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Existence and regularity of periodic solutions for a class of neutral evolution equation with delay

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Abstract: The purpose of this paper is to investigate the existence and C^1-regularity of ω-periodic mild solutions for a class of neutral evolution equation with two-constant delays in Banach space X

$$\frac{d}{dt}(z(t) - cz(t-\delta)) + A(z(t) - cz(t-\delta)) = f(t,\ z(t),\ z(t-\tau)), \quad t \in \mathbb{R},$$

where $|\ c\ |< 1$, the constants $\tau,\ \delta > 0$ are defined as time lags, $A : \mathcal{D}(A) \subset X \to X$ is a sectorial operator and has compact resolvent, that is, $-A$ generates exponentially stable, compact analytic operator semigroup $T(t)(t \geqslant 0)$, and $f : \mathbb{R} \times X \times X \to X$ is nonlinear mapping which is ω-periodic in t. By using the theory of analytic operator semigroups, fixed point theorems, and the fractional powers of the sectorial operator, we establish the existence and C^1-regularity results of ω-periodic mild solutions for the equation for the first time when f satisfies the appropriate growth conditions. In the end, we present an example to demonstrate the applications of our main results.

Keywords: neutral evolution equation with delay; periodic mild solution; analytic operator semigroup; existence and C^1-regularity; fixed point theorem
Mathematics Subject Classification: 34K30, 35K55, 35K90, 47D03

1. Introduction

Neutral delayed evolution equations have an extensive background in mathematical physics and can simulate a number of problems that arise in engineering, such as population systems, transmission lines, immune responses, and other fields, see [1–3] and relevant references. In some special models, periodic problems for this kinds of equations are of great significance. Thus, in this paper, we discuss the existence and C^1-regularity of ω-periodic mild solutions to the neutral delayed evolution equation in Banach space X

$$\frac{d}{dt}(z(t) - cz(t-\delta)) + A(z(t) - cz(t-\delta)) = f(t,\ z(t),\ z(t-\tau)), \quad t \in \mathbb{R}. \tag{1.1}$$

The equation can be regarded as the more general abstract form of the resistance-coupled transmission lines model [4, 5], and this study appropriately fills in the blanks in the theory of periodic solutions for functional differential equations, and provides important theoretical basis for designing low-loss and high-stability transmission lines. Thus, the research on it has theoretical significance and practical value.

Most of the work in the past has focused on the periodic problem of evolution equations without delay, see [6–8] and the references therein. Li [6] established the upper and lower solution theorem for the first time for the abstract evolution equation

$$z'(t) + Az(t) = f(t,\ z(t)).$$

In [9–11], by using a monotone iterative method, the authors investigated the periodic problem of some nonlinear parabolic equations without time lag, and the existence and uniqueness results of periodic solutions were obtained. The periodic problems of evolution equations with delay have also been studied by many scholars, see [12–15] and the references therein. The work by Li [16] also was concerned with the evolution equation with multiple delays in Hilbert space H

$$z'(t) + Az(t) = F(t,\ z(t),\ z(t-\tau_1), \ldots,\ z(t-\tau_n)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A : \mathcal{D}(A) \subset H \to H$ is a positive definite self-adjoint operator, $F : \mathbb{R} \times H^{n+1} \to H$ is a continuous mapping that is ω periodic in t, and $\tau_1, \tau_2, \ldots, \tau_n > 0$. Under the hypotheses that F satisfies suitable inequalities, the existence, regularity, and asymptotic stability results of periodic mild solutions to (1.2) was obtained through analytic semigroups, integral inequalities with delays, and the fixed point method. In [17–19], researchers considered the boundedness and attractivity of periodic problems for evolution equations with delay.

In the last few decades, the existence problem of solutions to neutral delayed evolution equations has been given much attention by numerous scholars (see [20–22]) as they are more valuable to study than evolution equations without neutral item. Chang [23] studied the equation

$$\begin{cases} \frac{d}{dt}(x(t) + G(t,\ x_t)) + Ax(t) = F(t,\ x_t),\ t \in [0, T], \\ x_0 = \varphi \in C_g, \end{cases} \tag{1.3}$$

where $-A$ generates compact analytic semigroups, $G, F : [0, T] \times C_g \to X$ is continuous, $x_t \in C_g$ satisfying $x_t(s) = x(t+s)(s \in (-\infty, 0])$, and C_g is the phase space of functions mapping $(-\infty, 0]$ into X defined by axiomatic conditions in [24]. By the Sadovskii fixed point theorem, the existence results of mild solutions to IVP (1.3) was obtained. In [25–28], some scholars further discussed regularity and stability analysis of solutions for the above problem. Specially, the existence of periodic solution to neutral evolution equations with delay has become a crucial topic of investigation, see [29–31]. Hernandez and Henriquez [32] further discussed (1.3). When F satisfies the Lipschitz condition, the existence result of periodic mild solution to (1.3) was obtained. Next, Ezzinbi et al. and Kyelem et al. [33, 34] researched the equation in the fading memory space

$$\frac{d}{dt}(u(t) - \mathcal{D}_0(u_t)) + A(u(t) - \mathcal{D}_0(u_t)) = F(t,\ u_t),\ t \in \mathbb{R}, \tag{1.4}$$

where $-A$ generates the analytic semigroup $\{T(t)\}_{t\geqslant 0}$ in Banach space X, $\mathcal{D}_0 : C_g \to X$ is a bounded linear operator, $F : \mathbb{R} \times C_g \to X$ is a continuous function, Lipschitzian in its second argument, σ-periodic in its first variable. By applying the Poincare mapping, the Hale-Lunel fixed point theorem, and a prior estimate of the solution to the corresponding IVP, the existence result of σ-periodic solutions to (1.4) was obtained.

