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Research article

A class of multivariate linear partial differential equations on C^p

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Abstract: We used a new type of characteristics to solve a class of homogeneous linear multivariate partial differential equations on C^p. For x in R^p and n in Z^p, set $\partial_x^n = \prod_{j=1}^p \left(\partial/\partial x_j\right)^{n_j}$. Given square matrices $\left\{N_j\right\}$ and $\{S_n\}$ in $C^{s\times s}$, set

$$Y(x) = \exp\left(\sum_{j=1}^p x_j N_j\right) \text{ in } C^{s\times s}$$

and $T_n(x) = Y(x)\, S_n\, Y(-x)$ in $C^{s\times s}$. When $\left\{N_j\right\}$ commute, we show that the linear partial differential equation

$$\sum_{n=0_p}^q T_n(x)\, \partial_x^n f(x) = 0_s \text{ for } f(x) \text{ in } C^s$$

has solutions $f(x) = f_n(x, \nu)$ for each admissible $n \le sq$ and any ν in C^p such that $d(\nu) = 0$, where

$$d(\nu) = \det D(\nu),\ D(\nu) = \sum_{n=0_p}^q S_n \prod_{j=1}^p \left(\nu_j I_s + N_j\right)^{n_j}.$$

The research aims to develop a new method, based on a novel type of characteristics, for solving a broad class of multivariate homogeneous linear partial differential equations with matrix coefficients of a specific exponential-conjugate form, extending classical Cauchy characteristic techniques beyond the univariate case and providing explicit basis solutions parameterized over complex surfaces.

Keywords: Bell polynomial; expansion; commute; linear partial differential equation
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1. Introduction

The method of Cauchy characteristics is used to solve second-order univariate linear partial differential equations on C^2 in terms of arbitrary constants. See, for example, Chapter 3 of Myint-U [1] or Williams [2, page 328]. See also Goorjian [3], Menikoff [4], and Zeman [5]. Section 19 of Treves [6] sketches its extension to univariate partial differential equations of any order and notes that multivariate results are more difficult. This paper uses a different type of characteristic to solve a rich class of multivariate partial differential equations.

Let R and C denote the real and complex numbers. Set $Z = \{0, 1, 2, \ldots\}$. Let p and s be positive integers. For $n \leq q \in Z^p$ and $x \in R^p$, set

$$x^n = \prod_{j=1}^{p} x_j^{n_j},\ \partial_x = \partial/\partial x,\ \partial_x^n = \prod_{j=1}^{p} \left(\partial/\partial x_j\right)^{n_j},\ n! = \prod_{j=1}^{p} n_j!,\ (q)_n = q!/(q-n)!,$$

and $(q)_n = 0$ for $n \not\leq q$. For $n, m \in Z^p$, set $\binom{n}{m} = \prod_{j=1}^{p} \binom{n_j}{m_j}$. Given $q \in Z^p$ and matrix coefficients $T_n(x):\ R^p \to C^{s\times s}$ for $n \in Z^p$, set

$$L = \sum_{n=0_p}^{q} T_n(x)\partial_x^n,\ m = |q| = \sum_{j=1}^{p} q_j, \tag{1.1}$$

where $0_p = (0, \ldots, 0)' \in Z^p$. So, m is the *order* of the linear differential operator L. In Chapter 18 [6], Treves assumes that for $n = (m, 0, \ldots, 0)$, $T_n(x) = I_s$. He considers the problem $Lf(x) = f_0(x)$ with Cauchy conditions $\left[\partial_{x_1}^k f(x)\right]_{x_1=0} = v_k(x)$ for $0 \leq k < m$, where f_0, v_k are given $s \times 1$ functions. He shows how to reduce the order m to $m-1$ and so to 1, by increasing s. It then follows from the Cauchy–Kovaleska theorem in Chapter 17 [6] that the problem has a unique solution. Chapter 19 [6] then sketches how the method of Cauchy characteristics can often be used to find an actual solution when $s = 1$.

We consider a special class of matrix coefficients $T_n(x)$. We use a new type of characteristics to solve the homogeneous linear equation

$$L\, f(x) = \sum_{n=0_p}^{q} T_n(x)\, f_{\cdot n}(x) = 0_s, \tag{1.2}$$

where

$$f_{\cdot n}(x) = \partial_x^n f(x),\ T_n(x) = Y(x) S_n Y(-x),\ Y(x) = \exp\left(\sum_{j=1}^{p} x_j N_j\right), \tag{1.3}$$

for any matrices $N_j, S_n \in C^{s\times s}$ such that $N_1, \ldots, N_p$ commute.

We shall see in Section 5, that typically $T_n(x)$ is a mixture of polynomials in $\sum_{j=1}^{p} x_j$ and factors $\exp\{\pm\lambda' x\}$ for certain $\lambda \in C^p$. For example, in Example 4.2, $T_n(x)$ is linear in $\sin\ 2\theta$ and $\cos\ 2\theta$ where θ has the form $\lambda' x$.

In Section 2, we give our main results: Solutions to (1.2) in terms of a free parameter ν that ranges over a surface in C^p. ν can be used to satisfy boundary conditions. Corollary 2.2 gives *s basis solutions* $f_n(x, \nu)$ for each admissible $n \leq sq$; ν is confined to a surface in C^p depending on n determined by

$$d(\nu) = \det D(\nu), \tag{1.4}$$

where

$$D(v)=\sum_{n=0_p}^{q} S_n\ (vI_s+N)^n \in C^{s\times s},\ (vI_s+N)^n=\prod_{j=1}^{p}\left(v_jI_s+N_j\right)^{n_j}. \tag{1.5}$$

We call $D(v)$ the *characteristic matrix* of the operator L of (1.1). Theorem 2.3 shows that for a given n, these s basis solutions can be derived from each other. Section 3 gives examples. In Section 4, we derive commuting N_j when $s = 2$. For a discussion on commuting matrices, see `https://en.wikipedia.org/wiki/Commuting_matrices`. These include circulants. Section 5 gives a way of constructing commuting N_j using the Jordan form.

The research aims to develop a novel method for solving a broad class of multivariate, linear, homogeneous partial differential equations by introducing a new type of characteristic distinct from the classical method of Cauchy characteristics. While earlier approaches (such as those outlined by Treves and others) focus on univariate partial differential equations and often rely on order reduction techniques or the Cauchy-Kovalevskaya theorem, this paper targets more complex multivariate systems by considering differential operators with specially structured matrix coefficients. These coefficients are formed via similarity transformations involving exponential matrices generated by commuting matrices N_j. The key innovation is the construction of explicit solutions parameterized by a complex vector v, constrained to lie on a characteristic surface defined by the determinant of a "characteristic matrix" $D(v)$. This method not only enables analytical solution construction for such systems but also extends to practical applications and examples, including matrix types like circulants and those derived via Jordan forms.

When $p = 1$, L is an *ordinary* differential operator, and was used in Withers and Nadarajah [7] to solve an ordinary d.e. for perturbation of the planets. Set $i = \sqrt{-1}$.

2. Characteristic solutions to $L\,f(x) = 0_s$

Fix $p \in Z$, $s \in Z$ and $q \in Z^p$. Take $\left\{N_j\right\}$, $\{S_n,\ T_n(x)\}$, $Y(x)$, L of (1.1)-(1.2) and $d(v), D(v)$ of (1.5). Suppose that $\left\{N_j\right\}$ commute.

Theorem 2.1. *For $v \in C^p$, set*

$$E(v) = D(v)a(v) \in C^s. \tag{2.1}$$

Choose v such that

$$d(v) = 0 \text{ and } a = a(v) \in C^s \text{ such that } E(v) = 0_s. \tag{2.2}$$

Then for $Y_j(x)$, the jth column of $Y = Y(x)$, a solution to $L\,f(x) = 0_s$ is

$$f(x, v) = e^{v'x}Y(x)\ a(v) = e^{v'x}\sum_{j=1}^{s} a_j(v)Y_j(x). \tag{2.3}$$

Proof. By Leibniz' theorem,

$$\partial_x^n\ \left(e^{v'x}Y(x)\right)=\sum_{m=0_p}^{n}\binom{n}{m}A_{n-m}B_m = e^{v'x}Y(x)\ \ (vI_s+N)^n,$$

where

$$A_{n-m} = \partial_x^{n-m}\, e^{\nu' x} = \nu^{n-m} e^{\nu' x},\ B_m = \partial_x^m\, Y(x) = Y(x) N^m.$$

So, since $Y(x)^{-1} = Y(-x)$,

$$L\, e^{\nu' x} Y(x) = e^{\nu' x} Y(x) D(\nu),\ L\, f(x,\nu) = e^{\nu' x} Y(x) E(\nu) = 0_s.$$

The proof is complete. □

We call Eq (2.3) a *primary characteristic solution* or p.c.s. of L. $D = D(\nu)$ and $d = d(\nu)$ are polynomials of degree q and sq in ν. The condition $d(\nu) = 0$ confines ν to the *primary characteristic surface* in C^p.

Theorem 2.2. *For $a = a(\nu)$ of (1.3)–(2.2), and $n, m \in Z^p$, set*

$$(x + \partial_\nu)^n = \prod_{j=1}^{p} \left(x_j + \partial_{\nu_j}\right)^{n_j},\ a_{.m}(\nu) = \partial_\nu^m a(\nu),$$

$$z_n = z_n(x,\nu) = (x + \partial_\nu)^n\, a(\nu) = \sum_{m=0_p}^{n} \binom{n}{m} x^{n-m} a_{.m}(\nu), \tag{2.4}$$

$$f_n = f_n(x,\nu) = e^{\nu' x} Y(x)\, z_n(x,\nu) = e^{\nu' x} \sum_{j=1}^{s} z_{n,j}(x,\nu) Y_j(x). \tag{2.5}$$

Then

$$L\, f_n(x,\nu) = e^{\nu' x} Y(x)\ (x + \partial_\nu)^n\, E(\nu),$$

where $E_{.m}(\nu) = \partial_\nu^m E(\nu)$. So, $f_n(x,\nu)$ is a solution to $Lf(x) = 0_s$ if $d(\nu) = 0$ and

$$E_{.m}(\nu) = 0_s \text{ for } 0_p \le m \le n. \tag{2.6}$$

Proof. By Leibniz' rule,

$$\partial_t^o\, (e^{\nu t} Y z_n) = \sum_{m=0_p}^{o} \binom{o}{m} e^{\nu t} Y\ (\nu I_s + N)^{o-m}\, \partial_x^m z_n,$$

$$L f_n = e^{\nu' x} Y G_n,$$

where

$$G_n = \sum_{o=0_p}^{q} S_o \sum_{m=0_p}^{o} \binom{o}{m} (\nu I_s + N)^{o-m}\, \partial_x^m\, (x + \partial_\nu)^n\ a = \sum_{m,r} \binom{n}{m}\binom{n-m}{r} x^{n-m-r} D_{.m} a_{.r}.$$

Transform from m to $c = m + r$. Then $\binom{n}{m}\binom{n-m}{r} = \binom{n}{c}\binom{c}{r}$. So,

$$G_n = \sum_{c} \binom{n}{c} x^{n-c} H_c,$$

where

$$H_c = \sum_r \binom{c}{r} D_{.c-r} a_{.r} = \partial_\nu^c E(\nu) = 0_s,$$

by (2.6). The proof is complete. □

So, $f_{0_p}(x,\nu)$ is the p.c.s. (2.3). We call the other $f_n(x,\nu)$ of (2.5), *secondary characteristic solutions.* We call n *admissible* if (2.6) holds for some ν. For $m, o \in Z^p$,

$$E_{.m}(\nu) = \sum_{o=0_p}^{m} D_{.o}(\nu) a_{.m-o}(\nu),\ D_{.o}(\nu) = \sum_{n=0_p}^{q} (n)_o S_n \left(\nu I_s + N\right)^{n-o}.$$

Corollary 2.1. *Consider the exceptional case when $D(\nu) = 0_{s\times s}$. Suppose that*

$$D_{.m}(\nu) = 0_{s\times s} \text{ for } 0_p \le m \le n.$$

Then for any $a \in C^s$, a characteristic solution is

$$f_n(x,\nu) = x^n e^{\nu' x} Y(x)\, a.$$

We can expand $D(\nu)$ and $d(\nu)$ as

$$D(\nu) = \sum_{k=0_p}^{q} \nu^k D_k,\ d(\nu) = \sum_{k=0_p}^{sq} \nu^k d_k,$$

where

$$D_k = \sum_{n=k}^{q} \binom{n}{k} S_n N^{n-k},\ D_{0_p} = D\left(0_p\right),\ D_q = S_q,$$
$$d_{0_p} = \det D\left(0_p\right),\ d_{sq} = \det S_q.$$

So,

$$D_{.m}(\nu) = \sum_{k=m}^{sq} (k)_m \nu^{k-m} D_k,\ d_{.m}(\nu) = \sum_{k=m}^{sq} (k)_m \nu^{k-m} d_n. \tag{2.7}$$

We now transfer the condition (2.6) from $E(\nu)$ to $d(\nu)$. Let M be the *adjoint* of D: $(-1)^{j+k} M_{k,j}$ is the determinant of D with its jth row and kth column deleted. If $d \neq 0$, then $M = dD^{-1}$.

