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Research article

Estimating the stress-strength reliability parameter of the inverse power Lomax distribution

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Abstract: This research focused on estimating the stress-strength parameter by considering stress and strength as distinct random variables, both characterized by the inverse power Lomax (IPL) distribution. The maximum likelihood estimate (MLE) for stress-strength reliability was then calculated using the Newton-Raphson method. Using the asymptotic normality of MLEs, this study developed approximate confidence intervals. Bootstrap confidence intervals for the stress-strength reliability parameter (R) were investigated. The Bayes estimator of R was considered. Furthermore, we utilized the Markov chain Monte Carlo (MCMC) method to create both symmetric and asymmetric loss functions, allowing for a more comprehensive analysis. The highest posterior density (HPD) credible intervals under a gamma prior distribution were calculated. The different approaches were assessed using a Monte Carlo simulation. Finally, a numerical example was given to show the effectiveness of the proposed methods.

Keywords: maximum likelihood estimator; the inverse power Lomax model; stress-strength reliability; symmetric and asymmetric loss functions; bootstrap resampling; Bayes estimator
Mathematics Subject Classification: 60E05, 62F10, 62F15

1. Introduction

Studies have indicated that certain parts or equipment endure because of their durability. These gadgets can withstand a given amount of stress, but when more force is placed on them, they malfunction because they are unable to handle it. The likelihood that these components will function properly under specified conditions and at a given stress level is what is known as reliability. Reliability engineering studies use this probability to control, assess, and estimate a device's capability and lifetime. The stress-strength model defines a component's life as follows: If stress surpasses strength, the component will fail. The reliability of single-component systems involving strength under stress is an issue shared by many aspects of agriculture engineering, biology, and medicine. Stress-strength model has numerous applications and can be used to quantify the differences between two populations.

In statistical literature, estimating a function $R = P(Y < X)$, considering the independent distributions of X and Y, is a frequently encountered problem. [1] discussed the estimation problem based on complete samples. While central to system reliability, the metric $P(Y < X)$ finds significant application in diverse fields. In biometrics, it allows for comparing treatment outcomes: If X and Y denote patient longevity under drugs A and B, a patient's decision may be informed by whether $P(Y < X)$ deviates from 0.5. In statistical tolerance analysis, $P(Y < X)$ quantifies the probability of successful assembly, such as a bearing (diameter Y) fitting a shaft (diameter X) without interference. Consequently, the study and inference of $P(Y < X)$ are of considerable importance.

A review of the stress-strength model in reliability was given by [2]. Using censored samples, [3] investigated the situation in which X and Y are independent Burr Type XII random variables. In the case where X and Y are two independent random variables with a Burr Type X distribution, [4] looked into this subject. [5] investigated estimating the stress-strength reliability model using a finite mixture of two-parameter Lindley distributions. The estimation of R under Type II progressive censoring for an exponential distribution was derived by [6]. In order to establish the asymptotic distribution of the MLE for the generalized exponential distribution, [7] assumed that X and Y were independently distributed. [8] calculated R using logistic and Laplace distributions. The estimate of R for an instance in which X and Y are Burr-X random variables was determined by [9]. [10] examined the class of lifespan distributions, focusing specifically on the gamma and exponential distributions. The estimator was developed by [11] for the situation in which X and Y exhibit independence and both fit into a three-parameter Weibull distribution with different scale parameters but the same shape and location parameters. Based on data that was progressively censored using Type-II, [12] estimated the stress-strength reliability for the alpha power exponential distribution. Using the generalized variable technique, [13] created interval estimation algorithms for R, particularly for the Weibull distribution. The estimation of R was discussed by [14] in relation to the proportional odds ratio model. The estimation of R where X and Y are the minimum of two exponential samples was studied by [15].

Rostamian and Nematollahi [16] studied stress-strength using the inverse Gaussian distribution under Type-II censoring. Ghanbari et al. [17] extended this to the Marshall–Olkin model, and Asadi and Panahi [18] applied it to coating reliability. Hu and Ren [19] used the inverse Weibull with adaptive censoring, while Elbatal et al. [20] introduced a binomial removal scheme for broader inference. Temraz [21] used the Exponentiated Generalized Marshall–Olkin-G distribution for skewed data. Xavier et al. [22] and Hassan et al. 23 focused on system reliability using Kumaraswamy-based models. Recently, Xu et al. [24] incorporated stress-strength concepts into federated learning by proposing an adaptive sampling strategy for predicting aircraft engine life, and Xu et al. [25] modeled heavy-tailed degradation paths using a multivariate Student-t process, aiding stress-strength analysis in multicomponent systems.

As highlighted in the cited references, previous studies have primarily focused on the MLE as a frequentist approach, along with Bayesian methods, for estimating R to evaluate the stress-strength reliability of the IPL distribution. We systematically analyze the performance of these estimators under varying sample sizes and parameter values, aiming to establish practical guidelines for selecting the most suitable method. Our findings are particularly relevant for applied statisticians and reliability engineers seeking robust estimation strategies.

Through extensive simulation studies and real-data analysis, we demonstrate that Bayesian methods can provide more desirable estimates than MLE, reinforcing their applicability in practical settings. Notably, this study is the first comprehensive effort to compare distinct estimation methods for IPL stress-strength reliability, offering new insights for the field. In terms of real data analysis, they showed that the IPL distribution has more flexibility than some well-known distributions like Weibull, Burr XII, and gamma distributions in modeling various types of data.

We consider independent random variables X and Y that follow a three-parameter IPL distribution, defined by the probability density function (pdf) and its cumulative distribution function (cdf). Our objective in this paper is to estimate the quantity $P(Y < X)$.

$$g(x;\gamma,\nu,\xi) = \gamma\nu\xi^{-1}x^{-\nu-1}(1+\xi^{-1}x^{-\nu})^{-(\gamma+1)}, \quad x \ge 0,\ \gamma,\nu,\xi > 0, \tag{1.1}$$

$$G(x;\gamma,\nu,\xi) = (1+\xi^{-1}x^{-\nu})^{-\gamma}, \quad x \ge 0,\ \gamma,\nu,\xi > 0. \tag{1.2}$$

Here ξ is the scale parameter and γ, ν are shape parameters. Studying situations when a failure rate that is not monotonic is genuinely present, the IPL model can be applied to numerous real-world data modeling and analysis scenarios, see [26], which studied several statistical features of the IPL distribution.

The IPL distribution is particularly adaptable, especially for scenarios involving non-monotonic failure rates. This flexibility makes the IPL model suitable for numerous real-world data modeling and analysis tasks, as demonstrated by Hassan and Abd-Allah [26], who investigated its statistical properties for engineering applications (see also Kumar and Sharma [27], Shi and Shi [28], and Ahmed and Mustafa [29]). The complexity of estimating the stress-strength reliability of the three unknown parameters of the IPL distribution under complete data has hindered research in this area. Specifically, no published studies exist on statistical inference of the stress-strength reliability for the IPL distribution under complete data. Therefore, this work focuses on providing the first treatment of both classical and Bayesian inference for the IPL distribution within the complete framework.

This paper estimates reliability $R = P(Y < X)$ within a stress-strength framework. We consider the case where strength (X) and stress (Y) are independent IPL random variables sharing the same parameters (ξ and ν) but differing in their parameters (γ and η) to obtain the closed form to R. The primary focus is on statistical inference for R under these conditions.

This research aims to analyze complete data from the IPL distribution to estimate the stress-strength parameter. The model parameters and the stress-strength function are estimated using the MLE method. The MLEs are numerically computed using the Newton-Raphson method. Interval estimation is performed using two approximation information matrix methods and the bootstrap method. Bayesian estimators and credible intervals are derived via the Metropolis-Hastings algorithm under squared error and linear-exponential loss functions, assuming independent gamma priors for parameters. Finally, Monte Carlo simulation is used to evaluate the performance of these estimators using mean squared error, average length, and probability coverage.

The paper's remaining portions are arranged as follows: The MLE is presented in Section 2. In Section 3, the bootstrap confidence intervals are covered. In Section 4, Bayes estimates are derived using the MCMC algorithm. Algorithms are developed, and both simulation studies and real data analysis are carried out in Sections 5 and 6. Finally, the conclusion is presented in Section 7.

