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Research article

A study of Caputo fractional differential equations of variable order via Darbo's fixed point theorem and Kuratowski measure of noncompactness

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Abstract: This paper investigated the existence and stability of solutions for boundary value problems involving Caputo fractional differential equations of variable order. Unlike constant-order models, variable-order equations allow the fractional order to change over time, enabling more flexible and accurate modeling of complex systems with evolving dynamics and memory. Using Darbo's fixed point theorem and the Kuratowski measure of noncompactness, we established new existence results for solutions within a Banach space of continuous functions. Our approach treated the variable order as piecewise constant, transforming the problem into a sequence of more manageable constant-order subproblems. Furthermore, we demonstrated Ulam-Hyers stability of the solutions, ensuring that small perturbations in the system did not lead to significant deviations in the results. To validate the theoretical findings, we provided a detailed example supported by numerical simulations. These results offered a solid foundation for future applications in science and engineering where system dynamics evolve over time.

Keywords: boundary value problem; fractional differential equations; variable order; measure of noncompactness; Darbo's fixed point; Ulam-Hyers
Mathematics Subject Classification: 26A33, 34K37

1. Introduction

Fractional calculus, a vibrant branch of mathematical analysis, extends the concepts of differentiation and integration to non-integer orders, offering powerful tools for modeling complex phenomena across diverse disciplines [15]. Fractional calculus has found diverse applications across several domains. In physics and chemistry, it models systems with memory and hereditary properties [7–9]. In biology and biophysics, it has been used to describe population dynamics and biological processes with nonlocal behavior [32, 33]. Applications in economics and control theory have leveraged fractional models for better optimization and stability analysis [17, 34]. Furthermore, in signal processing and aerodynamics, it provides tools for capturing complex system dynamics and wave propagation [18, 21, 31].

In particular, the Caputo fractional derivative has gained popularity in applications due to its physically interpretable formulation and compatibility with classical initial conditions [16]. This makes it especially suited for boundary value problems (BVPs) and initial value problems in applied contexts. Recent studies have extended the reach of fractional models to a variety of scientific and engineering domains. For instance, tempered and space-fractional reaction-diffusion models have been explored for combustion-like problems [19]. High-dimensional and multi-term time-fractional models with non-smooth solutions have also been addressed using advanced spectral techniques [19]. In the context of optimization and control, operational approaches to fractional variational problems with indefinite integrals have been proposed [12]. Furthermore, the preservation of physical properties, such as the discrete maximum principle in sub-diffusion equations, has been addressed by novel numerical methods [30].

Population dynamics and biological growth have also benefited from fractional formulations. For example, fractional logistic population models provide better accuracy in capturing nonlinear behaviors [11]. Similarly, recent advancements in the numerical treatment of multi-term time-fractional diffusion-wave equations, particularly in two-dimensional and delayed systems, illustrate the growing sophistication of numerical techniques [2].

Variable-order fractional differential equations, where the order of the derivative is a function rather than a constant, have emerged as a critical area of study due to their ability to model systems with evolving dynamics [1, 24, 25, 28, 29]. These equations generalize constant-order fractional differential equations, enabling the description of processes with time-dependent or spatially varying memory effects [14, 26]. Despite their potential, the theoretical analysis of variable-order problems, particularly concerning the existence and stability of solutions, remains underexplored compared to their constant-order counterparts [6, 10].

This study builds on and extends the constant-order results of Benchohra et al. [5], who analyzed implicit fractional differential equations using Caputo derivatives. Foundational definitions and properties of variable order fractional calculus were established by Samko [17, 18], while more recent contributions by Valério et al. [21] and Tavares et al. [20] have focused primarily on numerical methods.

Their study provided foundational insights into BVPs for such equations, paving the way for further exploration of implicit formulations. Building on this, recent studies have begun to address the complexities of variable-order fractional differential equations, leveraging advanced mathematical tools such as the Kuratowski measure of noncompactness and fixed point theorems. In this manuscript, we focus on the implicit BVP for Caputo fractional differential equations of variable order, a class of equations that introduces additional challenges due to the implicit dependence on the fractional derivative itself.

$$\begin{cases} {}^cD_{0^+}^{\omega(s)}\xi(s) = \Psi_1(s, \xi(s), {}^cD_{0^+}^{\omega(s)}\xi(s)), \quad s \in \Lambda := [0, M], \\ \xi(0) = 0, \quad \xi(M) = 0, \end{cases} \tag{1.1}$$

where $\Psi_1 : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Continuous functions (CF), $1 < \omega(s) \leq 2$, and ${}^cD_{0^+}^{\omega(s)}$ is the Caputo fractional derivative of variable order (CFDVO).

The main contributions of this article are summarized as follows:

1. Existence of solutions: We establish novel existence results for solutions to BVP involving Caputo fractional differential equations of variable order (VO-FDEs) within a Banach space of continuous functions. This extends previous work primarily focused on constant-order equations.

2. Piecewise constant approach: We address the inherent complexity of the variable order $\omega(s)$ by treating it as a piecewise constant function. This transforms the original problem into a sequence of more tractable constant-order subproblems over partitioned intervals.

3. Noncompactness and fixed point theory: We utilize Darbo's fixed point theorem combined with the Kuratowski measure of noncompactness as the primary mathematical tools to prove the existence results. This approach effectively handles the nonlinearity and implicit nature of the problem.

4. Stability analysis: We demonstrate the Ulam-Hyers stability of solutions to the considered VO-FDE BVP. This ensures that small perturbations in the system lead to proportionally small deviations in the solutions, a crucial property for robust modeling.

5. Numerical validation: We provide a concrete example supported by numerical simulations to illustrate the applicability and validity of the derived theoretical results.

The paper is organized as follows: Section 2 introduces fundamental notations, definitions of variable-order fractional operators, and essential preliminaries, including Darbo's fixed point theorem and the Kuratowski measure of noncompactness. Section 3 establishes the main existence results for solutions to the BVP (1.1) under appropriate hypotheses, after which Section 4 investigates the Ulam-Hyers stability of the solutions to ensure their robustness. Section 5 provides a concrete example supported by numerical simulations to illustrate these theoretical findings, while Section 6 concludes by summarizing the key conclusions and contributions of the study, highlighting its advancements in the analysis of variable-order fractional differential equations.

2. Preliminaries

In this section, we collect a few notions and results to be used later in the paper.

2.1. *Notations*

Table 1 lists the main acronyms and their corresponding meanings used in this work for ease of reference.

Table 1. Notations used throughout the paper.

Acronym	Meaning
BS	Banach Space
CF	Continuous Functions
CFICO	Caputo Fractional Integral of Constant Order
CFIVO	Caputo Fractional Integral of Variable Order
CFDCO	Caputo Fractional Derivative of Constant Order
CFDVO	Caputo Fractional Derivative of Variable Order
EC	Equi Continuous
PWCF	Piece Wise-Constant Function
UHS	Ulam Hyers Stable

By $C(\Lambda, \mathbb{R})$, we denote the Banach space (BS) of CF from Λ into $\mathbb{R}$ with the norm

$$\|\varkappa\| = \sup_{s\in\Lambda} |\varkappa(s)|.$$

For $-\infty < \nu_1 < \nu_2 < +\infty$, we consider the mappings $\omega(s) : [\nu_1, \nu_2] \to (0, +\infty)$ and $u(s) : [\nu_1, \nu_2] \to (n-1, n)$. Then, the left CFDVO for function $\Psi_2(s)$ [22, 23, 27] is

$$I_{\nu_1^+}^{\omega(s)}\Psi_2(s) = \int_{\nu_1}^{s} \frac{(s-\theta)^{\omega(s)-1}}{\Gamma(\omega(s))}\Psi_2(\theta)d\theta, \quad s > \nu_1, \tag{2.1}$$

and the left CFDVO for function $\Psi_2(s)$ [22, 23, 27] is

$${}^{c}D_{\nu_1^+}^{u(s)}\Psi_2(s) = \int_{\nu_1}^{s} \frac{(s-\theta)^{n-u(s)-1}}{\Gamma(n-u(s))}\Psi_2^{(n)}(\theta)d\theta, \quad s > \nu_1. \tag{2.2}$$

As anticipated, in case of $\omega(s)$ and $u(s)$ are constant, then Caputo fractional integral of variable order (CFIVO) and CFDVO coincide with Caputo fractional integral of constant order (CFICO) and Caputo fractional derivative of constant order (CFDCO), (see, e.g., [15, 22, 23]).

Lemma 2.1. *[4] Let ω_1, $\omega_2 > 0$, $\nu_1 > 0$, $\Psi_2 \in L^1(\nu_1, \nu_2)$, ${}^{c}D_{\nu_1^+}^{\omega_1}\Psi_2 \in L^1(\nu_1, \nu_2)$. Then, the differential equation*

$${}^{c}D_{\nu_1^+}^{\omega_1}\Psi_2 = 0$$

has unique solution

$$\Psi_2(s) = \beta_0 + \beta_1(s-\nu_1) + \beta_2(s-\nu_1)^2 + ... + \beta_{n-1}(s-\nu_1)^{n-1},$$

and

1. $I_{\nu_1^+}^{\omega_1}\, {}^{c}D_{\nu_1^+}^{\omega_1}\Psi_2(s) = \Psi_2(s) + \beta_0 + \beta_1(s-\nu_1) + \beta_2(s-\nu_1)^2 + ... + \beta_{n-1}(s-\nu_1)^{n-1}$.
2. ${}^{c}D_{\nu_1^+}^{\omega_1} I_{\nu_1^+}^{\omega_1}\Psi_2(s) = \Psi_2(s)$.
3. $I_{\nu_1^+}^{\omega_1} I_{\nu_1^+}^{\omega_2}\Psi_2(s) = I_{\nu_1^+}^{\omega_2} I_{\nu_1^+}^{\omega_1}\Psi_2(s) = I_{\nu_1^+}^{\omega_1+\omega_2}\Psi_2(s)$.

