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Research article

Dynamics of a detrital-based mangrove food chain system driven by Holling type II functional response and intraspecific competition

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Abstract: In this paper, we propose a detrital-based mangrove food chain system with Holling type II functional response, intraspecific competition, and delay effects. First, we prove the solution of this system is positive and bounded with the positive initial conditions. Second, we calculate the equilibria and investigate the asymptotical stability of equilibria with and without delays. Then, by taking the delay as the bifurcation parameter and using bifurcation theory, the bifurcation conditions for the system to undergo Hopf bifurcation at the interior equilibrium point are obtained. Furthermore, we also conduct the length estimation of delay to preserve the stability by using the Nyquist criterion. Finally, with the suitable choices of the parameters, numerical simulations have been carried out to substantiate our analytical results.

Keywords: detrital food chain system; stability; Hopf bifurcation; Nyquist criteria; delay length estimation
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1. Introduction

Ecological environmental protection is a hot issue of sustainable development in the world. In recent decades, due to global climate changes, ecosystems worldwide have encountered significant challenges. Among them, mangrove as one of the most productive natural wetland ecosystems, which performs ecological functions including biodiversity conservation, sequestration and storage of carbon, water and atmosphere purification, and so forth [1–6], the population changes in this system have gained increasing attention and become a global concern.

Delay differential equations are recognized as an effective approach to describe the population evolution systems that depend on not only the current state, but also the historical state, and have been widely applied in many fields such as biology, engineering, and epidemiology [7, 8]. To understand the complex population dynamics in the mangrove system, Bhattacharyya and Mukhopadhyay [9] proposed the following three-dimensional differential system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha - a_1x(t) - x(t)y(t) + \beta z(t-\tau_1), \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -a_2y(t) + b_2x(t)y(t) - c_1y(t)z(t), \\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}t} = -a_3z(t) + c_2y(t)z(t), \end{cases}$$

where α is the eternal input of detritus, a_1 is the washout rate of detritus, β is the recycling rate of detritus from dead predators of detritivores, a_2 is the death rate of detritivores, b_2 is the intake rate of detritus, c_1 is the uptake rate of detritivores, a_3 is the death rate of predators of detritivores, c_2 is the specific growth rate of predator, and τ_1 is the recycling delay of predators of detritivore. There is some literature studying detrital food chains through dynamic modeling, where the main research subjects are the existence of equilibrium, the stability, and the bifurcation analysis of the system. For instance, we refer to [9–11]. However, existing studies on the theoretical research of the mangrove ecosystem remain scarce. Therefore, there is still considerable scope for further research.

On account of the life history of detritivore predators, the past historical density affects the current density due to the gestation period [12, 13]. In addition, it is generally found that a prey-predator system achieves nonsynchronous changes in predation relationships. This fact indicates the delay feedback mechanism of the prey to the density growth of predators [14–16]. Hence, two discrete time delays described above be incorporated into the food chain system of mangrove. There are detritivores such as invertebrate shrimps and crabs and predators of detritivores such as carnivorous fish in a mangrove system. Therefore, it is more reasonable to modify the cooperative relation between preys and predators by Holling type II functional response which describes functional response for invertebrates and carnivorous fish [17–20]. Because of the limited resources in ecological environments, intraspecific competition occurs naturally between individuals within populations of predators of detritivores. Therefore, the investigation for the intraspecific competition rate of predators of detritivores is of great importance. It is to be noted that since detrital resources continue to be imported into the mangrove system, there is no consideration of intraspecific competition among detritivores [21–23].

Based on the above background, we build the following differential system with time delays:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha - a_1x(t) - b_1x(t)y(t) + \beta z(t-\tau_1), \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -a_2y(t) + b_2x(t)y(t) - \dfrac{c_1y(t)z(t)}{k+y(t)}, \\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}t} = -a_3z(t) + \dfrac{c_2y(t-\tau_2)z(t-\tau_2)}{k+y(t-\tau_2)} - \gamma z^2(t), \end{cases} \tag{1.1}$$

under the initial condition

$$x(t) = \phi_1(t) \geq 0,\ \ y(t) = \phi_2(t) \geq 0,\ \ z(t) = \phi_3(t) \geq 0 \quad \text{for } t \in [-\tau, 0], \tag{1.2}$$

where $\tau = \max\{\tau_1, \tau_2\}$.

The further explanations of system (1.1) are as follows (see Figure 1):

(i) x, y, and z are the densities of detritus, detritivore and predator of detritivore, respectively.

(ii) We believe that there is a detritus uptake rate b_1 by the detritivore, which is greater than the intake rate of detritus b_1.

(iii) The cooperative relation between detritivores and predators is represented by the Holling type II functional response $f(y) = \dfrac{yz}{k+y}$ with the half-saturation constant k. This type of functional response exhibits a saturation effect when the detritivores population is abundant.

(iv) We assume that the gestation time of detritivore predators and the feedback time of detritivores to their predators are the same for mathematical convenience, denoted by τ_2.

(v) Denote γ as the intraspecific competition rate of predators of detritivores, and express it as a limiting factor. We will analyze the impact of intraspecific competition rate in conjunction with the influence of natural mortality in Section 5.

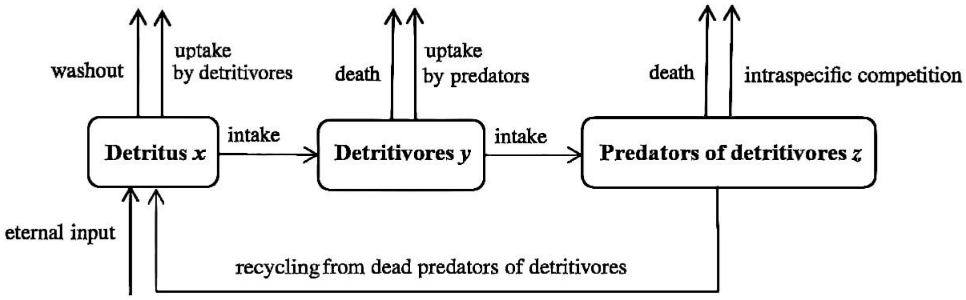

Figure 1. Schematic diagram of the interrelationships between detritus, detritivore, and predator of detritivore.

Literature [9] emphasises that predator mortality is a central factor for the stability of their single delay mangrove system. And, when the time delay required to turn from a dead predator into detritus exceeds a critical threshold, the system will experience a Hopf bifurcation. Here, this paper provides an in-depth analysis of the mangrove ecosystem based on a differential model with multiple time delays. This model also takes into account the Holling type II functional response between the detritivore and its predator, as well as interspecific competition among detritivore predators. The obtained results show that, in addition to the recycle delay of dead predator, gestation time of detritivore predator and feedback time of detritivore to their predator are also very important for the dynamic changes of mangrove populations. Moreover, along with the mortality rate of detritivore predator, the effects of Holling type II functional response and intraspecific competition are also investigated. The Holling type II functional response can affect the time for the detrital food chain to reach a stable state. Increasing the interspecific competition rate of detritivore predators can restore the instability of mangrove system caused by excessive detritivore predator density.

The objective of this article is to construct a model with multiple time delays concerning detrital food chain of the mangrove ecosystem, and to explore a series of key issues including the existence of equilibrium, stability analysis, Hopf bifurcation, and the estimation of time delay length. The rest of the article is arranged as follows: In Section 2, the positivity and boundedness of the system are first rigorously proved to lay the foundation for the subsequent analysis. Also, in the absence of delays, we discuss the equilibrium and the stability analysis. In Sections 3 and 4, taking the delay as the bifurcation parameter and using bifurcation theory, the existence of Hopf bifurcation is proved under the effect of single time delay and multiple time delays respectively. Moreover, the length range estimation of the time delay is also performed by using Nyquist's criterion. In Section 5, numerical simulations are carried out to support the theoretical findings. Finally, the paper ends with a conclusion.

2. Mathematical results for the non-delay model

Without delay, system (1.1) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha - a_1x(t) - b_1x(t)y(t) + \beta z(t), \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -a_2y(t) + b_2x(t)y(t) - \dfrac{c_1y(t)z(t)}{k + y(t)}, \\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}t} = -a_3z(t) + \dfrac{c_2y(t)z(t)}{k + y(t)} - \gamma z^2(t). \end{cases} \tag{2.1}$$

In this section, we begin with the positivity and boundedness for the solution of system (2.1), then we analyze the local stability of system (2.1) at the equilibrium.

2.1. Positivity and boundedness

Lemma 2.1. *For* $x(0) = \phi_1(0) > 0, y(0) = \phi_2(0) > 0,$ *and* $z(0) = \phi_3(0) > 0,$ *the solution of system* (2.1) *is positive and bounded.*

Proof. From system (2.1), we can obtain

$$\begin{aligned} x(t) &= x(0)e^{-\int_0^t (a_1+b_1y(s))ds} + \int_0^t (\alpha + \beta z(u))\, e^{-\int_u^t (a_1+b_1y(s))ds} du, \\ y(t) &= y(0)e^{-\int_0^t \left(a_2+\frac{c_1z(s)}{k+y(s)}\right)ds} + \int_0^t (b_2x(u)y(u))\, e^{-\int_u^t \left(a_2+\frac{c_1z(s)}{k+y(s)}\right)ds} du, \\ z(t) &= z(0)e^{-\int_0^t (a_3+\gamma z(s))ds} + \int_0^t \left(\frac{c_2y(u)z(u)}{k + y(u)}\right) e^{-\int_u^t (a_3+\gamma z(s))ds} du. \end{aligned}$$

Then, we see that the solution is positive immediately under the positive initial condition $x(0) = \phi_1(0) > 0, y(0) = \phi_2(0) > 0$, and $z(0) = \phi_3(0) > 0$.

