq 1$, *this equation has at most one solution for fixed* p, q.

Proof. First, we consider $p = 2$ with the equation

$$2^x = (z^n - q^y)(z^n + q^y). \tag{3.2}$$

Then, following the idea from [8, Eq (14)], we get

$$2q^y = 2^k(2^{x-2k} - 1).$$

Since $q \neq 2$, we get $k = 1$ and thus $2^{x-2} - q^y = 1$. If $x - 2 = 1$, we get $q^y = 1$ and thus $y = 0$. Hence, the solution is

$$(p, x, q, y, z, n) = (2, 3, q, 0, 3, 1).$$

In the case where $x-2 \geq 2$, if $y \geq 2$, by Theorem 3, the equation $2^{x-2} - q^y = 1$ has no solution. From this observation and the fact that the existence of Eq (3.2) is equivalent to the equation $2^{x-2} - q^y = 1$, so we can conclude that $2^x + q^{2y} = z^{2n}$ has no solution. In the case where $y = 0$, we have $x = 3$, which contradicts $x \geq 4$. Now, it remains to consider the case $y = 1$. Again, we are back to Eq (3.2) and thus we obtain $2^{x-2} - q = 1$. Hence, we have

$$(p, x, q, y, z, n) = \left(2, 2 + \log_2 (q+1), q, 1, (q+2)^{\frac{1}{n}}, n\right).$$

If $n \geq 2$, it is easy to see that $2^k - 1 = z^n - 2$, where $k = \log_2 (q+1)$. Then $2^k + 1 = z^n$. Applying Theorem 3, we get

$$(p, x, q, y, z, n) = (2, 5, 7, 1, 3, 2).$$

Consequently, all solutions are

$$(p, x, q, y, z, n) = (2, 2 + \log_2 (q+1), q, 1, q+2, 1), (2, 5, 7, 1, 3, 2),$$

and then we have the set

$$C = \{(2, 3, q, 0, 3, 1), (2, 2 + \log_2 (q+1), q, 1, q+2, 1) : q \text{ is an odd prime}\}.$$

Next, we assume that $p \neq 2$. If $n \geq 2$, then it follows directly that the solutions of the equation $p^x + q^{2y} = z^{2n}$ are $(p, x, q, y, z, n) \in \left\{(17, 1, 2, 3, 3, 2), \left(p, \log_p \left(2^{n+1} - 1\right), 2^n - 1, 1, 2, n\right) : n \text{ is a prime}\right\}$ which are from the set B and C in Proposition 7. Note that $x = \log_p \left(2^{n+1} - 1\right)$ must be an integer and thus the Diophantine equation $p^x + 1 = 2^{n+1}$ has a solution. Applying Theorem 3, the unique solution is $(p, x, n) = \left(2^{n+1} - 1, 1, n\right)$. Since $2^n - 1$ and $2^{n+1} - 1$ are primes, we get $n = 2$. So, under the condition that $x = \log_p \left(2^{n+1} - 1\right)$ is an integer, we can simplify a solution $(p, x, q, y, z, n) = (7, 1, 3, 1, 2, 2)$. Hence, for this case, we get all the solutions

$$(p, x, q, y, z, n) = (17, 1, 2, 3, 3, 2), (7, 1, 3, 1, 2, 2).$$

Finally, we investigate the case of $n = 1$ to complete the proof. With the case of $p \neq 2$, Eq (3.2) becomes

$$p^x = (z - q^y)(z + q^y). \tag{3.3}$$

This induces us to separate the proof into two cases as follows.