Corollary 2.2. *Let $e_{1,s},\ldots,e_{s,s}$ be any basis for R^s. For D and d of (1.5) and (1.4), set $a_{(j)} = Me_{j,s}$, $E_{(j)} = Da_{(j)} = de_{j,s}$. So for $m \in Z^p$, $E_{(j).m} = d_{.m} e_{j,s}$. Choose ν so that $d(\nu) = 0$. Given $1 \le j \le s$ and $a = a_{(j)}$, $f_{n,j}(x) = f_n(x,\nu)$ of (2.4), (2.5) is a characteristic solution if*

$$d_{.m}(\nu) = 0 \text{ for } 0_p \le m \le n. \tag{2.8}$$

We can say that (2.8) confines v to the *characteristic surface of order* n in C^p. By (2.7), (2.8) does not extend to $n = sq$ if $d_{sq} \neq 0$.

We now take $e_{j,s}$ as the jth unit vector in R^s. So for $1 \leq j \leq s$, $a_{(j)}$ is the jth column of M. For $1 \leq k \leq s$, its kth element is $a_{(j)k} = M_{k,j} = a_{j,k}$. Corollary 2.2 breaks the solution $f_n(x, v)$ into s *basis solutions* $f_{n,j}(x)$, $1 \leq j \leq s$. For example, if $s = 2$, then

$$d = D_{1,1}D_{2,2} - D_{1,2}D_{2,1},\ M = \begin{pmatrix} D_{2,2} & -D_{1,2} \\ -D_{2,1} & D_{1,1} \end{pmatrix},$$
$$e_{1,2} = (1, 0)',\ e_{2,2} = (0, 1)',$$

implying

$$a_{(1)} = (D_{2,2}, -D_{2,1})'\ ,\ a_{(2)} = (-D_{1,2}, D_{1,1})'\ ,\ d = (D_{2,2}, D_{1,2})\, a_{(1)} = (D_{2,1}, D_{2,2})\, a_{(2)}. \tag{2.9}$$

If $s = 3$, then

$$e_{1,3} = (1, 0, 0)',\ e_{2,3} = (0, 1, 0)',\ e_{3,3} = (0, 0, 1)',$$

implying

$$\begin{aligned} a_{(1)} &= (D_{2,2}D_{3,3} - D_{2,3}D_{3,2}, D_{2,3}D_{3,1} - D_{2,1}D_{3,3}, D_{2,1}D_{3,2} - D_{2,2}D_{3,1})'\ , \\ a_{(2)} &= (D_{1,3}D_{3,2} - D_{1,2}D_{3,3}, D_{1,1}D_{3,3} - D_{1,3}D_{3,1}, D_{1,2}D_{3,1} - D_{1,1}D_{3,2})'\ , \\ a_{(3)} &= (D_{1,2}D_{2,3} - D_{1,3}D_{2,2}, D_{1,3}D_{2,1} - D_{1,1}D_{2,3}, D_{1,1}D_{2,2} - D_{1,2}D_{2,1})'\ , \end{aligned} \tag{2.10}$$

and

$$d = (D_{1,1}, D_{1,2}, D_{1,3})\ a_{(1)} = (D_{2,1}, D_{2,2}, D_{2,3})\ a_{(2)} = (D_{3,1}, D_{3,2}, D_{3,3})\ a_{(3)}.$$

Theorem 2.3 shows that for $1 \leq j \leq s$ and each n, $f_{n,j}(x)$ is a linear combination of $\{f_{m,1}(x) :\ m \leq n\}$. So, $f_{0_p,j}(x) \propto f_{0_p,1}(x)$. Its proofs needs a lemma.

Lemma 2.1. *For $a = a(v)$, $b = b(v) \in C$,*

$$(x + \partial_v)^n\, ab = \sum_{k=0_p}^{n} \binom{n}{k} a_{.k}\ (x + \partial_v)^{n-k}\, b. \tag{2.11}$$

Proof. The left-hand side of (2.11) is equal to

$$\sum_{m=0_p}^{n} \binom{n}{m} x^{n-m} \partial_v^m ba = \sum_{m=0_p}^{n} \binom{n}{m} x^{n-m} \sum_{k=0_p}^{m} \binom{m}{k} b_{.m-k} a_{.k} = \sum_{k=0_p}^{n} A_{n,k} a_{.k},$$

where

$$A_{n,k} = \sum_{m=k}^{n} \binom{n}{m}\binom{m}{k} x^{n-m} b_{.m-k} = \binom{n}{k} (x + \partial_v)^{n-k}\, b.$$

The proof is complete. □

Theorem 2.3. *Suppose that for some k,* $a_{1,k} = M_{k,1} \neq 0$*. Then*

$$a_{(j)} = c_{j,k} a_{(k)},$$

where

$$c_{j,k} = a_{1,j}/a_{1,k} = M_{j,1}/M_{k,1} \tag{2.12}$$

and

$$f_{n,j}(x) = \sum_{l=0_p}^{n} \binom{n}{l} c_{j,k.n-l} f_{l,k}(x). \tag{2.13}$$

So,

$$f_{0_p,j}(x) \ = c_{j,k} f_{0_p,k}(x). \tag{2.14}$$

Proof. For $s = 2, 3$, $a_{(j)} = c_{j,k} a_{(k)}$ can be verified directly from (2.9) and (2.10). For example, if $s = 2$, then (2.12)–(2.14) hold for $(j, k) = (1, 2)$ if $D_{1,2} \neq 0$, and for $(j, k) = (2, 1)$ if $D_{2,2} \neq 0$. More generally, for $1 \leq k \leq s$, $MD = dI_s$ has kth column

$$\sum_{j=1}^{s} D_{j,k} a_{(j)} = de_{k,s} = 0.$$

These s equations imply that $a_{(j)} = c_{j,k} a_{(k)}$. Multiplying by $(x + \partial_\nu)^n$ gives

$$z_{n,j}(x) \ = (x + \partial_\nu)^n \, a_{(j)} = \sum_{l=0_p}^{n} \binom{n}{l} c_{j,k.n-l} z_{l,k}(x),$$

by Lemma 2.1. Now, multiply by $e^{\nu' x} Y(x)$. □

Note that, since $DM = dI_s$, $\det M = d^{s-1}$.

Equation (2.6) or (2.8) confines ν to a surface $\mathcal{S}_n$ say. So, for $\mu(.)$ a measure on $\mathcal{S}_n$,

$$f_n(x) = \int_{\mathcal{S}_n} f_n(x, \nu) d\mu(\nu)$$

is also a solution when well-defined. This can be used to satisfy boundary conditions, as in Chapter 6 of [8]. For example, $f_n\left(0_p\right) = f_0$ some given value in C^s, if

$$\int_{\mathcal{S}_n} a_{.n}(\nu) d\mu(\nu) = f_0,$$

since $f_n\left(0_p, \nu\right) = a_{.n}(\nu)$.

3. Examples

In the examples in this section, we explore the structure and solutions of a system of linear partial differential equations involving a differential operator L, matrices S_n, and matrices N_j that are scalar multiples of the identity or a matrix M. The operator L acts on vector-valued functions and includes derivatives up to a multi-index q, with coefficients S_n. When $N_j = \lambda_j I_s$, the matrix exponential $Y(x)$ simplifies to a scalar exponential times the identity, and the resulting differential equation $Lf(x) = 0$ becomes ordinary. The transformed variables $\overline{\nu} = \nu + \lambda$, and the characteristic polynomial $d(\nu)$, involve powers of $\overline{\nu}$ and determinants of matrices S_n. When the operator takes a particularly simple form with only the highest-order derivative and a matrix S, the characteristic surfaces are defined by $\overline{\nu}^q = \theta_k$, where θ_k are eigenvalues of $-S$. These surfaces determine where the characteristic solutions exist, and the solutions themselves can be expressed in terms of exponential-polynomial functions whose coefficients and structure are governed by these eigenvalues and the structure of S. In further examples, transformations using an invertible matrix P simplify the partial differential equation into a form where the transformed operator acts diagonally via eigenvalues of M, and solutions can again be constructed using exponential and polynomial terms, guided by transformed versions of the original coefficients. The case for $s = 2$ is treated in detail, showing how the determinant and eigenstructure of the transformed operator influence the construction and independence of characteristic solutions.

Example 3.1. *Take $\lambda_j \in C$ and $N_j = \lambda_j I_s$. Then*

$$Y(x) = e^{\lambda' x} I_s,\ T_n(x) = S_n,\ L = \sum_{n=0_p}^{q} \nu^n S_n \partial_x^n,$$

so that $Lf(x) = 0_s$ is an ordinary d.e. Also

$$\begin{aligned}
(\nu I_s + N)^n &= \overline{\nu}^n I_s,\ \overline{\nu} = \nu + \lambda \in C^p,\\
D &= \sum_{n=0_p}^{q} \overline{\nu}^n S_n,\ d(\nu) = \sum_{n=0_p}^{sq} \overline{\nu}^n \overline{d}_n,\ \overline{d}_{0_p} = \det S_{0_p},\ \overline{d}_{sq} = \det S_q,\\
d_{\cdot m}(\nu) &= \sum_{n=0_p}^{sq} (n)_m \overline{\nu}^{n-m} \overline{d}_n,
\end{aligned}$$

and characteristic solutions are given by Corollary 2.2.

Now take $S_n = 0_{s\times s}$ except for $S_q = I_s$ and $S = S_{0_p}$. Then

$$L\,f(x) = f_{\cdot q}(x) + S\,f(x),\ D(\nu) = S + \theta I_s,$$

where $\theta = \overline{\nu}^q$. Suppose that $d = 0$ and $Da = 0_s$. Then $Sa = -\theta a$. So θ is any eigenvalue of $-S$, say θ_k, (but otherwise $\overline{\nu}$ is arbitrary), and a is its eigenvector. So, ν in C^p lies in one of the s characteristic surfaces $\overline{\nu}^q = \theta_k$, $k = 1, \ldots, s$. As we illustrate in Example 3.2, $a(\nu)$ and $E(\nu)$ of (2.1) can be expanded as

$$a(\nu) = \sum_{j=0}^{s-1} a_j \theta^j,\ E(\nu) = \sum_{j=0}^{s} E_j \theta^j.$$

So,

$$E_{.m}(\nu) = \sum_{j=0}^{s} E_j \, (\theta^j)_{.m}, \; \left(\theta^j\right)_{.m} = (jq)_m \overline{\nu}^{jq-m}. \tag{3.1}$$

Example 3.2. *Consider Example 3.1 with $s = 2$. So,*

$$E = E(\nu) = Da = \begin{pmatrix} (S_{1,1} + \theta)\, a_1 & S_{1,2} a_2 \\ S_{2,1} a_1 & (S_{2,2} + \theta)\, a_2 \end{pmatrix},$$
$$d(\nu) = (S_{1,1} + \theta)(S_{2,2} + \theta) - S_{1,2} S_{2,1} = d_S + t_S \theta + \theta^2,$$

where $d_S = \det S$ and $t_S = \text{trace } S = S_{1,1} + S_{2,2}$. Take ν such that $\theta = -t_S/2 \pm \left(t_S^2/4 - d_S\right)^{1/2}$. So $d(\nu) = 0$ and (2.2) holds.