2. The maximum likelihood estimation

Let X and Y be two distinct, independent IPL random variables with the following parameters: (γ, ν, ξ) and (η, ν, ξ), respectively. R is

$$R = P(Y < X) = \int_0^\infty \gamma\nu\, \xi^{-1} x^{-\nu-1} (1 + \xi^{-1} x^{-\nu})^{-(\gamma+\eta)-1} dx = \frac{\gamma}{\gamma + \eta}. \tag{2.1}$$

To calculate the MLE of R, we deduce the MLEs of γ and η. Assume that there are two random samples from an IPL: Let one be from an IPL(γ, ν, ξ) as $X_1, X_2, \cdots, X_{n_1}$ and let another be from an IPL(η, ν, ξ) as $Y_1, Y_2, \cdots, Y_{n_2}$. Consequently, these are the observed samples' probability functions:

$$L(\underline{x}, \underline{y}; \gamma, \eta, \nu, \xi) \propto \prod_{i=1}^{n_1} \left[\gamma\nu\xi^{-1} x_i^{-\nu-1} (1 + \xi^{-1} x_i^{-\nu})^{-(\gamma+1)}\right] \prod_{i=1}^{n_2} \left[\eta\nu\xi^{-1} y_i^{-\nu-1} (1 + \xi^{-1} y_i^{-\nu})^{-(\eta+1)}\right]. \tag{2.2}$$

The natural logarithm of the likelihood function $\ell(\gamma, \eta, \nu, \xi) = \ln L(\underline{x}, \underline{y}; \gamma, \eta, \nu, \xi)$ is given by:

$$\begin{aligned} \ell(\gamma, \eta, \nu, \xi) \;&=\; n_1 \ln\gamma + n_2 \ln\eta - (n_1 + n_2)(\ln\xi - \ln\nu) - (\nu + 1)\left(\sum_{i=1}^{n_1} \ln x_i + \sum_{i=1}^{n_2} \ln y_i\right) \\ &\quad -(\gamma + 1)\sum_{i=1}^{n_1} \ln(1 + \xi^{-1} x_i^{-\nu}) - (\eta + 1)\sum_{i=1}^{n_2} \ln(1 + \xi^{-1} y_i^{-\nu}). \end{aligned} \tag{2.3}$$

By solving the following set of equations, the MLEs of $\Theta = (\gamma, \eta, \nu, \xi)$ can be obtained:

$$\begin{aligned} \frac{\partial \ell}{\partial \gamma} &= \frac{n_1}{\gamma} - \sum_{i=1}^{n_1} \ln(1 + \xi^{-1} x_i^{-\nu}) = 0, \\ \frac{\partial \ell}{\partial \eta} &= \frac{n_2}{\eta} - \sum_{i=1}^{n_2} \ln(1 + \xi^{-1} y_i^{-\nu}) = 0, \\ \frac{\partial \ell}{\partial \nu} &= \frac{n_1 + n_2}{\nu} - \sum_{i=1}^{n_2} \ln y_i - \sum_{i=1}^{n_1} \ln x_i + (\eta + 1)\sum_{i=1}^{n_2} \frac{\xi^{-1} y_i^{-\nu} \ln y_i}{1 + \xi^{-1} y_i^{-\nu}} + (\gamma + 1)\sum_{i=1}^{n_1} \frac{\xi^{-1} x_i^{-\nu} \ln x_i}{1 + \xi^{-1} x_i^{-\nu}} = 0, \\ \frac{\partial \ell}{\partial \xi} &= -\frac{n_1 + n_2}{\xi} + (\eta + 1)\sum_{i=1}^{n_2} \frac{\xi^{-2} y_i^{-\nu}}{1 + \xi^{-1} y_i^{-\nu}} + (\gamma + 1)\sum_{i=1}^{n_1} \frac{\xi^{-2} x_i^{-\nu}}{1 + \xi^{-1} x_i^{-\nu}} = 0. \end{aligned}$$

The MLEs of the parameters are the solution of the above nonlinear system that insures that the corresponding information matrix is positive definite. The system of four non-linear equations does not have an analytic solution and therefore statistical or mathematical software should be used to solve it numerically.

Consequently, the MLE of R is as follows:

$$\hat{R} = \frac{\hat{\gamma}}{\hat{\gamma} + \hat{\eta}}. \tag{2.4}$$

3. Bayesian estimation

This section presents the Bayes estimates for the unknown parameters and stress-strength parameter, derived from a complete data sample. Within the Bayesian framework, parameters are treated as random variables governed by prior distributions. A fundamental challenge in statistical inference is the selection of an appropriate prior distribution that encapsulates available parameter information. When considering the IPL distribution with all parameters unspecified, the lack of joint conjugate priors necessitates alternative strategies. Consequently, assigning independent gamma priors to the unknown parameters presents a viable approach. The appeal of gamma priors lies in their flexibility, encompassing non-informative priors, and their generality as a distribution family that includes exponential and chi-square distributions. We assume that γ and η have independent gamma priors. Also, the parameters ν and ξ have independent priors. Now, we can build the Bayes estimate (BE) of R. These presumptions are used to calculate the BE of R.

$$\pi(\gamma) \sim \text{Gamma}(c_1, d_1),\ \pi(\eta) \sim \text{Gamma}(c_2, d_2),\ \pi(\nu) = \frac{1}{\nu},\ \pi(\xi) = \frac{1}{\xi},\ \ \nu, \xi > 0, \tag{3.1}$$

where (c_1, d_1) and (c_2, d_2) are known hyperparameters. The prior described above is advantageous due to its greater flexibility and its ability to assume. The joint posterior density is given as

$$\begin{aligned}\pi^*(\gamma,\eta,\nu,\xi) &= \frac{L(\gamma,\eta,\nu,\xi)\,\pi(\gamma)\pi(\eta)\pi(\nu)\pi(\xi)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\gamma,\eta,\nu,\xi)\pi(\gamma)\pi(\eta)\pi(\nu)\pi(\xi)d\gamma d\eta d\nu d\xi}\\ &\propto \gamma^{n_1+c_1-1}\eta^{n_2+c_2-1}\nu^{n_1+n_2-1}\xi^{-(n_1+n_2)-1}e^{-\gamma\left(d_1+\sum_{i=1}^{n_1}\ln(1+\xi^{-1}x_i^{-\nu})\right)}e^{-\eta\left(d_2+\sum_{i=1}^{n_2}\ln(1+\xi^{-1}y_i^{-\nu})\right)}\times\\ &\quad e^{-\nu\left(\sum_{i=1}^{n_1}\ln x_i+\sum_{i=1}^{n_2}\ln y_i\right)}\prod_{i=1}^{n_1}\left(\frac{1}{1+\xi^{-1}x_i^{-\nu}}\right)\prod_{i=1}^{n_2}\left(\frac{1}{1+\xi^{-1}y_i^{-\nu}}\right).\end{aligned} \tag{3.2}$$

A closed-form solution for Eq (3.2) cannot be obtained. Therefore, the MCMC technique to create samples from the posterior distributions is used. The MCMC yields both a point estimate and an interval estimate of the parameters. MCMC can be thought of as an iterative sampling procedure that takes values from the parameter's posterior distributions in the relevant model. Gibbs sampling and the broader Metropolis-Hastings (MH)-within-Gibbs sampler are important subclasses of MCMC techniques. The conditional posterior density functions can be expressed as follows:

$$\pi_\gamma^*(\gamma|\eta,\nu,\xi) \propto \gamma^{n_1+c_1-1}e^{-\gamma\left(d_1+\sum_{i=1}^{n_1}\ln(1+\xi^{-1}x_i^{-\nu})\right)} \tag{3.3}$$

$$\pi_\eta^*(\eta|\gamma,\nu,\xi) \propto \eta^{n_2+c_2-1}e^{-\eta\left(d_2+\sum_{i=1}^{n_2}\ln(1+\xi^{-1}y_i^{-\nu})\right)}, \tag{3.4}$$

$$\pi_\nu^*(\nu|\gamma,\eta,\xi) \propto \nu^{n_1+n_2-1}e^{-\nu\left(\sum_{i=1}^{n_1}\ln x_i+\sum_{i=1}^{n_2}\ln y_i\right)}\prod_{i=1}^{n_1}(1+\xi^{-1}x_i^{-\nu})^{-(\gamma+1)}\prod_{i=1}^{n_2}(1+\xi^{-1}y_i^{-\nu})^{-(\eta+1)}, \tag{3.5}$$

$$\pi_\xi^*(\xi|\gamma,\eta,\nu) \propto \xi^{-n_1-n_2-1}\prod_{i=1}^{n_1}(1+\xi^{-1}x_i^{-\nu})^{-(\gamma+1)}\prod_{i=1}^{n_2}(1+\xi^{-1}y_i^{-\nu})^{-(\eta+1)}. \tag{3.6}$$

The distributions in Eqs (3.3)–(3.6) cannot be analytically transformed into well-known distributions, which prevents direct sampling with conventional techniques. However, the samples for γ and η in Eqs (3.3) and (3.4) are generated using the gamma distribution routine. Plots of Eqs (3.5) and (3.6) (see Figure 1) conditional posterior distributions of ν and ξ approximate normal distributions. Thus, we apply the MH approach with a normal proposal distribution to generate random integers from these distributions. Additionally, determining the marginal posterior distributions and finding the posterior distributions' means are generally important in Bayesian analysis.

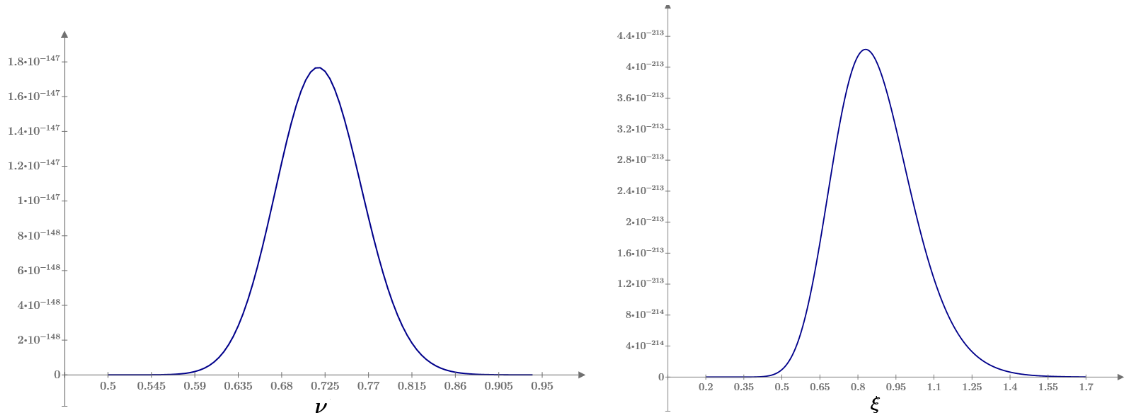

Figure 1. The probability density functions for ν and ξ, conditional on the observed data.