Case 1: $q = 2$. Here, Eq (3.3) becomes $p^x + 2^{2y} = z^2$. By Proposition 7, the solutions are $(p, x, y, z) = \left(p, \log_p \left(2^{y+1} + 1\right), y, 2^y + 1\right)$. It suffices to determine all values y such that $x = \log_p \left(2^{y+1} + 1\right)$ is an integer. This means that we have to solve the equation $2^{y+1} + 1 = p^x$. Applying Theorem 4 for the case $y + 1 \geq 2$, we find that all solutions of this equation are

$$(p, x, y+1) = (3, 2, 3), \left(2^{2^\ell} + 1, 1, 2^\ell\right),$$

where $\ell \in \mathbb{N}$. For the case $y + 1 = 1$, the solution includes in the case $(p, x, y+1) = \left(2^{2^\ell} + 1, 1, 2^\ell\right)$ when $\ell = 0$. These imply that all the solutions in this case are

$$(p, x, q, y, z, n) = (3, 2, 2, 2, 5, 1), \left(2^{2^\ell} + 1, 1, 2, 2^\ell - 1, 2^{2^{\ell}-1} + 1, 1\right),$$

where $\ell \in \mathbb{N}_0$.

Case 2: q is an odd prime. We consider Eq (3.3) in two cases as follows.

Case 2.1: If $z - q^y \neq 1$, then $z - q^y = p^k$ and $z + q^y = p^{x-k}$ for some $k \geq 1$. Consequently, $2q^y \equiv 0 \pmod{p}$, and we have $p \mid 2$ or $p \mid q$. This leads to a contradiction, since $p \neq 2$ and q are distinct primes.

Case 2.2: If $z - q^y = 1$, then $x = \log_p (2q^y + 1)$. Since p is an odd prime, $p \equiv 1, 3 \pmod{4}$. If $p \equiv 1 \pmod{4}$, then q is even, which is a contradiction. Thus, $p \equiv 3 \pmod{4}$. Moreover, x must be odd; otherwise $x = 2k$ for some $k \in \mathbb{N}$, and thus $2q^y = p^{2k} - 1 \equiv 0 \pmod{4}$, which is a contradiction. Hence,

$$(p, x, q, y, z, n) = \left(p, \log_p (2q^y + 1), q, y, q^y + 1, 1\right),$$

where $p \equiv 3 \pmod{4}$ and q is an odd prime. From now on, we would like to simplify the term $\log_p (2q^y + 1)$ to the solution above.

First, if $y = 0$, then the solution (p, x, q, y, z, n) is $(3, 1, q, 0, 2, 1)$. Thus, the set

$$B = \{(3, 1, q, 0, 2, 1)\} \cup \left\{\left(2^{2^\ell} + 1, 1, 2, 2^\ell - 1, 2^{2^{\ell}-1} + 1, 1\right) : \ell \in \mathbb{N}\right\}$$

has been done already. Next, we consider $y \geq 1$.

Case 2.2.1: $q \equiv 3 \pmod{4}$. Then, by Theorem 6, all solutions of the equation $z^2 = p^x + q^{2y}$ are in the form

$$(p, x, q, y, z, n) = (3, 5, 11, 2, 122, 1) \text{ or } (2q^y + 1, 1, q, y, q^y + 1, 1).$$

So, both of the sets

$$A = \{(2, 5, 7, 1, 3, 2), (3, 2, 2, 2, 5, 1), (3, 5, 11, 2, 122, 1), (7, 1, 3, 1, 2, 2), (17, 1, 2, 3, 3, 2)\},$$

and

$$D = \{(2q^y + 1, 1, q, y, q^y + 1, 1) : q \equiv 3 \pmod{4} \text{ and } y \geq 1\}$$

are completely collected. The remaining sets E and F are explored in the following case.

Case 2.2.2: $q \equiv 1 \pmod{4}$. Recall that x is odd and $z - q^y = 1$. By Eq (3.3), we have

$$2q^y = p^x - 1 = (p - 1)(p^{x-1} + \cdots + p + 1), \tag{3.4}$$

and this leads us to conclude that $p \equiv 1 \pmod{q}$ if $p > 3$. If $p = 3$, then Eq (3.4) becomes $3^x = 2q^y + 1$ and it has no solution in the case where y is even by Theorem 2. Furthermore, for $q \equiv 1 \pmod{4}$, $\left(\frac{3}{q}\right) = 1$ if and only if $q \equiv 1 \pmod{12}$. Though we can not simplify the logarithmic term, under the condition $q \equiv 1 \pmod{4}$, we have additional information about the necessary condition for the existence of solutions as follows:

$$(p, x, q, y, z, n) = \begin{cases} (3, \log_3 (2q^y + 1), q, y, q^y + 1, 1); y \text{ is odd and } q \equiv 1 \pmod{12} \\ \left(p, \log_p (2q^y + 1), q, y, q^y + 1, 1\right); \begin{array}{l} p > 3,\ p \equiv 1 \pmod{q}, \\ q \equiv 1 \pmod{4} \text{ and } y \geq 1. \end{array} \end{cases}$$

□

Next, we select equations from [8, 11, 12] to apply to Theorem 8 in the remarks.