First choose $a = a(\nu) = a_{(1)} = \left(S_{2,2} + \theta, -S_{2,1}\right)'$, where $\theta = \overline{\nu}^q$. So, $E(\nu) = (1, 0)' d(\nu)$, and for this choice of $a(\nu)$, a p.c.s. is

$$f_{(1)}(x, \nu) = e^{\overline{\nu}' x} \left(S_{2,2} + \theta, -S_{2,1}\right)'.$$

For $m \in Z^p$ and $m \neq 0_p$,

$$d_{.m}(\nu) = t_S \, \theta_{.m} + \left(\theta^2\right)_{.m}, \; (x + \partial_\nu)^n \, 1 = x^n. \tag{3.2}$$

Set

$$\theta_{n,x} = (x + \partial_\nu)^n \, \theta = \sum_{m=0_p}^{n} \binom{n}{m} x^{n-m} (q)_m \overline{\nu}^{q-m}. \tag{3.3}$$

The case $t_S \neq 0$. *Take $0 \leq n \leq q$, $n \neq 0_p, q$. If $d_S = 0$, then $\theta = 0$ or $-t_S$, $0 = d_{.n} = \overline{\nu}^{q-n} \left[t_S (q)_n + (2q)_n \overline{\nu}^q\right]$. So either $\overline{\nu}^{q-n} = 0$, $\theta = 0$, $d_S = 0$, or $\theta = -t_S (q)_n/(2q)_n$. If this last condition holds for $n = e_{j,p}$, then $\theta = -t_S/2$. But this rules out it holding for any other n. Now suppose that $\overline{\nu}^{q-n} = 0$. This holds if we choose ν such that $\overline{\nu}_j = 0$ for one or more j in $1, \ldots, p$ such that $n_j \neq q_j$. For example, if $q - n = (1, 1)$, we can choose $\nu_1 = 0$ and arbitrary ν_2, or vice versa. Then, (2.8) holds by (3.2) and (3.1). By (3.3), $\theta_{n,x} = 0$, giving*

$$z_n(x, \nu) = x^n \left(S_{2,2}, -S_{2,1}\right)', \; f_n(x, \nu) = x^n e^{\overline{\nu}' x} \left(S_{2,2}, -S_{2,1}\right)'.$$

This does not extend to $n = q$ as $d_{.q}(\nu) = t_S q! \neq 0$. This gives $o(q) - 2$ solutions where

$$o(q) = \prod_{j=1}^{p} \left(q_j + 1\right) \text{ is the number of } n \text{ such that } 0_p \leq n \leq q.$$

Now choose $a = a_{(2)} = (-S_{1,2}, S_{1,1} + \theta)'$. Then, $E(\nu) = (0, 1)' d(\nu)$, a p.c.s. is

$$f_{(2)}(x, \nu) = e^{\overline{\nu}' x} \left(-S_{1,2}, S_{1,1} + \theta\right)',$$

and another $o(q) - 2$ solutions are

$$z_n(x, \nu) = x^n \left(-S_{1,2}, S_{1,1}\right)', \; f_n(x, \nu) = x^n e^{\overline{\nu}' x} \left(-S_{1,2}, S_{1,1}\right)'$$

for $n \neq 0_p$. *But by (2.13), these are linearly dependent on the first choice.*

The case $t_S = 0$. *Take* $0_p \leq n \leq 2q$, $n \neq 0_p, 2q$. *Suppose that* $\overline{\nu}^{2q-n} = 0$. *This holds if we choose* ν *such that* $\overline{\nu}_j = 0$ *for one or more* j *in* $1, \dots, p$ *such that* $n_j \neq 2q_j$. *Then (2.6) holds by (3.2) and (3.1). The choice* $a = a_{(1)}$ *gives* $o(2q) - 2$ *solutions,* $f_n(x,\nu) = e^{\overline{\nu}' x} z_n(x,\nu)$, *where* $z_n(x,\nu) = x^n \left(S_{2,2}, -S_{2,1}\right)' + (1,0)'\theta_{n,x}$. *A variety of* $\theta_{n,x}$ *are possible. If* $p = 2$ *and* $q_2 = 0$, *then* $n_2 = 0$, $1 \leq n_1 < 2q_1$, $0 = \overline{\nu}^{2q-n} = \overline{\nu}_1^{2q_1 - n_1}$ *implying* $\overline{\nu}_1 = 0$, $\theta = \theta_{n,x} = 0$.

If $p = 2$, $q_2 = 1$, $q_1 \geq 1$, $n = (0,1)$, *then* $\overline{\nu}_1 = 0$ *implying* $\theta_{n,x} = 0$, *but* $\overline{\nu}_2 = 0$ *implying* $\theta_{n,x} = q_1! \overline{\nu}_1^{q_1}$.

If $p = 2$, $q_1 \geq 1$, $\overline{\nu}_1 = 0$, *then*

$$\theta_{q,x} = q_1! \sum_{m_2=0}^{q_2} x_2^{q_2 - m_2} \left(q_2\right)_{m_2} \overline{\nu}_2^{q_2 - m_2};$$

for example, if $\overline{\nu}_2 = 0$, *then* $\theta_{q,x} = q!$.

The choice $a = a_{(2)}$ *gives another* $o(2q) - 2$ *solutions,*

$$f_n(x,\nu) = e^{\overline{\nu}' x} z_n(x,\nu),$$

where

$$z_n(x,\nu) = x^n \left(-S_{1,2}, S_{1,1}\right)' + (0,1)'\theta_{n,x}.$$

But by (2.13), these are linearly dependent on the first choice.

In the next example, we transform to

$$\overline{L} = P^{-1}L,\ g(x) = P^{-1}f(x),\ b = P^{-1}a,\ \overline{E} = P^{-1}E,\ \overline{D} = P^{-1}DP,\ \overline{M} = P^{-1}MP,$$

for a certain $P \in C^{s \times s}$, as this puts the partial differential equation in a simpler form.

Example 3.3. *Suppose that* $L f(x) = f_{.q}(x) + S f(x)$ *and* $N_j = \lambda_j M$ *for* $\lambda_j \in C$, $M = P \Lambda P^{-1}$, $\Lambda = \operatorname{diag}\left(\tau_1, \dots, \tau_p\right)$. *Set*

$$\theta_{j,k} = \nu_j + \lambda_j \tau_k,\ \overline{\Lambda}_j = \nu_j I_s + \lambda_j \Lambda = \operatorname{diag}\left(\theta_{j,1}, \dots, \theta_{j,s}\right),\ \theta_k = \prod_{j=1}^{p} \theta_{j,k}^{q_j}.$$

Then, $\nu_j I_s + N_j = P \overline{\Lambda}_j P^{-1}$ *and*

$$(\nu I_s + N)^q = P \overline{\Lambda}^q P^{-1},\ \overline{\Lambda}^q = \prod_{j=1}^{p} \overline{\Lambda}_j^{q_j} = \operatorname{diag}\left(\theta_1, \dots, \theta_s\right).$$

Set

$$w_x = \lambda' x = \sum_{j=1}^{p} \lambda_j x_j,\ \overline{S} = P^{-1} S P.$$

Then

$$\sum_{j=1}^{p} x_j N_j = \lambda' x\, M,\ Y(x) = \exp(\lambda' x\, M) = P \exp(w_x \Lambda) P^{-1},$$

$$T(x) = P \exp(w_x \Lambda) \overline{S} \exp(-w_x \Lambda) P^{-1}.$$

Set $g(x) = P^{-1} f(x)$, $b(\nu) = P^{-1} a(\nu)$. *Then,*

$$P^{-1} L f(x) = g_{.q}(x) + \exp(w_x \Lambda) \overline{S} \exp(-w_x \Lambda) g(x) = \widetilde{L} g(x) \text{ say}, \tag{3.4}$$

and $Lf(x) = \widetilde{L} g(x) = 0_s$ *has solution*

$$g_n(x, \nu) = P^{-1} f_n(x, \nu) = e^{\nu' x} e^{w_x \Lambda} (x + \partial_\nu)^n b(\nu) \tag{3.5}$$

at ν *such that* $d = \det \overline{D} = 0$ *and (2.6) holds with* E *replaced by* $\overline{E} = \overline{D} b$, *where* $\overline{D} = P^{-1} D P = \overline{S} + \overline{\Lambda}^q$, $b = b(\nu) = P^{-1} a(\nu)$. *Also,* $0_s = P^{-1} D a = \overline{D} b$ *if and only if* $\overline{S} b(\nu) = -\overline{\Lambda}^q b(\nu)$, *so that* -1 *is an eigenvalue of* $\overline{\Lambda}^{-q} \overline{S}$ *for* ν *such that* $\prod_{k=1}^{s} \theta_k \neq 0$.

Suppose that $s = 2$. *Then*

$$d(\nu) = \det \overline{D} = \det \overline{S} + \theta_1 \overline{S}_{2,2} + \theta_2 \overline{S}_{1,1} + \theta_1 \theta_2,$$

where

$$\theta_k = \prod_{j=1}^{p} \theta_{j,k}^{q_j}.$$

First, choose $b(\nu) = \left(\overline{D}_{2,2}, -\overline{D}_{2,1}\right)' = b_{(1)}$ *say. If* $d(\nu) = 0$, *then* $\overline{E} = (1, 0)' d(\nu)$ *and a p.c.s. of (3.4) is (3.5) with* $n = 0_p$. *Now, choose* $b(\nu) = \left(-\overline{D}_{1,2}, \overline{D}_{1,1}\right)' = b_{(2)}$. *If* $d(\nu) = 0$, *then* $\overline{E} = (0, 1)' d(\nu)$ *and a second p.c.s. is (3.5) with* $n = 0_p$. *These two p.c.s. are proportional to each other by (2.14) with D,* $a_{(j)}$, $f_{n,j}$ *replaced by* $\overline{D}$, $b_{(j)}$, $g_{n,j} = P^{-1} f_{n,j}$.

Now take $0 \leq n \leq q$, $n \neq 0_p, q$. *Then*

$$d_{.n}(\nu) = \theta_{1.n} \left(\theta_2 + \overline{S}_{2,2}\right) + \theta_{2.n} \left(\theta_1 + \overline{S}_{1,1}\right),\ \theta_{k.n} = (q)_n \prod_{j=1}^{p} \theta_{j,k}^{q_j - n_j}.$$

So, $\nu_j = -\lambda_j \tau_k$ *if and only if* $\theta_{j,k} = 0$. *Further,*

$$\theta_{j_1,1}^{q_{j_1} - n_{j_1}} = \theta_{j_2,2}^{q_{j_2} - n_{j_2}} = 0$$

implies

$$\theta_{1.n} = \theta_{2.n} = 0,$$

which implies

$$d_{.n}(\nu) = 0, \tag{3.6}$$

so that applying Corollary 2.2 to the transformed problem, $g_{n,j}$ *and* $f_{n,j}$ *are characteristic solutions. For example, (3.6) holds if* $\theta_{1,1} = 0$, $q_1 \geq 2$ *and* $\theta_{j,2} = 0$ *for some* $j \neq 1$ *such that* $q_j \geq 1$, *or if* $\theta_{1,2} = 0$, $q_2 \geq 2$ *and* $\theta_{j,1} = 0$ *for some* $j \neq 2$ *such that* $q_j \geq 1$, *or if* $\theta_{j_1,1} = \theta_{j_2,2} = 0$ *where* $q_{j_1} \geq 1$, $q_{j_2} \geq 1$, $1 \leq j_1 \neq j_2 \leq p$.

We can extend this to $s \geq 2$ *by applying Corollary 2.2 with* L *and* f *replaced by* $\widetilde{L}$ *and* g.