3.1. Bayesian estimators for loss functions

A symmetric loss functions (squared error (SE)) in life and reliability testing have been extensively studied recently by numerous writers (see [30]). First introduced by [31], the linear-exponential (LINEX) loss function is one of the most-often-utilized asymmetric loss functions. Several papers employ it, including [32] and [33]. On one side, this function is roughly linear, while on the other, it rises approximately to zero. Estimators can be used to represent the LINEX loss function if the minimal loss occurs at $\hat{\Theta} = \Theta$,

$$L(\Psi) \propto e^{c\Psi} - c\Psi - 1, \;\; c \neq 0, \tag{3.7}$$

where $\Psi = (\hat{\Theta} - \Theta)$, and $\hat{\Theta}$ is the approximation estimate of Θ.

The direction and degree of symmetry are represented by the value of c. In particular, suggests that $c > 0$ overestimation is more dangerous than underestimating $c < 0$, but it actually suggests the reverse. The LINEX loss function approximates the SE loss when c is near zero.

Under the LINEX loss function, the value of $\hat{\Theta}$, the Bayes estimator, is

$$\hat{\Theta}_{\text{LINEX}} = -\frac{1}{c} \ln\left(E_{\Theta}\left[e^{-c\Theta}\right]\right), \tag{3.8}$$

such that $E_{\hat{\Theta}}[e^{-c\hat{\Theta}}]$ exists.

A general entropy (GE) loss function is an asymmetric loss function that was proposed by [34]. The $\hat{\Theta}_{\text{GE}}$ under the GE loss Bayes estimate of Θ is

$$\hat{\Theta}_{\text{GE}} = \left[E_{\Theta}(\Theta^{-\rho})\right]^{-1/\rho} \tag{3.9}$$

such that $E_{\Theta}(\Theta^{-\rho})$ exists and is finite. The minimum occurs at $\hat{\Theta} = \Theta$. The SE loss function results from $\rho = -1$. When $\rho > 0$, the repercussions of a positive error are greater than those of a negative error. To generate γ, η, ν, and ξ, we compute from the posterior density functions, we now propose the following method, which will enable us to find the pertinent credible intervals for the Bayes estimates of R.

Algorithm 1:

(1) Start with $(\gamma^{(0)}, \eta^{(0)}, \nu^{(0)}, \xi^{(0)})$.
(2) Create $j = 1$.
(3) Both $\gamma^{(j)}$ and $\eta^{(j)}$ create and generate from the gamma, $\gamma^{(j)}$ from Gamma$(n_1 + c_1, d_1 + \sum_{i=1}^{n_1} \ln(1 + \xi^{-1} x_i^{-\nu}))$, and $\eta^{(j)}$ from Gamma$(n_2 + c_2, d_2 + \sum_{i=1}^{n_2} \ln(1 + \xi^{-1} y_i^{-\nu}))$.
(4) Generate $\nu^{(j)}$ from $\pi_\nu^*(\nu|\gamma, \eta, \xi)$, with the proposal distribution using MH.

MH algorithm:

(i) Generate a starting point $\nu^{(*)}$ for which $\pi_\nu^*\left(\nu^{(i-1)}, \mathrm{var}(\hat{\nu})\right)$.
(ii) Calculate the acceptance probability:

$$q_\nu = \min\left[1, \frac{\pi_\nu^*\left(\nu^{(*)}, \mathrm{var}(\hat{\nu})\right)}{\pi_\nu^*\left(\nu^{(i-1)}, \mathrm{var}(\hat{\nu})\right)}\right], \tag{3.10}$$

and create $U \sim U(0,\ 1)$.
(iii) If you agree with the proposal, $U \leq q_\nu$ and make a set $\nu^{(*)} = \nu^{(i)}$. If not, dismiss the idea.

(5) Using MH to generate $\xi^{(j)}$ from $\pi_\xi^*(\xi|\gamma, \eta, \nu)$, with the $N(\xi^{(j-1)}, \mathrm{var}(\hat{\xi}))$ proposal distribution.
(6) Determine R_1 using Eq (2.1).
(7) Assign $j = j + 1$.
(8) Go back and repeat steps three through six N times.
(9) Obtain the Bayes point estimators of R with regard to SE, LINEX, and GE loss functions as follows:

$$\hat{R}_{\mathrm{SE}} = \frac{1}{N-M} \sum_{i=M+1}^{N} R^{(i)}, \tag{3.11}$$

$$\hat{R}_{\mathrm{LINEX}} = -\frac{1}{c} \ln\left[\frac{1}{N-M} \sum_{i=M+1}^{N} e^{-cR^{(i)}}\right], \tag{3.12}$$

$$\hat{R}_{\mathrm{GE}} = \left[\frac{1}{N-M} \sum_{i=M+1}^{N} \left(R^{(i)}\right)^{-\rho}\right]^{-1/\rho}, \tag{3.13}$$

where M is burn-in.

4. Interval estimation

4.1. Asymptotic confidence bounds

Since the MLEs cannot be derived analytically, we do not have their actual distributions that can allow us to derive the exact confidence intervals of the parameters. Alternatively, we can derive the asymptotic confidence intervals using the asymptotic behavior of the MLEs. It is well known that the MLE of Θ, say $\hat{\Theta}$, follows approximately a multivariate normal distribution with a mean of Θ and a variance-covariance matrix equaling the inverse of the observed Fisher information matrix, see [35]. That is, $\hat{\Theta} = (\hat{\gamma}, \hat{\eta}, \hat{\nu}, \hat{\xi}) \sim \boldsymbol{N}_4(\Theta, \boldsymbol{V})$, where

$$\boldsymbol{V} = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \gamma^2} & -\frac{\partial^2 \ell}{\partial \gamma \partial \eta} & -\frac{\partial^2 \ell}{\partial \gamma \partial \nu} & -\frac{\partial^2 \ell}{\partial \gamma \partial \xi} \\ -\frac{\partial^2 \ell}{\partial \eta \partial \gamma} & -\frac{\partial^2 \ell}{\partial \eta^2} & -\frac{\partial^2 \ell}{\partial \eta \partial \nu} & -\frac{\partial^2 \ell}{\partial \eta \partial \xi} \\ -\frac{\partial^2 \ell}{\partial \nu \partial \gamma} & -\frac{\partial^2 \ell}{\partial \nu \partial \eta} & -\frac{\partial^2 \ell}{\partial \nu^2} & -\frac{\partial^2 \ell}{\partial \nu \partial \xi} \\ -\frac{\partial^2 \ell}{\partial \xi \partial \gamma} & -\frac{\partial^2 \ell}{\partial \xi \partial \eta} & -\frac{\partial^2 \ell}{\partial \xi \partial \nu} & -\frac{\partial^2 \ell}{\partial \xi^2} \end{bmatrix}^{-1}_{(\hat{\Theta})}.$$

From Eq (2.3), the second derivatives can be computed as follows:

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \gamma^2} &= -\frac{n_1}{\gamma^2}, \\
\frac{\partial^2 \ell}{\partial \gamma \partial \eta} &= \frac{\partial^2 \ell}{\partial \eta \partial \gamma} = 0, \\
\frac{\partial^2 \ell}{\partial \gamma \partial \xi} &= \frac{\partial^2 \ell}{\partial \xi \partial \gamma} = \sum_{i=1}^{n_1} \frac{\xi^{-1} x_i^{-\nu} \ln x_i}{1 + \xi^{-1} x_i^{-\nu}}, \\
\frac{\partial^2 \ell}{\partial \gamma \partial \xi} &= \frac{\partial^2 \ell}{\partial \xi \partial \gamma} = \sum_{i=1}^{n_1} \frac{\xi^{-2} x_i^{-\nu}}{1 + \xi^{-1} x_i^{-\nu}}, \\
\frac{\partial^2 \ell}{\partial \eta^2} &= \frac{-n_2}{\eta^2}, \\
\frac{\partial^2 \ell}{\partial \nu \partial \eta} &= \frac{\partial^2 \ell}{\partial \eta \partial \nu} = \sum_{i=1}^{n_2} \frac{\xi^{-1} y_i^{-\nu} \ln y_i}{1 + \xi^{-1} y_i^{-\nu}}, \\
\frac{\partial^2 \ell}{\partial \eta \partial \xi} &= \frac{\partial^2 \ell}{\partial \xi \partial \eta} = \sum_{i=1}^{n_2} \frac{\xi^{-2} y_i^{-\nu}}{1 + \xi^{-1} y_i^{-\nu}}, \\
\frac{\partial^2 \ell}{\partial \nu^2} &= -\frac{n_1 + n_2}{\nu^2} - (\eta + 1) \sum_{i=1}^{n_2} \frac{\xi^{-1} y_i^{-\nu} \ln^2 y_i}{\left(1 + \xi^{-1} y_i^{-\nu}\right)^2} - (\gamma + 1) \sum_{i=1}^{n_1} \frac{\xi^{-1} x_i^{-\nu} \ln^2 x_i}{\left(1 + \xi^{-1} x_i^{-\nu}\right)^2}, \\
\frac{\partial^2 \ell}{\partial \nu \partial \xi} &= \frac{\partial^2 \ell}{\partial \xi \partial \nu} = -(\gamma + 1) \sum_{i=1}^{n_1} \frac{\xi^{-2} x_i^{-\nu} \ln x_i}{\left(1 + \xi^{-1} x_i^{-\nu}\right)^2} - (\eta + 1) \sum_{i=1}^{n_2} \frac{\xi^{-2} y_i^{-\nu} \ln y_i}{\left(1 + \xi^{-1} y_i^{-\nu}\right)^2}, \\
\frac{\partial^2 \ell}{\partial \xi^2} &= \frac{n_1 + n_2}{\xi^2} - (\gamma + 1) \sum_{i=1}^{n_1} \frac{\xi^{-3} x_i^{-\nu} (2 + \xi^{-1} x_i^{-\nu})}{\left(1 + \xi^{-1} x_i^{-\nu}\right)^2} - (\eta + 1) \sum_{i=1}^{n_2} \frac{\xi^{-3} y_i^{-\nu} (2 + \xi^{-1} y_i^{-\nu})}{\left(1 + \xi^{-1} y_i^{-\nu}\right)^2}.
\end{aligned}$$

A $100(1 - \alpha)\%$ confidence interval of $\Theta = (\gamma, \eta, \nu, \xi)$ can be approximated by

$$\hat{\gamma} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\gamma})}, \ \ \hat{\eta} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\eta})}, \ \ \hat{\nu} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\nu})}, \ \text{ and } \hat{\xi} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\xi})},$$

where $z_{\alpha/2}$ is the percentile of the standard normal distribution with a probability of the right tail, and $\text{var}(\hat{\Theta}_i)$, $i = 1, 2, 3, 4$, are the elements on the major diagonal in the covariance matrix $\boldsymbol{V}$.