Next, we prove the boundedness. System (2.1) derives that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\{x(t) + y(t) + z(t)\} &= \alpha - a_1x(t) - b_1x(t)y(t) + \beta z(t) - a_2y(t) + b_2x(t)y(t) - \frac{c_1y(t)z(t)}{k + y(t)} \\ &\quad - a_3z(t) + \frac{c_2y(t)z(t)}{k + y(t)} - \gamma z^2(t) \\ &= \alpha + (b_2-b_1)x(t)y(t) + (c_2-c_1)\frac{y(t)z(t)}{k+y(t)} - a_1x(t) - a_2y(t) + (\beta-a_3)z(t) - \gamma z^2(t) \end{aligned}$$

$$\leq \alpha + M\{x(t) + y(t) + z(t)\},$$

where $M = \max\{-a_1, -a_2, \beta - a_3\}$. Through integral calculation, we can get

$$x(t) + y(t) + z(t) \leq (x(0) + y(0) + z(0))e^{Mt} + \frac{\alpha}{M}(e^{Mt} - 1).$$

Due to the constant $M < 0$, we see that

$$x(t) + y(t) + z(t) \leq (x(0) + y(0) + z(0)) - \frac{\alpha}{M},$$

which is the boundedness of solution of system (2.1), hence the lemma.

2.2. *Equilibria and stability*

There are three types of equilibria of system (2.1): (i) The axial equilibrium $E_A = (\frac{\alpha}{a_1}, 0, 0)$, which represents the extinction of detritivores and their predators in the mangrove ecosystem; (ii) The detritivore predators free equilibrium $E_B = (\frac{a_2}{b_2}, \frac{\alpha b_2 - a_1 a_2}{a_2 b_1}, 0)$, provided that $H = \alpha b_2 - a_1 a_2 > 0$; (iii) The interior equilibrium $E^* = (x^*, y^*, z^*)$, where the form of x^*, y^*, and z^* can be obtained by solving the equations:

$$\begin{aligned}
\alpha - a_1 x^* - b_1 x^* y^* + \beta z^* &= 0,\\
a_2 y^*(k + y^*) - b_2 x^* y^*(k + y^*) + c_1 y^* z^* &= 0,\\
a_3 z^*(k + y^*) - c_2 y^* z^* + \gamma(k + y^*)(z^*)^2 &= 0.
\end{aligned}$$

In fact,

$$\begin{aligned}
x^* &= \frac{[\beta(c_2 - a_3) + \alpha\gamma]y^* + (\alpha\gamma - \beta a_3)k}{\gamma(a_1 + b_1 y^*)(k + y^*)},\\
z^* &= \frac{(a_1 + b_1 y^*)x^* - \alpha}{\beta},
\end{aligned}$$

and y^* is the positive root of the cubic equation

$$ay^3 + by^2 + cy + d = 0.$$

Here, the coefficients are calculated by $a = \beta\gamma a_2 b_1$, $b = \beta(c_2 - a_3)(b_1 c_1 - \beta b_2) + \beta\gamma[(a_1 a_2 - \alpha b_2) + 2a_2 b_1 k]$, $c = 2\beta\gamma(a_1 a_2 - \alpha b_2)k - \beta a_3(b_1 c_1 - \beta b_2)k + \beta a_1 c_1(c_2 - a_3) + \beta k(\gamma a_2 b_1 k - \beta b_2 c_2)$ and $d = \beta\gamma(a_1 a_2 - \alpha b_2)k^2 + \beta a_3(\beta b_2 k - a_1 c_1)k$. System (2.1) has a unique positive internal equilibrium if any of the following assumptions are satisfied:

(a) $p > 0, q < 0$; (b)$p = 0, q < 0$; (c)$p < 0, q > 0, -2\sqrt{3|p|^3}/9 + q = 0$;
(d) $p < 0, q = 0$; (e)$p < 0, q < 0$,
where $p = c/a - b^2/3a^2$ and $q = 2b^3/27a^3 - bc/3a^2 + d/a$.

Now, let us consider the stability of the equilibrium of system (2.1). The Jacobian matrix around the axial equilibrium E_A is given by

$$J_{E_A} = \begin{pmatrix} -a_1 & -\frac{\alpha b_1}{a_1} & \beta \\ 0 & \frac{\alpha b_2 - a_1 a_2}{a_1} & 0 \\ 0 & 0 & -a_3 \end{pmatrix}.$$

The eigenvalues of J_{E_A} are

$$\lambda_1 = -a_1, \quad \lambda_2 = \frac{\alpha b_2 - a_1 a_2}{a_1} = \frac{H}{a_1}, \quad \lambda_3 = -a_3.$$

The following theorem can be obtained.

Theorem 2.2. *If $H > 0$, then the axial equilibrium E_A is unstable. If $H < 0$, then the axial equilibrium E_A is locally asymptotically stable.*

Remark 2.3. *Theorem 2.2 shows that when the growth rate of detritivores due to constant input of detritus is greater than the death rate, the simultaneous extinction of detritivores and their predators can not occur. And, when the growth rate of detritivores due to constant input of detritus is less than the death rate, detritivores go locally extinct and then predators of detritivores also go locally extinct.*

Calculating the Jacobian matrix of boundary equilibrium E_B, we have

$$J_{E_B} = \begin{pmatrix} -\frac{\alpha b_2}{a_2} & -\frac{a_2 b_1}{b_2} & \beta \\ a_2 & 0 & \frac{(\alpha b_2 - a_1 a_2)c_1}{a_2 b_1 k + \alpha b_2 - a_1 a_2} \\ 0 & 0 & -a_3 + \frac{(\alpha b_2 - a_1 a_2)c_2}{a_2 b_1 k + \alpha b_2 - a_1 a_2} \end{pmatrix}.$$

The eigenvalues of J_{E_B} are

$$\lambda_1 = \frac{-\frac{\alpha b_2}{a_2} + \sqrt{(\frac{\alpha b_2}{a_2})^2 - \frac{4a_2^2 b_1}{b_2}}}{2}, \ \lambda_2 = \frac{-\frac{\alpha b_2}{a_2} - \sqrt{(\frac{\alpha b_2}{a_2})^2 - \frac{4a_2^2 b_1}{b_2}}}{2}, \ \lambda_3 = \frac{c_2 H}{a_2 b_1 k + H} - a_3.$$

Hence, we have the following theorem.

Theorem 2.4. *Assume $H > 0$ holds. If $a_3 < \frac{c_2 H}{a_2 b_1 k + H}$, then the boundary equilibrium E_B is unstable; If $a_3 > \frac{c_2 H}{a_2 b_1 k + H}$, then the boundary equilibrium E_B is locally asymptotically stable.*

Proof. When $H > 0$, if $(\frac{\alpha b_2}{a_2})^2 - \frac{4 a_2{}^2 b_1}{b_2} > 0$, then $\lambda_1 = (-\frac{\alpha b_2}{a_2} + \sqrt{(\frac{\alpha b_2}{a_2})^2 - \frac{4 a_2{}^2 b_1}{b_2}})/2 < (-\frac{\alpha b_2}{a_2} + \frac{\alpha b_2}{a_2})/2 = 0$, and it is obvious that $\lambda_2 < 0$. If $(\frac{\alpha b_2}{a_2})^2 - \frac{4 a_2{}^2 b_1}{b_2} \leq 0$, then $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = -\frac{\alpha b_2}{a_2} < 0$. Hence, E_B is unstable when $a_3 < \frac{c_2 H}{a_2 b_1 k + H}$, and E_B is locally asymptotically stable when $a_3 > \frac{c_2 H}{a_2 b_1 k + H}$. □

Remark 2.5. *In the case that $H < 0$, according to the expression of E_B, it does not exist. Theorem 2.4 indicates that when the mortality rate of detritivore predators exceeds a threshold, predators of detritivores become locally extinct. For a mangrove ecosystem with given natural conditions, this threshold can be calculated.*

Finally, we consider the stability of internal equilibrium E^*. The Jacobian matrix around E^* is calculated as

$$J_{E^*} = \begin{pmatrix} -a_1 - b_1 y^* & -b_1 x^* & \beta \\ b_2 y^* & \frac{c_1 y^* z^*}{(k+y^*)^2} & -\frac{c_1 y^*}{k+y^*} \\ 0 & \frac{c_2 k z^*}{(k+y^*)^2} & -\gamma z^* \end{pmatrix}.$$

We can get the characteristic equation

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0,$$

where

$$\begin{aligned}
A_1 &= \gamma z^* + (a_1 + b_1 y^*) - \frac{c_1 y^* z^*}{(k+y^*)^2},\\
A_2 &= \gamma(a_1+b_1y^*)z^* - \gamma\frac{c_1 y^*(z^*)^2}{(k+y^*)^2} - (a_1+b_1y^*)\frac{c_1 y^* z^*}{(k+y^*)^2} + b_1 b_2 x^* y^* + \frac{c_1 c_2 k y^* z^*}{(k+y^*)^3},\\
A_3 &= -\beta\frac{b_2 c_2 k y^* z^*}{(k+y^*)^2} - \gamma(a_1+b_1y^*)\frac{c_1 y^*(z^*)^2}{(k+y^*)^2} + \gamma b_1 b_2 x^* y^* z^* + (a_1+b_1y^*)\frac{c_1 c_2 k y^* z^*}{(k+y^*)^3}.
\end{aligned}$$

Here, we make the following assumptions

$$(\mathrm{H}_1)\quad \mathrm{A}_1 > 0,\ \mathrm{A}_3 > 0 \text{ and } \mathrm{A}_1\mathrm{A}_2 - \mathrm{A}_3 > 0.$$

Then, the well-known Routh Hurwitz criteria ensures the following result.