Example 3.4. *Take* $N_j = P\Lambda_j P^{-1}$, *where* $\Lambda_j = \operatorname{diag}\left(\lambda_{1,j}, \ldots, \lambda_{s,j}\right)$. *So,*

$$\sum_{j=1}^{p} x_j N_j = P\Lambda(x)P^{-1},\ Y(x) = Pe^{\Lambda(x)}P^{-1},\ e^{\Lambda(x)} = \operatorname{diag}\left(e^{\lambda_1' x}, \ldots, e^{\lambda_s' x}\right),\ \lambda_k' x = \sum_{j=1}^{p} \lambda_{k,j} x_j,$$

where

$$\Lambda(x) = \sum_{j=1}^{p} x_j \Lambda_j.$$

Set $\overline{L} = Y(-x)L$, $g(x) = P^{-1}f(x)$ *and* $\overline{S}_n = S_n P$ *or* $P^{-1}S_n P$. *Then*

$$Lf(x) = 0_s,$$

if and only if

$$\overline{L}f(x) = 0_s,$$

if and only if

$$\sum_{n=0_p}^{q} \overline{S}_n e^{-\Lambda(x)} g_{.n}(x) = 0_s,$$

if and only if

$$\sum_{n=0_p}^{q} \sum_{k=1}^{s} \overline{S}_{n,j,k} e^{-\lambda_k' x} g_{k.n}(x) = 0_s \text{ for } j = 1, \ldots, s.$$

A characteristic solution is

$$P^{-1} f_n(x, \nu) = g_n(x, \nu) = e^{\nu' x} e^{\Lambda(x)} P^{-1} z_n(x, \nu),$$

if (2.4) and (2.6) hold, where

$$D(\nu) = \sum_{n=0_p}^{q} S_n P (\nu I_s + \Lambda)^n P^{-1},\ D_{.m}(\nu) = \sum_{n=0_p}^{q} (n)_m S_n P (\nu I_s + \Lambda)^{n-m}\ P^{-1},$$

$$(\nu I_s + \Lambda)^n = \operatorname{diag}\left(\overline{\lambda}_1^n, \ldots, \overline{\lambda}_s^n\right),\ \overline{\lambda}_k^n = \prod_{j=1}^{p} \overline{\lambda}_{k,j}^{n_j},\ \overline{\lambda}_{k,j} = \nu_j + \lambda_{k,j}.$$

Example 3.5. *Take* $N_j = \lambda_j N_0$ *for* λ_j *a scalar. Then* $D(\nu)$ *is given by (1.5) with*

$$(\nu I_s + N)^n = \prod_{j=1}^{p} \left(\nu_j I_s + \lambda_j N_0\right)^{n_j} = (\nu I_s + \lambda N_0)^n.$$

Also,

$$\sum_{j=1}^{p} x_j N_j = \lambda' x\, N_0,\ Y(x) = \exp\left(\lambda' x\, N_0\right),\ T_n = Y(x) S_n Y(-x),\ L = Y(x)\overline{L},$$

for

$$\overline{L} = \sum_{n=0_p}^{q} S_n Y(-x)\, \partial_x^n$$

and

$$f(x,\nu) = e^{\nu' x} e^{\lambda' x N_0}\ a(\nu)$$

is the p.c.s. if (2.2) holds. An alternative solution is

$$f(x) = Y(x) f\left(0_p\right),$$

if $F(\lambda, N_0) = 0_{s\times s}$*, where*

$$F(\lambda, N_0) = \sum_{n=0_p}^{q} S_n N_0^{|n|} \lambda^n,\ |n| = \sum_{j=1}^{p} n_j. \tag{3.7}$$

For, $f_{.n}(x) = \lambda^n N_0^{|n|} f(x)$, $\overline{L}\ f(x) = F(\lambda, N_0) f\left(0_p\right)$*. Since* $D\left(0_p\right) = F(\lambda, N_0)$*, this is an example of Corollary 2.1 with* $\nu = 0_p$.

Example 3.6. *Suppose that* $p = 2$*,* S *and* N_0 *are any matrices in* $C^{s\times s}$*,* $Y = e^{x_2 N_0}$*,* $T(x_2) = YSY^{-1}$ *and*

$$Lf(x) = I_s f_{.q_1,0}(x) + T(x_2)\, f_{.0,q_2}(x).$$

Then,

$$D = \nu_1^{q_1} + S\ \ (\nu_2 + N_0)^{q_2},$$

so that $\nu_1^{q_1}$ *is any eigenvalue of* $-S\ \ (\nu_2 + N_0)^{q_2}$*. In this case,* $N_1 = 0_{s\times s}$ *and* $N_2 = N_0$.

4. Commuting $\left\{N_j\right\}$ when $s = 2$

Here, we derive sets of commuting $\left\{N_j\right\}$ when $s = 2$. In Example 4.1, $Lf = L_0 f + \cosh 2w\ L_1 f + \sinh 2w\ L_2 f$, where L_j are linear differential operators with *constant* coefficients, and w is linear in x, y. The same is true of Example 4.1, except that $\cosh 2w$ and $\sinh 2w$ is replaced by $\cos 2w$ and $\sin 2w$.

We begin by computing $(\nu I_2 + N)^n$ and e^{xN} when $p = 1$ for a special N and then for general N. From Withers and Nadarajah [7], we have

Theorem 4.1. *Take* $s = 2$, $x \in C$, $N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. *Then*

$$e^{xN} = I_2\ \cos x + N \sin x = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix},\ (\nu I_2 + N)^n = g_n + h_n N,$$

where

$$g_n = g_n(\nu) = \sum_j \binom{n}{2j} (-1)^j \nu^{n-2j} = \text{Real}(\nu + i)^n,$$
$$h_n = h_n(\nu) = \sum_j \binom{n}{2j+1} (-1)^j \nu^{n-2j-1} = \text{Imag}(\nu + i)^n.$$

For $S \in C^{2\times2}$, *set* $T(x, S) = e^{xN} S e^{-xN}$. *Then*

$$-2T(x, S) = B(S) + C(S) \cos 2x + G(S) \sin 2x = B(S) + G(S) N e^{2xN},$$

where

$$B(S) = \begin{pmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{pmatrix},\ C(S) = \begin{pmatrix} C_1 & -C_2 \\ -C_2 & -C_1 \end{pmatrix} = G(S)N,\ G(S) = \begin{pmatrix} C_2 & C_1 \\ C_1 & -C_2 \end{pmatrix},$$
$$B_1 = S_{1,1} + S_{2,2},\ B_2 = S_{1,2} - S_{2,1},\ C_1 = S_{1,1} - S_{2,2},\ C_2 = S_{1,2} + S_{2,1}.$$

Also, $\exp\{yT(x, S)\} = T\left(x, e^{yS}\right)$ *for* $y \in C$.

Theorem 4.2. *Take* $x \in C$. *Consider the general* 2×2 *matrix*

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then,

$$e^{xN} = e^{xu} \left[(c_x - us_x) I_2 + s_x N\right] = e^{xu} \begin{pmatrix} c_x + \nu s_x & bs_x \\ cs_x & c_x - \nu s_x \end{pmatrix}, \tag{4.1}$$

where

$$u = (a + d)/2,\ \nu = (a - d)/2,\ t = (\alpha - \beta)/2 = \pm\left[-\det(N - uI_2)\right]^{1/2},$$
$$c_x = \cosh tx,\ s_x = t^{-1} \sinh tx \text{ for } t \neq 0,$$

and α, β *are the eigenvalues of* N. *Also for* $m \in Z$,

$$N^m = \sum_{n=0}^{m} \binom{m}{n} u^{m-n} F_n = \begin{pmatrix} a_m(\nu) & b_m \\ c_m & a_m(-\nu) \end{pmatrix}, \tag{4.2}$$

where

$$F_{2n} = t^{2n} I_2,\ F_{2n+1} = t^{2n} (N - uI_2),$$

$$a_m(v) = \sum_{n=0}^{m} \binom{m}{n} u^{m-n} F_{n,1,1} = s_{m,0} + vt^{-1} s_{m,1},$$

$$b_m/b = c_m/c = \sum_{n=0}^{m} \binom{m}{n} u^{m-n} F_{n,1,2} = t^{-1} s_{m,1},$$

where

$$F_{2n,1,1} = t^{2n},\ F_{2n+1,1,1} = vt^{2n},\ F_{2n,1,2} = 0,\ F_{2n+1,1,2} = t^{2n},$$
$$2s_{m,0} = (u+t)^m + (u-t)^m,\ 2s_{m,1} = (u+t)^m - (u-t)^m.$$

Further,

$$(vI_2 + N)^n = \sum_m \binom{n}{m} v^{n-m} \begin{pmatrix} a_m(v) & b_m \\ c_m & a_m(-v) \end{pmatrix} = \begin{pmatrix} \widetilde{a}_n(v) & \widetilde{b}_n \\ \widetilde{c}_n & \widetilde{a}_n(-v) \end{pmatrix}, \tag{4.3}$$

where

$$2\widetilde{a}_n(v) = \left(1 + vt^{-1}\right)(\overline{v} + t)^n + \left(1 - vt^{-1}\right)(\overline{v} - t)^n,\ \overline{v} = v + u,$$
$$2\widetilde{b}_n/b = 2\widetilde{c}_n/c = t^{-1}\left[(\overline{v}+t)^n - (\overline{v}-t)^n\right].$$

If $t = 0$, then replace s_x by x and $t^{-1}s_{m,1}$ by mu^{m-1}, so that

$$e^{xN} = e^{xu}\left[I_2 + x\begin{pmatrix} v & b \\ c & -v \end{pmatrix}\right],\ N^m = u^m + mu^{m-1}\begin{pmatrix} v & b \\ c & -v \end{pmatrix}$$

and (4.3) holds with

$$\widetilde{a}_n(v) = \overline{v}^n + vn\overline{v}^{n-1},\ \widetilde{b}_n/b = \widetilde{c}_n/c = n\overline{v}^{n-1}.$$

Now, suppose that $d = a$. Then for $t = (bc)^{1/2} \neq 0$ and $x \in C$, $v = 0$,

$$e^{xN} = e^{ax}\begin{pmatrix} \cosh tx & bt^{-1}\sinh tx \\ ct^{-1}\sinh tx & \cosh tx \end{pmatrix},\ N^m = \begin{pmatrix} a_m & b_m \\ c_m & a_m \end{pmatrix}, \tag{4.4}$$

where

$$2a_m = (a+t)^m + (a-t)^m,\ 2b_m/b = 2c_m/c = \left[(a+t)^m - (a-t)^m\right]/t, \tag{4.5}$$

and (4.3) holds with $2\widetilde{a}_n(0) = (\overline{v}+t)^n + (\overline{v}-t)^n$.

If $d = a$ and $bc = 0$ so that $t = 0$, then

$$e^{xN} = e^{xu}\begin{pmatrix} 1 & bx \\ cx & 1 \end{pmatrix},$$

N^m is given by (4.4) with $a_m = a^m$, $b_m/b = c_m/c = ma^{m-1}$ and (4.3) holds with $\widetilde{a}_n(0) = \overline{v}^n$.

Proof. Equation (4.1) is given by `https://en.wikipedia.org/wiki/Matrix_exponential`. Expanding and taking the power of x^m gives (4.2). The rest follows. □

When $p = 2$, we replace x by (x, y), n by (m, n), N_1, N_2 by M, N and set $Y = Y(x, y)$, $\overline{Y} = Y(-x, -y)$, and N_1, N_2 by M, N. We construct 2×2 commuting matrices in the following examples. The examples describe a family of matrix-valued differential operators and their characteristic solutions under various algebraic constraints. The setup involves two 2×2 matrices M and N, where N is related to the complex conjugate of M's entries and constructed so that $MN = NM$ under a specific condition involving a parameter g. When M and N are in a special form (with equal diagonal elements and off-diagonal elements related by conjugation and scaling), they generate commuting exponentials e^{xM} and e^{yN}, and these are used to build matrix functions like $Y = e^{xM+yN}$. Through substitutions and simplifications (for example, using $b = -c$ to create trigonometric forms), the matrix exponentials take the form of rotation or hyperbolic rotation matrices. These lead to compact representations of solutions to a partial differential equation $Lf = 0$, where L is constructed via bilinear forms involving matrices M_m and N_n raised to powers and parameterized by spectral variables. The characteristic solutions are then expressed as exponential-scaled rotations (or hyperbolic rotations) acting on vector-valued functions, illustrating an elegant algebraic structure underpinning solutions to such operator equations.