Here, $\beta_0, \beta_1, ..., \beta_{n-1} \in \mathbb{R}$.

Remark 2.1. *[20, 32, 34] In the general case, note that*

$$I_{\nu_1^+}^{\omega(s)} I_{\nu_1^+}^{u(s)} \Psi_2(s) \neq I_{\nu_1^+}^{\omega(s)+u(s)} \Psi_2(s).$$

Example 2.1. *Let*

$$\omega(s) = s, \quad s \in [0,4], \quad u(s) = \begin{cases} 2, & s \in [0,1], \\ 3, & s \in]1,4], \end{cases} \qquad \Psi_2(s) = 2, \quad s \in [0,4],$$

$$\begin{aligned}
I_{0^+}^{\omega(s)} I_{0^+}^{u(s)} \Psi_2(s) &= \int_0^s \frac{(s-\theta)^{\omega(s)-1}}{\Gamma(\omega(s))} \int_0^\theta \frac{(\theta-\alpha)^{u(\theta)-1}}{\Gamma(u(\theta))} \Psi_2(\alpha) d\alpha d\theta \\
&= \int_0^s \frac{(s-\theta)^{s-1}}{\Gamma(s)} [\int_0^1 \frac{(\theta-\alpha)}{\Gamma(2)} 2d\alpha + \int_1^\theta \frac{(\theta-\alpha)^2}{\Gamma(3)} 2d\alpha] d\theta \\
&= \int_0^s \frac{(s-\theta)^{s-1}}{\Gamma(s)} [2\theta - 1 + \frac{(\theta-1)^3}{3}] d\theta,
\end{aligned}$$

and

$$I_{0^+}^{\omega(s)+u(s)} \Psi_2(s)| = \int_0^s \frac{(s-\theta)^{\omega(s)+u(s)-1}}{\Gamma(\omega(s)+u(s))} \Psi_2(\theta) d\theta.$$

So, we get

$$\begin{aligned}
I_{0^+}^{\omega(s)} I_{0^+}^{u(s)} \Psi_2(s)|_{s=3} &= \int_0^3 \frac{(3-\theta)^2}{\Gamma(3)} [2\theta - 1 + \frac{(\theta-1)^3}{3}] d\theta \\
&= \frac{21}{10},
\end{aligned}$$

$$\begin{aligned}
I_{0^+}^{\omega(s)+u(s)} \Psi_2(s)|_{s=3} &= \int_0^3 \frac{(3-\theta)^{\omega(s)+u(s)-1}}{\Gamma(\omega(s)+u(s))} \Psi_2(\theta) d\theta \\
&= \int_0^1 \frac{(3-\theta)^4}{\Gamma(5)} 2d\theta + \int_1^3 \frac{(3-\theta)^5}{\Gamma(6)} 2d\theta \\
&= \frac{1}{12} \int_0^1 (\theta^4 - 12\theta^3 + 54\theta^2 - 108\theta + 81) d\theta \\
&\quad + \frac{1}{60} \int_1^3 (-\theta^5 + 15\theta^4 - 90\theta^3 + 270\theta^2 - 405\theta + 243) d\theta \\
&= \frac{665}{180}.
\end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{\omega(s)} I_{0^+}^{u(s)} \Psi_2(s)|_{s=3} \neq I_{0^+}^{\omega(s)+u(s)} \Psi_2(s)|_{s=3}.$$

Lemma 2.2. *[13] Let $\omega \in C(\Lambda, (1,2])$ and $0 \leq \sigma \leq 1$; then for $\Psi_2 \in C_\sigma(\Lambda, \mathbb{R})$, where*

$$C_\sigma(\Lambda, \mathbb{R}) = \{\Psi_2(s) \in C(\Lambda, \mathbb{R}),\ s^\sigma \Psi_2(s) \in C(\Lambda, \mathbb{R}), 0 \leq \sigma \leq 1\},$$

the $I_{0^+}^{\omega(s)} \Psi_2(s)$ exists for any $s \in \Lambda$.

Lemma 2.3. *[9] If $\omega \in C(\Lambda, (1,2])$, then*

$$I_{0^+}^{\omega(s)} \Psi_2(s) \in C(\Lambda, \mathbb{R}) \text{ for any } \Psi_2 \in C(\Lambda, \mathbb{R}).$$

Definition 2.1. *[14, 26, 33] Let $I \subset \mathbb{R}$ be a generalized interval (i.e., an interval, a singleton $\{b_1\}$, or the empty set $\emptyset$).*
A partition $\mathcal{P}$ of I is a finite family of nonoverlapping generalized intervals $\{\Pi_i\}_{i=1}^n$ such that every $x \in I$ belongs to exactly one $\Pi_i \in P$.
A function $\hbar : I \to \mathbb{R}$ is called piecewise-constant with respect to $\mathcal{P}$ if $\hbar$ is constant on each $\Pi_i \in \mathcal{P}$.

2.2. Measure of noncompactness

Definition 2.2. *[3] Let B be a BS and Ω_B the bounded subsets of B. The Kuratowski measure of noncompactness) is a mapping $\Phi : \Omega_B \to [0, \infty]$, which is constructed as follows:*

$$\Phi(D) = \inf\{\epsilon > 0 : \exists (D_\vartheta)_{\vartheta=1,2,\ldots,n} \subset B,\ D \subseteq \cup_{\vartheta=1}^n D_\vartheta,\ diam(D_\vartheta) \leq \epsilon\},$$

where

$$diam(D_\vartheta) = sup\{\|\xi - y\| :\ \xi,\ y \in D_\vartheta\}.$$

Proposition 2.1. *[3, 4] Let B be a BS, and $D,\ D_1,\ D_2$ are bounded subsets of B, then*

1. $\Phi(D) = 0 \Longleftrightarrow D$ *is relatively compact.*
2. $\Phi(\phi) = 0$.
3. $\Phi(D) = \Phi(\overline{D}) = \Phi(convD)$.
4. $D_1 \subset D_2 \Longrightarrow \Phi(D_1) \leq \Phi(D_2)$.
5. $\Phi(D_1 + D_2) \leq \Phi(D_1) + \Phi(D_2)$.
6. $\Phi(\lambda D) = |\lambda| \Phi(D), \lambda \in \mathbb{R}$.
7. $\Phi(D_1 \cup D_2) = Max\{\Phi(D_1), \Phi(D_2)\}$.
8. $\Phi(D_1 \cap D_2) = Min\{\Phi(D_1), \Phi(D_2)\}$.
9. $\Phi(D + x_0) = \Phi(D)$ *for any* $x_0 \in B$.

Lemma 2.4. *[13] If $U \subset C(\Lambda, B)$ is an Equi continuous (EC) and bounded set, then*

(i) the function $\Phi(U(s))$ is continuous for $s \in \Lambda$, and

$$\widehat{\Phi}(U) = \sup_{s \in \Lambda} \Phi(U(s)).$$

(ii) $\Phi\left(\int_0^M \xi(t)dt : \xi \in U\right) \leq \int_0^M \Phi(U(t))dt,$

where

$$U(t) = \{\xi(t) :\ \xi \in U\},\ t \in \Lambda.$$

Lemma 2.5. *(Darbo's fixed point theorem) [3] Let T be convex, closed, bounded, and nonempty of a BS (B) and $F : T \longrightarrow T$ is a continuous operator satisfying*

$$\Phi(F(S)) \leq k\Phi(S),\ for\ any\ S(\neq \emptyset) \subset T,\ k \in [0, 1).$$

Then, F has at least one fixed point in T.

Lemma 2.6. *[10] The (1.1) is Ulam Hyers stable (UHS) if there exists $c_{\Psi_1} > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(\Lambda, \mathbb{R})$ of the following inequality*

$$|{}^c D_{0^+}^{\omega(s)} z(s) - \Psi_1(s, z(s), {}^c D_{0^+}^{\omega(s)} z(s))| \leq \epsilon,\ \ s \in \Lambda, \tag{2.3}$$

there exists a solution $\xi \in C(\Lambda, \mathbb{R})$ of (1.1) with

$$|z(s) - \xi(s)| \leq c_{\Psi_1}\epsilon,\ \ s \in \Lambda.$$

3. Existence of solutions

To complete the main results, some assumptions are needed:

(H1) Let $\wp = \{\Lambda_1 := [0, M_1], \Lambda_2 := (M_1, M_2], \Lambda_3 := (M_2, M_3], ...\Lambda_n := (M_{n-1}, M]\}$ be a partition of the interval Λ, and let $\omega(s) : \Lambda \to (1, 2]$ be a piece wise-constant function (PWCF) with respect to $\mathcal{P}$, i.e.,

$$\omega(s) = \sum_{\vartheta=1}^{n} \omega_\vartheta I_\vartheta(s) = \begin{cases} \omega_1, & if\ s \in \Lambda_1, \\ \omega_2, & if\ s \in \Lambda_2, \\ \ \ . \\ \ \ . \\ \ \ . \\ \omega_n, & if\ s \in \Lambda_n, \end{cases}$$

where $1 < \omega_\vartheta \leq 2$ are constants, and I_ϑ is the indicator of the interval $\Lambda_\vartheta := (M_{\vartheta-1}, M_\vartheta]$, $\vartheta = 1, 2, ..., n$, (with $M_0 = 0,\ M_n = M$) such that

$$I_\vartheta(s) = \begin{cases} 1, & for\ s \in \Lambda_\vartheta, \\ 0, & for\ elsewhere. \end{cases}$$

Remark 3.1. *[5, p 20] Under the condition (H2), the following inequality holds for any bounded sets $M_1, M_2 \subset T$ and for each $s \in \Lambda$:*

$$\Phi(s^\sigma |\Psi_1(s, M_1, M_2)|) \leq \eta\Phi(M_1) + \gamma\Phi(M_2).$$

This inequality is equivalent to the Lipschitz-like condition in (H2), ensuring the applicability of the Kuratowski measure of noncompactness in our analysis.