Remark 1. *Consider the equation*

$$p^x + 4^y = z^{2n},$$

where $p \geq 3$ is a prime and $y \geq 1$. Applying Theorem 8, all solutions of this equation contained in $A \cup B$. We can then see that p must be $3, 17$ or any prime in the form $2^{2^\ell} + 1$ where $\ell \in \mathbb{N}$. On the other hand, this means that the equations

$$7^x + 4^y = z^{2n}, 11^x + 4^y = z^{2n}, 13^x + 4^y = z^{2n} \text{ and } p^x + 4^y = z^{2n}$$

have no solution for any prime $p \in \bigcup_{\ell \in \mathbb{N}} (2^{2^\ell} + 1, 2^{2^{\ell+1}} + 1) = (5, 17) \cup (17, 257) \cup \ldots$.

Additionally, if we would like to use the explicit solutions of the equation

$$2^x + q^y = z^2,$$

from Theorem 5, we must solve for them in the proof of those theorems in [8]. In this regard, Theorem 8 is more convenient to use than Theorem 5 when x or y is even.

Remark 2. *Here, we present examples where we can apply Theorem 8 to solve certain equations.*

(1) The equation $47^x + 7^{2y} = z^2$ has no solution; see [11].
This article closely matches our solutions for the set

$$D = \{(2q^y + 1, 1, q, y, q^y + 1, 1) : q \equiv 3 \pmod 4 \text{ and } y \geq 1\}.$$

We can let $q = 7$, and then we only consider whether the value $47 = 2(7^y) + 1$ is an integer or not, to examine the existence of solutions for the equation $47^x + 7^{2y} = z^2$. However, $y = \log_7 23 \notin \mathbb{N}_0$, so this equation has no solution.

(2) The equation $2^{3x} + 3^{2y} = z^2$ has no solution if $x, y, z \geq 1$; see [12].
With the assumption $x, y, z \geq 1$ for $2^{3x} + 3^{2y} = z^2$, by considering p and q in Theorem 8, we only have $(p, x, q, y, z^n) = \left(2, 2 + \log_2(q + 1), q, 1, (q + 2)^1\right)$ where $q = 3$ from the equation $p^x + q^{2y} = z^2$ in Theorem 8. That is

$$2^{2+\log_2(3+1)} + 3^{2(1)} = (3 + 2)^{2(1)}$$

or

$$2^4 + 3^2 = 5^2$$

which does not match to $2^{3x} + 3^{2y} = z^2$. This leads us to conclude that if $x, y, z \geq 1$, then $2^{3x} + 3^{2y} = z^2$ has no solution. Actually, from the set C in Theorem 8, we know that $(x, y, z) = (1, 0, 2)$ is the only one solution of the equation $2^{3x} + 3^{2y} = z^2$ for $x, y, z \geq 0$.

Although our primary goal is to consider the case where $p \not\equiv q \pmod 4$, an alternative result is suggested by the following propositions.

Proposition 9. *Let $p \equiv 1 \pmod 4$ and q be an odd prime such that $p^{\frac{q-1}{2}} \equiv -1 \pmod q$. If x or y is odd, then the equation $p^x + q^y = z^{2n}$ has no solution.*

Proof. Let $p \equiv 1 \pmod 4$. By (1) and (4) from Theorem 1, the following are equivalent:

- $p^{\frac{q-1}{2}} \equiv -1 \pmod{q}$;
- $\left(\frac{p}{q}\right) = -1$;
- $\left(\frac{q}{p}\right) = -1$;
- $q^{\frac{p-1}{2}} \equiv -1 \pmod{p}$.