We apply the delta approach to determine the approximate estimator of the variance of $\hat{R}$. A generic method for calculating confidence intervals for functions of MLEs is the delta method.

Let $Q = (\frac{\partial R}{\partial \gamma} \quad \frac{\partial R}{\partial \eta} \quad \frac{\partial R}{\partial \nu} \quad \frac{\partial R}{\partial \xi})$ where

$$\left.\begin{array}{l} \frac{\partial R}{\partial \gamma} = \frac{\eta}{(\gamma+\eta)^2}, \\ \frac{\partial R}{\partial \eta} = -\frac{\gamma}{(\gamma+\eta)^2}, \\ \frac{\partial R}{\partial \nu} = \frac{\partial R}{\partial \xi} = 0. \end{array}\right\} \tag{4.1}$$

Subsequently, the approximate estimator of $\text{var}(\hat{R})$ is provided by

$$\text{var}(\hat{R}) \simeq \left[Q I_0^{-1} Q^T\right]\Big|_{(\hat{\Theta})},$$

where Q^T is the transpose of Q. The approximate confidence ranges of R are obtained from these data as

$$\hat{R} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{R})}. \tag{4.2}$$

4.2. Bootstrap confidence intervals

Normal approximations provide accurate confidence intervals when sample sizes are large. However, this approach is unreliable when the normality assumption is violated, as is often the case with small sample sizes. Resampling techniques, such as the bootstrap, are more suitable for approximating confidence intervals in such scenarios. Consequently, this section introduces bootstrap resampling approaches for estimating confidence intervals for the parameters and stress-strength parameter within the context of the IPL distribution.

For tiny sample sets, confidence intervals based on asymptotic results usually perform poorly. To address this confidence issue, we propose to use confidence intervals derived from the non-parametric percentile bootstrap approach as a solution to this issue of confidence [36].

Algorithm 2:

(1) Compute the MLEs $\hat{\gamma}, \hat{\eta}, \hat{\nu}, \hat{\xi}$, as well as $\hat{R}$ using the first two samples of $\{x_1, x_2, \cdots, x_{n_1}\}$ and $\{y_1, y_2, \cdots, y_{n_2}\}$ for the IPL distribution.

(2) Generate bootstrap samples as $\{x_1^*, x_2^*, \cdots, x_{n_1}^*\}$ and $\{y_1^*, y_2^*, \cdots, y_{n_2}^*\}$ using $(\hat{\gamma}, \hat{\nu}, \hat{\xi})$ and $(\hat{\eta}, \hat{\nu}, \hat{\xi})$ and calculate the bootstrap value of R, which we will call $\hat{R}^*$.

(3) Step 3 should be repeated N times.

(4) Put $\hat{R}_1^*, \hat{R}_2^*, \cdots, \hat{R}_N^*$ in ascending order to become $\hat{R}_{(1)}^*, \hat{R}_{(2)}^*, \cdots \hat{R}_{(N)}^*$. The approximate $100(1-\alpha)\%$ confidence interval for R is given by

$$(\hat{R}^*_{boot\left(N\frac{\alpha}{2}\right)}, \ \hat{R}^*_{boot\left(N\left(1-\frac{\alpha}{2}\right)\right)}). \tag{4.3}$$

4.3. MCMC credible intervals

A Bayesian credible interval and HPD interval are derived from the posterior distribution that quantifies the uncertainty about a random parameter. We apply the following procedure to determine credible intervals for R.

Algorithm 3:

(1) **Credible interval**:
Perform the sequence of steps 1 to 9 from Algorithm 1. To determine R's credible intervals (CIs), order $\hat{R}^{(M+1)}, \hat{R}^{(M+2)}, \cdots, \hat{R}^{(M-N)}$, and the $100(1-\alpha)\%$ CIs of R become

$$\left(\hat{R}_{(N-M)\frac{\alpha}{2}}, \hat{R}_{(N-M)(1-\frac{\alpha}{2})}\right). \tag{4.4}$$

(2) **HPD interval**:
For a given credibility level $100(1-\alpha)\%$, the HPD interval is defined as:

$$[LR_i, (M-N-L)R_i], [LR_i, (M-N-L)R_i] \tag{4.5}$$

where:

- L is an index running from 1 to $\alpha(M-N)$.
- R_i represents ordered posterior statistics (e.g., quantiles or sorted posterior samples).
- M and N are parameters that may relate to sample sizes or distributional thresholds.

The HPD interval is the shortest width among all possible $100(1-\alpha)\%$ CIs, making it the most precise estimate, which includes the region where the posterior density is highest, ensuring no point inside has a lower density than any point outside. Uses for HPD Intervals:

- Unlike symmetric CIs, the HPD interval adapts to skewed or multimodal distributions, capturing the most probable values efficiently.
- It is useful in Bayesian inference when a tight, probability-concentrated interval is preferred over equal-tailed alternatives.

5. Numerical comparison study

This section presents the findings from Monte Carlo simulations, comparing the effectiveness of various techniques. The confidence intervals derived from the asymptotic distributions of the MLE, bootstrap, and HPD CIs are compared in terms of average confidence lengths (ACL) and coverage percentages (CP). Two sets of parameter values $(\gamma, \eta, \nu, \xi) = (1.0, 1.5, 1.0, 0.5)$ and $(\gamma, \eta, \nu, \xi) = (1.0, 1.5, 2.0, 0.7)$ with different sample sizes as (20, 20), (40, 40), (60, 60), (80, 80), and (100, 100) are used. We provide the average mean and mean square error (MSE) of the MLE, in addition to the Bayes estimates of R, for various parameter values. For prior knowledge, two different priors are considered for the four parameters (denoted as (γ, η, ν, ξ)):

(1) Non-informative (diffuse) priors
Hyperparameters: Prior 0 (P^0) ($c_i = d_i = 0$ for $i = 1, 2$). These priors are "uninformative", meaning they introduce minimal prior knowledge. They let the observed data dominate the posterior distribution. It is often used when no strong prior information is available or to avoid bias.

(2) Informative (conjugate) priors
Hyperparameters: The informative priors are chosen such that $c_i/d_i \cong E(\Theta_i)$, ensuring prior means match true means under SE, LINEX, and GE loss functions. Three hyperparameter sets are used:
Prior 1 (P^1): $c_i = d_i = 0.1,\ i = 1, 2$.
Prior 2 (P^2): $c_i = d_i = 1,\ i = 1, 2$.

This alignment ensures consistent Bayesian estimation across loss functions. These are chosen so that the prior expectations match the true parameter values. For example, the prior is gamma-distributed (as is common for positive parameters).

We generated the estimates using 1000 MCMC samples, and distinct shape parameters ($c = -5$ and 5) are employed for the LINEX loss functions. The GE loss functions employ different values of ρ, ($\rho = -5$ and 5). The results are reported in Tables 1–3. Also, the MSEs are plotted in Figures 2–4.

We examine and contrast the MSE-based performance of the ML, boot-p, and BEs. It is clear that the average and MSE of the Bayes estimates provide more accurate results than those of the boot-p and MLEs. Additionally, we compare different 95% confidence intervals regarding ACL and CP, which are estimated to minimize randomness and are presented in Table 3. These confidence intervals are produced using bootstrap confidence intervals, CIs, and asymptotic distributions of the MLEs.

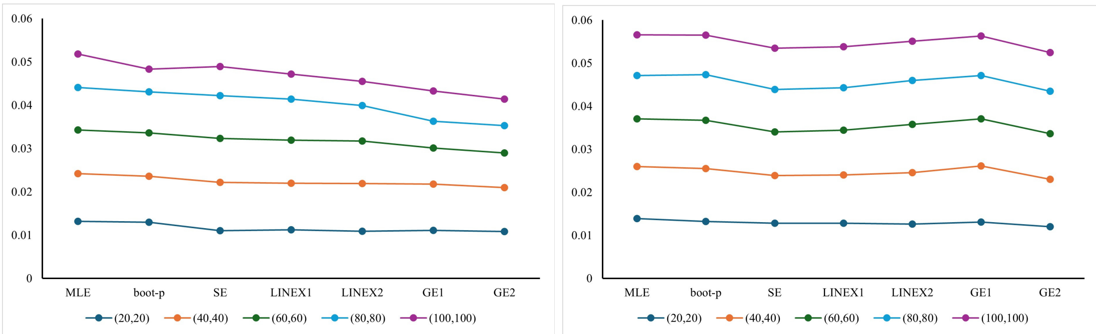

Figure 2. The MSE of $R = P(Y < X)$ for (γ, η, ν, ξ) for P^0.