In addition, let us indicate by for a given set U of functions $u:\Lambda\to T$,

$$U(s)=\{u(s),u\in U\},\ \ s\in\Lambda,$$

and

$$U(\Lambda)=\{U(s):u\in U,\ \ s\in\Lambda\}.$$

For each $\vartheta\in\{1,2,...,n\}$, the symbol $\Pi_\vartheta=C(\Lambda_\vartheta,\mathbb{R})$, indicated the BS of CF ($\xi:\Lambda_\vartheta\to\mathbb{R}$) equipped with the norm

$$\|\xi\|_{\Pi_\vartheta}=\sup_{s\in\Lambda_\vartheta}|\xi(s)|,$$

For any $s\in\Lambda_\vartheta,\ \vartheta=1,2,...,n$, the CFDVO for $\xi(s)\in C(\Lambda,\mathbb{R})$, given by (2.2), is the sum of CFDCO, i.e.,

$${}^cD_{0^+}^{\omega(s)}\xi(s)=\int_0^{M_1}\frac{(s-\theta)^{1-\omega_1}}{\Gamma(2-\omega_1)}\xi^{(2)}(\theta)d\theta+...+\int_{M_{\vartheta-1}}^{s}\frac{(s-\theta)^{1-\omega_\vartheta}}{\Gamma(2-\omega_\vartheta)}\xi^{(2)}(\theta)d\theta. \tag{3.1}$$

Thus, according to (3.1), (1.1) can be written for any $s\in\Lambda_\vartheta,\ \vartheta=1,2,...,n$ in the form

$$\begin{aligned}&\int_0^{M_1}\tfrac{(s-\theta)^{1-\omega_1}}{\Gamma(2-\omega_1)}\xi^{(2)}(\theta)d\theta &&+...+\\ &\int_{M_{\vartheta-1}}^{s}\tfrac{(s-\theta)^{1-\omega_\vartheta}}{\Gamma(2-\omega_\vartheta)}\xi^{(2)}(\theta)d\theta &&=\Psi_1(s,\xi(s),{}^cD_{0^+}^{\omega_\vartheta}\xi(s))\ \ s\in\Lambda_\vartheta.\end{aligned} \tag{3.2}$$

Definition 3.1. *BVP (1.1) has a solutio; if there are functions $\xi_\vartheta,\vartheta=1,2,...,n$, so that $\xi_\vartheta\in C([0,M_\vartheta],\mathbb{R})$ fulfilling Eq (3.2) and $\xi_\vartheta(0)=0=\xi_\vartheta(M_\vartheta)$.*

Let the function $\xi\in C(\Lambda,\mathbb{R})$ be such that $\xi(s)\equiv0$ on $s\in[0,M_{\vartheta-1}]$, and it solves integral Eq (3.2). Then, (3.2) is reduced to

$${}^cD_{M_{\vartheta-1}^+}^{\omega_\vartheta}\xi(s)=\Psi_1(s,\xi(s),{}^cD_{M_{\vartheta-1}^+}^{\omega_\vartheta}\xi(s)),\quad s\in\Lambda_\vartheta.$$

We shall deal with following implicit BVPs of CFDCO

$$\begin{cases}{}^cD_{M_{\vartheta-1}^+}^{\omega_\vartheta}\xi(s)=\Psi_1(s,\xi(s),{}^cD_{M_{\vartheta-1}^+}^{\omega_\vartheta}\xi(s)),\quad s\in\Lambda_\vartheta,\\ \xi(M_{\vartheta-1})=0,\xi(M_\vartheta)=0.\end{cases} \tag{IBVPCFDCO}$$

Lemma 3.1. *Assume that $\Psi_1:\Lambda_\vartheta\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is CF, and there exists a number $\sigma\in(0,1)$ such that $s^\sigma\Psi_1\in C(\Lambda_\vartheta\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. The solution of the integral equation, given by*

$$\xi(s)=-(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})I_{M_{\vartheta-1}^+}^{\omega_\vartheta}y(M_\vartheta)+I_{M_{\vartheta-1}^+}^{\omega_\vartheta}y(s), \tag{3.3}$$

where

$$y(s)\ \ =\ \ \Psi_1\Big(s,-(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})I_{M_{\vartheta-1}^+}^{\omega_\vartheta}y(M_\vartheta)+I_{M_{\vartheta-1}^+}^{\omega_\vartheta}y(s),y(s)\Big),\quad s\in\Lambda_\vartheta,$$

solves the (IBVPCFDCO).

Proof. We presume that $\xi \in \Pi_{\vartheta}$ is a solution of IBVPCFDCO and we take ${}^{c}D^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}}\xi(s) = y(s)$. Employing the operator $I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}}$ to both sides of IBVPCFDCO and regarding Lemma 2.1, we find

$$\xi(s) = \beta_1 + \beta_2(s - M_{\vartheta-1}) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s), \quad s \in \Lambda_{\vartheta}.$$

By $\xi(M_{\vartheta-1}) = 0$, we get $\beta_1 = 0$.

Let $\xi(s)$ satisfy $\xi(M_{\vartheta}) = 0$. So, we observe that

$$\beta_2 = -(M_{\vartheta} - M_{\vartheta-1})^{-1} I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(M_{\vartheta}).$$

Then, we find

$$\xi(s) = -(M_{\vartheta} - M_{\vartheta-1})^{-1}(s - M_{\vartheta-1}) I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(M_{\vartheta}) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s),$$

where

$$y(s) = \Psi_1\Big(s, -(M_{\vartheta} - M_{\vartheta-1})^{-1}(s - M_{\vartheta-1}) I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(M_{\vartheta}) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s), y(s)\Big), \quad s \in \Lambda_{\vartheta}.$$

Conversely, we can easily show that ξ solves the (IBVPCFDCO) CF of $s^{\sigma}\Psi_1$ and Lemma 2.1.

The following result is based on the Kuratowski measure noncompactness and Darbo's fixed point theorem.

Theorem 3.1. *Let the conditions of Lemma 3.1 be satisfied, and there exist a constants* $\eta, \gamma > 0$, *such that*

$$s^{\sigma}|\Psi_1(s, y_1, z_1) - \Psi_1(s, y_2, z_2)| \leq \eta|y_1 - y_2| + \gamma|z_1 - z_2|, \text{ for any } y_i,\ z_i \in \mathbb{R}, i = 1, 2, s \in \Lambda_{\vartheta},$$

and the inequality

$$\frac{2(M_{\vartheta} - M_{\vartheta-1})^{\omega_{\vartheta}-1}(M_{\vartheta}^{1-\sigma} - M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_{\vartheta})}(2\eta\frac{(M_{\vartheta} - M_{\vartheta-1})^{\omega_{\vartheta}}}{\Gamma(\omega_{\vartheta}+1)} + \gamma) < 1, \tag{3.4}$$

holds.

Then, IBVPCFDCO possesses at least one solution in Π_{ϑ}.

Proof. Consider the operator

$$S : \Pi_{\vartheta} \to \Pi_{\vartheta}$$

defined by:

$$Sy(s) = -(M_{\vartheta} - M_{\vartheta-1})^{-1}(s - M_{\vartheta-1}) I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(M_{\vartheta}) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s), \tag{3.5}$$

where

$$y(s) = \Psi_1\Big(s, -(M_{\vartheta} - M_{\vartheta-1})^{-1}(s - M_{\vartheta-1}) I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(M_{\vartheta}) + I^{\omega_{\vartheta}}_{M^{+}_{\vartheta-1}} y(s), y(s)\Big), \quad s \in \Lambda_{\vartheta}.$$

Let the set

$$B_{r_{\vartheta}} = \{y \in \Pi_{\vartheta},\ \|y\|_{\Pi_{\vartheta}} \leq r_{\vartheta}\},$$

where

$$r_\vartheta \geq \frac{\frac{2\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}}{1-\frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}(M_\vartheta^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)},$$

and

$$\Psi^\star = \sup_{s\in\Lambda_\vartheta}|\Psi_1(s,0,0)|.$$

Clearly B_{r_ϑ} is convex, closed, bounded, and nonempty.

Step 1. Claim: $S(B_{r_\vartheta}) \subseteq (B_{r_\vartheta})$.