Assume that x is odd. Since $-1 = (-1)^x \equiv \left(p^{\frac{q-1}{2}}\right)^x \equiv \left(\frac{p}{q}\right)^x \equiv \left(\frac{p^x}{q}\right) \pmod{p}$, we see that $z^{2n} \equiv p^x + q^y \pmod{p}$ has no solution. Hence, the equation $p^x + q^y = z^{2n}$ has no solution. Similarly, if y is odd, then $-1 = (-1)^y \equiv \left(q^{\frac{p-1}{2}}\right)^y \equiv \left(\frac{q}{p}\right)^y \equiv \left(\frac{q^y}{p}\right) \pmod{p}$. Therefore, the equation $p^x + q^y = z^{2n}$ has no solution. □

The following table (see Table 2) presents equations with no solutions, derived from applying Proposition 9.

Table 2. Equations derived from Proposition 9.

Conditions	Equations
$p \equiv 1 \pmod{4}$ and $q = 7 \equiv 3 \pmod{4}$	$41^{2x+1} + 7^y = z^{2n}$, $73^{2x+1} + 7^y = z^{2n}$, $89^{2x+1} + 7^y = z^{2n}$, $101^{2y+1} + 7^y = z^{2n}$.
$p = 5 \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$	$5^x + 3^{2y+1} = z^{2n}$, $5^x + 7^{2y+1} = z^{2n}$, $5^x + 23^{2y+1} = z^{2n}$, $5^x + 43^{2y+1} = z^{2n}$.

It is well known that the generalized Diophantine equation can be applied in various ways. One such form is $a^x + b^y = c^z$. For instance, in 1956, W. Sierpiński [14] demonstrated that the equation $3^x + 4^y = 5^z$ has no solution for any positive integers x, y, and z, except for the case $x = y = z = 2$. In the same year, L. Jeśmanowicz [15] published an article extending Sierpiński's work by considering other equations, such as $5^x + 12^y = 13^z$, $7^x + 24^y = 25^z$, $9^x + 40^y = 41^z$, and $11^x + 60^y = 61^z$. Notably, the exponential numbers on the right-hand side of these equations, denoted by c, are nearly prime. In 2005, Acu [16] examined three cases:

- $a = b = c = p$;
- $a = b = p$ and $c = 2p$;
- $a = p$, $b = q$, and $c = pq$, where p and q are prime numbers.

Additionally, works from 2015 to 2025, including [5, 17–24], have continued to explore these types of equations.

Lastly, we discuss all solutions of Eq (3.1) when z is restricted to be a prime. We then introduce a powerful theorem first proven by Z. F. Cao in 1986, which was later re-proven by R. Scott and R. Styer [25] in 2004, as no published proof existed for the original version. The theorem is stated as follows.

Theorem 10. *[25] Let p, q be two distinct primes, y be an even positive integer, x be a positive integer, and $n > 1$. The equation $2^x + p^y = q^n$ has only four solutions, namely*

$$2 + 5^2 = 3^3, \quad 2^2 + 11^2 = 5^3, \quad 2^4 + 3^2 = 5^2, \quad 2^5 + 7^2 = 3^4.$$

If we combine our main theorems (Theorems 8 and 10), they yield our last theorem in this section.

Theorem 11. *Let p and q be two distinct primes. If z is a prime, all solutions of Eq (3.1) are $(p, x, q, y, z, n) \in A \cup B \cup C$, where*

- $A = \{(3, 2, 2, 2, 5, 1), (7, 1, 3, 1, 2, 2), (17, 1, 2, 3, 3, 2)\}$;
- $B = \{(3, 1, q, 0, 2, 1), (5, 1, 2, 1, 3, 1)\}$;
- $C = \{(2, 4, 3, 1, 5, 1), (2, 5, 7, 1, 3, 2), (2, 3, q, 0, 3, 1)\}$.

Compared with Theorem 8, if z is a prime, Eq (3.1) has most one solution (x, y, z) for fixed p, q, n, except $(p, q, n) = (2, 3, 1), (3, 2, 1)$. For each exceptional case, there are exactly two solutions.