Theorem 2.6. *If (H_1) holds, then the interior equilibrium E^* is locally asymptotically stable.*

3. Mathematical results for a single delay effect

In this section, we investigate the stability of interior equilibrium, the existence of Hopf bifurcation, and the delay length estimation for a single delay system.

3.1. Stability of interior equilibrium and local Hopf bifurcation

First, we focus on analyzing the stability of the interior equilibrium and the existence of local Hopf bifurcations occurring at the interior equilibrium.

Case (a) When $\tau_1 > 0$, $\tau_2 = 0$, the characteristic equation for system (1.1) is given by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 + B_4 e^{-\lambda\tau_1} = 0, \tag{3.1}$$

where $B_1 = A_1$, $B_2 = A_2$, $B_3 = -\frac{\gamma(a_1+b_1y^*)c_1y^*(z^*)^2}{(k+y^*)^2} + \gamma b_1 b_2 x^* y^* z^* + \frac{(a_1+b_1y^*)c_1c_2ky^*z^*}{(k+y^*)^3}$, and $B_4 = -\frac{b_2c_2k\beta y^*z^*}{(k+y^*)^2}$. Let $\lambda = \mu + i\omega$. Note that μ and ω are functions of delay τ_1. Substituting $\lambda = \mu + i\omega$ into the characteristic equation and separating the real and imaginary parts, we obtain

$$\mu^3 - 3\mu\omega^2 + B_1\left(\mu^2 - \omega^2\right) + B_2\mu + B_3 + B_4 e^{-\mu\tau_1}\cos\omega\tau_1 = 0, \tag{3.2}$$

$$-\omega^3 + 3\mu^2\omega + 2B_1\mu\omega + B_2\omega - B_4 e^{-\mu\tau_1}\sin\omega\tau_1 = 0. \tag{3.3}$$

Suppose that $\mu(\tau_1) = 0$ and $\omega(\tau_1) = \omega$. Then, from (3.2) and (3.3), we have

$$B_1\omega^2 - B_3 = B_4\cos\omega\tau_1, \tag{3.4}$$

$$-\omega^3 + B_2\omega = B_4\sin\omega\tau_1. \tag{3.5}$$

Let $z = \omega^2$. We then know that z satisfies the following equation from (3.4) and (3.5):

$$f(z) = z^3 + \left({B_1}^2 - 2B_2\right)z^2 + \left({B_2}^2 - 2B_1B_3\right)z + \left({B_3}^2 - {B_4}^2\right) = 0. \tag{3.6}$$

If Eq (3.6) has no positive roots, in other words, the characteristic equation (3.1) has no pure imaginary roots, then the positive equilibrium E^* of system (1.1) is always locally asymptotically stable when $\tau_1 \geq 0$.

Without loss of generality, we assume that there are j ($j = 1, 2, 3$) positive roots of Eq (3.6), marked as $z_1 < z_2 < z_3$, and the solutions for ω are accordingly ω_{1j} ($j = 1, 2, 3$). For each fixed ω_{1j}, (3.4), and (3.5), we derive that

$$\sin \omega_{1j}\tau_1 = \frac{\omega_{1j}\left(B_2 - {\omega_{1j}}^2\right)}{B_4}, \quad \cos \omega_{1j}\tau_1 = \frac{B_1{\omega_{1j}}^2 - B_3}{B_4}.$$

Hence, the corresponding critical values with respect to τ_1 are

$$\tau_{1j}^k = \begin{cases} \dfrac{1}{\omega_{1j}}\left(\arccos \dfrac{B_1{\omega_{1j}}^2 - B_3}{B_4} + 2k\pi\right), \quad k = 0, 1, 2, \cdots, & \text{if } B_2 \geq {\omega_{1j}}^2, \\ \dfrac{1}{\omega_{1j}}\left(2\pi - \arccos \dfrac{B_1{\omega_{1j}}^2 - B_3}{B_4} + 2k\pi\right), \quad k = 0, 1, 2, \cdots, & \text{if } B_2 < {\omega_{1j}}^2. \end{cases} \tag{3.7}$$

When $\tau_1 = \tau_{1j}^k$, the characteristic equation (3.1) has a pure imaginary root $\lambda_{1j} = i\omega_{1j}$. By basic computation, we get

$$\begin{aligned} M_1 \frac{\mathrm{d}\mu}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} - M_2 \frac{\mathrm{d}\omega}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} &= M_3, \\ M_2 \frac{\mathrm{d}\mu}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} + M_1 \frac{\mathrm{d}\omega}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} &= M_4, \end{aligned}$$

where

$$\begin{aligned} M_1 &= -3{\omega_{1j}}^2 + B_2 - B_4\tau_1 \cos \omega_{1j}\tau_{1j}, \quad & M_2 &= 2B_1\omega_{1j} + B_4\tau_1 \sin \omega_{1j}\tau_{1j}, \\ M_3 &= B_4\omega_{1j} \sin \omega_{1j}\tau_{1j}, & M_4 &= B_4\omega_{1j} \cos \omega_{1j}\tau_{1j}. \end{aligned}$$

It turns out that

$$\begin{aligned} \frac{\mathrm{d}\mu}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} &= \frac{M_1M_3 + M_2M_4}{{M_1}^2 + {M_2}^2} \\ &= \frac{{\omega_{1j}}^2\left(3{\omega_{1j}}^4 + 2\left({B_1}^2 - 2B_2\right){\omega_{1j}}^2 + \left({B_2}^2 - 2B_1B_3\right)\right)}{{M_1}^2 + {M_2}^2}. \end{aligned}$$

Hence, if the inequalities ${B_1}^2 - 2B_2 > 0$ and ${B_2}^2 - 2B_1B_3 > 0$ are satisfied, then we have the following transversality condition:

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau_1}\bigg|_{\tau_1=\tau_{1j}^k} > 0 \quad \text{for} \quad j = 1, 2, 3 \text{ and } k = 0, 1, 2 \cdots.$$

Since τ_{1j}^k monotonically increases with k, we have $\tau_{1j}^0 = \min \tau_{ij}^k \; j = 1, 2, 3, k = 0, 1, 2, \cdots$. The following conclusion can be obtained.

Theorem 3.1. *Suppose that* (H_1) *holds. If* ${B_1}^2 - 2B_2 > 0$ *and* ${B_2}^2 - 2B_1B_3 > 0$, *then we have the following statements:*

(i) If Eq (3.6) *has no positive roots, then the positive equilibrium* E^* *of system* (1.1) *is always locally asymptotically stable when* $\tau_1 \geq 0$.

(ii) If Eq (3.6) *has only one positive root* z_1*, namely, the characteristic equation of system* (1.1) *has a unique pair of pure imaginary roots* $i\omega_{11}$*, where* $\omega_{11} = \sqrt{z_1}$*, then when* $\tau_1 \in [0, \tau_{11}^0]$*, the equilibrium* E^* *is locally asymptotically stable and unstable when* $\tau_1 > \tau_{11}^0$*. The equilibrium* E^* *of system* (1.1) *undergoes Hopf bifurcation at* $\tau_1 = \tau_{11}^k$ $(k = 0, 1, 2, \cdots)$*, where* τ_{11}^k *is given by Eq* (3.7)*.*

(iii) If Eq (3.6) *has two positive roots* z_1 *and* z_2*, where we assume* $z_1 < z_2$*, then there is a positive integer m*
such that the positive equilibrium E^* *of system* (1.1) *is stable when* $\tau_1 \in [0, \tau_{12}^0] \cup (\tau_{11}^0, \tau_{12}^1) \cup (\tau_{11}^1, \tau_{12}^2) \cup \cdots \cup (\tau_{11}^{m-1}, \tau_{12}^m)$ *and unstable when* $\tau_1 \in (\tau_{12}^0, \tau_{11}^0) \cup (\tau_{12}^1, \tau_{11}^1) \cup \cdots \cup (\tau_{12}^{m-1}, \tau_{11}^{m-1}) \cup (\tau_{12}^m, \infty)$*.*

(iv) If Eq (3.6) *has three positive roots, then the stability switching phenomenon is similar to the case of (iii).*

Case (b) When $\tau_1 = 0$, $\tau_2 > 0$, the characteristic equation of system (1.1), evaluated at E^* is

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 + \left(C_4\lambda^2 + C_5\lambda + C_6\right)e^{-\lambda\tau_2} = 0, \tag{3.8}$$

where

$$\begin{aligned} C_1 &= -\left(b_{11} + b_{22} + b_{32}\right), & C_2 &= b_{11}\left(b_{22} + b_{32}\right) - b_{12}b_{21} + b_{22}b_{32}, \\ C_3 &= \left(b_{12}b_{21} - b_{11}b_{22}\right)b_{32}, & C_4 &= -b_{33}, \\ C_5 &= \left(b_{11} + b_{22}\right)b_{33} - b_{23}b_{31}, & C_6 &= \left(b_{11}b_{23} - b_{13}b_{21}\right)b_{31} - \left(b_{11}b_{22} - b_{12}b_{21}\right)b_{33}, \end{aligned}$$

in which

$$\begin{aligned} b_{11} &= -a_1 - b_1y^*, & b_{12} &= -b_1x^*, & b_{13} &= \beta, \\ b_{21} &= b_2y^*, & b_{22} &= \frac{c_1y^*z^*}{(k + y^*)^2}, & b_{23} &= -\frac{c_1y^*}{k + y^*}, \\ b_{31} &= \frac{c_2kz^*}{(k + y^*)^2}, & b_{32} &= -a_3 - 2\gamma z^*, & b_{33} &= \frac{c_2y^*}{k + y^*}. \end{aligned}$$