For $y \in B_{r_\vartheta}$, we have

$$\begin{aligned}
&|(Sy)(s)|\\
\leq\ & \frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)\\
&\qquad +I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
\leq\ & \frac{2}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
\leq\ & \frac{2}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\\
&\qquad -\Psi_1(\theta,0,0)\Big|d\theta+\frac{2}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}|\Psi_1(\theta,0,0)|d\theta\\
\leq\ & \frac{2}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\theta^{-\sigma}(\eta|-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)\\
&\qquad +I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta)|+\gamma|y(\theta)|)d\theta+\frac{2\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\\
\leq\ & \frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}(\eta|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta)|+\gamma|y(\theta)|)d\theta+\frac{2\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\\
\leq\ & \frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(2\eta\|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y\|_{\Pi_\vartheta}+\gamma\|y\|_{\Pi_\vartheta})\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta+\frac{2\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\\
\leq\ & \frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}(M_\vartheta^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)r_\vartheta+\frac{2\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\\
\leq\ & r_\vartheta.
\end{aligned}$$

Step 2. Claim: S is continuous.

Let's consider (y_n) as a sequence converging to y in Π_ϑ. Then,

$$\begin{aligned}
&|(Sy_n)(s)-(Sy)(s)|\\
\leq\ & \frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y_n(M_\vartheta)\\
&+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y_n(\theta),y_n(\theta)\Big)-\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y_n(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y_n(\theta),y_n(\theta)\Big)\\
&-\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
\leq\ & \frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\theta^{-\sigma}\Big(\eta|-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})\\
&\times I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(M_\vartheta)-y(M_\vartheta))+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(\theta)-y(\theta))|+\gamma|(y_n(\theta)-y(\theta))|\Big)\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{\theta}(s-\theta)^{\omega_\vartheta-1}\theta^{-\sigma}\Big(\eta|-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(M_\vartheta)-y(M_\vartheta))\\
&+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(\theta)-y(\theta))|+\gamma|(y_n(\theta)-y(\theta))|\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \tfrac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}\Big(\eta|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(M_\vartheta)-y(M_\vartheta))+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n(\theta)-y(\theta))|\\
&\quad+\gamma|(y_n(\theta)-y(\theta))|\Big)\\
&\le \tfrac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(2\eta\|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(y_n-y)\|_{\Pi_\vartheta}+\gamma\|y_n-y\|_{\Pi_\vartheta})\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta\\
&\le \tfrac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}(M_\vartheta^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|y_n-y\|_{\Pi_\vartheta}.
\end{aligned}
$$

By (3.4), the coefficient is finite, and $\|y_n-y\|_{\Pi_\vartheta}\to 0$ implies $\|(Sy_n)-(Sy)\|_{\Pi_\vartheta}\to 0$ as $n\to\infty$. Ergo, the operator S, is a continuous on Π_ϑ.

Step 3. Claim: S is relatively compact.

By Step 1, we have $\|S(y)\|_{\Pi_\vartheta}\le r_\vartheta$ for each $y\in B_{r_\vartheta}$, which gives the boundedness of $S(B_{r_\vartheta})$. Now we will show that $S(B_{r_\vartheta})$ is EC. For $s_1,s_2\in\Lambda_\vartheta,\ s_1<s_2$, and $y\in B_{r_\vartheta}$, estimate

$$
\begin{aligned}
&|(Sy)(s_2)-(Sy)(s_1)|\\
=\ &\Big|\frac{-(M_\vartheta-M_{\vartheta-1})^{-1}(s_2-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)\\
&+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)d\theta+\frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s_1-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\\
&\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)d\theta\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s_2}(s_2-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)d\theta\\
&-\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s_1}(s_1-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)d\theta\Big|
\end{aligned}
$$

$$
\begin{aligned}
\le\ &\frac{(M_\vartheta-M_{\vartheta-1})^{-1}}{\Gamma(\omega_\vartheta)}\Big((s_2-M_{\vartheta-1})-(s_1-M_{\vartheta-1})\Big)\\
&\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s_1}\Big((s_2-\theta)^{\omega_\vartheta-1}-(s_1-\theta)^{\omega_\vartheta-1})\Big)\\
&\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{1}{\Gamma(\omega_\vartheta)}\int_{s_1}^{s_2}(s_2-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(s,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
\le\ &\frac{(M_\vartheta-M_{\vartheta-1})^{-1}}{\Gamma(\omega_\vartheta)}\Big((s_2-M_{\vartheta-1})-(s_1-M_{\vartheta-1})\Big)\\
&\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{(s_2-s_1)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s_1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta\\
&+\frac{(s_2-s_1)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\int_{s_1}^{s_2}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta)\Big)\Big|d\theta
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{(M_\vartheta - M_{\vartheta-1})^{-1}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \\
& \int_{M_{\vartheta-1}}^{M_\vartheta} (M_\vartheta - \theta)^{\omega_\vartheta - 1}\Big|\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta), y(\theta)\Big)\Big| d\theta \\
& + \frac{(s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s_2} \Big|\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta), y(\theta)\Big)\Big| d\theta \\
\leq\ & \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 2}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \\
& \int_{M_{\vartheta-1}}^{M_\vartheta} \Big|\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta), y(\theta)\Big) - \Psi_1(\theta, 0, 0)\Big| d\theta \\
& + \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 2}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \int_{M_{\vartheta-1}}^{M_\vartheta} |\Psi_1(\theta, 0, 0)| d\theta \\
& + \frac{(s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s_2} \Big|\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) \\
& + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta), y(\theta)\Big) - \Psi_1(\theta, 0, 0)\Big| d\theta + \frac{(s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s_2} |\Psi_1(\theta, 0, 0)| d\theta \\
\leq\ & \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 2}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \\
& \int_{M_{\vartheta-1}}^{M_\vartheta} \theta^{-\sigma}\Big(\eta| - (M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta)| + \gamma|y(\theta)|)\Big) d\theta \\
& + \frac{\Psi^\star (M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \\
& + \frac{(s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s_2} \theta^{-\sigma}\Big(\eta| - (M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) \\
& + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta)| + \gamma|y(\theta)|)\Big) d\theta + \frac{\Psi^\star (s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)}(s_2 - M_{\vartheta-1}) \\
\leq\ & \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 2}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \int_{M_{\vartheta-1}}^{M_\vartheta} \theta^{-\sigma}\Big(\eta| I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) \\
& + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta)| + \gamma|y(\theta)|)\Big) d\theta + \frac{\Psi^\star (M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big) \\
& + \frac{(s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s_2} \theta^{-\sigma}\Big(\eta| I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(M_\vartheta) + I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y(\theta)| + \gamma|y(\theta)|)\Big) d\theta \\
& + \frac{\Psi^\star (s_2 - s_1)^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)}(s_2 - M_{\vartheta-1}) \\
\leq\ & \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 2}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big)(2\eta\|I^{\omega_\vartheta}_{M^+_{\vartheta-1}} y\|_{\Pi_\vartheta} + \gamma\|y\|_{\Pi_\vartheta}) \int_{M_{\vartheta-1}}^{M_\vartheta} \theta^{-\sigma} d\theta \\
& + \frac{\Psi^\star (M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 1}}{\Gamma(\omega_\vartheta)}\Big((s_2 - M_{\vartheta-1}) - (s_1 - M_{\vartheta-1})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(s_2-s_1)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(2\eta\|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y\|_{\Pi_\vartheta}+\gamma\|y\|_{\Pi_\vartheta})\int_{M_{\vartheta-1}}^{s_2}\theta^{-\sigma}d\theta+\frac{\Psi^\star(s_2-s_1)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(s_2-M_{\vartheta-1})\\
\leq\ &\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-2}(M_\vartheta^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}\Big((s_2-M_{\vartheta-1})-(s_1-M_{\vartheta-1})\Big)(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|y\|_{\Pi_\vartheta}\\
&+\frac{\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\Big((s_2-M_{\vartheta-1})-(s_1-M_{\vartheta-1})\Big)\\
&+\frac{(s_2-s_1)^{\omega_\vartheta-1}(s_2^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|y\|_{\Pi_\vartheta}\\
&+\frac{\Psi^\star(s_2-s_1)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(s_2-M_{\vartheta-1})\\
\leq\ &\Big[\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-2}(M_\vartheta^{1-\sigma}-M_{\vartheta-1}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|y\|_{\Pi_\vartheta}+\frac{\Psi^\star(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\Big]\\
&\Big((s_2-M_{\vartheta-1})-(s_1-M_{\vartheta-1})\Big)\\
&+\Big[\frac{s_2^{1-\sigma}-M_{\vartheta-1}^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|y\|_{\Pi_\vartheta}+\frac{\Psi^\star(s_2-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\Big](s_2-s_1)^{\omega_\vartheta-1}.
\end{aligned}
$$

Hence, $\|(Sy)(s_2)-(Sy)(s_1)\|_{\Pi_\vartheta}\to 0$ as $|s_2-s_1|\to 0$. It implies that $S(B_{r_\vartheta})$ is EC.

Step 4. Claim: S is K-set contractions.