Proof. We will conclude all solutions of the equation $p^x + q^{2y} = z^{2n}$ when we restrict the condition z to be a prime. For $p \neq 2$, by Theorem 8, all solutions of the equation $p^x + q^{2y} = z^{2n}$ are $(p, x, q, y, z, n) \in A \cup B \cup D \cup E \cup F$ where

- $A = \{(3, 2, 2, 2, 5, 1), (3, 5, 11, 2, 122, 1), (7, 1, 3, 1, 2, 2), (17, 1, 2, 3, 3, 2)\}$;
- $B = \{(3, 1, q, 0, 2, 1)\} \cup \left\{\left(2^{2^\ell} + 1, 1, 2, 2^\ell - 1, 2^{2^{\ell-1}} + 1, 1\right) : \ell \in \mathbb{N}\right\}$;
- $D = \{(2q^y + 1, 1, q, y, q^y + 1, 1) : q \equiv 3 \pmod 4 \text{ and } y \geq 1\}$;
- $E = \{(3, \log_3(2q^y + 1), q, y, q^y + 1, 1) : q \equiv 1 \pmod{12} \text{ and } y \text{ is odd}\}$;
- $F = \left\{\left(p, \log_p(2q^y + 1), q, y, q^y + 1, 1\right) : \begin{array}{l} p \equiv 3 \pmod 4 \text{ with } p \neq 3, q \equiv 1 \pmod 4 \\ \text{and } p \equiv 1 \pmod q, y \geq 1 \end{array}\right\}$.

In the set A, $(3, 5, 11, 2, 122, 1)$ is impossible, since 122 is not a prime. Hence,

$$A = \{(3, 2, 2, 2, 5, 1), (7, 1, 3, 1, 2, 2), (17, 1, 2, 3, 3, 2)\}.$$

In the set B, since $z = 2^{2^{\ell-1}} + 1$ is prime and $2^\ell - 1$ is odd, we obtain $z = 3$ and $\ell = 1$. Hence,

$$B = \{(3, 1, q, 0, 2, 1)\} \cup \{(5, 1, 2, 1, 3, 1)\}.$$

In the sets D, E, and F, since $z = q^y + 1$ is an even prime, we get $z = 2$, and then $y = 0$, which is impossible.

For $p = 2$, by Theorem 10, for $x \neq 0$ and $y \neq 0$, we get $(p, x, q, y, z, n) = (2, 4, 3, 1, 5, 1)$ or $(2, 5, 7, 1, 3, 2)$. For $x = 0$, by Theorem 3, there is no solution. Finally, for $y = 0$, by Theorem 3 again, $(p, x, q, y, z, n) = (2, 3, q, 0, 3, 1)$. Hence, $C = \{(2, 4, 3, 1, 5, 1), (2, 5, 7, 1, 3, 2), (2, 3, q, 0, 3, 1)\}$. □

Table 3 highlights the key features of Theorem 11 under the condition that z is a prime, which ensures that only eight forms of the equations have solutions. This implies that the other equations of these eight forms have no solutions. For example, the equations $2^x + 49^y = 5^{2n}$, $3^x + 4^y = 7^{2n}$, $7^x + 9^y = 11^{2n}$, and $11^x + 169^y = 17^{2n}$ do not have solutions, and we state the result in Corollary 12.

Table 3. Eight equations derived from Theorem 11.

Equations	Solutions
$2^x + 9^y = 5^{2n}$	$(x, y, n) = (4, 1, 1)$ is the only one solution.
$2^x + 49^y = 3^{2n}$	$(x, y, n) = (5, 1, 2)$ is the only one solution.
$2^x + q^{2y} = 3^{2n}$	$(x, y, n) = (3, 0, 1)$ is the only one solution for any prime q.
$3^x + 4^y = 5^{2n}$	$(x, y, n) = (2, 2, 1)$ is the only one solution.
$3^x + q^{2y} = 2^{2n}$	$(x, y, n) = (1, 0, 1)$ is the only one solution for any prime q.
$5^x + 4^y = 3^{2n}$	$(x, y, n) = (1, 1, 1)$ is the only one solution.
$7^x + 9^y = 2^{2n}$	$(x, y, n) = (1, 1, 2)$ is the only one solution.
$17^x + 4^y = 3^{2n}$	$(x, y, n) = (1, 3, 2)$ is the only one solution.