For $\omega > 0$, $i\omega$ is a root of (3.8), which leads to

$$\omega^6 + \left({C_1}^2 - 2C_2 - {C_4}^2\right)\omega^4 + \left({C_2}^2 - {C_5}^2 + 2C_4C_6 - 2C_1C_3\right)\omega^2 + {C_3}^2 - {C_6}^2 = 0.$$

We rewrite the above equation as

$$g(z) = z^3 + \left({C_1}^2 - 2C_2 - {C_4}^2\right)z^2 + \left({C_2}^2 - {C_5}^2 + 2C_4C_6 - 2C_1C_3\right)z + {C_3}^2 - {C_6}^2 = 0. \tag{3.9}$$

Through the same calculation process as delay τ_1, we derive the corresponding critical value with respect to delay τ_2 as follows:

$$\tau_{2j}^k = \begin{cases} \dfrac{1}{\omega_{2j}}\left(\arccos\dfrac{D_1D_3 + D_2D_4}{{D_1}^2 + {D_2}^2} + 2k\pi\right), \ k = 0, 1, 2\cdots, & \text{if } D_1D_4 \geq D_2D_3, \\[2ex] \dfrac{1}{\omega_{2j}}\left(2\pi - \arccos\dfrac{D_1D_3 + D_2D_4}{{D_1}^2 + {D_2}^2} + 2k\pi\right), \ k = 0, 1, 2\cdots, & \text{if } D_1D_4 < D_2D_3, \end{cases} \tag{3.10}$$

where $D_1 = C_4\omega^2 - C_6, D_2 = C_5\omega, D_3 = -C_1\omega^2 + C_3$, and $D_4 = \omega^3 - C_2\omega$. When $\tau_2 = \tau_{2j}^k$, characteristic equation (3.8) has pure imaginary root $\lambda_j = i\omega_{2j}$. From (3.8), we can calculate that

$$\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{2j}^k} = \frac{P_1P_3 + P_2P_4}{{P_3}^2 + {P_4}^2} + \frac{Q_1Q_3 + Q_2Q_4}{{Q_3}^2 + {Q_4}^2},$$

where

$$\begin{aligned}
&P_1 = -3{\omega_{2j}}^2 + C_2, && P_2 = 2C_1\omega_{2j}, && P_3 = -{\omega_{2j}}^4 + C_2{\omega_{2j}}^2, && P_4 = C_1{\omega_{2j}}^3 - C_3\omega_{2j},\\
&Q_1 = C_5, && Q_2 = 2C_4\omega_{2j}, && Q_3 = -C_5{\omega_{2j}}^2, && Q_4 = -C_4{\omega_{2j}}^3 + C_6\omega_{2j}.
\end{aligned}$$

With respect to the delay τ_2, we have the following results.

Theorem 3.2. *Suppose that (H_1) holds. If* $\operatorname{Re}(\mathrm{d}\lambda/\mathrm{d}\tau_2)^{-1}|_{\tau_2=\tau_{2j}^k} \neq 0$ ($j = 1, 2, 3$ *and* $k = 0, 1, 2\cdots$)*, then the following conclusions hold:*

(i) If Eq (3.9) *has no positive roots, then the interior equilibrium E^* of system* (1.1) *is always locally asymptotically stable when $\tau_2 \geq 0$.*

(ii) If Eq (3.9) *has only one positive root z_1, namely, the characteristic equation of system* (1.1) *has a unique pair of pure imaginary roots $i\omega_{21}$, where $\omega_{21} = \sqrt{z_1}$, then when $\tau_1 \in [0, \tau_{21}^0]$, the interior equilibrium E^* is locally asymptotically stable and unstable when $\tau_2 > \tau_{21}^0$. The equilibrium E^* of system* (1.1) *undergoes Hopf bifurcation at $\tau_2 = \tau_{21}^k$ ($k = 0, 1, 2, \cdots$), where τ_{21}^k is given by Eq* (3.10).

(iii) If Eq (3.9) *has two positive roots z_1 and z_2, where we assume that $z_1 < z_2$, then there exists a positive integer n such that the interior equilibrium E^* of system* (1.1) *is stable when* $\tau_2 \in [0, \tau_{22}^0] \cup (\tau_{21}^0, \tau_{22}^1) \cup (\tau_{21}^1, \tau_{22}^2) \cup \cdots \cup (\tau_{21}^{n-1}, \tau_{22}^n)$ *and unstable when* $\tau_1 \in (\tau_{22}^0, \tau_{21}^0) \cup (\tau_{22}^1, \tau_{21}^1) \cup \cdots \cup (\tau_{22}^{n-1}, \tau_{21}^{n-1}) \cup (\tau_{22}^n, \infty)$.

(iv) If Eq (3.9) *has three positive roots, then the stability switching phenomenon is similar to the case of (iii).*

3.2. Estimation of the length of delay for stability

Now we will study the length estimation of delay of system (1.1). Let $u_1 = x - x^*$, $u_2 = y - y^*$, and $u_3 = z - z^*$. For $\tau_1 > 0, \tau_2 = 0$, the linearized form of system (1.1) around the interior equilibrium E^* can be written as

$$\begin{aligned}
\frac{\mathrm{d}u_1}{\mathrm{d}t} &= a_{11}u_1 + a_{12}u_2 + a_{13}u_3(t - \tau_1)\\
\frac{\mathrm{d}u_2}{\mathrm{d}t} &= a_{21}u_1 + a_{22}u_2 + a_{23}u_3\\
\frac{\mathrm{d}u_3}{\mathrm{d}t} &= a_{31}u_1 + a_{32}u_2 + a_{33}u_3
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&a_{11} = -a_1 - b_1y^*, && a_{12} = -b_1x^*, && a_{13} = \beta,\\
&a_{21} = b_2y^*, && a_{22} = \frac{c_1y^*z^*}{(k + y^*)^2}, && a_{23} = -\frac{c_1y^*}{k + y^*},
\end{aligned}$$

$$a_{31} = 0, \qquad a_{32} = \frac{c_2 k z^*}{(k + y^*)^2}, \quad a_{33} = -\gamma z^*.$$

Taking the Laplace transform of the system (3.11), we get

$$\begin{aligned}
&(s - a_{11})\,\overline{u}_1(s) = u_1(0) + a_{12}\overline{u}_2(s) + a_{13}e^{-s\tau_1}\overline{u}_3(s) + a_{13}e^{-s\tau_1}K_1(s),\\
&(s - a_{22})\,\overline{u}_2(s) = u_2(0) + a_{21}\overline{u}_1(s) + a_{23}\overline{u}_3(s),\\
&(s - a_{33})\,\overline{u}_3(s) = u_3(0) + a_{32}\overline{u}_2(s),
\end{aligned} \tag{3.12}$$

where $K_1(s) = \int_{-\tau_1}^{0} e^{-s\tau_1} u_3(t)\,\mathrm{d}t$, and $\overline{u}_1(s)$, $\overline{u}_2(s)$, and $\overline{u}_3(s)$ are the Laplace transforms of $u_1(t)$, $u_2(t)$, and $u_3(t)$, respectively. The characteristic equation of (3.12) is given as

$$H(s) = s^3 + B_1 s^2 + B_2 s + B_3 + B_4 e^{-s\tau_1} = 0,$$

in which B_1, B_2, B_3, and B_4 have the same expressions as in Subsection 3.1 after Eq (3.1). By setting $s = i\eta$ and separating the real and imaginary parts, one has

$$\begin{aligned}
\mathrm{Re}H(i\eta) &= -B_1\eta^2 + B_3 + B_4\cos\eta\tau_1 = 0,\\
\mathrm{Im}H(i\eta) &= -\eta^3 + B_2\eta - B_4\sin\eta\tau_1 = 0.
\end{aligned}$$

By the Nyquist criterion [24, 25], the condition for local asymptotic stability of E^* is

$$\mathrm{Re}H(i\eta_0) = 0, \tag{3.13}$$

$$\mathrm{Im}H(i\eta_0) > 0, \tag{3.14}$$

where η_0 is a positive solution of (3.13). That is, if

$$B_1{\eta_0}^2 = B_4\cos\eta_0\tau_1 + B_3, \tag{3.15}$$

$$-{\eta_0}^3 > B_4\sin\eta_0\tau_1 - B_2\eta_0 \tag{3.16}$$

hold simultaneously, then system (1.1) is stable. Using (3.15) and (3.16), we can obtain estimation of the delay τ_1. First, we seek the upper bound η_+ of η_0 which is independent of τ_1 and satisfies the relations (3.16) for all η, $0 \le \eta \le \eta_+$ at $\eta = \eta_0$. From (3.15), we obtain

$$|B_1|{\eta_0}^2 \le |B_3| + |B_4|, \tag{3.17}$$

which leads to

$$\eta_0 \le \eta_+ = \sqrt{\frac{|B_3| + |B_4|}{|B_1|}}.$$

Equality (3.16) can be rewritten as

$${\eta_0}^2 < B_2 - \frac{B_4\sin\eta_0\tau_1}{\eta_0}. \tag{3.18}$$

Substituting (3.15) in (3.18), we get

$$B_4(\cos\eta_0\tau_1 - 1) + \frac{B_1B_4\sin\eta_0\tau_1}{\eta_0} < B_1B_2 - (B_3 + B_4). \tag{3.19}$$

Through a simple reduction, one has

$$B_4\left(1-\cos\eta_0\tau_1\right)=2B_4\sin^2\frac{\eta_0\tau_1}{2}\leq\frac{1}{2}|B_4|{\eta_+}^2{\tau_1}^2$$

and

$$\frac{B_1B_4\sin\eta_0\tau_1}{\eta_0}\leq|B_1B_4|\tau_1.$$

Let

$$K_{11}=\frac{1}{2}|B_4|{\eta_+}^2,\quad K_{22}=|B_1B_4|,\quad K_{33}=B_1B_2-\left(B_3+B_4\right).$$

It is apparent that if

$$K_{11}{\tau_1}^2+K_{22}\tau_1<K_{33},$$

then (3.15) holds. Hence, τ_1 is estimated by

$$0\leq\tau_1<\tau_{1+}=\frac{1}{2K_{11}}\left(-K_{22}+\sqrt{{K_{22}}^2+4K_{11}K_{33}}\right).$$

Therefore, we can determine as above the length of delay τ_1 that preserves the stability of system (1.1).