Example 4.1. *Take* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $N = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$. *Then,* $MN = NM$ *if*

$$\overline{b}/b = \overline{c}/c = \left(\overline{a} - \overline{d}\right)/(a - d) = g,$$

that is,

$$N = \begin{pmatrix} \overline{a} & gb \\ gc & \overline{a} + g(d - a) \end{pmatrix}.$$

So, for a given M*,* N *has two free parameters,* $\overline{a}$ *and* g*.*

Now suppose that $d = a$ *and* $\overline{d} = \overline{a}$*. So,* $M = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, $N = \begin{pmatrix} \overline{a} & gb \\ gc & \overline{a} \end{pmatrix}$*. We give the partial differential equation* $Lf = 0_2$ *and its characteristic solutions.*

Replacing $M, m, a, b, c, x, t = (bc)^{1/2}$ *in Theorem 4.2 by* N*,* n*,* $\overline{a}$*,* $\overline{b} = gb$*,* $\overline{c} = gc$*,* y*,* gt *gives*

$$N^n = \begin{pmatrix} \overline{a}_n & \overline{b}_n \\ \overline{c}_n & \overline{a}_n \end{pmatrix}, \; e^{yN} = e^{\overline{a}y} \begin{pmatrix} \cosh gty & bt^{-1} \sinh gty \\ ct^{-1} \sinh gty & \cosh gty \end{pmatrix}, \tag{4.6}$$

where $2\overline{a}_n = (\overline{a} + gt)^n + (\overline{a} - gt)^n$ *and* $2\overline{b}_n/\overline{b} = 2\overline{c}_n/\overline{c} = \left[(\overline{a} + gt)^n - (\overline{a} - gt)^n\right]/gt$*. Set*

$$w = t(x + gy), \; u = \cosh w, \; v = \sinh w, \tag{4.7}$$

$$Z(w) = \begin{pmatrix} u & bt^{-1}v \\ ct^{-1}v & u \end{pmatrix}, \; \overline{Z}(w) = Z(w)^{-1} = \begin{pmatrix} u & -bt^{-1}v \\ -ct^{-1}v & u \end{pmatrix}. \tag{4.8}$$

Then,

$$Y = e^{xM+yN} = e^{xa+y\overline{a}} Z(w), \; \overline{Y} = e^{-xM-yN} = e^{-xa-y\overline{a}} \overline{Z}(w). \tag{4.9}$$

Set $\tau = b/c$*. For* $m, n \in Z$*, and any* $S = S_{m,n} \in C^{2\times 2}$*, by (1.3),*

$$T_{m,n}(x, y) = Q\left(w, S_{m,n}\right),$$

where

$$Q(w,S) = Su^2 + \tau^{-1/2}E_1(S)uv + \tau^{-1}E_2(S)v^2,$$
$$E_1(S) = \begin{pmatrix} -S_{1,2} + \tau S_{2,1} & -\tau\left(S_{1,1} - S_{2,2}\right) \\ S_{1,1} - S_{2,2} & S_{1,2} - \tau S_{2,1} \end{pmatrix},$$
$$E_2(S) = -\begin{pmatrix} \tau S_{2,2} & \tau^2 S_{2,1} \\ S_{1,2} & \tau S_{1,1} \end{pmatrix}.$$

So,

$$2Q(w,S) = S\,(\cosh 2w + 1) + \tau^{-1/2}E_1(S)\sinh 2w + \tau^{-1}E_2(S)\,(\cosh 2w - 1)\,.$$

Set

$$\overline{\nu}_1 = \nu_1 + a,\ \overline{\nu}_2 = \nu_2 + \overline{a},\ \beta = \overline{\nu}_1 + t,\ \overline{\beta} = \overline{\nu}_1 - t,\ \gamma = \overline{\nu}_2 + gt,\ \overline{\gamma} = \overline{\nu}_2 - gt.$$

For $D(\nu)$, we need

$$M_m = (\nu_1 I_2 + M)^m\,,\ N_n = (\nu_2 I_2 + N)^n\,. \tag{4.10}$$

The elements of M_m are given by

$$2M_{m,1,1} = 2M_{m,2,2} = \beta^m + \overline{\beta}^m,\ 2M_{m,1,2}/b = 2M_{m,2,1}/c = \beta^m - \overline{\beta}^m,$$

N is M with a, b, c, t replaced by $\overline{a}$, gb, gc, gt. So,

$$2N_{n,1,1} = 2N_{n,2,2} = \gamma^n + \overline{\gamma}^n,\ 2N_{n,1,2}/gb = 2N_{n,2,1}/gc = \gamma^n - \overline{\gamma}^n,$$
$$D(\nu) = \sum_{m=0}^{q_1}\sum_{n=0}^{q_2} S_{m,n}O_{m,n}, \tag{4.11}$$

where $O_{m,n} = M_m N_n$. For example,

$$O_{0,0} = I_2,$$
$$O_{0,1} = \begin{pmatrix} \overline{\nu}_2 & -bg^2t \\ cg^2t & \overline{\nu}_2 \end{pmatrix},$$
$$O_{1,0} = \begin{pmatrix} \overline{\nu}_1 & bt \\ ct & \overline{\nu}_1 \end{pmatrix},\ O_{1,1} = \begin{pmatrix} \overline{\nu}_1\overline{\nu}_2 + bcg^2t & b\left(g^2\overline{\nu}_1 + \overline{\nu}_2\right) \\ c\left(g^2\overline{\nu}_1 + \overline{\nu}_2\right) & \overline{\nu}_1\overline{\nu}_2 + bcg^2t \end{pmatrix}.$$

Note that

$$L\,f(x,y) = \sum_{m=0}^{q_1}\sum_{n=0}^{q_2} Q\left(w, S_{m,n}\right)\ f_{\cdot m,n}(x,y)$$

is linear in $\cosh 2w$ *and* $\sinh 2w$*. By (4.9), for ν and $a(\nu) \in C^2$ such that $d(\nu) = 0$ and (2.6) holds, a characteristic solution of $L\,f(x,y) = 0_2$ is*

$$f_n(x,y,\nu) = \exp\left\{\overline{\nu}_1 x + \overline{\nu}_2 y\right\}\ Z(w)\ z_n(x,y,\nu)$$

for $Z(w)$ of (4.7)-(4.8) and z_n of (2.4).

Taking $b = -c$ and setting $\theta = -iw$ gives $u = \cos\theta$, $v = -i\sin\theta$.

Example 4.2. *Take*

$$M = \begin{pmatrix} a & -c \\ c & a \end{pmatrix},\ N = \begin{pmatrix} \overline{a} & -gc \\ gc & \overline{a} \end{pmatrix},\ \theta = c(x + gy).$$

Then

$$e^{xM} = e^{xa} \begin{pmatrix} \cos cx & -\sin cx \\ \sin cx & \cos cx \end{pmatrix},\ e^{yN} = e^{y\overline{a}} \begin{pmatrix} \cos gy & -\sin gcy \\ \sin gy & \cos gy \end{pmatrix},$$

M^m and N^n are given by (4.5) and (4.6) with

$$a_m = \text{Real}(a + ic)^m,\ -b_m/c = c_m/c = \text{Imag}(a + ic)^m,$$
$$\overline{a}_n = \text{Real}\left(\overline{a} + igc\right)^n,\ -\overline{b}_n/gc = \overline{c}_n/gc = \text{Imag}\left(\overline{a} + igc\right)^m,$$

where the real and imaginary parts are calculated as if a, c, $\overline{a}$, gc were real.

$D(v)$ is given by (4.11) in terms of

$$O_{m,n,1,1} = \left[(1 + g)\left(\beta^m\gamma^n - \overline{\beta}^m\overline{\gamma}^n\right) + (1 - g)\left(\beta^m\overline{\gamma}^n - \overline{\beta}^m\gamma^n\right)\right]/4,$$

where $\beta = \overline{v}_1 + ic$, $\overline{\beta} = \overline{v}_1 - ic$, $\gamma = \overline{v}_2 + igc$, and $\overline{\gamma} = \overline{v}_2 - igc$.

For example, if $a = \overline{v}_1\overline{v}_2 + g^2c^2$, $b = ic^2\left(\overline{v}_2 + g^2\overline{v}_1\right)$ then

$$O_{0,0} = I_2,\ O_{0,1} = \begin{pmatrix} \overline{v}_2 & -ig^2c^2 \\ ig^2c^2 & \overline{v}_2 \end{pmatrix},\ O_{1,0} = \begin{pmatrix} \overline{v}_1 & -ic^2 \\ ic^2 & \overline{v}_1 \end{pmatrix},\ O_{1,1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$
$$T_{m,n}(x, y) = Q\left(\theta, S_{m,n}\right),$$
$$Lf(x, y) = \sum_{m=0}^{q_1}\sum_{n=0}^{q_2} Q\left(\theta, S_{m,n}\right)\ f_{.m,n}(x, y), \tag{4.12}$$

where

$$Q(\theta, S) = S\ \cos^2\theta + E_1(S)\ \cos\theta\sin\theta + E_2(S)\ \sin^2\theta, \tag{4.13}$$
$$2Q(\theta, S) = S\ (\cos 2\theta + 1) + E_1(S)\ \sin 2\theta + E_2(S)\ (1 - \cos 2\theta), \tag{4.14}$$

and

$$E_1(S) = \begin{pmatrix} -S_{1,2} - S_{2,1} & -S_{1,1} + S_{2,2} \\ S_{1,1} - S_{2,2} & S_{1,2} + S_{2,1} \end{pmatrix},\ E_2(S) = \begin{pmatrix} S_{2,2} & -S_{2,1} \\ -S_{1,2} & S_{1,1} \end{pmatrix}.$$

Y and $\overline{Y}$ are given by (4.9) with

$$Z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ \overline{Z}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.15}$$

So, for v and $a(v) \in C^2$ such that $d(v) = 0$ and (2.6) holds, a characteristic solution of $L\,f(x, y) = 0_2$ is

$$f_n(x, y, v) = \exp\left\{x\overline{v}_1 + y\overline{v}_2\right\}\ Z(\theta)\ z_n(x, y, v)$$

for z_n of (2.4). $L\,f(x, y)$ is linear in $\cos 2\theta$ and $\sin 2\theta$.

Taking $a = c = g = 1$, $\overline{a} = 0$, $t = i$, gives

Example 4.3. *Take* $s = 2$, $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I_2 + N$, $N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. *Then,*

$$e^{xM} = e^x Z(x),\ e^{yN} = Z(y),$$

where $Y(x,y) = e^x Z(\theta)$, $\theta = x + y$ *for* $Z(\theta)$ *of (4.15).* $T_{m,n}(x,y)$ *and* L *are given by (4.13) and (4.14). The elements of* M_m *and* N_n *of (4.10) are given by*

$$\begin{aligned} &2M_{m,1,1} = 2M_{m,2,2} = \beta^m + \overline{\beta}^m,\ -2M_{m,1,2} = 2M_{m,2,1} = \beta^m - \overline{\beta}^m, \\ &2N_{n,1,1} = 2N_{n,2,2} = \gamma^n + \overline{\gamma}^n,\ -2N_{n,1,2} = 2N_{n,2,1} = \gamma^n - \overline{\gamma}^n, \\ &\beta = \overline{\nu}_1 + i,\ \overline{\beta} = \overline{\nu}_1 - i,\ \overline{\nu}_1 = \nu_1 + 1,\ \gamma = \nu_2 + i,\ \overline{\gamma} = \nu_2 - i. \end{aligned}$$

$D(\nu)$ *is now given by (4.11), where*

$$O_{m,n,1,1} = O_{m,n,2,2} = \left(\beta^m \overline{\gamma}^n + \overline{\beta}^m \gamma^n\right)/2,\ O_{m,n,2,1} = -O_{m,n,1,2} = \left(\beta^m \gamma^n - \overline{\beta}^m \overline{\gamma}^n\right)/2.$$

For example,

$$\begin{aligned} &O_{0,0} = I_2,\ O_{0,1} = \begin{pmatrix} \overline{\nu}_2 & -i \\ i & \overline{\nu}_2 \end{pmatrix}, \\ &O_{1,0} = \begin{pmatrix} \overline{\nu}_1 & -i \\ i & \overline{\nu}_1 \end{pmatrix},\ O_{1,1} = \begin{pmatrix} \overline{\nu}_1 \nu_2 + 1 & -i\left(\overline{\nu}_1 + \nu_2\right) \\ i\left(\overline{\nu}_1 + \overline{\nu}_2\right) & \overline{\nu}_1 \overline{\nu}_2 + 1 \end{pmatrix}. \end{aligned}$$

Set

$$L\, f(x,y) = \sum_{m=0}^{q_1} \sum_{n=0}^{q_2} T_{m,n}(x,y)\, f_{.m,n}(x,y) \in C^2.$$

If $d(\nu) = 0$ *and (2.6) holds, a characteristic solution of* $L\, f(x,y) = 0_2$ *is*

$$f_n(x,y,\nu) = \exp\left\{(\nu_1 + 1)\, x + \nu_2 y\right\}\ Z(\theta)\ z_n(x,y,\nu),$$

for $Z(\theta)$ *of (4.15) and* z_n *of (2.4). Also,* $T_{m,n}(x,y)$ *and* L *are given by (4.12) and (4.13). If instead we choose* $M = I_2$*, then the only change is that* $\theta = y$*.*

If $\left\{N_j\right\}$ commute, then so do the polynomials and power series in them. This can be used to construct other examples.