For $U\in B_{r_\vartheta}$, $s\in\Lambda_\vartheta$, we get

$$
\begin{aligned}
\Phi(S(U)(s)) &= \Phi((Sy)(s),y\in U)\\
&\leq \Big\{\tfrac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\\
&\quad\times\textstyle\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Phi\Psi_1(\theta,-(M_\vartheta-M_{\vartheta-1})^{1-\omega_\vartheta}(\theta-M_{\vartheta-1})^{\omega_\vartheta-1}I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta))\\
&\quad+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta))d\theta\\
&\quad+\tfrac{1}{\Gamma(\omega_\vartheta)}\textstyle\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Phi\Psi_1(\theta,-(M_\vartheta-M_{\vartheta-1})^{1-\omega_\vartheta}(\theta-M_{\vartheta-1})^{\omega_\vartheta-1}I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(M_\vartheta))\\
&\quad+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}y(\theta),y(\theta))d\theta,\xi\in U\Big\}.
\end{aligned}
$$

For each $s\in\Lambda_\vartheta$, Remark 3.1 implies that

$$
\begin{aligned}
\Phi(S(U)(s)) &\leq \Big\{\tfrac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\\
&\quad\textstyle\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big[2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\widehat{\Phi}(U)\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta+\gamma\widehat{\Phi}(U)\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta\Big]\\
&\quad+\tfrac{1}{\Gamma(\omega_\vartheta)}\textstyle\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Big[2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\widehat{\Phi}(U)\int_{M_{\vartheta-1}}^{s}\theta^{-\sigma}d\theta\\
&\quad+\gamma\widehat{\Phi}(U)\textstyle\int_{M_{\vartheta-1}}^{s}\theta^{-\sigma}d\theta\Big],\xi\in U\Big\}.\\
&\leq \Big\{\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-2}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\textstyle\int_{M_{\vartheta-1}}^{M_\vartheta}\Big[2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\widehat{\Phi}(U)\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta\\
&\quad+\gamma\widehat{\Phi}(U)\textstyle\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta\Big]\\
&\quad+\tfrac{(s-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\textstyle\int_{M_{\vartheta-1}}^{s}\Big[2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\widehat{\Phi}(U)\int_{M_{\vartheta-1}}^{s}\theta^{-\sigma}d\theta\\
&\quad+\gamma\widehat{\Phi}(U)\textstyle\int_{M_{\vartheta-1}}^{s}\theta^{-\sigma}d\theta\Big],\xi\in U\Big\}.\\
&\leq \tfrac{({M_\vartheta}^{1-\sigma}-{M_{\vartheta-1}}^{1-\sigma})(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-2}(s-M_{\vartheta-1})}{(1-\sigma)\Gamma(\omega_\vartheta)}\Big(2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma\Big)\widehat{\Phi}(U)\\
&\quad+\tfrac{(s^{1-\sigma}-{M_{\vartheta-1}}^{1-\sigma})(s-M_{\vartheta-1})^{\omega_\vartheta-1}}{(1-\sigma)\Gamma(\omega_\vartheta)}\Big(2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma\Big)\widehat{\Phi}(U)\\
&\leq \tfrac{2({M_\vartheta}^{1-\sigma}-{M_{\vartheta-1}}^{1-\sigma})(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{(1-\sigma)\Gamma(\omega_\vartheta)}\Big(2\eta\tfrac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma\Big)\widehat{\Phi}(U).
\end{aligned}
$$

Thus,

$$\widehat{\Phi}(SU) \le \frac{2({M_\vartheta}^{1-\sigma} - {M_{\vartheta-1}}^{1-\sigma})(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta - 1}}{(1-\sigma)\Gamma(\omega_\vartheta)}\Big(2\eta \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta + 1)} + \gamma\Big)\widehat{\Phi}(U).$$

Consequently, by (3.4), S is a K-set contraction.
Thus, by Theorem 2.5, the IBVPCFDCO possesses a solution $\widetilde{\xi_\vartheta}$ in B_{r_ϑ}. Since $B_{r_\vartheta} \subset \Pi_\vartheta$, the claim of Theorem 3.1 is verified.

We impose the following assumption:

(H2) Let $\Psi_1 \in C(\Lambda \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$; there exists a number $\sigma \in (0, 1)$ such that $s^\sigma \Psi_1 \in C(\Lambda \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist a constants $\eta,\ \gamma > 0$, such that
$s^\sigma|\Psi_1(s, y_1, z_1) - \Psi_1(s, y_2, z_2)| \le \eta|y_1 - y_2| + \gamma|z_1 - z_2|$, for any $y_1,\ y_2,\ z_1,\ z_2 \in \mathbb{R}$ and $s \in \Lambda$.

Theorem 3.2. *Let (H1) and (H2) and inequality (3.4) be satisfied. Then, the problem (1.1) possesses at least one solution in $C(\Lambda, \mathbb{R})$.*

Proof. According to Theorem 3.1, the IBVPCFDCO possesses at least one solution $\widetilde{\xi_\vartheta} \in \Pi_\vartheta$.
We define the function

$$\xi_\vartheta = \begin{cases} 0, & s \in [0, M_{\vartheta-1}], \\ \widetilde{\xi}_\vartheta, & s \in \Lambda_\vartheta, \end{cases}$$

Thus, the function $\xi_\vartheta \in C([0, M_\vartheta], \mathbb{R})$ solves the integral equation (3.2) for $s \in \Lambda_\vartheta$ with $\xi_\vartheta(0) = 0,\ \xi_\vartheta(M_\vartheta) = \widetilde{\xi}_\vartheta(M_\vartheta) = 0$.

Then, the function

$$\xi(s) = \begin{cases} \xi_1(s), & s \in \Lambda_1, \\ \xi_2(s) = \begin{cases} 0, & s \in \Lambda_1, \\ \widetilde{\xi}_2, & s \in \Lambda_2, \end{cases} \\ . \\ . \\ . \\ . \\ \xi_n(s) = \begin{cases} 0, & s \in [0, M_{\vartheta-1}], \\ \widetilde{\xi}_\vartheta, & s \in \Lambda_\vartheta \end{cases} \end{cases} \tag{3.6}$$

is a solution of (1.1) in $C(\Lambda, \mathbb{R})$.

4. Ulam Hyers stability

Theorem 4.1. *Let the conditions (H1), (H2) and inequality (3.4) be satisfied. Then,* (1.1) *is UHS.*

Proof. Let $\epsilon > 0$ an arbitrary number and the function $z(s)$ from $z \in C(\Lambda_\vartheta, \mathbb{R})$ satisfy inequality (2.3), where

$$z_\vartheta(s) = \begin{cases} 0, & s \in [0, M_{\vartheta-1}], \\ z(s), & s \in \Lambda_\vartheta. \end{cases}$$

We have

$$^{c}D^{\omega(s)}_{M_{\vartheta-1}{}^{+}} z_\vartheta(s) = \int_{M_{\vartheta-1}}^{s} \frac{(s-\theta)^{1-\omega_\vartheta}}{\Gamma(2-\omega_\vartheta)} z^{(2)}(\theta) d\theta.$$

We obtain

$$\begin{aligned}
\Big| z_\vartheta(s) \quad &+ \quad \frac{(M_\vartheta - M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)} \\
&\int_{M_{\vartheta-1}}^{M_\vartheta} (M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} z_\vartheta(M_\vartheta) + I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} z_\vartheta(\theta), z_\vartheta(\theta)\Big) d\theta \\
&- \frac{1}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s} (s-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} z_\vartheta(M_\vartheta) \\
&+ I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} z_\vartheta(\theta), z_\vartheta(\theta)\Big) d\theta \Big| \\
\leq \quad & \epsilon \int_{M_{\vartheta-1}}^{s} \frac{(s-\theta)^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)} d\theta \\
\leq \quad & \epsilon \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}.
\end{aligned}$$

According to Theorem 3.2, (1.1) has a solution $\xi \in C(\Lambda, \mathbb{R})$ defined by $\xi(s) = \xi_\vartheta(s)$ for $s \in \Lambda_\vartheta$, $\vartheta = 1, 2, ..., n$, where

$$\xi_\vartheta = \begin{cases} 0, & s \in [0, M_{\vartheta-1}], \\ \widetilde{\xi}_\vartheta, & s \in \Lambda_\vartheta, \end{cases} \tag{4.1}$$

and $\widetilde{\xi}_\vartheta \in \Pi_\vartheta$ is a solution of IBVPCFDCO. According to Lemma 3.1, the integral equation

$$\widetilde{\xi}_\vartheta(s) = -\frac{(M_\vartheta - M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}.$$

$$\int_{M_{\vartheta-1}}^{M_\vartheta} (M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(M_\vartheta) + \ I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(\theta), \widetilde{\xi}_\vartheta(\theta)\Big) d\theta$$

$$+ \ \frac{1}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s} (s-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(M_\vartheta) + I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(\theta), \widetilde{\xi}_\vartheta(\theta)\Big) d\theta, \tag{4.2}$$

holds.