For $\tau_1=0,\tau_2>0$, let $v_1=x-x^*$, $v_2=y-y^*$, and $v_3=z-z^*$. Then, we can linearize system (3.1) as

$$\begin{aligned}\frac{\mathrm{d}v_1}{\mathrm{d}t}&=b_{11}v_1+b_{12}v_2+b_{13}v_3,\\\frac{\mathrm{d}v_2}{\mathrm{d}t}&=b_{21}v_1+b_{22}v_2+b_{23}v_3,\\\frac{\mathrm{d}v_3}{\mathrm{d}t}&=b_{31}v_2\left(t-\tau_2\right)+b_{32}v_3+b_{33}v_3\left(t-\tau_2\right),\end{aligned}$$

where b_{ij} ($i=1,2,3;j=1,2,3$) are given in Subsection 3.1 after Eq (3.8). Using the Nyquist criterion and similar derivations as in the case of the delay τ_1, we estimate that

$$0\leq\tau_2<\tau_{2+}=\frac{1}{2L_{11}}\left(-L_{22}+\sqrt{{L_{22}}^2+4L_{11}L_{33}}\right),$$

in which $L_{11}=|C_6-C_4{\eta_+}^2-C_1C_5|{\eta_+}^2/2$, $L_{22}=\left(C_5-C_1C_4\right){\eta_+}^2+C_1C_6$, $L_{33}=C_1C_2-C_3-C_6+|C_4|{\eta_+}^2+C_1C_5$, and $\eta_+=\sqrt{{C_5}^2+4\left(C_1-C_4\right)\left(|C_3|+|C_6|\right)}/2\left(C_1-C_4\right)$. We omit the proof.

4. Mathematical results for system (1.1) with multiple delays

In this section, we study the mangrove food chain system (1.1) proposed in Section 1 as follows:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=\alpha-a_1x(t)-b_1x(t)y(t)+\beta z(t-\tau_1),\\[2ex]\dfrac{\mathrm{d}y}{\mathrm{d}t}=-a_2y(t)+b_2x(t)y(t)-\dfrac{c_1y(t)z(t)}{k+y(t)},\\[2ex]\dfrac{\mathrm{d}z}{\mathrm{d}t}=-a_3z(t)+\dfrac{c_2y(t-\tau_2)z(t-\tau_2)}{k+y(t-\tau_2)}-\gamma z^2(t).\end{cases}\tag{1.1}$$

The local stability of interior equilibrium and the existence of Hopf bifurcation are established. Under the influence of multiple delays τ_1 and τ_2, the delay length for maintaining stability is also presented.

4.1. Stability of interior equilibrium and bifurcation analysis

Linearizing system (1.1) at the equilibrium point E^*, we get

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= c_{11}x(t) + c_{12}y(t) + c_{13}z(t-\tau_1), \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= c_{21}y(t) + c_{22}y(t) + c_{23}z(t), \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= c_{31}y(t-\tau_2) + c_{32}z(t) + c_{33}z(t-\tau_2),
\end{aligned} \tag{4.1}$$

in which

$$\begin{aligned}
&c_{11} = -a_1 - b_1y^*, && c_{12} = -b_1x^*, && c_{13} = \beta, \\
&c_{21} = b_2y^*, && c_{22} = \frac{c_1y^*z^*}{(k+y^*)^2}, && c_{23} = -\frac{c_1y^*}{k+y^*}, \\
&c_{31} = \frac{c_2kz^*}{(k+y^*)^2}, && c_{32} = -a_3 - 2\gamma z^*, && c_{33} = \frac{c_2y^*}{k+y^*}.
\end{aligned}$$

The characteristic equation corresponding to system (4.1) is given by

$$\lambda^3 + E_1\lambda^2 + E_2\lambda + E_3 + \left(E_4\lambda^2 + E_5\lambda + E_6\right)e^{-\lambda\tau_2} + E_7e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{4.2}$$

Here,

$$\begin{aligned}
&E_1 = -(c_{11} + c_{22} + c_{32}), && E_2 = c_{11}(c_{22} + c_{32}) - c_{12}c_{21} + c_{22}c_{32}, \\
&E_3 = c_{12}c_{21}c_{32} - c_{11}c_{22}c_{32}, && E_4 = -c_{33}, \\
&E_5 = (c_{11} + c_{22})c_{33} - c_{23}c_{31}, && E_6 = (c_{12}c_{21} - c_{11}c_{22})c_{33} + c_{11}c_{23}c_{31}, \\
&E_7 = -c_{13}c_{21}c_{31}.
\end{aligned}$$

We suppose that the delay τ_1 is in the stable interval and take τ_2 as the bifurcation parameter. Let $i\omega(\omega > 0)$ be a root of (4.2). Then, one has

$$N_1\cos\omega\tau_2 + N_2\sin\omega\tau_2 = N_3, \tag{4.3}$$

$$N_1\sin\omega\tau_2 - N_2\cos\omega\tau_2 = N_4, \tag{4.4}$$

with $N_1 = -E_4\omega^2 + E_6 + E_7\cos\omega\tau_1$, $N_2 = E_5\omega - E_7\sin\omega\tau_1$, $N_3 = E_1\omega^2 - E_3$, and $N_4 = -\omega^3 + E_2\omega$. From (4.3) and (4.4), we obtain

$$\begin{aligned}
\omega^6 + \left(E_1{}^2 - 2E_2 - E_4{}^2\right)\omega^4 + \left(E_2{}^2 - 2E_1E_3 + 2E_4E_6 + 2E_4E_7\cos\omega\tau_1 - E_5{}^2\right)\omega^2 \\
+E_3{}^2 + 2E_5E_7\omega\sin\omega\tau_1 - 2E_6E_7\cos\omega\tau_1 - E_6{}^2 - E_7{}^2 = 0.
\end{aligned}$$

Generally, we assume that there exist three positive roots ω_{2j} $(j = 1, 2, 3)$ for this cubic equation of ω^2. For each fixed ω_{2j}, it follows from (4.3) and (4.4) that

$$\sin\omega_{2j}\tau_2 = \frac{N_1N_4 + N_2N_3}{N_1{}^2 + N_2{}^2}, \quad \cos\omega_{2j}\tau_2 = \frac{N_1N_3 - N_2N_4}{N_1{}^2 + N_2{}^2},$$

and hence,

$$\tau_{2j}^{k} = \begin{cases} \dfrac{1}{\omega_{2j}}\left(\arccos\dfrac{N_1N_3 - N_2N_4}{N_1{}^2 + N_2{}^2} + 2k\pi\right), \ k = 0, 1, 2\cdots, \quad \text{if } N_1N_4 + N_2N_3 \geq 0, \\ \\ \dfrac{1}{\omega_{2j}}\left(2\pi - \arccos\dfrac{N_1N_3 - N_2N_4}{N_1{}^2 + N_2{}^2} + 2k\pi\right), \ k = 0, 1, 2\cdots, \quad \text{if } N_1N_4 + N_2N_3 < 0. \end{cases}$$

When $\tau_2 = \tau_{2j}^{k}$, the characteristic equation for system (1.1) has the pure imaginary root $\lambda_j = i\omega_{2j}$. From (4.2), we have

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{2j}^{k}} = \frac{F_1F_3 + F_2F_4}{F_3{}^2 + F_4{}^2} + \frac{G_1G_3 + G_2G_4}{G_3{}^2 + G_4{}^2},$$

where $F_1 = -3\omega_{2j}{}^2 + E_2 + E_5, F_2 = 2(E_1 + E_4)\omega_{2j}$, $F_3 = -\omega_{2j}{}^4 + E_2\omega_{2j}{}^2, F_4 = E_1\omega_{2j}{}^3 - E_3\omega_{2j}$, $G_1 = E_7\tau_1\cos\omega_{2j}(\tau_1 + \tau_2), G_2 = E_7\tau_1\sin\omega_{2j}(\tau_1 + \tau_2), G_3 = \omega_{2j}{}^4 - E_2\omega_{2j}{}^2$, and $G_4 = E_3\omega_{2j} - E_1\omega_{2j}{}^3$. Let $\tau_1^* = \min_{1\leq j\leq 3}\{\tau_{1j}^0\}$ and $\tau_2^* = \min_{1\leq j\leq 3}\{\tau_{2j}^0\}$. Then, we arrive at the following conclusion.