For $h_j \in C$, and $h = (h_1, h_2, \ldots)$, the *partial* and *complete exponential Bell polynomials* $B_{k,j}(h)$ and $B_k(h)$, are defined in terms of $B = \sum_{k=1}^{\infty} h_k t^k/k!$ by

$$B^j/j! = \sum_{k=j}^{\infty} B_{k,j}(h) t^k/k!,\ B_k(h) = \sum_{j=0}^{k} B_{k,j}(h), \tag{4.16}$$

for $j, k \in Z$. Also

$$e^B = \sum_{k=0}^{\infty} B_k(h) t^k/k!. \tag{4.17}$$

So, $B_{k,0}(h) = \delta_{k,0}$, $B_{k,1}(h) = h_k$, $B_{k,k}(h) = h_1^k$, $B_0(h) = 1$, and $B_1(h) = h_1$, where $\delta_{k,j} = 1$ or 0 for $j = k$ or $j \neq k$. Other $B_{k,j}(h)$ are shown in Comtet [9, pages 307-308]. Set

$$(1^{a_1}, \ldots, p^{a_p}) = h_1^{a_1} \cdots h_p^{a_p}. \tag{4.18}$$

Example 4.4. *Suppose that for* $1 \leq j \leq p$, $f_j(t) = \sum_{k=0}^{\infty} f_{j,k} t^k/k! : C \to C$ *with an extension* $N_j = f_j(Q) : C^{s\times s} \to C^{s\times s}$ *and that* Q *has diagonal Jordan form* $Q = P\Lambda P^{-1}$. *Then*

$$(\nu I_s + N)^n = P\,(\nu I_s + F)^n\, P^{-1},\ (\nu I_s + F)^n = \prod_{j=1}^{p} \left(\nu_j I_s + F_j\right)^{n_j},$$

$$F_j = f_j(\Lambda),\ D = P\overline{D}P^{-1},\ \overline{D} = \sum_{n=0_p}^{q} \overline{S}_n\,(\nu I_s + F)^n,\ \overline{S}_n = P^{-1} S_n P.$$

Set

$$h_k = \sum_{j=1}^{p} x_j f_{j,k},\ H = \sum_{j=1}^{p} x_j F_j = \sum_{k=0}^{\infty} h_k \Lambda^k/k!. \tag{4.19}$$

Then,

$$h_0 = \sum_{j=1}^{p} x_j f_j(0),\ Y(x) = P\overline{Y}(x)P^{-1},$$

where

$$\overline{Y}(x) = e^H = e^{h_0} \sum_{k=0}^{\infty} B_k(h)\Lambda^k/k!$$

for $B_k(h)$ *and* $B_{k,j}(h)$ *of (4.16). Also*

$$\overline{Y}(-x) = e^{-h_0} \sum_{k=0}^{\infty} B_k(-h)\Lambda^k/k!,\ B_{k,j}(-h) = (-1)^j B_{k,j}(h).$$

For example, take $s = 2$, $Q = N$ *of Example 4.3. Then* $Q = P\Lambda P^{-1}$ *for* P, P^{-1} *of Example 5.1, and* $\Lambda = \operatorname{diag}(i, -i)$ *so that* $\Lambda^2 = -I_2$. *For Example 4.3,* $p = 2$, $f_1(t) = 1 + t$ *(or* 1 *if* $M = I_2$*),* $f_2(t) = t$, $h_0 = x_1$, $h_1 = x_1 + x_2$ *(or* x_2 *if* $M = I_2$*),* $h_k = 0$ *for* $k \geq 2$, $B_k(h) = h_1^k$, $\overline{Y}(x) = e^{x_1} e^{h_1\Lambda}$, $e^{h_1\Lambda} = I_2 \cos h_1 + \Lambda \sin h_1$, $\Lambda^2 = -I_2$.

Example 4.5. *Suppose in Example 4.4 that for some* $r \in Z$ *and* $\tau \in C$, $\Lambda^r = \tau I_s$. *So, each element of the diagonal of* Λ *is an rth root of* $\tau 1$, $e^{2i\pi n/r} w$ *for* $w = \tau^{1/r}$ *and some integer n. Then for* h_k *of (4.19),*

$$H = \sum_{k=0}^{\infty} h_k \Lambda^k/k! = \sum_{a=0}^{r-1} t_a \Lambda^a/a!,$$

where

$$t_a/a! = \sum_{k=0}^{\infty} (\tau 1)^k h_{kr+a}/(kr + a)!$$

for $0 \leq a < r$. *Then* $H = t_0 I_s + H_0$, *where*

$$t_0 = \sum_{k=0}^{\infty} \tau^k h_{rk}/(rk)!$$

and

$$H_0 = \sum_{a=1}^{r-1} t_a \Lambda^a/a! = \sum_{a=1}^{\infty} t_a \Lambda^a/a!,\ H_0^k/k! = \sum_{a=k}^{\infty} \Lambda^a B_{a,k}(t)/a!,\ e^{H_0} = \sum_{a=0}^{\infty} \Lambda^a B_{a(t)}/a!,$$

where $t_a = 0$ *for* $a \geq r$. *So,*

$$e^{H_0} = \sum_{k=0}^{r-1} s_k \Lambda^k,$$

for

$$s_k = \sum_{a=0}^{\infty} \tau^a B_{ar+k}/(ar+k)!,$$

where $B_a = B_a(t)$. s_k *are interesting functions of* $t_0, \ldots, t_{r-1}$ *and* τ.

The case $\Lambda^2 = \tau I_s$**, that is,** $r = 2$**.**

Set $w = \gamma = wt_1$. *Then*

$$t_0 = \sum_{k=0}^{\infty} \tau^k h_{2k}/(2k)!,\ t_1 = \sum_{k=0}^{\infty} \tau^k h_{2k+1}/(2k+1)!,\ H_0 = \Lambda t_1,\ B_k = t_1^k,$$
$$\overline{Y} = e^{t_0} e^{H_0},\ s_0 = \cosh\gamma,\ s_1 = w^{-1}\sinh\gamma,\ e^{H_0} = I_s \cosh\gamma + \Lambda w^{-1}\sinh\gamma.$$

So, if $\tau = -1$, *then* $s_0 = \cos t_1\ s_1 = \sin t_1$, $e^{H_0} = I_s \cos t_1 + \Lambda \sin t_1$.

The case $\Lambda^3 = \tau I_s$**, that is,** $r = 3$**.**

Replacing h *by* t *in (4.18),*

$$H_0 = t_1\Lambda + t_2\Lambda^2/2,\ B_1 = (1) = t_1,\ B_2 = (2) + \left(1^2\right) = t_2 + t_1^2,\ B_3 = 3(12) + \left(1^3\right),$$
$$B_4 = 3\left(2^2\right) + 6\left(1^2 2\right) + \left(1^4\right),\ B_5 = 15\left(12^2\right) + 10\left(1^3 2\right) + \left(1^5\right),$$
$$B_6 = 15\left(2^3\right) + 45\left(1^2 2^2\right) + 15\left(1^4 2\right) + \left(1^6\right),$$
$$B_7 = 105\left(12^3\right) + 105\left(1^3 2^2\right) + 21\left(1^5 2\right) + \left(1^7\right).$$

We now write e^{H_0} *in terms of the Hermite polynomials, defined by*

$$e^{u(x,y)} = \sum_{k=0}^{\infty} He_k(x) y^k/k!,$$

where $u(x,y) = xy + y^2/2$. *For* $t_2 \neq 0$, *set* $x = t_1 t_2^{-1/2}$. *Then* $H_0 = u\left(x, t_2^{1/2}\Lambda\right)$. *So,*

$$e^{H_0} = \sum_{k=0}^{\infty} He_k(x)\ \left(t_2^{1/2}\Lambda\right)^k/k! = \sum_{k=0}^{\infty} b_k \Lambda^k = \sum_{a=0}^{2} s_a \Lambda^a,$$

where

$$b_k = He_k(x)t_2^{k/2}/k!,\ s_a = \sum_{k=0}^{\infty} \tau^k b_{3k+a}.$$

The case $\Lambda^4 = -I_s$, that is, $r = 4$.

Replacing h by t in (4.18),

$$\begin{aligned}
&H_0 = t_1\Lambda + t_2^2/2 + t_3\Lambda^3/3!,\ B_1 = (1),\ B_2 = (2) + \left(1^2\right),\\
&B_3 = (3) + 3(12) + \left(1^3\right),\ B_4 = 4(13) + 3\left(2^2\right) + 6\left(1^2 2\right) + \left(1^4\right),\\
&B_5 = 10(23) + 10\left(1^2 3\right) + 15\left(12^2\right) + 10\left(1^3 2\right) + \left(1^5\right),\\
&B_6 = 10\left(3^2\right) + 60(123) + 15\left(2^3\right) + 20\left(1^3 3\right) + 45\left(1^2 2^2\right) + 15\left(1^4 2\right) + \left(1^6\right),\\
&B_7 = 70\left(13^2\right) + 105\left(2^2 3\right) + 210\left(1^2 23\right) + 105\left(12^3\right) + 35\left(1^4 3\right) + 105\left(1^3 2^2\right) + 21\left(1^5 2\right) + \left(1^7\right).
\end{aligned}$$

5. Use of the Jordan form

Example 4.1 solved $N_1N_2 = N_2N_1 \in C^{2\times 2}$. An easier method that gives a wide class of permuting $\left\{N_j\right\}$, follows from *the Jordan form of a square matrix*.

Suppose that N has *Jordan form* $N = PJP^{-1}$, where $J = \mathrm{diag}(J_1, \ldots, J_r)$, $J_j = J_{m_j}\left(\lambda_j\right)$, $J_m(\lambda) = \lambda I_m + U_m$, and for $m \in Z$, U_m is the $m \times m$ matrix of zeros except for 1s on the first super-diagonal, that is, $(U_m)_{j,k} = \delta_{k,j+1}$ for $1 \le j < m$. $J_1, \ldots, J_r$ are the *Jordan blocks* of N. So, $s = \sum_{j=1}^{r} m_j$ and for $0 \le n < m$, U_m^n is the $m \times m$ matrix of zeros except for ones on the nth super-diagonal, that is, $(U_m^n)_{j,k} = \delta_{k,j+n}$ for $1 \le j < m - n$. For $n \ge m$, $U_m^n = 0_{m\times m}$.

For $f(z) : C \to C$ a function with finite derivatives of order $m \in Z$ at λ,

$$f\left(J_m(\lambda)\right) = \sum_{k=0}^{m-1} f_{.k}(\lambda)U_m^k/k!,$$

where $U_m^0 = I_m$, as noted in `https://en.wikipedia.org/wiki/Jordan_matrix`. In particular, for $m \ge 1$ and $n \ge 0$,

$$J_m(\lambda)^n = \sum_{k=0}^{\min(n,m-1)} \binom{n}{k} \lambda^{n-k}U_m^k,\ \exp\left\{tJ_m(\lambda)\right\} = e^{\lambda t}V_m(t),\ V_m(t) = \sum_{k=0}^{m-1} t^k U_m^k/k!,\ t \in C. \tag{5.1}$$

So, $V_m(t)$ has zeros on its subdiagonals, and for $n \ge 0$, the elements of its nth superdiagonal are all $t^n/n!$. That is, $V_m(t)_{j,j+n} = t^n/n!$ for $1 \le j < m - n$. Note that $J_m\left(\lambda_j\right)$ commute

$$\prod_{j=1}^{p} J_m\left(\lambda_j\right) = \sum_{a=0}^{p} s_{p-a}U_m^a,$$

where

$$s_0 = 1,\ s_1 = \sum_{j=1}^{p} \lambda_j,\ s_p = \prod_{j=1}^{p} \lambda_j,$$

$$\sum_{j=0}^{p} t^j s_j = \prod_{k=1}^{p} (1 + t\lambda_k)\,,\ s_j = \sum \lambda_{k_1} \cdots \lambda_{k_j}$$

summed over all $\binom{p}{j}$ combinations $k_1, \ldots, k_j$ of $1, 2, \ldots, p$. Similarly, for $n \in Z^p$ and

$$J_m(\lambda)^n = \prod_{j=1}^{p} \ J_m\left(\lambda_j\right)^{n_j},$$

we note

$$J_m(\lambda)^n = \sum_{a=0}^{m-1} s_{n,a}(\lambda) U_m^a, \tag{5.2}$$

where

$$\sum_{a=0}^{\infty} s_{n,a}(\lambda) t^a = \prod_{j=1}^{p} \left(\lambda_j + t\right)^{n_j},$$

for $|k|$ of (3.7). For example, $s_{n,0} = \lambda^n$ and if $\lambda_1 \cdots \lambda_p \neq 0$, then

$$\lambda^{-n} s_{n,1} = \sum_{j=1}^{p} n_j \lambda_j^{-1},\ \lambda^{-n} s_{n,2} = \sum_{j=1}^{p} \binom{n_j}{2} \lambda_j^{-2} + \sum_{j_1 \neq j_2} n_{j_1} n_{j_2} \left(\lambda_{j_1} \lambda_{j_2}\right)^{-1}.$$

For $m \in Z^p$, the mth derivative of the last expression in (5.2) is

$$\sum_{a=0}^{\infty} s_{n,a.m}(\lambda) t^a = \prod_{j=1}^{p} (n_j)_{m_j} \left(\lambda_j + t\right)^{n_j - m_j} = (n)_m \sum_{a=0}^{\infty} s_{n-m,a}(\lambda) t^a,$$

so

$$s_{n,a.m}(\lambda) = (n)_m s_{n-m,a}(\lambda). \tag{5.3}$$

We now give a simpler form for $s_{n,a}$ using Bell polynomials.