Let $s \in \Lambda_\vartheta$, $\vartheta = 1, 2, ..., n$. Then, by Eqs (4.1) and (4.2), we get

$$\begin{aligned}
& |z(s) - \xi(s)| = |z(s) - \xi_\vartheta(s)| = |z_\vartheta(s) - \widetilde{\xi}_\vartheta(s)| \\
= \quad & \Big| z_\vartheta(s) + \frac{(M_\vartheta - M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{M_\vartheta} (M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) \\
& \times I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(M_\vartheta) + I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(\theta), \widetilde{\xi}_\vartheta(\theta)\Big) d\theta \\
& - \frac{1}{\Gamma(\omega_\vartheta)} \int_{M_{\vartheta-1}}^{s} (s-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta, -(M_\vartheta - M_{\vartheta-1})^{-1}(\theta - M_{\vartheta-1}) I^{\omega_\vartheta}_{M^{+}_{\vartheta-1}} \widetilde{\xi}_\vartheta(M_\vartheta)
\end{aligned}$$

$$
\begin{aligned}
& +I^{\omega_\vartheta}_{M^+_{\vartheta-1}}\widetilde{\xi}_\vartheta(\theta),\widetilde{\xi}_\vartheta(\theta)\Big)d\theta\Big| \\
\leq\ & \Big|z_\vartheta(\theta)+\frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1}) \\
& \times I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(\theta),z_\vartheta(\theta)\Big)d\theta \\
& -\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(M_\vartheta) \\
& +I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(\theta),z_\vartheta(\theta)\Big)d\theta\Big| \\
& +\frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(M_\vartheta) \\
& +I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(\theta),z_\vartheta(\theta)\Big)d\theta-\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}\widetilde{\xi}_\vartheta(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}\widetilde{\xi}_\vartheta(\theta),\widetilde{\xi}_\vartheta(\theta)\Big)\Big|d\theta \\
& +\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\Big|\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}z_\vartheta(\theta),z_\vartheta(\theta)\Big)d\theta \\
& -\Psi_1\Big(\theta,-(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})I^{\omega_\vartheta}_{M^+_{\vartheta-1}}\widetilde{\xi}_\vartheta(M_\vartheta)+I^{\omega_\vartheta}_{M^+_{\vartheta-1}}\widetilde{\xi}_\vartheta(\theta),\widetilde{\xi}_\vartheta(\theta)\Big)\Big|d\theta \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\frac{(M_\vartheta-M_{\vartheta-1})^{-1}(s-M_{\vartheta-1})}{\Gamma(\omega_\vartheta)} \\
& \int_{M_{\vartheta-1}}^{M_\vartheta}(M_\vartheta-\theta)^{\omega_\vartheta-1}\theta^{-\sigma}\Big(\eta\Big|(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(M_\vartheta)-\widetilde{\xi}_\vartheta(M_\vartheta))) \\
& +(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta)))\Big|+\gamma\Big|(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta))\Big|\Big)d\theta \\
& +\frac{1}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{s}(s-\theta)^{\omega_\vartheta-1}\theta^{-\sigma}\Big(\eta\Big|(M_\vartheta-M_{\vartheta-1})^{-1}(\theta-M_{\vartheta-1})(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(M_\vartheta)-\widetilde{\xi}_\vartheta(M_\vartheta))) \\
& +(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta)))\Big|+\gamma\Big|(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta))\Big|\Big)d\theta \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}\Big(\eta\Big|(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(M_\vartheta)-\widetilde{\xi}_\vartheta(M_\vartheta))) \\
& +(I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta)))\Big|+\gamma\Big|(z_\vartheta(\theta)-\widetilde{\xi}_\vartheta(\theta))\Big|\Big)d\theta \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}}{\Gamma(\omega_\vartheta)}(2\eta\|I^{\omega_\vartheta}_{M^+_{\vartheta-1}}(z_\vartheta-\widetilde{\xi}_\vartheta)\|_{\Pi_\vartheta}+\gamma\|z_\vartheta-\widetilde{\xi}_\vartheta\|_{\Pi_\vartheta})\int_{M_{\vartheta-1}}^{M_\vartheta}\theta^{-\sigma}d\theta \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}({M_\vartheta}^{1-\sigma}-{M_{\vartheta-1}}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\|z_\vartheta-\widetilde{\xi}_\vartheta\|_{\Pi_\vartheta} \\
& +\gamma\|z_\vartheta-\widetilde{\xi}_\vartheta\|_{\Pi_\vartheta}) \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\frac{2(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta-1}({M_\vartheta}^{1-\sigma}-{M_{\vartheta-1}}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\gamma)\|z_\vartheta-\widetilde{\xi}_\vartheta\|_{\Pi_\vartheta} \\
\leq\ & \epsilon\frac{(M_\vartheta-M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}+\mu\|z-\xi\|,
\end{aligned}
$$

where

$$\mu = \max_{\vartheta=1,2,\ldots,n} \frac{2(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta-1}({M_\vartheta}^{1-\sigma} - {M_{\vartheta-1}}^{1-\sigma})}{(1-\sigma)\Gamma(\omega_\vartheta)}(2\eta\frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)} + \gamma).$$

Then,

$$\|z-\xi\|(1-\mu) \le \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}\epsilon.$$

We obtain, for each $s \in \Lambda_\vartheta$,

$$|z(s)-\xi(s)| \le \|z-\xi\| \le \frac{(M_\vartheta - M_{\vartheta-1})^{\omega_\vartheta}}{(1-\mu)\Gamma(\omega_\vartheta+1)}\epsilon := c_{\Psi_1}\epsilon.$$

Therefore, (1.1) is UHS.

5. Example

Consider the following fractional problem,

$$\begin{cases} {}^cD_{0^+}^{\omega(s)}\xi(s) = \frac{s^{-\frac{1}{3}}e^{-s}}{(e^{e^{\frac{s^2}{1+s}}}+4e^{2s}+1)(1+|x(s)|+|{}^cD_{0^+}^{\omega(s)}\xi(s)|)}, & s \in \Lambda := [0,2], \\ \xi(0) = 0, \ \ \xi(2) = 0. \end{cases} \tag{5.1}$$

Let

$$\Psi_1(s,y,z) = \frac{s^{-\frac{1}{3}}e^{-s}}{(e^{e^{\frac{s^2}{1+s}}} + 4e^{2s} + 1)(1+y+z)}, \ (s,y,z) \in [0,2] \times [0,+\infty) \times [0,+\infty).$$

$$\omega(s) = \begin{cases} \frac{3}{2}, & s \in \Lambda_1 := [0,1], \\ \frac{9}{5}, & s \in \Lambda_2 :=]1,2]. \end{cases} \tag{5.2}$$

Then, we have

$$\begin{aligned} s^{\frac{1}{3}}|\Psi_1(s,y_1,z_1) - \Psi_1(s,y_2,z_2)| &= \left|\frac{e^{-s}}{(e^{e^{\frac{s^2}{1+s}}} + 4e^{2s} + 1)}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\ &\le \frac{e^{-s}(|y_1-y_2| + |z_1-z_2|)}{(e^{e^{\frac{s^2}{1+s}}} + 4e^{2s} + 1)(1+y_1+z_1)(1+y_2+z_2)} \\ &\le \frac{e^{-s}}{(e^{e^{\frac{s^2}{1+s}}} + 4e^{2s} + 1)}(|y_1-y_2| + |z_1-z_2|) \\ &\le \frac{1}{(e+5)}|y_1-y_2| + \frac{1}{(e+5)}|z_1-z_2|. \end{aligned}$$

Thus, assumption (H2) is satisfied with $\sigma = \frac{1}{3}$ and $\eta = \gamma = \frac{1}{e+5}$.
According to IBVPCFDCO, by (5.2) we consider two auxiliary implicit BVPs of CFDCO

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}}\xi(s) = \frac{s^{-\frac{1}{3}}e^{-s}}{(e^{\frac{s^2}{1+s}}+4e^{2s}+1)(1+|\xi(s)|+|{}^cD^{\frac{3}{2}}\xi(s)|)}, \quad s \in \Lambda_1, \\ \\ \xi(0) = 0, \ \ \xi(1) = 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{9}{5}}\xi(s) = \frac{s^{-\frac{1}{3}}e^{-s}}{(e^{\frac{s^2}{1+s}}+4e^{2s}+1)(1+|\xi(s)|+|{}^cD^{\frac{9}{5}}\xi(s)|)}, \quad s \in \Lambda_2, \\ \\ \xi(1) = 0, \ \ \xi(2) = 0. \end{cases} \tag{5.4}$$

We shall check that assumption (3.4) is fulfilled for $\vartheta = 1$. Indeed,

$$\frac{2({M_1}^{1-\sigma} - {M_0}^{1-\sigma})(M_1 - M_0)^{\omega_1-1}}{(1-\sigma)\Gamma(\omega_1)}\Big(\frac{2\eta(M_1 - M_0)^{\omega_1}}{\Gamma(\omega_1 + 1)} + \gamma\Big) = \frac{1}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})}\Big(\frac{2}{\Gamma(\frac{5}{2})} + 1\Big) \simeq 0.3664 < 1.$$

Hence, assumption (3.4) is satisfied. By Theorem 3.1, the problem (5.3) has a solution $\widetilde{\xi_1} \in \Pi_1$.
We prove that the condition (3.4) is fulfilled for $\vartheta = 2$. Indeed,

$$\frac{2({M_2}^{1-\sigma} - {M_1}^{1-\sigma})(M_2 - M_1)^{\omega_2-1}}{(1-\sigma)\Gamma(\omega_2)}\Big(\frac{2\eta(M_2 - M_1)^{\omega_2}}{\Gamma(\omega_2 + 1)} + \gamma\Big) = \frac{2^{\frac{2}{3}} - 1}{\frac{2}{3}\Gamma(\frac{9}{5})}\frac{1}{e+5}\Big(\frac{2}{\Gamma(\frac{14}{5})} + 1\Big) \simeq 0.2682 < 1.$$

Thus, the condition (3.4) is satisfied.