Theorem 4.1. *Suppose that* (H_1) *holds. Let* $\tau_1 \in (0, \tau_1^*)$. *If* $\mathrm{Re}\,(\mathrm{d}\lambda/\mathrm{d}\tau_2)^{-1}|_{\tau_2=\tau_{2j}^k} \neq 0$ $(j = 1, 2, 3$ *and* $k = 0, 1, 2\cdots)$*, then system* (1.1) *undergoes a Hopf bifurcation at the interior equilibrium* E^* *when* $\tau_2 = \tau_{2j}^k$ $(j = 1, 2, 3$ *and* $k = 0, 1, 2\cdots)$. *The interior equilibrium point* E^* *of system* (1.1) *is locally asymptotically stable for* $\tau_2 \in (0, \tau_2^*)$ *and is unstable for* $\tau_2 > \tau_2^*$.

4.2. Delay length estimation for stability preservation

We take $w_1 = x - x^*$, $w_2 = y - y^*$, and $w_3 = z - z^*$, and then linearize system (1.1) as

$$\begin{aligned} \frac{\mathrm{d}w_1}{\mathrm{d}t} &= c_{11}w_1 + c_{12}w_2 + c_{13}w_3(t - \tau_1), \\ \frac{\mathrm{d}w_2}{\mathrm{d}t} &= c_{21}w_1 + c_{22}w_2 + c_{23}w_3, \\ \frac{\mathrm{d}w_3}{\mathrm{d}t} &= c_{31}w_2(t - \tau_2) + c_{32}w_3 + c_{33}w_3(t - \tau_2), \end{aligned} \tag{4.5}$$

in which c_{ij} $(i = 1, 2, 3, j = 1, 2, 3)$ are given in Subsection 4.1. The Laplace transforms of the system (4.5) can be obtained by

$$\begin{aligned} (s - c_{11})\overline{w}_1(s) &= w_1(0) + c_{12}\overline{w}_2(s) + c_{13}e^{-s\tau_1}\overline{w}_3(s) + c_{13}e^{-s\tau_1}J_1(s), \\ (s - c_{22})\overline{w}_2(s) &= w_2(0) + c_{21}\overline{w}_1(s) + c_{23}\overline{w}_3(s), \\ (s - c_{32} - c_{33}e^{-s\tau_2})\overline{w}_3(s) &= w_3(0) + c_{31}e^{-s\tau_2}\overline{w}_2(s) + c_{31}e^{-s\tau_2}J_2(s) + c_{33}e^{-s\tau_2}J_3(s), \end{aligned}$$

with $J_1(s) = \int_{-\tau_1}^{0} e^{-s\tau_1}w_3(t)\,\mathrm{d}t$, $J_2(s) = \int_{-\tau_2}^{0} e^{-s\tau_2}w_2(t)\,\mathrm{d}t$, $J_3(s) = \int_{-\tau_2}^{0} e^{-s\tau_2}w_3(t)\,\mathrm{d}t$, and $\overline{w}_1(s)$, $\overline{w}_2(s)$, and $\overline{w}_3(s)$ are the Laplace transform of $w_1(t)$, $w_2(t)$, and $w_3(t)$, respectively. The characteristic equation of (4.5) is

$$I(s) = s^3 + E_1s^2 + E_2s + E_3 + \left(E_4s^2 + E_5s + E_6\right)e^{-s\tau_2} + E_7e^{-s(\tau_1+\tau_2)} = 0, \tag{4.6}$$

where E_i $(i = 1, 2, \cdots, 7)$ has the same expression as (4.2). Now we consider the condition that τ_1 is in its stable interval $(0, \tau_1^*)$ and $\tau_2 > 0$. Substituting $s = i\eta$ into (4.6), then the real part and imaginary parts give

$$\begin{aligned}&\left(-E_4\omega^2 + E_6 + E_7\cos\omega\tau_1\right)\cos\omega\tau_2 + (E_5\omega - E_7\sin\omega\tau_1)\sin\omega\tau_2 = E_1\omega^2 - E_3,\\&\left(-E_4\omega^2 + E_6 + E_7\cos\omega\tau_1\right)\sin\omega\tau_2 - (E_5\omega - E_7\sin\omega\tau_1)\cos\omega\tau_2 = -\omega^3 + E_2\omega.\end{aligned}$$

The Nyquist criteria for local stability shows that $\operatorname{Re}(i\eta_0) = 0$ and $\operatorname{Im}(i\eta_0) > 0$, that is,

$$-E_1{\eta_0}^2 + E_3 + \left(-E_4{\eta_0}^2 + E_6\right)\cos\eta_0\tau_2 + E_5\eta_0\sin\eta_0\tau_2 + E_7\cos\eta_0(\tau_1 + \tau_2) = 0, \tag{4.7}$$

and

$$-{\eta_0}^3 + E_2\eta_0 + E_5\eta_0\cos\eta_0\tau_2 + \left(E_4{\eta_0}^2 - E_6\right)\sin\eta_0\tau_2 - E_7\sin\eta_0(\tau_1 + \tau_2) > 0 \tag{4.8}$$

hold simultaneously, and so system (1.1) is stable. Here, η_0 is a positive solution of $\operatorname{Re}(i\eta_0) = 0$. Using (4.7) and (4.8), the upper bound η_+ of η_0 can be obtained, which is independent of τ_1 and τ_2, and satisfies the relationship (4.7) for all η, $0 \le \eta \le \eta_+$ at $\eta = \eta_0$. In fact, it follows from (4.7) that

$$(E_1 + E_4){\eta_0}^2 \le E_5\eta_0 + E_3 + |E_6| + |E_7|.$$

Hence, we get

$$\eta_+ = \frac{1}{2(E_1 + E_4)}\left(E_5 + \sqrt{{E_5}^2 + 4(E_1 + E_4)(|E_3| + |E_6| + |E_7|)}\right).$$

By basic deduction from (4.7) and (4.8), we have

$$\begin{aligned}&\left(-E_4{\eta_0}^2 - E_1E_5 + E_6 + E_7\cos\eta_0\tau_1 + \frac{E_1E_7\sin\eta_0\tau_1}{\eta_0}\right)(\cos\eta_0\tau_2 - 1)\\&+\left((E_5 - E_1E_4)\eta_0 - E_7\sin\eta_0\tau_1 + \frac{E_1(E_6 + E_7)}{\eta_0}\right)\sin\eta_0\tau_2\\&\le |E_4|{\eta_+}^2 + E_1(E_2 + E_5) - E_3 - E_6 - E_7 - \frac{|E_1E_7|}{\eta_+}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(-E_4{\eta_0}^2 - E_1E_5 + E_6 + E_7\cos\eta_0\tau_1 + \frac{E_1E_7\sin\eta_0\tau_1}{\eta_0}\right)(1 - \cos\eta_0\tau_2)\\&\le\frac{1}{2}(-E_4{\eta_+}^4 + |E_6 + E_7 - E_1E_5|{\eta_+}^2 + |E_1E_7|\eta_+){\tau_2}^2,\end{aligned}$$

and

$$\begin{aligned}&\left((E_5 - E_1E_4)\eta_0 - E_7\sin\eta_0\tau_1 + \frac{E_1(E_6 + E_7)}{\eta_0}\right)\sin\eta_0\tau_2\\&\le\left(|E_5 - E_1E_4|{\eta_+}^2 - E_7\eta_+ + |E_1E_6 + E_1E_7|\right)\tau_2.\end{aligned}$$

We use the following notations:

$$J_{11} = \frac{1}{2}(-E_4{\eta_+}^4 + |E_6 + E_7 - E_1E_5|{\eta_+}^2 + |E_1E_7|\eta_+){\tau_2}^2,$$
$$J_{22} = \left(|E_5 - E_1E_4|{\eta_+}^2 - E_7\eta_+ + |E_1E_6 + E_1E_7|\right)\tau_2,$$
$$J_{33} = |E_4|{\eta_+}^2 + E_1\left(E_2 + E_5\right) - E_3 - E_6 - E_7 - \frac{|E_1E_7|}{\eta_+}.$$

Then, τ_2 is estimated from

$$J_{11}{\tau_2}^2 + J_{22}\tau_2 < J_{33}$$

as follows:

$$0 \le \tau_2 < \tau_{2+} = \frac{1}{2J_{11}}\left(-J_{22} + \sqrt{{J_{22}}^2 + 4J_{11}J_{33}}\right).$$

Thus, we complete the delay length estimation for preserving the stability of system (1.1) subjected to multiple discrete time delays.

5. Numerical simulations

This section performs some numerical simulations to illustrate the our obtained theoretical results. Based on the actual situation and parametric values in the literature [26–30], we select appropriate parameters, calculate the interior equilibrium E^* of system (1.1), simulate the changes in the density of detritus, detritivores, and predators of detritivores, and present effects of some important parameters on the stability of a detrital-based food chain in the mangrove ecosystem.

Based on the above analysis, we take the parameters as follows (see Table 1):

$$\begin{aligned}&\alpha = 1.78, \beta = 0.30, \gamma = 0.15, a_1 = 0.70, a_2 = 0.15,\\ &a_3 = 0.10, b_1 = 0.45, b_2 = 0.15, c_1 = 0.70, c_2 = 0.50, k = 0.50.\end{aligned} \tag{5.1}$$

$$\begin{aligned}&\alpha = 1.78, \beta = 0.30, \gamma = 0.15, a_1 = 0.75, a_2 = 0.20,\\ &a_3 = 0.13, b_1 = 0.30, b_2 = 0.15, c_1 = 0.70, c_2 = 0.50, k = 0.50.\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\alpha = 1.78, \beta = 0.30, \gamma = 0.15, a_1 = 0.80, a_2 = 0.15,\\ &a_3 = 0.10, b_1 = 0.45, b_2 = 0.15, c_1 = 0.70, c_2 = 0.50, k = 0.50.\end{aligned} \tag{5.3}$$

Table 1. Parameters of system (5.1) with default values.