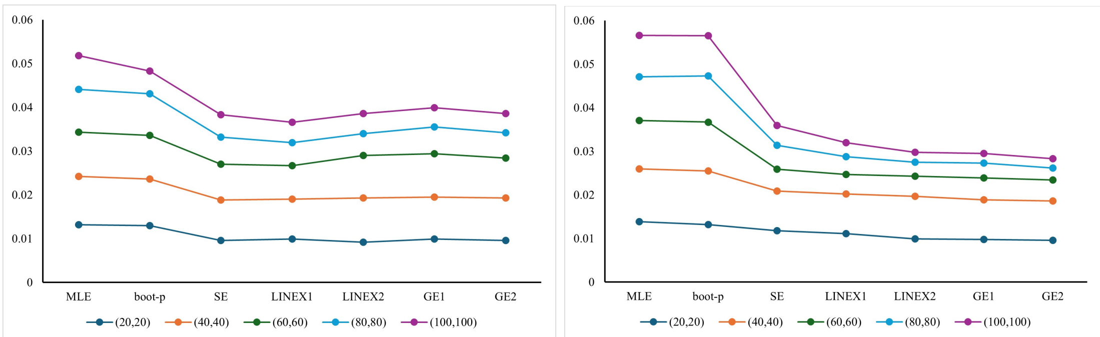

Figure 3. The MSE of $R = P(Y < X)$ for (γ, η, ν, ξ) for P^1.

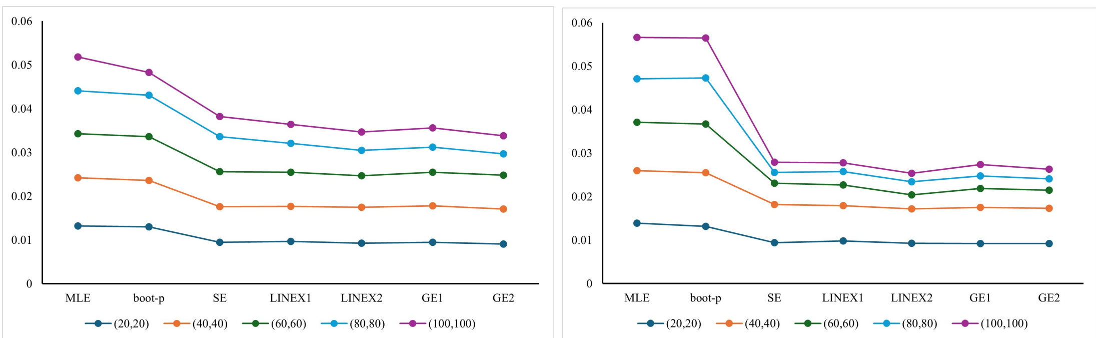

Figure 4. The MSE of $R = P(Y < X)$ for (γ, η, ν, ξ) for P^2.

Table 1. Average mean and MSE of $R = P(Y < X)$ for $(\gamma, \eta, \nu, \xi) = (1.0, 1.5, 1.0, 0.5)$.

				Bayes: P^0				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.6104	0.6177	0.588	0.5958	0.581	0.5847	0.5937
	MSE	0.0132	0.013	0.011	0.0112	0.0109	0.0111	0.0108
(40,40)	Mean	0.5991	0.589	0.59037	0.5957	0.5991	0.5687	0.5922
	MSE	0.011	0.0106	0.0112	0.0108	0.011	0.0107	0.0102
(60,60)	Mean	0.588	0.6019	0.5934	0.5827	0.6047	0.5867	0.5909
	MSE	0.0101	0.01	0.0101	0.0099	0.0098	0.0083	0.008
(80,80)	Mean	0.5907	0.5963	0.5867	0.5768	0.5883	0.5837	0.5995
	MSE	0.0098	0.0095	0.0099	0.0095	0.0082	0.0062	0.0063
(100,100)	Mean	0.587	0.6023	0.5883	0.5867	0.6085	0.5834	0.588
	MSE	0.0077	0.0052	0.0067	0.0058	0.0056	0.007	0.0061
				Bayes: P^1				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.6104	0.6177	0.599	0.593	0.596	0.6084	0.9007
	MSE	0.0132	0.013	0.0096	0.0099	0.0092	0.0099	0.0096
(40,40)	Mean	0.5991	0.589	0.6007	0.5957	0.9055	0.587	0.6043
	MSE	0.011	0.0106	0.0092	0.0091	0.0101	0.0096	0.0097
(60,60)	Mean	0.588	0.6019	0.5837	0.586	0.6043	0.5895	0.5729
	MSE	0.0101	0.01	0.0082	0.0077	0.0097	0.0099	0.0091
(80,80)	Mean	0.5907	0.5963	0.6107	0.6028	0.6022	0.6084	0.601
	MSE	0.0098	0.0095	0.0062	0.0052	0.005	0.0061	0.0058
(100,100)	Mean	0.587	0.6023	0.6014	0.599	0.602	0.5982	0.597
	MSE	0.0077	0.0052	0.0051	0.0047	0.0046	0.0044	0.0044
				Bayes: P^2				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.6104	0.6177	0.5883	0.5951	0.5907	0.6044	0.6026
	MSE	0.0132	0.013	0.0095	0.0097	0.0093	0.0095	0.0091
(40,40)	Mean	0.5991	0.589	0.607	0.5927	0.5995	0.5687	0.5925
	MSE	0.011	0.0106	0.0081	0.008	0.0082	0.0083	0.008
(60,60)	Mean	0.588	0.6019	0.593	0.586	0.5957	0.588	0.5959
	MSE	0.0101	0.01	0.008	0.0078	0.0072	0.0077	0.0077
(80,80)	Mean	0.5907	0.5963	0.5996	0.5928	0.59931	0.6016	0.599
	MSE	0.0098	0.0095	0.008	0.0066	0.0058	0.0057	0.0049
(100,100)	Mean	0.587	0.6023	0.597	0.588	0.6095	0.6063	0.599
	MSE	0.0077	0.0052	0.0046	0.0043	0.0042	0.0044	0.0041

Table 2. Average mean and MSE of $R = P(Y < X)$ for $(\gamma, \eta, \nu, \xi) = (1.0, 1.5, 2.0, 0.7)$.

				Bayes: P^0				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.5982	0.6089	0.599	0.5929	0.6028	0.5968	0.5896
	MSE	0.0139	0.0132	0.0128	0.0128	0.0126	0.0131	0.012
(40,40)	Mean	0.5851	0.6013	0.5982	0.5887	0.5909	0.589	0.5957
	MSE	0.0121	0.0123	0.0111	0.0112	0.012	0.013	0.011
(60,60)	Mean	0.591	0.6002	0.5829	0.5899	0.615	0.5908	0.5996
	MSE	0.0111	0.0112	0.0101	0.0104	0.0112	0.011	0.0106
(80,80)	Mean	0.5883	0.5995	0.5939	0.5958	0.5807	0.5938	0.5875
	MSE	0.01	0.0106	0.0099	0.0099	0.0102	0.01	0.0099
(100,100)	Mean	0.5993	0.5886	0.5967	0.6028	0.6011	0.588	0.5893
	MSE	0.0095	0.0092	0.0096	0.0095	0.0091	0.0092	0.009
				Bayes: P^1				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.5982	0.6089	0.5988	0.592	0.596	0.612	0.6007
	MSE	0.0139	0.0132	0.0118	0.0111	0.0099	0.0098	0.0096
(40,40)	Mean	0.5851	0.6013	0.5927	0.591	0.5912	0.5896	0.6125
	MSE	0.0121	0.0123	0.0091	0.0091	0.0098	0.0091	0.009
(60,60)	Mean	0.591	0.6002	0.5879	0.593	0.6011	0.5993	0.5829
	MSE	0.0111	0.0112	0.005	0.0045	0.0046	0.005	0.0048
(80,80)	Mean	0.5883	0.5995	0.608	0.587	0.5992	0.6006	0.5996
	MSE	0.01	0.0106	0.0055	0.0041	0.0032	0.0034	0.0028
(100,100)	Mean	0.5993	0.5886	0.5939	0.5911	0.5868	0.5866	0.5871
	MSE	0.0095	0.0092	0.0045	0.0032	0.0023	0.0022	0.0021
				Bayes: P^2				
					LINEX		GE	
(n_1, n_2)		MLE	boot-p	SE	$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
(20,20)	Mean	0.5982	0.6089	0.589	0.5883	0.5898	0.6011	0.6108
	MSE	0.0139	0.0132	0.0094	0.0098	0.0093	0.0092	0.0092
(40,40)	Mean	0.5851	0.6013	0.6109	0.5995	0.5915	0.5993	0.5883
	MSE	0.0121	0.0123	0.0088	0.0081	0.0079	0.0083	0.0081
(60,60)	Mean	0.591	0.6002	0.5928	0.5933	0.5911	0.609	0.5886
	MSE	0.0111	0.0112	0.0049	0.0048	0.0032	0.0044	0.0042
(80,80)	Mean	0.5883	0.5995	0.5867	0.5933	0.5839	0.6	0.6016
	MSE	0.01	0.0106	0.0025	0.0031	0.003	0.0029	0.0026
(100,100)	Mean	0.5993	0.5886	0.6081	0.5908	0.598	0.6058	0.6008
	MSE	0.0095	0.0092	0.0023	0.002	0.002	0.0026	0.0022

Table 3. Average length 95% coverage probabilities for $R = P(Y < X)$.