According to Theorem 3.1, the implicit BVPs of CFDCO (5.4) possess a solution $\widetilde{\xi_2} \in \Pi_2$.
Then, by Theorem 3.2, (5.1) has a solution

$$\xi(s) = \begin{cases} \widetilde{\xi_1}(s), \quad s \in \Lambda_1, \\ \\ \xi_2(s), \quad s \in \Lambda_2, \\ \end{cases}$$

where

$$\xi_2(s) = \begin{cases} 0, \quad s \in \Lambda_1, \\ \\ \widetilde{\xi_2}(s), \quad s \in \Lambda_2. \\ \end{cases}$$

According to Theorem 4.1, (5.1) is UHS.

In Figures 1–3, we presented the plot of the function $\Psi_1(s, y, z)$ in the two intervals $[0, 1]$ and $[1, 2]$.

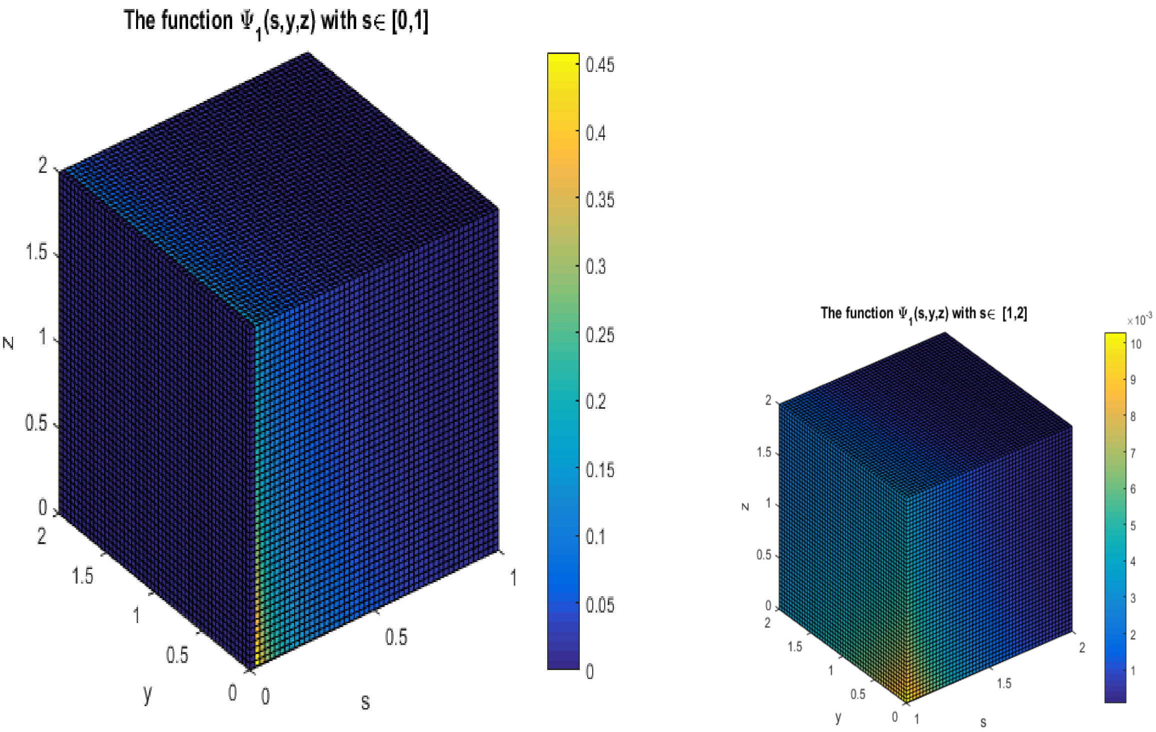

Figure 1. A plot 3D of $\Psi_1(s, y, z)$ with $s \in [0, 1]$ and $s \in [1, 2]$.

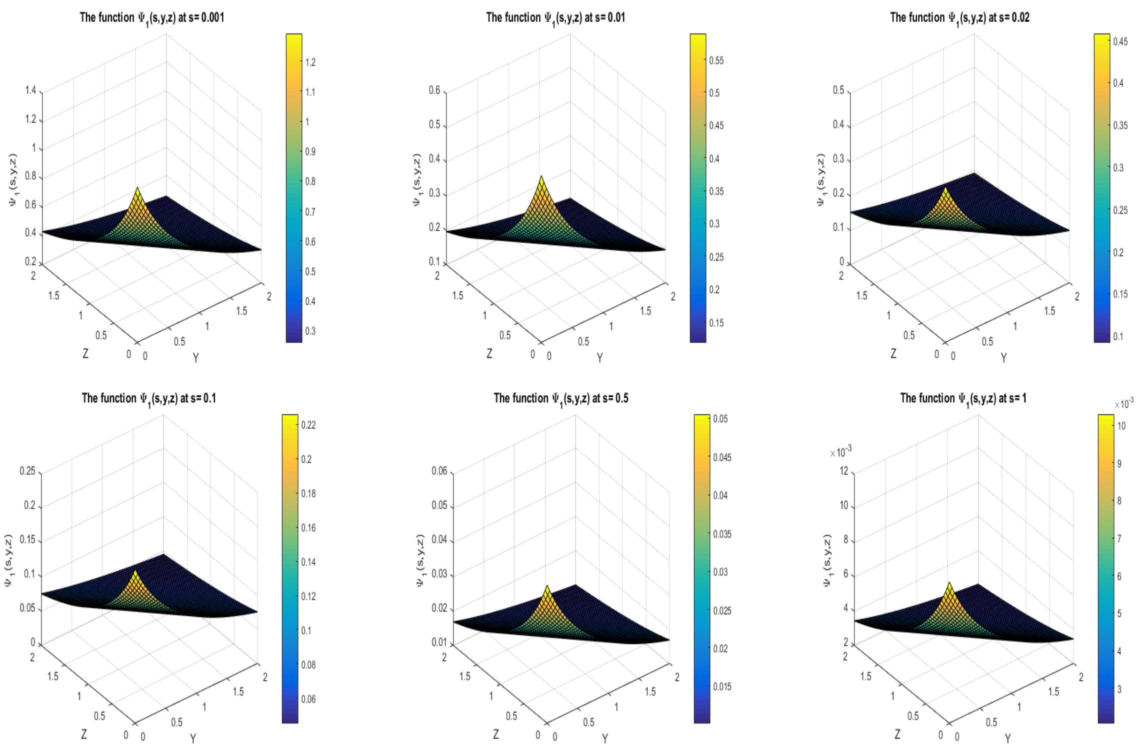

Figure 2. A plot of $\Psi_1(s, y, z)$ for different $s \in [0, 1]$.

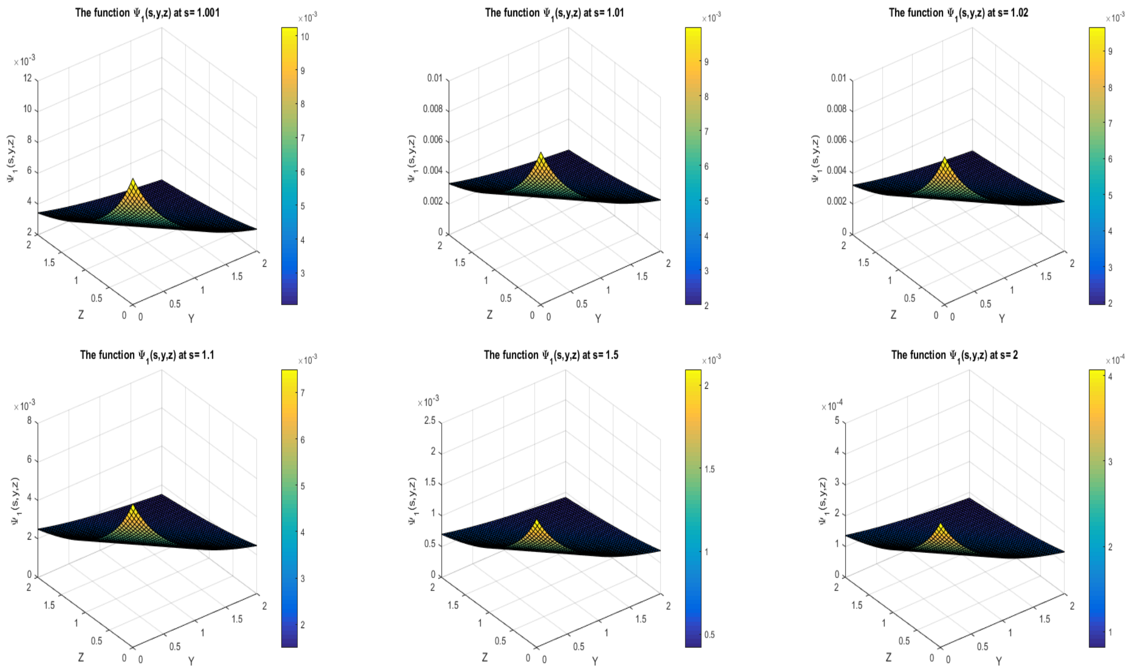

Figure 3. A plot of $\Psi_1(s, y, z)$ for different $s \in [1, 2]$.