Theorem 5.1. *For $\prod_{j=1}^{p} \lambda_j \neq 0$, set*

$$u_k = \sum_{j=1}^{p} n_j \lambda_j^{-k},\ v_k = (-1)^{k-1} (k-1)!\, u_k.$$

Then (5.2) holds with $\lambda^{-n} s_{n,a}(\lambda) = B_a(v)/a!$.

Proof. Set $t_j = t/\lambda_j$, $A = \ln \lambda^n$. Then

$$\ln \prod_{j=1}^{p} \left(\lambda_j + t\right)^{n_j} = \sum_{j=1}^{p} n_j \left(\lambda_j + t\right) = A + B,$$

where

$$B=\sum_{j=1}^{p} n_j \ln\left(1+t_j\right)=\sum_{k=1}^{\infty} v_k t^k/k!.$$

This implies

$$\prod_{j=1}^{p}\left(\lambda_j+t\right)^{n_j}=\lambda^n e^B.$$

$s_{n,a}$ is the coefficient of t^a in $\lambda^n e^B$ given by (4.17) with $h_k=v_k$. □

$J_m(\lambda)$ only commutes with $\Lambda_m=\operatorname{diag}(\lambda_1,\ldots,\lambda_m)$ if $\lambda_j\equiv\lambda_1$. By (5.1),

$$\exp\left\{\sum_{j=1}^{p} x_j J_m\left(\lambda_j\right)\right\}=\prod_{j=1}^{p} e^{\lambda_j x_j} V_m\left(x_j\right)=e^{\lambda' x}\overline{V}_m(x),$$

where

$$\overline{V}_m(x)=\prod_{j=1}^{p} V_m\left(x_j\right)=\sum_{|n|<m} x^n U_m^{|n|}/n!=\sum_{n_0=0}^{m-1}\left(x_0\right)^{n_0} U_m^{n_0}/n_0!=V_m\left(x_0\right),\ x_0=\sum_{j=1}^{p} x_j. \tag{5.4}$$

Now suppose that for $1\le j\le p$, $N_j\in C^{s\times s}$ has Jordan form

$$N_j=P\operatorname{diag}\left(J_{1,j},\ldots,J_{r,j}\right)P^{-1},\ J_{k,j}=J_{m_k}\left(\lambda_{k,j}\right),\ s=\sum_{k=1}^{r} m_k. \tag{5.5}$$

Then $N_1,\ldots,N_p$ commute, as their Jordan blocks commute,

$$N^n=N_1^{n_1}\cdots N_p^{n_p}=P\operatorname{diag}\left(J_1^n,\ldots,J_r^n\right)P^{-1},$$

where

$$J_k^n=\prod_{j=1}^{r} J_{k,j}^{n_j},\ (\nu I_s+N)^n=P\operatorname{diag}\left(\overline{J}_1^n,\ldots,\overline{J}_r^n\right)P^{-1},$$

where

$$\overline{J}_k^n=\prod_{j=1}^{p}\overline{J}_{k,j}^{n_j},$$

and

$$\overline{\nu}_{k,j}=\nu_j+\lambda_{k,j},\ \overline{J}_{k,j}=\nu_j I_{m_k}+J_{k,j}=\overline{\nu}_{k,j} I_{m_k}+U_{m_k}=J_{m_k}\left(\overline{\nu}_{k,j}\right),$$

since $\left(\nu_j I_s+N_j\right)^{n_j}=P\operatorname{diag}\left(\overline{J}_{1,j}^n,\ldots,\overline{J}_{r,j}^n\right)P^{-1}$. Also,

$$D(\nu)=D^*(\nu)P^{-1}, D^*(\nu)=\sum_{n=0_p}^{q} S_n\, P\operatorname{diag}\left(\overline{J}_1^n,\ldots,\overline{J}_r^n\right), \tag{5.6}$$

and for $m \in Z^p$,

$$D_{.m}(\nu) = \sum_{n=0_p}^{q} S_n \, P \,\mathrm{diag}\left(\left(\overline{J}_1^n\right)_{.m}, \ldots, \left(\overline{J}_r^n\right)_{.m}\right) P^{-1},$$

where, by (5.3),

$$\left(\overline{J}_k^n\right)_{.m} = \sum_{a=0}^{m_k-1} (n)_m s_{n-m,a}\left(\overline{\nu}_{k,j}\right) U_{m_k}^a.$$

J_k^n and $\overline{J}_k^n$ are given by (5.2) with $m = m_k$, $\lambda_j = \lambda_{k,j}$ and $\lambda_j = \overline{\nu}_{k,j}$.

The partial differential Eq (1.2).

$$\sum_{j=1}^{p} x_j N_j = PM(x)P^{-1},$$

where

$$M(x) = \mathrm{diag}\left(M_{1,x}, \ldots, M_{r,x}\right),\ M_{k,x} = \sum_{j=1}^{p} x_j J_{k,j}.$$

For $V_m(t)$ of (5.1), x_0 of (5.4), and $\lambda_{k,j}$ of (5.5), set

$$\lambda_k = \left(\lambda_{k,1}, \ldots, \lambda_{k,p}\right)',\ \lambda_k' x = \sum_{j=1}^{p} \lambda_{k,j}\, x_j,\ F_k(x) = \exp\left(M_{k,x}\right) = e^{\lambda_k' x}\, V_{m_k}\left(x_0\right),\ g(x) = P^{-1} f(x).$$

Then,

$$Y(x) = P\,\overline{Y}(x),\ T_n(x) = P\,\overline{T}_n(x),\ L = P\,\overline{L},$$

where

$$\overline{Y}(x) = F(x)\,P^{-1},\ F(x) = \mathrm{diag}\left(F_1(x), \ldots, F_r(x)\right), \tag{5.7}$$

$$\overline{T}_n(x) = \overline{Y}(x)\,S_n\,Y(-x) = F(x)Q_nF(-x)P^{-1},\ Q_n = P^{-1}S_nP, \tag{5.8}$$

$$\overline{L} = \sum_{n=0_p}^{q} \overline{T}_n(x)\,\partial_x^n,\ \overline{L}f(x) = \sum_{n=0_p}^{q} F(x)Q_nF(-x)g_{.n}(x).$$

So, $T_n(x)$ is a mixture of polynomials in x_0 and factors $\exp\{(\lambda_c - \lambda_d)'\,x\}$. Partition N_j, P, P^{-1} as $m_a \times m_b$ blocks,

$$N_j = \left(N_{j,a,b}\right),\ P = \left(P_{a,b}\right),\ P^{-1} = \left(P^{a,b}\right),$$

for $1 \le a, b \le r$. Partition $Y(x)$ and $T_n(x)$ similarly. Then

$$\left(N^n\right)_{j,k} = \sum_{a=1}^{r} P_{j,a} J_a^n P^{a,k},\ Y(x)_{j,k} = \sum_{c=1}^{r} P_{j,c} F_c(x) P^{c,k},$$

$$T_n(x)_{j,k} = \sum_{c,d=1}^{r} P_{j,c} F_c(x) Q^c_{d,n} F_d(-x) P^{d,k},\ Q^c_{d,n} = \sum_{a,b=1}^{r} P^{c,a} S_{n,a,b} P_{b,d}. \tag{5.9}$$

Write $f(x) \in C^s$ as $(f_1(x)', \ldots, f_r(x)')$ with $f_k(x) \in C^{m_k}$. Then the partial differential equation $Lf(x) = 0_s$ for L of (1.2) can be written

$$\sum_{n=0_p}^{q} \sum_{k=1}^{r} T_n(x)_{j,k}\ f_{k.n}(x) = 0_{m_j},\ 1 \leq j \leq r.$$

Consider the two extremes when N_j has 1 or s Jordan blocks.

One Jordan block: $N_j = PJ_s\left(\lambda_j\right)P^{-1}$. Then $r = 1$, $m_1 = s$, P is scalar, say 1. Take $s_{n,a}(\lambda)$ of Theorem 4.1, x_0 of (5.4), and $\overline{\nu}_j = \nu_j + \lambda_j$. Then

$$Y(x) = F_1(x) = \exp\left(\sum_{j=1}^{p} \lambda_j x_j\right) V_s(x_0),\ T_n(x) = V_s(x_0) S_n V_s(-x_0), \tag{5.10}$$

and

$$T_n(x)_{j,k} = \sum_{a=j}^{s} \sum_{b=1}^{k} \left[x_0^{a-j}/(a-j)!\right] S_{n,a,b}\ (-x_0)^{k-b}/(k-b)!, \tag{5.11}$$

$$(\nu I_s + N)^n = \sum_{a=0}^{s-1} s_{n,a}(\overline{\nu})\ U_s^a,\ D(\nu) = \sum_{a=0}^{s-1} D_a U_s^a,\ D_a = \sum_{n=0_p}^{q} s_{na}(\overline{\nu})\ S_n. \tag{5.12}$$

Column j of $D_a U_s^a$ is 0_s for $1 \leq j \leq a$, and is column $j - a$ of D_a if $j \geq a + 1$.

So for $m, n \in Z^p$, $D_{.m}(\nu)$ needed for the characteristic solution of Theorem 2.2, is given in terms of $s_{n,a.m}(\overline{\nu}) = (n)_m s_{n-m,a}(\overline{\nu})$ by (5.3).

Diagonal Jordan form: $N_j = P\ \mathrm{diag}\left(\lambda_{1,j}, \ldots, \lambda_{s,j}\right)P^{-1}$. Then $r = s$, $m_j \equiv 1$, $J_c = \lambda_c$ and for $n, m \in Z^p$, and $Q^c_{d,n}$ of (5.9),

$$\nu_j I_s + N_j = P\mathrm{diag}\left(\nu_{1,j}, \ldots, \nu_{s,j}\right),\ \nu_{k,j} = \nu_j + \lambda_{k,j},$$

$$(\nu I_s + N)^n = P\mathrm{diag}\left(\theta_{1,n}, \ldots, \theta_{s,n}\right)P^{-1},\ \theta_{k,n} = \prod_{j=1}^{p} \nu_{k,j}^{n_j} = \overline{J}_k^n,$$

$$D_{.m}(\nu) = \sum_{n=0_p}^{q} S_n P\mathrm{diag}\left(\theta_{1,n.m}, \ldots, \theta_{s,n.m}\right)P^{-1},\ \theta_{k,n.m} = (n)_m \theta_{k,n-m},$$

$$Y(x) = PF(x)P^{-1},\ F(x) = \mathrm{diag}\left(e^{\lambda_1' x}, \ldots, e^{\lambda_s' x}\right),$$

$$T_n(x)_{j,k} = \sum_{c,d=1}^{s} e^{(\lambda_c - \lambda_d)' x} P_{j,c} Q^c_{d,n} P^{d,k}, \tag{5.13}$$

with all components scalar. For example, if $P = I_s$, then $T_n(x)_{j,k} = e^{\left(\lambda_j - \lambda_k\right)' x} S_{n,j,k}$. In the examples of Section 3, N_j have Jordan form with the same P.