Corollary 12. *Let p, q be two distinct primes and z be a prime such that $\{2, 3\} \nsubseteq \{p, q, z\}$. Then Eq (3.1) has no solution.*

Proof. This corollary follows directly from Theorem 11. $\square$

3.2. Consequences of Theorem 8

In this section, we focus on the equation

$$2^x + q^y = z^{2n}, \tag{3.5}$$

where q is an odd prime. This equation was previously considered in [8], which is restated as Theorem 5 in Section 2. However, the explicit form of the solutions was not clearly established. With our main theorem, we have addressed almost all cases, except for those where both x and y are odd numbers, as shown in the following proposition.

Proposition 13. *Let x be even or let y be even. All solutions of Eq (3.5) are as follows:*

(1) If $n = 1$, then $(x, q, y, z) \in A \cup B \cup C \cup D$, where

- $A = \{(0, 3, 1, 2), (4, 3, 2, 5), (5, 7, 2, 9)\}$;
- $B = \left\{\left(2^{\ell+1} - 2, 2^{2^\ell} + 1, 1, 2^{2^\ell - 1} + 1\right) : \ell \in \mathbb{N}\right\}$;
- $C = \{(3, q, 0, 3) : q \text{ is an odd prime}\}$;
- $D = \left\{\left(2 + k, 2^k - 1, 2, 2^k + 1\right) : 2^k - 1 \text{ is a Mersenne prime with } k \geq 3\right\}$.

(2) If $n \geq 2$, then $(x, q, y, z, n) \in E = \{(5, 7, 2, 3, 2), (6, 17, 1, 3, 2)\}$.

Proof. This follows directly from Theorem 8 by focusing only on the sets A, B, and C. $\square$

By applying Proposition 13, we can solve the equations $p^x + (p + 1)^y = z^2$ [2] and $2^x + 7^y = z^2$ [4], in Table 1, as follows.

Corollary 14. *The equation*

$$p^x + (p + 1)^y = z^2$$

has only three solutions $(p, x, y, z) = (3, 1, 0, 2), (3, 2, 2, 5)$, and $(7, 0, 1, 3)$ where $p = 2^t - 1$ is a Mersenne prime.

Proof. Since $p = 2^t - 1$ is a Mersenne prime, we get $t \geq 2$. If both x and y are odd, then $z^2 = p^x + (p + 1)^y \equiv 3 \pmod 4$, which is impossible. Then x is even or y is even. Applying Proposition 13 to the equation

$$z^2 = p^x + (p + 1)^y = (2^t - 1)^x + 2^{ty},$$

where $t \geq 2$, we have only seven possibilities as follows:

Proposition 13	$(ty, 2^t - 1, x, z)$	The value of t	(p, x, y, z)
A	$(0, 3, 1, 2)$	2	$(3, 1, 0, 2)$
A	$(4, 3, 2, 5)$	2	$(3, 2, 2, 5)$
A, E	$(5, 7, 2, 9)$	Impossible	
B	$\left(2^{\ell+1} - 2, 2^{2^\ell} + 1, 1, 2^{2^{\ell}-1} + 1\right)$	Impossible	
C	$(3, q, 0, 3)$	3	$(7, 0, 1, 3)$
D	$\left(2 + k, 2^k - 1, 2, 2^k + 1\right)$	2	$(3, 2, 2, 5)$
E	$(6, 17, 1, 9)$	Impossible	

Hence, $p^x + (p + 1)^y = z^2$ has exactly three solutions. □

Corollary 15. *The equation*

$$2^x + 7^y = z^2$$

has only two solutions $(x, y, z) = (3, 0, 3), (5, 2, 9)$ *where* $x \neq 1$.