Parameters	Biological significance	Values	source
α	Eternal input of detritus	1.78	set
β	Recycling rate of detritus from dead detritivores predators	0.3	set
γ	Intraspecific competition rate of detritivores predators	0.15	Naji et al. (2016)
a_1	Washout rate of detritus	0.7-0.8	set
a_2	Death rate of detritivores	0.15-0.2	set
a_3	Death rate of detritivores predators	around 0.1	Meng et al. (2014)
b_1	Detritus uptake rate by detritivores	around 0.447	Li et al. (2016)
b_2	Intake rate of detritus	around 0.117	Li et al. (2016)
c_1	Uptake rate of detritivores	0.70	Ganjisaffar et al. (2016)
c_2	Specific growth rate of predator	0.5	Pandey et al. (2016)
k	Half-saturation constant	0.5	set

For parameters (5.1), system (1.1) has a unique interior equilibrium $E^* = (2.362, 0.175, 0.197)$. We find that the assumptions of Theorem 2.6 hold. Hence, when $\tau_1 = \tau_2 = 0$, the interior equilibrium E^* is locally asymptotically stable, meaning that detrital-based food chain is self-regulating in the sense that detritus, detritivores, and predators of detritivores can change from an unstable state to a stable state after a period of time. The number of different populations changing over time is shown in Figure 2.

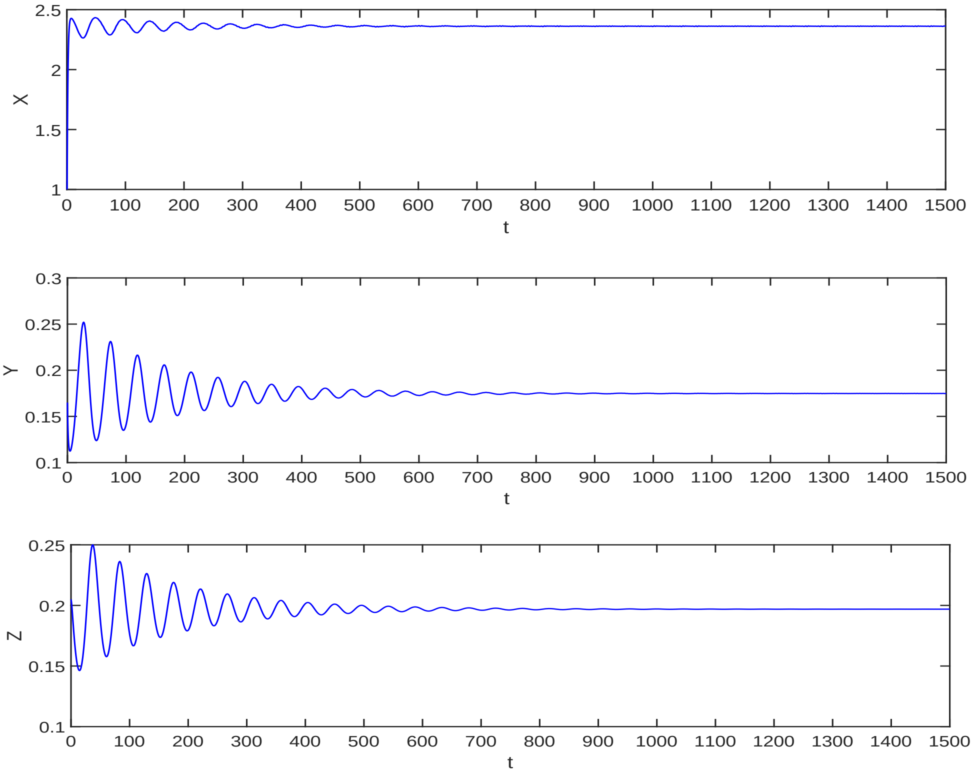

Figure 2. For parameters (5.1), when $\tau_1 = \tau_2 = 0$, the equilibrium E^* of system (1.1) is locally asymptotically stable.

By changing the relationship between detritivores and predators from Holling type II to a linear type and keeping all the other parameters the same as in (5.1), we find that system (1.1) shows a stable state of the populations in a short period of time, as shown in the Figure 3. The Holling type II functional response describes the more realistic interaction between detritivores and predators as well as the saturation predation phenomenon. Under this dynamic mechanism, the system may have more richer dynamic behavioral performance. In contrast, the linear type is suitable for describing the proportional relationships in simple systems, which makes the dynamic behaviors of the system relatively simple as well.

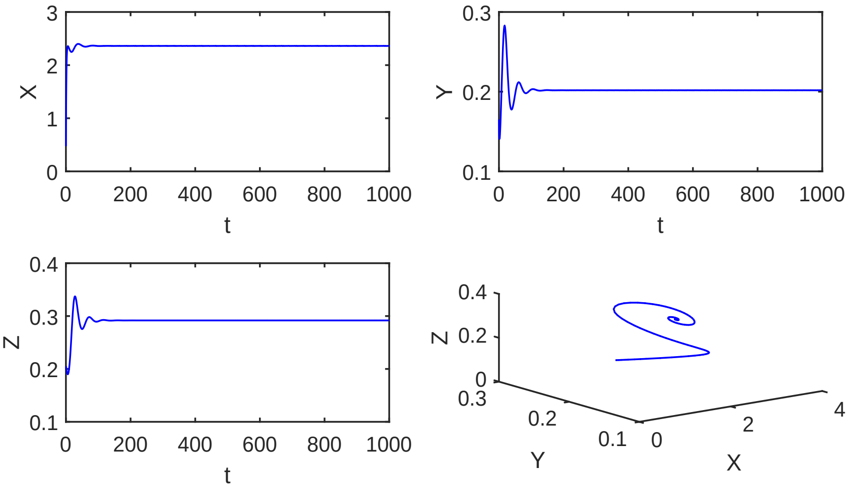

Figure 3. For parameters (5.1), when the cooperative relation between detritivores and predators is described by the linear relationship c_1yz, and system (1.1) with $\tau_1 = \tau_2 = 0$ exhibits more faster local asymptotic stabilization behavior.

When $\tau_1 > 0, \tau_2 = 0$, for parameters (5.2), there is a unique interior equilibrium $E^* = (2.236, 0.215, 0.138)$ of system (1.1). We solve the critical value $\tau_{11}^0 = 32.424$. According to Theorem 3.1, the interior equilibrium E^* is locally asymptotically stable for $\tau_1 \in (0, \tau_{11}^0)$ and is unstable for $\tau_1 > \tau_{11}^0$, see Figure 4. It is observed that the oscillation occurs when the time delay τ_1 is large. This shows that in the mangrove ecosystem, if dead predators are decomposed into detritius too slowly, this will affect the beginning of the detrital food chain and thus hinder its normal circulation, resulting in ecosystem imbalance.

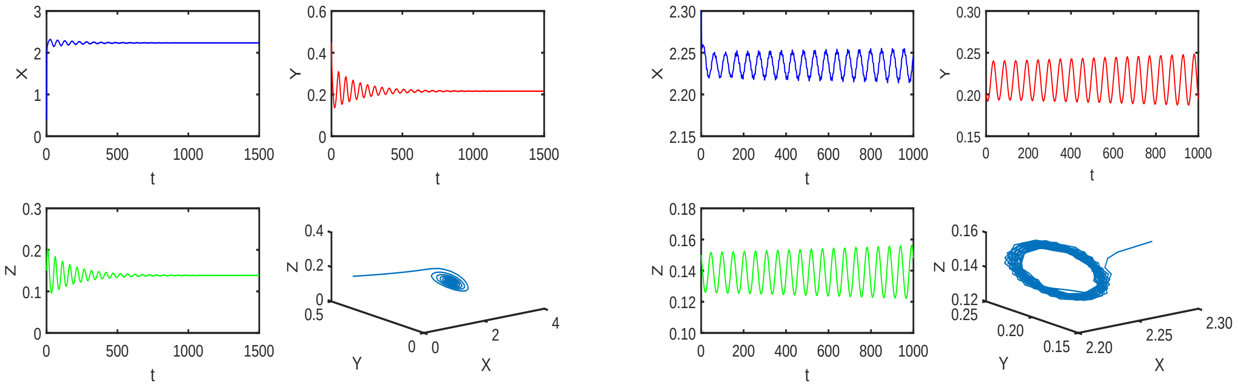

Figure 4. (left) When $\tau_1 < \tau_{11}^0, \tau_2 = 0$, the interior equilibrium E^* of system (1.1) is locally asymptotically stable; (right) When $\tau_1 > \tau_{11}^0, \tau_2 = 0$, the interior equilibrium E^* of system (1.1) is unstable.

When $\tau_1 = 0, \tau_2 > 0$, for parameters (5.3), we can obtain the only interior equilibrium $E^* = (2.091, 0.164, 0.155)$ of system (1.1) and the critical value $\tau_{21}^0 = 0.573$. Theorem 3.2 shows that the interior equilibrium E^* is locally asymptotically stable for $\tau_2 \in (0, \tau_{21}^0)$ and is unstable for $\tau_2 > \tau_{21}^0$, see Figure 5. From the figures, we can see that the long feedback time of detritivores to their predators may lead to the untimely response of predators to the detritivores population changes and the rapid fluctuating increase in detritivores numbers. Meanwhile, if the gestation period of predators is long, population renewal rate will be limited and the predation ability will be reduced. Consequently, detritivores significantly increase and gradually destroy the ecological balance of detrital food chain. This is consistent with reality.