The figures visualize the function $\Psi_1(s, y, z)$ for a Caputo fractional differential equation of variable order over $s \in [0, 2]$. Figure 1 is a 3D plot showing Ψ_1's behavior across $s \in [0, 1]$ ($\omega(s) = \frac{3}{2}$) and $s \in (1, 2]$ ($\omega(s) = \frac{9}{5}$), highlighting its decay with s, y, and z. Figure 2 plots Ψ_1 for fixed $s \in [0, 1]$, showing larger values near $s = 0$ that decrease with y and z. Figure 3 focuses on $s \in (1, 2]$, displaying smaller values due to further decay and the higher fractional order. These figures support the existence and stability results by illustrating Ψ_1's smooth, bounded behavior.

6. Conclusions

In this paper, we investigated a class of BVPs involving (VO-FDE), a model framework that allows the fractional order to vary over the domain. By employing Darbo's fixed point theorem in conjunction with the Kuratowski measure of noncompactness, we derived new existence results under appropriate conditions. A piecewise constant approximation of the variable order allowed us to reduce the problem to a series of manageable subproblems with constant fractional orders. This approach effectively addressed the analytical challenges associated with variable-order operators.

In addition to proving existence results, we established the Ulam-Hyers stability of the solutions, ensuring robustness with respect to perturbations—an essential property for practical applications. Theoretical findings were further validated through a numerical example, demonstrating both the applicability and accuracy of the proposed method.

Compared to existing literature, this work significantly advances the theoretical understanding of VO-FDEs by integrating noncompactness measures and fixed-point theory in a novel framework. While the current analysis focuses on Caputo derivatives, future work may explore broader classes of variable-order operators, nonlinear perturbations, and systems defined on multidimensional domains.

Furthermore, the development of more efficient numerical schemes tailored for variable-order systems remains an important avenue for future research.

Author contributions

Mohammed Said Souid: Writing-original draft, project administration, validation; Souhila Sabit: Software, data curation, conceptualization; Zoubida Bouazza: Resources, investigation, writing-review and editing; Kanokwan Sitthithakerngkiet: Funding acquisition, visualization, methodology. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. F. G. Aguilar, Analytical and numerical solutions of nonlinear alcoholism model via variable-order fractional differential equations, *Physica A*, **494** (2018), 52–57. https://doi.org/10.1016/j.physa.2017.12.007
2. M. M. Alsuyuti, E. H. Doha, B. I. Bayoumi, S. S. E. Eldien, Robust spectral treatment for time-fractional delay partial differential equations, *Comput. Appl. Math.*, **42**, (2023), 159. https://doi.org/10.1007/s40314-023-02287-w
3. J. Banaṡ, K. Goebel, *Measures of noncompactness in Banach spaces*, New York: Marcel Dekker, 1980.
4. J. Banaṡ, L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math. (Prace Mat.)*, **41** (2001), 13–23.
5. H. Benali, M. S. Souid, H. Günerhan, U. F. Gamiz, Estimates related to Caputo derivatives using generalized modified convex functions, *AIMS Math.*, **9** (2024), 28813–28827. https://doi.org/10.3934/math.20241398
6. A. Benkerrouche, S. Etemad, M. S. Souid, S. Rezapour, H. Ahmad, T. Botmart, Fractional variable order differential equations with impulses: A study on the stability and existence properties, *AIMS Math.*, **8** (2023), 775–791. https://doi.org/10.3934/math.2023038

7. A. Benkerrouche, M. S. Souid, A. Karapinar, A. Hakem, On the boundary value problems of Hadamard fractional differential equations of variable order, *Math. Method. Appl. Sci.*, 2022, 1–17. https://doi.org/10.1002/mma.8306
8. A. Benkerrouche, M. S. Souid, G. Stamov, I. Stamova, Multiterm impulsive Caputo-Hadamard type differential equations of fractional variable order, *Axioms*, **11** (2022), 1–14. https://doi.org/10.3390/axioms11110634
9. Z. Bouazza, M. S. Souid, H. Günerhan, A. Shokri, Problem involving the Riemann-Liouville derivative and implicit variable-order nonlinear fractional differential equations, *Complexity*, **2024** (2024), 1–14. https://doi.org/10.1155/2024/5595720
10. Z. Bouazza, S. Souhila, S. Etemad, M. S. Souid, A. Akgül, S. Rezapour, et al., On the Caputo-Hadamard fractional IVP with variable order using the upper-lower solutions technique, *AIMS Math.*, **8** (2023), 5484–5501. https://doi.org/10.3934/math.2023276
11. S. S. E. Eldien, On solving fractional logistic population models with applications, *Comput. Appl. Math.*, **37** (2018), 6392–6409. https://doi.org/10.1007/s40314-018-0693-4
12. S. S. E. Eldien, E. H. Doha, A. H. Bhrawy, A. A. E. Kalaawy, J. T. Machado, A new operational approach for solving fractional variational problems depending on indefinite integrals, *Commun. Nonlinear Sci.*, **57** (2018), 246–263. https://doi.org/10.1016/j.cnsns.2017.08.026
13. D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear integral equations in abstract spaces*, Dordrecht: Kluwer Academic Publishers, 1996.
14. A. Jiahui, C. Pengyu, Uniqueness of solutions to initial value problem of fractional differential equations of variable-order, *Dynam. Syst. Appl.*, **28** (2019), 607–623.
15. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Amsterdam: Elsevier Science B.V., 2006.
16. T. Liu, H. Zhang, X. Yang, The ADI compact difference scheme for three-dimensional integro-partial differential equation with three weakly singular kernels, *J. Appl. Math. Comput.*, **71** (2025), 3861–3889. https://doi.org/10.1007/s12190-025-02386-3
17. M. Manigandan, K. Manikandan, H. A. Hammad, M. D. L. Sen, Applying fixed point techniques to solve fractional differential inclusions under new, *AIMS Math.*, **9** (2024), 15505–15542. https://doi.org/10.3934/math.2024750
18. A. Morsy, C. Anusha, K. S. Nisar, C. Ravichandran, Results on generalized neutral fractional impulsive dynamic equation over time scales using nonlocal initial condition, *AIMS Math.*, **9** (2024), 8292–8310. https://doi.org/10.3934/math.2024403
19. H. Moussa, M. A. Saker, M. A. Zaky, M. Babatin, S. S. E. Eldien, Mapped Legendre-spectral method for high-dimensional multi-term time-fractional diffusion-wave equation with non-smooth solution, *Comput. Appl. Math.*, **44**, (2025), 167. https://doi.org/10.1007/s40314-025-03123-z
20. A. Refice, M. S. Souid, I. Stamova, On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique, *Mathematics*, **9** (2021), 1134. https://doi.org/10.3390/math9101134
21. A. Salim, S. T. M. Thabet, I. Kedim, M. V. Cortez, On the nonlocal hybrid (k,φ)-Hilfer inverse problem with delay and anticipation, *AIMS Math.*, **9** (2024), 22859–22882. https://doi.org/10.3934/math.20241112

22. S. G. Samko, Fractional integration and differentiation of variable order, *Anal. Math.*, **21** (1995), 213–236. https://doi.org/10.1007/BF01911126
23. S. G. Samko, B. Boss, Integration and differentiation to a variable fractional order, *Integr. Transf. Spec. F.*, **1** (1993), 277–300. https://doi.org/10.1080/10652469308819027
24. H. Sun, W. Chen, H. Wei, Y. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, *Eur. Phys. J.-Spec. Top.*, **193** (2011), 185–192. https://doi.org/10.1140/epjst/e2011-01390-6
25. D. Tavares, R. Almeida, D. F. M. Torres, Caputo derivatives of fractional variable order: Numerical approximations, *Commun. Nonlinear Sci.*, **35** (2016), 69–87. https://doi.org/10.1016/j.cnsns.2015.10.027
26. B. Telli, M. S. Souid, J. Alzabut, H. Khan, Existence and uniqueness theorems for a variable-order fractional differential equation with delay, *Axioms*, **12** (2023), 339. https://doi.org/10.3390/axioms12040339
27. D. Valerio, J. S. Costa, Variable-order fractional derivatives and their numerical approximations, *Signal Process*, **91** (2011), 470–483. https://doi.org/10.1016/j.sigpro.2010.04.006
28. J. V. D. C. Sousa, E. C. D. Oliveira, Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation, *Comput. Appl. Math.*, **37** (2018), 5375–5394. https://doi.org/10.1007/s40314-018-0639-x
29. J. Yang, H. Yao, B. Wu, An efficient numerical method for variable order fractional functional differential equation, *Appl. Math. Lett.*, **76** (2018), 221–226. https://doi.org/10.1016/j.aml.2017.08.020
30. X. Yang, Z. Zhang, Analysis of a new NFV scheme preserving DMP for two-dimensional sub-diffusion equation on distorted meshes, *J. Sci. Comput.*, **99** (2024), 80. https://doi.org/10.1007/s10915-024-02511-7
31. M. A. Zaky, A. A. kenany, S. E. Alhazmi, S. S. E. Eldien, Numerical treatment of tempered space-fractional Zeldovich-Frank-Kamenetskii equation, *Rom. Rep. Phys.*, 2025.
32. S. Zhang, L. Hu, Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis, *Mathematics*, **7** (2019), 1–23. https://doi.org/10.3390/math7030286
33. S. Zhang, L. Hu, The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order, *AIMS Math.*, **5** (2020), 2923–2943. https://doi.org/10.3934/math.2020189
34. S. Zhang, L. Hu, The existeness and uniqueness result of solutions to initial value problems of nonlinear di usion equations involving with the conformable variable, *Azerbaijan J. Math.*, **9** (2019), 22–45.