Proof. Assume that $x \neq 1$. For an odd x and an odd y, we have $z^2 = 2^x + 7^y \equiv 3 \pmod 4$, which is impossible. Hence, x is even or y is even. Then, focusing the value $q = 7$ in Proposition 13, all solutions will be in the sets A, C, D with $k = 3$, and E. □

The following proposition displays all explicit solutions of the results in [8] for $q \not\equiv 1 \pmod 8$.

Proposition 16. *All solutions of Eq (3.5) are as follows:*

(1) If $q \equiv 3 \pmod 8$*, then all solutions are* $(x, q, y, z, n) = (0, 3, 1, 2, 1), (4, 3, 2, 5, 1), (3, q, 0, 3, 1)$.

(2) If $q \equiv 5 \pmod 8$*, then all solutions are* $(x, q, y, z, n) = (2, 5, 1, 3, 1), (3, q, 0, 3, 1)$.

(3) If $q \equiv 7 \pmod 8$*,* $x \neq 1$*, and* y *is an odd positive integer, then the equation has no solution.*

In other words, if $q \equiv 7 \pmod 8$*,* y *is an odd positive integer, and the equation has a solution, then* $x = 1$.

(4) If $q \equiv 7 \pmod 8$ *and* y *is an even positive integer, then all solutions are* $(x, q, y, z, n) = (5, 7, 2, 9, 1), (3, q, 0, 3, 1), \left(2 + k, 2^k - 1, 2, 2^k + 1, 1\right), (5, 7, 2, 3, 2)$*, where* $2^k - 1$ *is a Mersenne prime with* $k \geq 3$.

Proof. To prove the first two statements, we assume that $q \equiv \pm 3 \pmod 8$. If $x = 0$, then $1 + q^y = z^{2n}$, and then $q^y = (z^n - 1)(z^n + 1)$. It forces the case that $2 = q^k(q^{y-2k} - 1)$ for some $k \in \mathbb{N}_0$. If $k \geq 1$, then $q^k(q^{y-2k} - 1) \geq 3$, which is impossible. Thus, $k = 0$ and thus $(x, q, y, z, n) = (0, 3, 1, 2, 1)$. If $x = 1$, then $2 + q^y = z^{2n}$, which implies that z is odd, and thus $q^y = z^{2n} - 2 \equiv 7 \pmod 8$, which contradicts $q \equiv \pm 3 \pmod 8$. If $x = 2$, then $4 + q^y = z^{2n}$, and then $q^y = (z^n - 2)(z^n + 2)$. It follows that $4 = q^k(q^{y-2k} - 1)$ for some $k \in \mathbb{N}_0$. If $k \geq 1$, then $q \mid 4$, which is a contradiction. Thus, $k = 0$ and hence, $(x, q, y, z, n) = (2, 5, 1, 3, 1)$. If $x \geq 3$, then z is odd and then $q^y = z^{2n} - 2^x \equiv 1 \pmod 8$. This means that y is even. Applying Theorem 8, we obtain

$$(x, q, y, z, n) = (3, q, 0, 3, 1), (2 + \log_2 (q + 1), q, 2, (q + 2), 1).$$

Since $\log_2(q+1)$ must be an integer, it implies that $q = 2^a - 1$ for some $a \in \mathbb{N}_0$. If $a \geq 3$, then $q \equiv -1 \pmod 8$, which is a contradiction. So a has to be $0, 1$, or 2. Since q is a prime, we get $q = 3$. Thus, $(x, q, y, z, n) = (3, q, 0, 3, 1), (4, 3, 2, 5, 1)$.

Next, we assume that $q \equiv 7 \pmod 8$ and y is odd. If $x \geq 2$, then, $z^{2n} = 2^x + q^y \equiv 0 + 3 \pmod 4$, which is a contradiction. In the case where $x = 0$, the equation becomes to $1 + q^y = z^{2n}$. By Theorem 3, this equation has no solution and then we have (3), as desired.

Finally, assume that $q \equiv 7 \pmod 8$ and y is even. By Proposition 13, all solutions follow from the sets A, C, D, and E. □

Remark 3. *From Proposition 16, we provide the open problem for the case $2^x + q^y = z^{2n}$, where $q \equiv 1 \pmod 8$.*

4. Conclusions and discussions

To summarize, we investigate the equation

$$p^x + q^y = z^{2n},$$

where x, y, z are non-negative integers as follows.