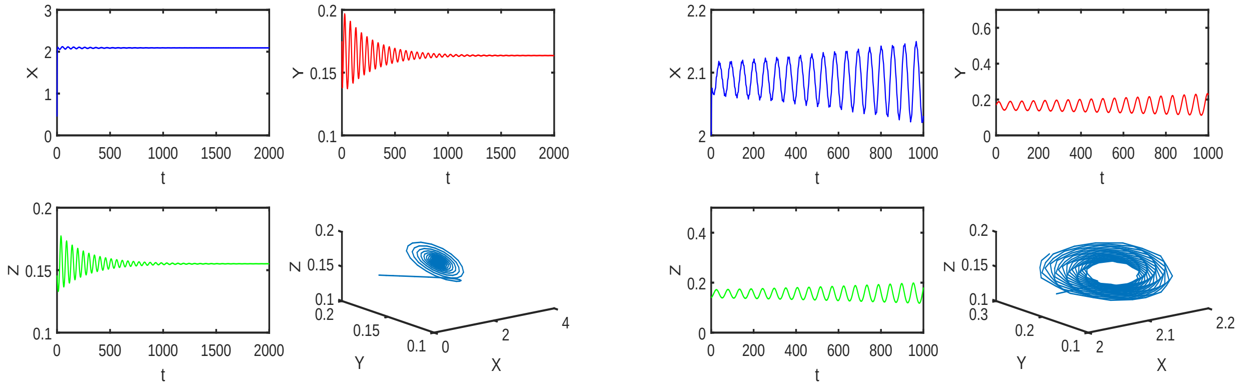

Figure 5. (left) When $\tau_1 = 0, \tau_2 < \tau_{21}^0$, the equilibrium E^* of system (1.1) is locally asymptotically stable; (right) When $\tau_1 = 0, \tau_2 > \tau_{21}^0$, the equilibrium E^* of system (1.1) is unstable.

Under parameters (5.2), we take $\tau_1 = 20 \in (0, 32.424)$, and then we have the critical value $\tau_2^* = 0.96$. When $\tau_1 = 20$ and $\tau_2 = 0.1 \in (0, \tau_2^*)$, the unique interior equilibrium $E^* = (2.236, 0.215, 0.138)$ of system (1.1) is locally asymptotically stable (Figure 6). When $\tau_1 = 20$ and $\tau_2 = 1 > \tau_2^*$, the interior equilibrium E^* is unstable (Figure 7). We have considered the intraspecific competition among detritivore predators in system (1.1), so we will study the effect of intraspecific competition rate on this system and compare it with the effect of detritivore predators mortality on the system. Increasing the mortality of predators of detritivores a_3 from 0.1 to 0.3, according to the simulations shown in the left of Figure 8, we see that the large mortality of detritivore predators allow system to regain the stability in a relatively short time. From the simulations performed in the right of Figure 8, we find that to return the system to a stable state in the same period of time, a relatively large intraspecific competition rate is required comparing with the mortality. In this situation, we have changed the intraspecific competition rate $\gamma = 0.15$ by $\gamma = 0.45$.

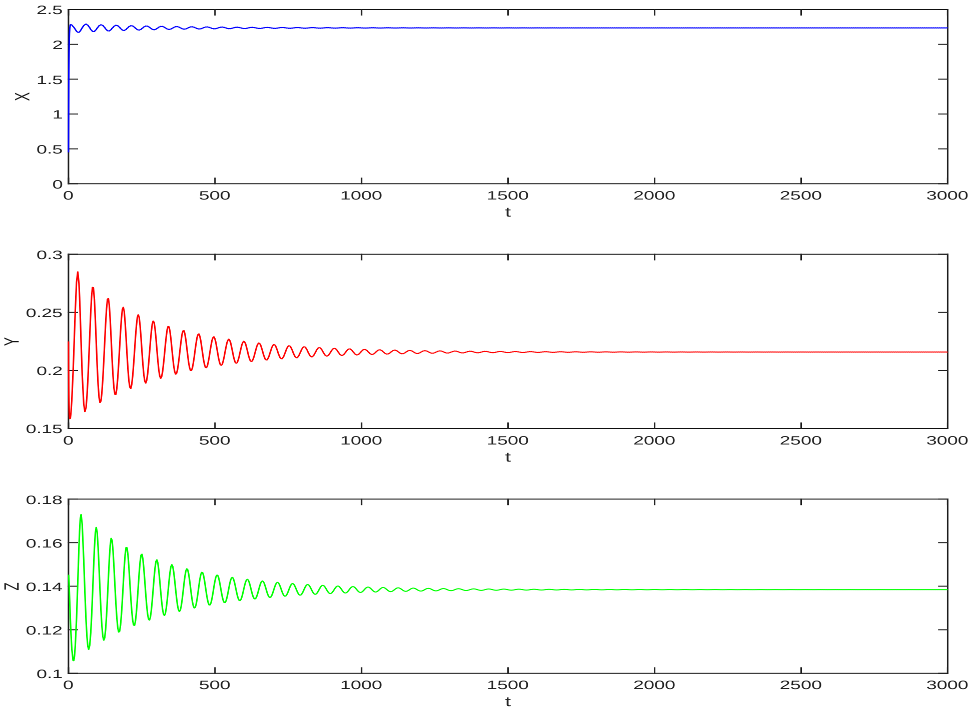

Figure 6. When $\tau_1 = 20, \tau_2 = 0.1$, the equilibrium E^* of system (1.1) is locally asymptotically stable.

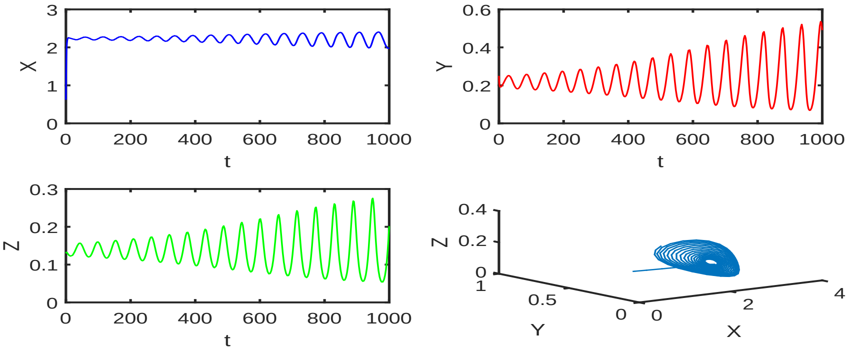

Figure 7. When $\tau_1 = 20, \tau_2 = 1$, the equilibrium E^* of system (1.1) is unstable.

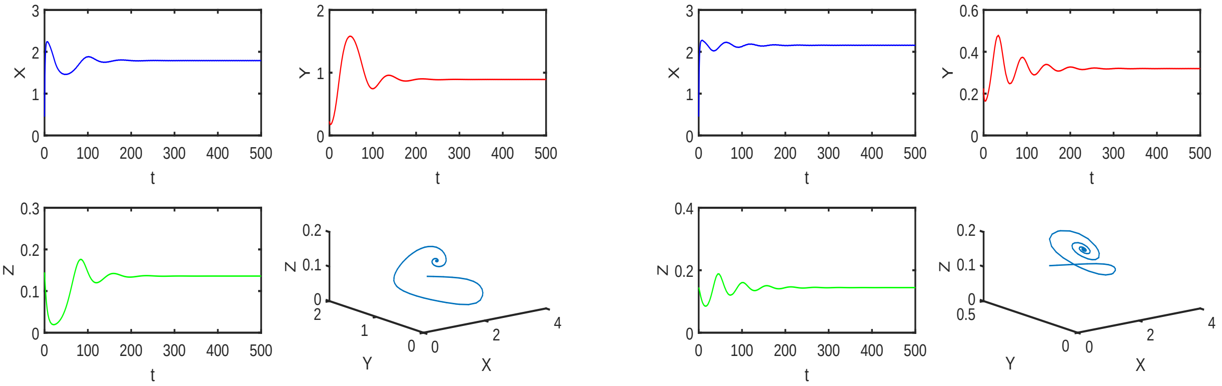

Figure 8. (left) When $\tau_1 = 20, \tau_2 = 1$, $a_3 = 0.3 > 0.13$, system (1.1) has a new unique interior equilibrium which is locally asymptotically stable; (right) When $\tau_1 = 20, \tau_2 = 1$, $\gamma = 0.45 > 0.15$, system (1.1) has a new interior unique equilibrium which is locally asymptotically stable.

6. Conclusions

A detrital-based food chain model in the mangrove ecosystem with time delays has been established in this paper. First, it is shown that the solutions of this model are positive and bounded. Then, we analyzed the stability of the interior equilibrium and the Hopf bifurcation. In the presence of delay, we also obtained the delay length estimation using the Nyquist criterion. Some numerical simulations were conducted as a support to our analytical results. The effects of Holling type II functional response, death rate of detritivores, intraspecific competition rate among detritivore predators, and detritivore predator mortality were discussed. We found that the relationship between detritivores and predators affect the time for the detrital food chain to reach a balanced state. In addition, increased mortality of detritivore predators and increased intraspecific competition rate among detritivore predators both contributed to the decline in detritivore predator numbers. If the detrital food chain is destabilized by an overabundance of detritivore predators, it can be restored by increasing detritivore predators mortality or intraspecific competition among detritivore predators. In contrast to the indirect reduction in the number of detritivore predators caused by intraspecific competition, the increase in the direct death of detritivore predators can stabilize the detrital food chain in a short period of time.

Author contributions

Hanqi Zhu: Methodology, writing-original draft, writing-review and editing; Yan Yan: Conceptualization, methodology, writing-original draft, language editing, funding acquisition. Both the authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest regarding the publication of this paper.

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