In Example 5.1, matrix functions like $T_n(x)$, $\overline{T}_n(x)$, and $Y(x)$ are constructed using a diagonalization of skew-symmetric matrices N_j via a unitary matrix P, where solutions are expressed in terms of exponentials involving differences of eigenvalues, trigonometric functions of $\theta = \sum \theta_j x_j$, and combinations of structure matrices Q_n and S_n.

Example 5.1. *Take $s = 2$. Then for $Q^c_{d,n}$ of (5.9),*

$$T_n(x)_{j,k} = \sum_{c,d=1}^{2} e^{(\lambda_c-\lambda_d)'x} P_{j,c} Q^c_{d,n} P^{d,k} = \sum_{c=1}^{2} P_{j,c} Q^c_{c,n} P^{c,k} + e^{\delta' x} P_{j,1} Q^1_{2,n} P^{2,k} + e^{-\delta' x} P_{j,2} Q^2_{1,n} P^{1,k},$$

where $\delta = \lambda_1 - \lambda_2$. Also

$$\overline{T}_n(x) = H(\delta' x, Q_n) P^{-1},$$

where

$$H(t, Q) = \begin{pmatrix} Q_{n,1,1} & e^t Q_{n,1,2} \\ e^{-t} Q_{n,2,1} & Q_{n,2,2} \end{pmatrix}. \tag{5.14}$$

Take

$$N_j = \theta_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $\theta = \sum_{j=1}^{2} \theta_j x_j$. Then

$$N_j = P \operatorname{diag}\left(\lambda_{1,j}, \lambda_{2,j}\right) P^{-1},$$

where $\lambda_{1,j} = -\lambda_{2,j} = i\theta_j$ and

$$P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} / \sqrt{2}, \ \det P = i, \ P^{-1} = P^* = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} / \sqrt{2}, \tag{5.15}$$

and $\overline{Y}(x)$ is given by (5.7) with

$$F_1(x) = e^{i\theta}, \ F_2(x) = e^{-i\theta}, \ Y(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and $\overline{T}_n(x)$ is given by (5.8) and (5.14) with

$$2Q_n = \begin{pmatrix} S_{1,1} - iS_{1,2} + iS_{2,1} + S_{2,2} & S_{1,1} + iS_{1,2} + iS_{2,1} - S_{2,2} \\ S_{1,1} - iS_{1,2} - iS_{2,1} - S_{2,2} & S_{1,1} + iS_{1,2} - iS_{2,1} + S_{2,2} \end{pmatrix}$$

at $S = S_n$. $D(\nu)$ is given by (5.6) with

$$\overline{J}^n_1 = \prod_{j=1}^{p} \left(\nu_j + i\theta_j\right)^{n_j}, \ \overline{J}^n_2 = \prod_{j=1}^{p} \left(\nu_j - i\theta_j\right)^{n_j}.$$

An easy way to create commuting N_j is to find an N whose P does not have one of its parameters, and then to vary that parameter. The next examples illustrate this. Example 5.2 describes a complex matrix M with complex conjugate eigenvalues and corresponding eigenvectors, and shows how it can be diagonalized using a similarity transformation $M = PJP^{-1}$, extended to a family of commuting matrices N_j, leading to explicit forms for matrix exponentials and powers, and ultimately yielding characteristic solutions involving products of diagonal Jordan form components. Example 5.3 presents a matrix M with eigenvalues $a \pm \tau b$ and corresponding eigenvectors, showing it can be diagonalized as $M = PJP^{-1}$, extended to a family of commuting matrices N_j, and leading to expressions for matrix functions and characteristic solutions as in Example 5.2 with modified spectral parameters. Example 5.4 considers a matrix M with eigenvalues $a \pm gbc$ and eigenvectors depending on b and c, showing that M can be diagonalized via $M = PJP^{-1}$, extended to a family of commuting matrices N_j, and used in expressions for matrix functions and characteristic solutions as in Example 5.2 with modified parameters involving b, c and g_j. In Example 5.5, a specific matrix P is used to diagonalize the Jordan block $J_2\left(\lambda_j\right)$ into a matrix N_j, and the characteristic solutions and related functions $Y(x)$, $T_n(x)$, $(\nu I_2 + N)^n$ and $D(\nu)$ are expressed through structured sums involving matrices S_n, shift operators U_2, and coefficients $S_{n,a}(\overline{\nu})$.

Example 5.2. *Consider*

$$M = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

of Example 4.2. Its eigenvalues are $\lambda_1 = a + ic$ and $\lambda_2 = a - ic$, with eigenvectors $p_1 = (1, -i)'$ and $p_2 = (1, i)'$. So, M has diagonal Jordan form PJP^{-1} for P and P^{-1} of (5.15), and $J = \text{diag}\ (\lambda_1, \lambda_2)$. So, a choice of commuting N_j is

$$N_j = \begin{pmatrix} a_j & -c_j \\ c_j & a_j \end{pmatrix} = PJ_jP^{-1},\ J_j = \text{diag}\left(a_j + ic_j, a_j - ic_j\right).$$

Also,

$$\sum_{j=1}^{p} x_j N_j = M,$$

with

$$a = \sum_{j=1}^{p} x_j a_j,\ c = \sum_{j=1}^{p} x_j c_j.$$

Also

$$Y(x) = Pe^J P^{-1},\ (\nu I_2 + N)^n = P\,\text{diag}\left(\overline{J}_1^n, \overline{J}_2^n\right)\ P^{-1},$$

where

$$e^J = \text{diag}\left(e^{\lambda_1}, e^{\lambda_2}\right),\ \overline{J}_k^n = \prod_{j=1}^{p} \overline{\nu}_{k,j}^{n_j},\ \overline{\nu}_{1,j} = \nu_j + a_j + ic_j,\ \overline{\nu}_{2,j} = \nu_j + a_j - ic_j.$$

$D(\nu)$ is given by (5.6) with $r = 2$, and $T_n(x)$ by (5.13) with $s = 2$. Characteristic solutions are given by Example 3.1 in terms of

$$D_{j,k.m} = \sum_{n=0_p}^{q} \sum_{a,b=1}^{2} S_{n,j,a} P_{a,b} \left(\overline{J}_b^n\right)_{.m} P^{b,k},$$

for $m \in Z^p$, where $\left(\overline{J}_b^n\right)_{.m} = (n)_m \overline{J}_b^{n-m}$.

Example 5.3. *Write M of Example 4.1 with $d = a$ as*

$$M = \begin{pmatrix} a & b \\ \tau^2 b & a \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = a + \tau b$ and $\lambda_2 = a - \tau b$, with eigenvectors $p_1 = (1, \tau)'$ and $p_2 = (1, -\tau)'$. So, M has diagonal Jordan form PJP^{-1}, where

$$P = \begin{pmatrix} 1 & 1 \\ \tau & -\tau \end{pmatrix}, \ P^{-1} = 2^{-1} \begin{pmatrix} 1 & \tau^{-1} \\ 1 & -\tau^{-1} \end{pmatrix}, \ J = \operatorname{diag}(\lambda_1, \lambda_2).$$

So, a choice of commuting N_j is

$$N_j = \begin{pmatrix} a_j & b_j \\ \tau^2 b_j & a_j \end{pmatrix} = PJ_jP^{-1}, \ J_j = \operatorname{diag}\left(a_j + \tau b_j, a_j - \tau b_j\right)$$

and

$$\sum_{j=1}^{p} x_j N_j = M, \ a = \sum_{j=1}^{p} x_j a_j, \ b = \sum_{j=1}^{p} x_j b_j.$$

Also, $Y(x)$, $(\nu I_2 + N)^n$, $D(\nu)$, $T_n(x)$, and characteristic solutions are given by Example 5.2, where now $\overline{\nu}_{1,j} = \nu_j + a_j + \tau b_j$, $\overline{\nu}_{2,j} = \nu_j + a_j - \tau b_j$.

Example 5.4. *For a variation of this, consider*

$$M = \begin{pmatrix} a & gb^2 \\ gc^2 & a \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = a + gbc$ and $\lambda_2 = a - gbc$, with eigenvectors $p_1 = \left(b^2, c\right)'$ and $p_2 = \left(b^2, -c\right)'$. So, M has diagonal Jordan form PJP^{-1}, where

$$P = \begin{pmatrix} b^2 & b^2 \\ c & -c \end{pmatrix}, \ P^{-1} = \begin{pmatrix} b^{-2} & c^{-1} \\ b^{-2} & -c^{-1} \end{pmatrix} /2, \ J = \operatorname{diag}(\lambda_1, \lambda_2).$$

So, a choice of commuting N_j is

$$N_j = \begin{pmatrix} a_j & g_j b^2 \\ g_j c^2 & a_j \end{pmatrix} = PJ_jP^{-1}, \ J_j = \operatorname{diag}\left(a_j + g_j bc, a_j - g_j bc\right),$$

$$\sum_{j=1}^{p} x_j N_j = M \text{ with } a = \sum_{j=1}^{p} x_j a_j, \ g = \sum_{j=1}^{p} x_j g_j.$$

$Y(x)$, $(\nu I_2 + N)^n$, $D(\nu)$, $T_n(x)$ and characteristic solutions are given by Example 5.2, where now $\overline{\nu}_{1,j} = \nu_j + a_j + g_j bc$, $\overline{\nu}_{2,j} = \nu_j + a_j - g_j bc$.

Example 5.5. *Take* $s = 2$,

$$P = \begin{pmatrix} 1 & 1 \\ -g & g \end{pmatrix}, \ P^{-1} = 2^{-1} \begin{pmatrix} 1 & -g^{-1} \\ 1 & g^{-1} \end{pmatrix}$$

and

$$N_j = PJ_2\left(\lambda_j\right)P^{-1} = \begin{pmatrix} \lambda_j + 1/2 & g^{-1}/2 \\ -g/2 & \lambda_j - 1/2 \end{pmatrix}.$$

Then, $Y(x)$, $T_n(x)$, $(\nu I_2 + N)^n$, $D(\nu)$ *are given by (5.10)–(5.12). Characteristic solutions are given by Example 3.1 in terms of*

$$D_{j,k} = \sum_{n=0_p}^{q} \sum_{a=0}^{1} s_{n,a}(\overline{\nu}) \ (S_n U_2^a)_{j,k}, \ D_{j,k,m} = \sum_{n=0_p}^{q} \sum_{a=0}^{1} (n)_m s_{n-m,a}(\overline{\nu}) \ (S_n U_2^a)_{j,k},$$

for $m \in Z^p$. *If* $S_{n,1}$ *is the first column of* S_n, *then* $S_n U = (0_2, S_{n,1})$.

N_1 and N_2 commute if they have Jordan forms with the same P and matching Jordan blocks, but different eigenvalues. We now weaken this condition. Suppose that $J = \text{diag}\ (J_1, \ldots, J_r)$ and $L = \text{diag}\ (L_1, \ldots, L_r)$, where $J_k = \lambda_k I_m$ or $J_m(\lambda_k)$, and $L_k = \nu_k I_m$ or $J_m(\nu_k)$. Then $LK = KL$ and $MN = NM$ for $M = PLP^{-1}$, $N = PKP^{-1}$. For $m > 1$, $\lambda_k I_m$ is not a Jordan block, but is composed of m scalar Jordan blocks, λ_k.

6. Conclusions

This paper extends the classical method of characteristics to a novel framework capable of solving a wide class of multivariate, linear, homogeneous partial differential equations with structured matrix coefficients. By leveraging exponential similarity transformations involving commuting matrices and introducing a new type of characteristic parameterized by a complex vector ν, the method provides explicit basis solutions that are both analytically tractable and adaptable to boundary conditions. The approach overcomes limitations of existing techniques that are primarily suited for univariate cases or depend heavily on order reduction, and it opens new avenues for tackling complex systems involving matrix operators-offering not only theoretical insights but also practical tools applicable to structured matrices like circulants and those derived via Jordan forms.

Future work will aim to: i) add numerical analysis and stability considerations making the methods more practical to implement, ii) analyze what happens in near-commuting cases, iii) expand on how boundary value problems are handled.

Author contributions

Both authors of this article have been contributed equally. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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