(1) The explicit solutions of $2^x + q^y = z^{2n}$, where q is any odd prime and xy is an even integer (see Proposition 13).

(2) All solutions of $p^x + q^{2y} = z^{2n}$, where p and q are two distinct primes and $n \in \mathbb{N}$ (see Theorem 8).

(3) $p^x + q^y = z^{2n}$ has no solution if $p \equiv 1 \pmod 4$, $p^{\frac{q-1}{2}} \equiv -1 \pmod q$, q is an odd prime, and x or y is odd (see Proposition 9).

(4) All solutions of $p^x + q^{2y} = z^{2n}$, where p, q, and z are prime numbers with $p \neq q$ (see Theorem 11).

(5) $p^x + q^{2y} = z^{2n}$ has no solution if p, q, and z are prime numbers with $\{2, 3\} \not\subseteq \{p, q, z\}$ (see Corollary 12).

We aim to contribute by identifying all explicit solutions to the equation $p^x + q^y = z^{2n}$, where p and q are primes and $x, y, z \in \mathbb{N}_0$, $n \in \mathbb{N}$. Some cases for p and q have been addressed in [8, 10]. However, in [8], the results for the equation $2^x + q^y = z^2$ are presented in terms of the existence or non-existence of solutions. In contrast, we seek to provide the exact solutions for $2^x + q^y = z^2$, as detailed in Section 3.2. We have successfully solved almost all cases, except when both x and y are odd.

Building on this foundation, we now shift our focus to a broader class of equations, specifically $p^x + q^y = z^{2n}$, where p and q are distinct odd prime numbers. To gain deeper insights into this equation, we categorize the analysis into two primary cases based on the congruence of p and q modulo 4.

To analyze the equation $p^x + q^y = z^{2n}$, where p and q are distinct odd prime numbers, we consider two cases based on their congruence modulo 4.

Case 1: $p \equiv q \pmod 4$.

All solutions for this case are completely known for $n = 1$ (Theorem 6 or [10]). Applying Theorems 3 and 6, it is not difficult to show that

$$(p, x, q, y, z, n) = \left(2^n - 1, 2, 2^{n+1} - 1, 1, 2, n\right) = (3, 2, 7, 1, 2, 2)$$

is a unique solution for $n \geq 2$.

Case 2: $p \not\equiv q \pmod 4$.

When xy is even, Theorem 8 can be applied directly. For an odd xy, we investigate the nature of solutions—whether none, finitely many, or infinitely many exist. In Proposition 9, assuming $p \equiv 1 \pmod 4$, we establish the following: If $p^{\frac{q-1}{2}} \equiv -1 \pmod q$ with x being odd or with y being odd, then the equation has no solution.

Under the condition that xy is odd and $p \equiv 1 \pmod 4$, which $p^{\frac{q-1}{2}} \not\equiv -1 \pmod q$ does not satisfy, we reveal the intriguing behaviors of the solutions in the equation $p^x + q^y = z^{2n}$, as shown by the following examples.

No solution: The equation $173^x + 19^y = z^2$ has no solution for an odd xy. In fact, the Diophantine equation $n^x + 19^y = z^2$ admits only one solution, $(n, x, y, z) = (2, 3, 0, 3)$, under the condition $n \equiv 2 \pmod{57}$ with n being a positive integer and x, y, z being non-negative integers (see [26]).

Solutions exist: Interestingly, the equation $5^x + 11^y = z^2$ has at least three solutions, and at least two of them satisfy the condition of an odd xy, as discussed in [27, Theorem 3.2].

These contrasting outcomes—ranging from no solutions to finitely or even infinitely many—underscore the delicate interplay between the primes involved and their congruence properties. Therefore, the case where xy is odd presents intriguing challenges that merit further exploration. This motivates our focus on the special case where $p \not\equiv q \pmod 4$ and xy is even, providing a more tractable framework for analysis while leaving exciting directions open for future research.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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