		$(\gamma,\eta,\nu,\xi) = (1, 1.5, 1.0, 0.5)$					$(\gamma,\eta,\nu,\xi) = (1, 1.5, 2.0, 0.7)$				
				Bayes					Bayes		
(n_1, n_2)		MLE	boot-p	P^0	P^1	P^2	MLE	boot-p	P^0	P^1	P^2
(20,20)	ACL	0.3012	0.3104	0.2934	0.3002	0.3096	0.3015	0.2961	0.2938	0.3006	0.2992
	CP	0.902	0.914	0.932	0.934	0.934	0.912	0.914	0.942	0.95	0.946
(40,40)	ACL	0.1816	0.1767	0.1868	0.181	0.1783	0.188	0.1813	0.1807	0.1767	0.176
	CP	0.912	0.934	0.942	0.95	0.948	0.916	0.93	0.95	0.95	0.952
(60,60)	ACL	0.141	0.1243	0.1114	0.111	0.1101	0.1412	0.1363	0.1201	0.1109	0.1131
	CP	0.94	0.942	0.954	0.946	0.954	0.934	0.942	0.946	0.952	0.956
(80,80)	ACL	0.133	0.102	0.0915	0.0912	0.0905	0.104	0.11	0.1001	0.0912	0.0827
	CP	0.942	0.946	0.954	0.948	0.964	0.94	0.946	0.95	0.956	0.95
(100,100)	ACL	0.0884	0.075	0.0799	0.07	0.0701	0.0881	0.0801	0.0789	0.07	0.071
	CP	0.948	0.954	0.958	0.956	0.968	0.954	0.958	0.958	0.964	0.962

For varying sample sizes, the MCMC credible intervals provide more accurate findings than the approximation. We discover that when the value of the shape parameter c rises, the MSEs of the BE using the LINEX loss function decrease. However, when evaluating all recommended methods (ML, Boot-p, and HPD), the HPD-based CIs demonstrate the best overall performance. Specifically, they tend to have shorter ACLs at the nominal level and CPs that are closer to the target level. Moreover, Bayesian CIs constructed using informative priors consistently outperform those based on non-informative priors. This superiority holds across all estimated parameters, including R. In addition, point and interval estimation under different combinations of sample sizes (n_1, n_2) and censoring schemes reveal that the Bayesian method with an informative prior delivers highly effective results. Parameter estimates also show varying performance depending on the loss function applied. Therefore, when prior knowledge about the unknown parameters is available, the Bayesian approach with informative priors is preferable. In situations where such prior information is lacking — particularly when the sample size is small — the Bayesian method with non-informative priors is often the more suitable alternative.

6. Real data application

This part discusses the suggested methodology's practical application from the preceding sections. The data provided shows the customer service wait times, expressed in minutes, for two distinct banks, as partially examined in [37], considering that $n_1 = 100$ and $n_2 = 60$. The parameter of stress-strength needs to be estimated, where X or Y reflect the service times to clients at Bank A and Bank B, respectively:

Data I (X): 0.8, 2.1, 3.5, 4.3, 4.9, 6.1, 7.1, 8.0, 8.9, 10.9, 11.9, 13.1, 17.3, 20.6, 33.1, 0.8, 2.6, 3.6, 4.4, 4.9, 6.2, 7.1, 8.2, 8.9, 11.0, 12.4, 13.3, 18.1, 21.3, 38.5, 1.3, 2.7, 4.0, 4.4, 5.0, 6.2, 7.1, 8.6, 9.5, 11.0, 13.6, 13.9, 18.2, 21.4, 1.5, 2.9, 4.1, 4.6, 5.3, 6.2, 7.1, 8.6, 9.6, 11.1, 13.7, 14.1, 18.4, 21.9, 1.8, 3.1, 4.2, 4.7, 5.5, 6.3, 7.4, 8.6, 9.7, 11.2, 12.5, 15.4, 18.9, 23.0, 1.9, 3.2, 4.2, 4.7, 5.7, 6.7, 7.6, 8.8, 9.8, 11.2, 12.9, 15.4, 19.0, 27.0, 1.9, 3.3, 4.3, 4.8, 5.7, 6.9, 7.7, 8.8, 10.7, 11.5, 13.0, 17.3, 19.9, 31.6.

Data II (Y): 0.1, 0.9, 1.9, 2.3, 2.7, 3.1, 3.5, 4.5, 5.6, 6.8, 7.7, 8.5, 10.9, 12.8, 14.5, 0.2, 1.1, 2.0, 2.3, 2.7, 3.2, 3.9, 4.7, 6.2, 7.3, 8.0, 8.7, 11.0, 12.9, 16.0, 0.3, 1.2, 2.2, 2.5, 2.9, 3.4, 4.0, 5.3, 6.3, 7.5, 8.0, 9.5, 12.1, 13.2, 16.5, 0.7, 1.8, 2.3, 2.6, 3.1, 3.4, 4.2, 5.6, 6.6, 7.7, 8.5, 10.7, 12.3, 13.7, 28.0.

Under transformation $(\frac{data}{10})^{1.5}$, we confirmed that modeling these data with the IPL distribution is a feasible approach. Figure 5 exhibits the fitted and empirical survival functions based on data X and Y, respectively.

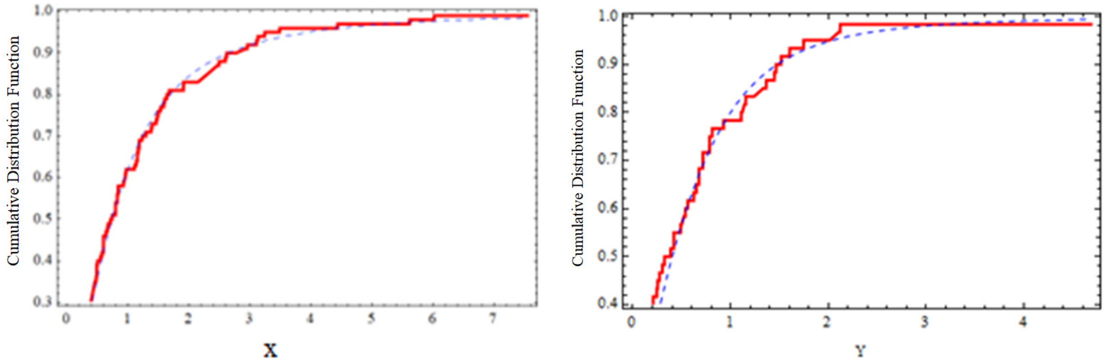

Figure 5. Fitted cumulative distribution and empirical functions of data X and data Y, respectively.

The charts show that the IPL distributions fit the provided data sets well. Table 4 shows the results of the p-values and the Kolmogorov-Smirnov (K-S) test for the new data.

Table 4. The K-S and p-values for the new data.

New data	K-S	p-value
X	0.0520	0.9497
Y	0.0854	0.7740

It specifies that every modified data set is taken at the significance level (0.05) from the IPL distribution.

The parametric percentile bootstrap method is used to compute the Boot-p estimations. The estimations from Bayes, when the hyperparameters are $c_i = d_i = 0.0001$, $i = 1, 2$, are computed. The MCMC algorithm to generate a sequence of 22,000 times is conducted and as a "burn-in", we discard the first 2000 values. Table 5 presents the BEs under the SE loss function of γ, η, ν, ξ, and R as well as the 95% CIs of R based on Bayes MCMC. Table 6 presents the BEs of R based on LINEX and GE loss functions with different values of c and ρ, respectively. The values of c and ρ will be chosen as positive and negative to maintain overestimation and underestimation, so that the estimator's output is close to the Bayesian method at SE.

Table 5. The point and 95% CIs estimating γ, η, ν, ξ, and R for MLE, Boot-p, and Bayes SE.

					Bayes			
	MLE		Boot-p		CIs		HPD CIs	
Parameters	Point	Interval	Point	Interval	Point	Interval	Point	Interval
γ	0.5863	(0.2452, 0.9274)	0.5795	(0.2410, 0.9391)	0.5320	(0.3034, 0.8805)	0.5271	(0.3045, 0.8747)
η	0.3277	(0.1465, 0.5089)	0.3166	(0.1419, 0.5099)	0.3095	(0.1041, 0.4421)	0.3114	(0.1340, 0.4552)
ν	1.9774	(1.3503, 2.6046)	1.9063	(1.2022, 2.6660)	1.9217	(1.0230, 2.1011)	1.9034	(1.1040, 2.1080)
ξ	0.8582	(0.0317, 1.6847)	0.8012	(0.0311, 1.6278)	0.5193	(0.0412, 1.6110)	0.5211	(0.0440, 1.6124)
R	0.6415	(0.5672, 0.7141)	0.6371	(0.5561, 0.6902)	0.6498	(0.5582, 0.7012)	0.6489	(0.5590, 0.7002)

Table 6. Bayesian approximations of R using loss functions.

LINEX		GE	
$c = -5$	$c = 5$	$\rho = -5$	$\rho = 5$
0.640327	0.63106	0.641189	0.630723

Figures 6–10 show the estimations of the marginal posterior density of γ, η, ν, ξ, and R along with their histograms.

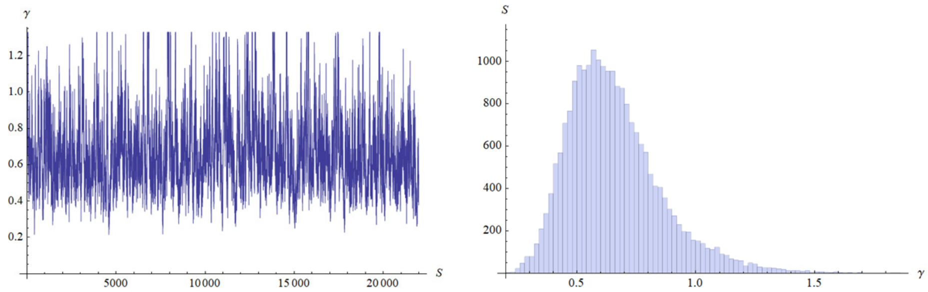

Figure 6. The actual number and histogram produced by the real MCMC of γ.

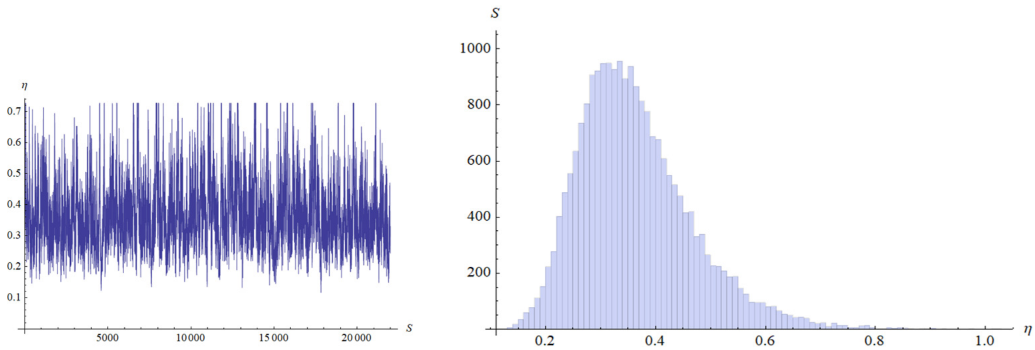

Figure 7. The actual number and histogram produced by the real MCMC of η.

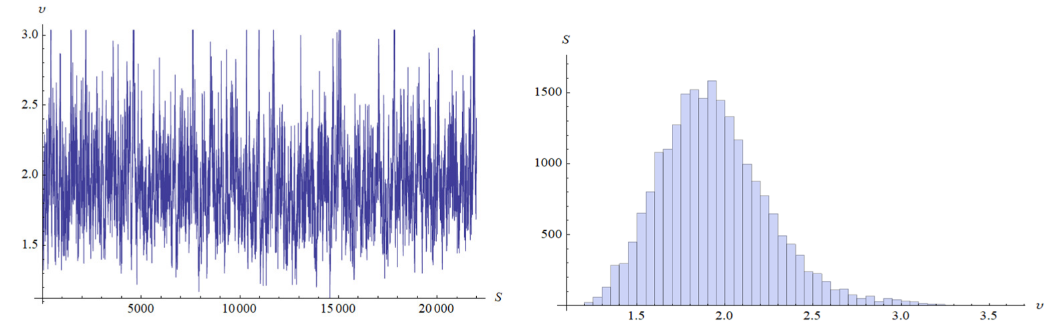

Figure 8. The actual number and histogram produced by the real MCMC of ν.

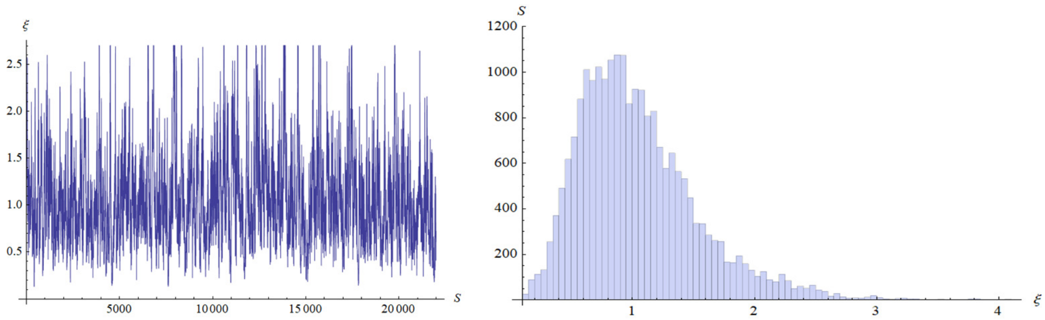

Figure 9. The actual number and histogram produced by the real MCMC of ξ.

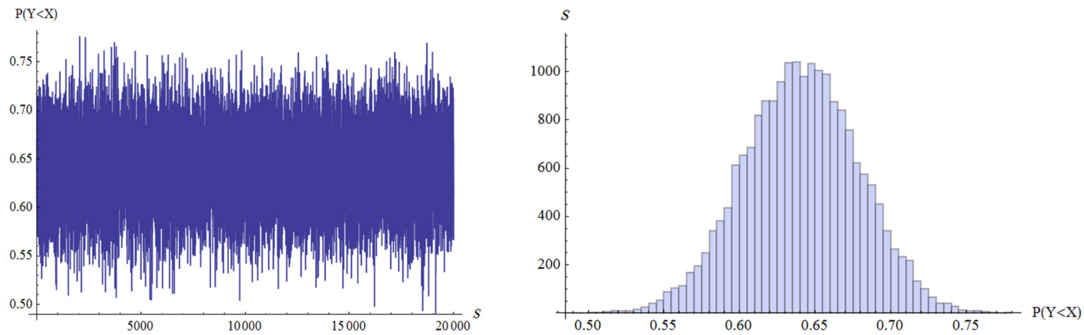

Figure 10. The actual number and histogram produced by the real MCMC of R.

The estimates show that there is symmetry in the marginal distribution. The trace of the iteration number versus the value of the draw of R of the iteration, which is displayed in Figure 10 together with the sample mean and 95% CIs, is used to track the convergence of MCMC samples. The plots deliver a strong, varied performance. The profile of the log likelihood function is plotted in Figures 11 and 12 and display unique results.

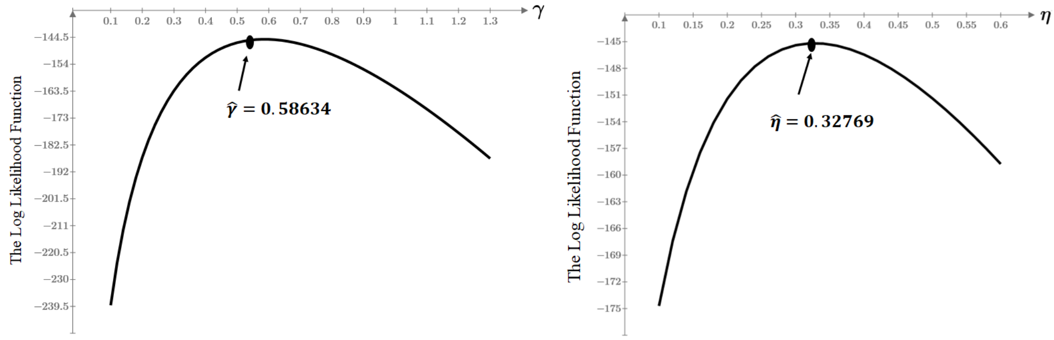

Figure 11. The depiction of the log likelihood function for γ and η.

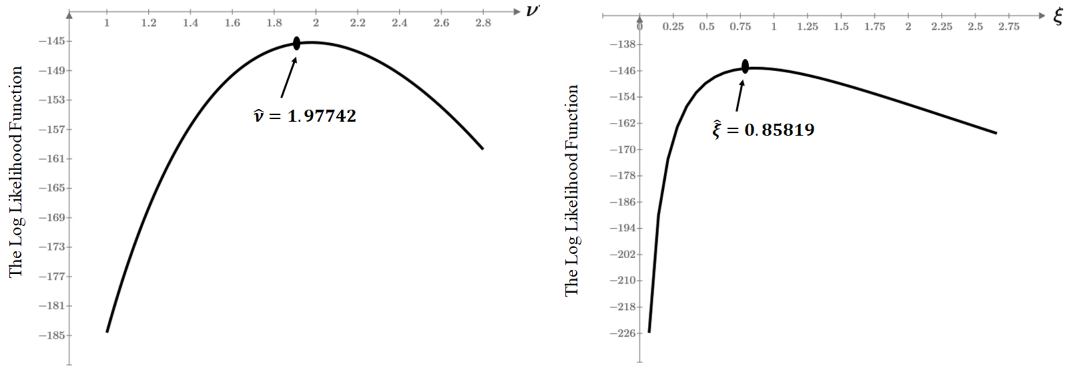

Figure 12. The depiction of the log likelihood function for ν and ξ.

From Figures 11 and 12, it is clear that the logarithm of the maximum likelihood function is convex upward and has only one solution at the parameter values $\hat{\gamma}$, $\hat{\eta}$, $\hat{\nu}$, and $\hat{\xi}$ obtained using the data [30].

7. Conclusions

This research examines several methods for estimating stress-strength reliability $R = P(Y < X)$, where X and Y are treated as independent random variables. It is assumed that X and Y follow IPL distributions with different scale parameters. The MLE for R is derived using an iterative process. Additionally, bootstrap confidence intervals for R are calculated and contrasted with those obtained using the delta method. A Bayesian approach to estimating R under GE, LINEX, and SE loss functions is also explored, with CIs constructed via the Gibbs sampling method by using non-informative and informative priors. When there is no subjective information, it was found that the MLEs perform rather well. With subjective information available, the Bayesian estimators perform better than the MLEs, as predicted. The applicability of these point and interval estimation methods is shown through several numerical examples. A simulation study and real data are then incorporated to investigate the performance of the proposed techniques across varying sample sizes. The results demonstrate the practical implementation of the proposed methods.

This study highlights several key areas for future research. Specifically, the design of optimal censoring strategies instead of complete, the prediction of statistical outcomes under complete or censoring conditions, and the development of more comprehensive inference methods to handle complex failure models are of considerable interest. The application of data mining techniques to these data may also prove beneficial, providing a means to identify variations in patient survival patterns and to estimate associated confidence intervals. These research avenues are worthy of further study.

Author contributions

Samah M. Ahmed and Abdelfattah Mustafa: Formal analysis, Validation, Writing–original draft and editing, Visualization, Software, Methodology, Data curation; Abdelfattah Mustafa and I. Khan: Conceptualization, Investigation, Writing–review and editing, Supervision, Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. Z. W. Birnbaum, On a use of the Mann-Whitney statistic, In: *Proceedings of the third Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*, 1956, 13–17.
2. R. A. Johnson, Stress strength models for reliability, *Handbook of statistics*, **7** (1988), 27–54. https://doi.org/10.1016/S0169-7161(88)07005-1
3. A. M. Awad, M. K. Charraf, Estimation of $P(Y < X)$ in the Burr case: a comparative study, *Commun. Stat.-Simul. Comput.*, **15** (1986), 389–403. http://doi.org/10.1080/03610918608812514
4. K. E. Ahmed, M. E. Fakhry, Z. F. Jaheen, Empirical Bayes estimation of $P(Y < X)$ and characterizations of Burr-Type X model, *J. Stat. Plan. Infer.*, **64** (1997), 297–308. https://doi.org/10.1016/S0378-3758(97)00038-4
5. A. H. Khan, T. R. Jan, Estimation of stress-strength reliability model using finite mixture of two parameter Lindley distributions, *J. Stat. Appl. Pro.*, **4** (2015), 147–159.
6. B. Saraçoğlu, I. Kinaci, D. Kundu, On estimation of $R = P(Y < X)$ for exponential distribution under progressive Type-II censoring, *J. Stat. Comput. Sim.*, **82** (2012), 729–744. https://doi.org/10.1080/00949655.2010.551772

7. D. Kundu, R. D. Gupta, Estimation of $P(X < Y)$ for generalized exponential distribution, *Metrika*, **61** (2005), 291–308. https://doi.org/10.1007/s001840400345
8. S. Nadarajah, Reliability for Laplace distributions, *Math. Probl. Eng.*, **2004** (2004), 169–183. https://doi.org/10.1155/S1024123X0431104X
9. J. G. Surles, W. J. Padgett, Inference for reliability and stress-strength for a scaled Burr-type X distribution, *Lifetime Data Anal.*, **7** (2001), 187–200. https://doi.org/10.1023/A:1011352923990
10. S. Nadarajah, Reliability for lifetime distributions, *Math. Comput. Model.*, **37** (2003), 683–688. https://doi.org/10.1016/S0895-7177(03)00074-8
11. D. Kundu, M. Z. Raqab, Estimation of $R = P[Y < X]$ for three-parameter Weibull distribution, *Stat. Probab. Lett.*, **79** (2009), 1839–1846. https://doi.org/10.1016/j.spl.2009.05.026
12. M. Nassar, R. Alotaibi, C. Zhang, Product of spacing estimation of stress–strength reliability for alpha power exponential progressively Type-II censored data, *Axioms*, **12** (2023), 752. http://doi.org/10.3390/axioms12080752
13. K. Krishnamoorthy, Y. Lin, Confidence limits for stress-strength reliability involving Weibull models, *J. Stat. Plan. Infer.*, **140** (2010), 1754–1764. https://doi.org/10.1016/j.jspi.2009.12.028
14. R. C. Gupta, C. Peng, Estimating reliability in proportional odds ratio models, *Comput. Stat. Data Anal.*, **53** (2009), 1495–1510. https://doi.org/10.1016/j.csda.2008.10.014
15. D. K. Al-Mutairi, M. E. Ghitany, R. C. Gupta, Estimation of reliability in a series system with random sample size, *Comput. Stat. Data Anal.*, **55** (2011), 964–972. https://doi.org/10.1016/j.csda.2010.07.027
16. S. Rostamian, N. Nematollahi, Estimation of stress–strength reliability in the inverse Gaussian distribution under progressively Type-II censored data, *Math. Sci.*, **13** (2019), 175–191. https://doi.org/10.1007/s40096-019-0289-1
17. S. Ghanbari, A. Rezaei Roknabadi, M. Salehi, Estimation of stress–strength reliability for Marshall–Olkin distributions based on progressively Type-II censored samples, *J. Appl. Stat.*, **49** (2022), 1913–1934. https://doi.org/10.1080/02664763.2021.1884207
18. S. Asadi, H. Panahi, Estimation of stress–strength reliability based on censored data and its evaluation for coating processes, *Qual. Technol. Quant. Manag.*, **19** (2022), 379–401. https://doi.org/10.1080/16843703.2021.2001129
19. X. Hu, H. Ren, Statistical inference of the stress-strength reliability for inverse Weibull distribution under an adaptive progressive type-II censored sample, *AIMS Math.*, **8** (2023), 28465–28487. https://doi.org/10.3934/math.20231457
20. I. Elbatal, A. S. Hassan, L. S. Diab, A. Ben Ghorbal, M. Elgarhy, A. R. El-Saeed, Stress–strength reliability analysis for different distributions using progressive Type-II censoring with binomial removal, *Axioms*, **12** (2023), 1054. https://doi.org/10.3390/axioms12111054
21. N. S. Y. Temraz, Inference on the stress strength reliability with exponentiated generalized Marshall Olkin-G distribution, *PLOS ONE*, **18** (2023), e0280183. https://doi.org/10.1371/journal.pone.0280183
22. T. Xavier, J. K. Jose, S. C. Bagui, Stress-strength reliability estimation of a series system with cold standby redundancy based on Kumaraswamy half-Logistic distribution, *American Journal of Mathematical and Management Sciences*, **42** (2023), 183–201. https://doi.org/10.1080/01966324.2023.2213835

23. A. S. Hassan, N. Alsadat, M. Elgarhy, H. Ahmad, H. F. Nagy, On estimating multi stress strength reliability for inverted Kumaraswamy under ranked set sampling with application in engineering, *J. Nonlinear Math. Phys.*, **31** (2024), 30. https://doi.org/10.1007/s44198-024-00196-y
24. A. Xu, R. Wang, X. Weng, Q. Wu, L. Zhuang, Strategic integration of adaptive sampling and ensemble techniques in federated learning for aircraft engine remaining useful life prediction, *Appl. Soft Comput.*, **175** (2025), 113067. https://doi.org/10.1016/j.asoc.2025.113067
25. A. Xu, G. Fang, L. Zhuang, C. Gu, A multivariate student-t process model for dependent tail-weighted degradation data, *IISE Trans.*, **57** (2025), 1071–1087. https://doi.org/10.1080/24725854.2024.2389538
26. A. S. Hassan, M. Abd-Allah, On the inverse power Lomax distribution, *Ann. Data Sci.*, **6** (2019), 259–278. https://doi.org/10.1007/s40745-018-0183-y
27. P. Kumar, H. Sharma, Prior preferences for the inverse power Lomax distribution: Bayesian method, *Journal of Statistics Applications & Probability Letters*, **10** (2023), 49–62. https://doi.org/10.18576/jsapl/100104
28. X. Shi, Y. Shi, Inference for inverse power Lomax distribution with progressive first-failure censoring, *Entropy*, **23** (2021), 1099. https://doi.org/10.3390/e23091099
29. S. M. Ahmed, A. Mustafa, Estimation of the coefficients of variation for inverse power Lomax distribution, *AIMS Math.*, **9** (2024), 33423–33441. https://doi.org/10.3934/math.20241595
30. A. P. Basu, N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, *J. Stat. Plan. Infer.*, **29** (1991), 21–31. https://doi.org/10.1016/0378-3758(92)90118-C
31. H. R. Varian, A Bayesian approach to real estate assessment, In: *Studies in Bayesian econometrics and statistics: in honor of L. J. Savage*, North Holland, Amsterdam, 1975, 195–208.
32. U. Balasooriya, N. Balakrishnan, Reliability sampling plans for log-normal distribution based on progressively censored samples, *IEEE Trans. Reliab.*, **49** (2000), 199–203. https://doi.org/10.1109/24.877338
33. G. Prakash, D. C. Singh, Shrinkage estimation in exponential Type-II censored data under LINEX loss, *J. Korean Stat. Soc.*, **37** (2008), 53–61. https://doi.org/10.1016/j.jkss.2007.07.002
34. R. Calabria, G. Pulcini, Bayes 2-sample prediction for the inverse Weibull distribution, *Commun. Stat.-Theor. Method.*, **23** (1994), 1811–1824. https://doi.org/10.1080/03610929408831356
35. J. F. Lawless, *Statistical models and methods for lifetime data*, New York: John Wiley and Sons, 2003. https://doi.org/10.1002/9781118033005
36. B. Efron, R. J. Tibshirani, *An introduction to the bootstrap*, 1 Eds., New York: Chapman and Hall/CRC, 1994. https://doi.org/10.1201/9780429246593
37. M. E. Ghitany, B. Atieh, S. Nadarajah, Lindley distribution and its application, *Math. Comput. Simul.*, **78** (2008), 493–506. https://doi.org/10.1016/j.matcom.2007.06.007

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