For all we know, most of the results of the above periodic problem for neutral delayed evolution equations in Banach space have great limitations. On the one hand, the most common method is the use of the boundedness or ultimate boundedness of solutions and using some tight embeddings to achieve the compactness of Poincar$\acute{e}$ mappings. However, in some practical models, it is hard to select appropriate initial conditions to guarantee the boundedness of the solution. On the other hand, the C^1-regularity results of periodic solutions of the form Eq (1.1) are rarely studied.

Therefore, based on the theory of analytic operator semigroup, fixed point theorems, and the fractional powers of the sectorial operator, we research the periodic problems of Eq (1.1). When the f satisfies some easily verifiable growth conditions, the existence and C^1-regularity results of ω-periodic mild solutions for Eq (1.1) are obtained, which promotes and supplies the relevant results in this area.

2. Preliminaries and basic definitions

Let $(X, \|\cdot\|)$ be a Banach space. Assume that $A : \mathcal{D}(A) \subset X \to X$ is a sectorial operator and has a compact resolvent, namely, $-A$ generates the exponentially stable, compact, and analytic operator semigroup $T(t)(t \geqslant 0)$. For more concepts and properties of the C_0-semigroup, see [3]. Let $C_\omega(\mathbb{R}, X)$ be the Banach space of all continuous ω-periodic functions from $\mathbb{R}$ to X with the maximum norm $\| z \|_C = \max_{t\in I} \| z(t) \|$, where $I = [0, \omega]$.

To obtain the regularity of periodic mild solution for Eq (1.1), first, for every $h \in C_\omega(\mathbb{R}, X)$, we consider the regularity of neutral linear evolution equation

$$\frac{d}{dt}(z(t) - cz(t-\delta)) + A(z(t) - cz(t-\delta)) = h(t),\ t \in \mathbb{R}. \tag{2.1}$$

Let $C^\mu(I, X)$ be the Banach space consisting of all H$\ddot{o}$lder continuous functions with exponent $0 < \mu < 1$ mapping I to X with norm

$$\| z \|_{C^\mu} = \max_{t\in I} \| z(t) \| + \sup_{t_1\neq t_2} \frac{\| z(t_1) - z(t_2) \|}{| t_1 - t_2 |^\mu},\ t_1, t_2 \in I.$$

Assume that $z \in C^\mu(I, X)$. Then, there exists a constant $C > 0$ such that

$$\| z(t_1) - z(t_2) \| \leqslant C \mid t_1 - t_2 \mid^\mu .$$

If $0 < \mu_1 < \mu_2$, then $C^{\mu_2}(I, X) \hookrightarrow C^{\mu_1}(I, X)$.

Definition 2.1. *[3] The operator A is said to be sectorial operator in X, if X is a Banach space, $A : \mathcal{D}(A) \subset X \to X$ is dense and closed linear operator, and there exist constants $\theta \in (0, \frac{\pi}{2})$ and $M > 0$ such that for*

$$\lambda \in \Sigma_\theta := \{\lambda :| \arg \lambda |< \frac{\pi}{2} + \theta\},$$

$\lambda I + A$ has a bounded inverse operator and

$$\| (\lambda I + A)^{-1} \| \leqslant \frac{M}{1+ | \lambda |}, \ \lambda \in \Sigma_\theta.$$

By [3], A is a sectorial operator if and only if $-A$ generates exponentially stable analytic operator semigroup $T(t)(t \geqslant 0)$. Namely, there exist constants $M \geqslant 1$ and $\nu > 0$ such that

$$\| T(t) \| \leqslant Me^{-\nu t}, \ t \geqslant 0.$$

Definition 2.2. *[3] Let A be sectorial operator. For $\alpha > 0$, we define the bounded operator $A^{-\alpha}$ expressed by*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt.$$

Definition 2.3. *[3] Let A be sectorial operator. For $\alpha \geqslant 0$, we define $A^\alpha = (A^{-\alpha})^{-1}$ and $\mathcal{D}(A^\alpha) = A^{-\alpha}X$. Specially, if $\alpha = 0$, then $A^\alpha = I$.*

Lemma 2.1. *[3] Let A be sectorial operator. then, A^α satisfies the following properties:*
(i) A^α is a dense and closed linear operator for $\alpha > 0$ in X;
(ii) $\mathcal{D}(A^\beta) \subset \mathcal{D}(A^\alpha)$ for $0 \leqslant \alpha < \beta$;
(iii) $A^\beta x \in \mathcal{D}(A^\alpha)$ and $A^{\alpha+\beta} x = A^\alpha(A^\beta x)$ for $\alpha, \ \beta \geqslant 0$ and $x \in \mathcal{D}(A^{\alpha+\beta})$.

Definition 2.4. *[3] Let A be a sectorial operator. For $\alpha \geqslant 0$, if X_α is a Banach space of $\mathcal{D}(A^\alpha)$ endowed with the norm $\| x \|_\alpha = \| A^\alpha x \|$ for all $x \in \mathcal{D}(A^\alpha)$, then it is called the interpolation space. Specially, $X_0 = X, \ X_1 = \mathcal{D}(A)$. If $0 < \alpha < 1$, X_α is called the interpolation space between X_0 and X_1.*

Lemma 2.2. *[3] Let A be a sectorial operator and $X_\alpha(\alpha \geqslant 0)$ be the fractional power space of A. Then, $X_\beta \hookrightarrow X_\alpha$ for $0 \leqslant \alpha < \beta$. Furthermore, if A has a compact resolvent, then $X_\beta \hookrightarrow X_\alpha$ is a compact embedding.*

Lemma 2.3. *[3] Let A be a sectorial operator, $X_\alpha(\alpha \geqslant 0)$ be the interpolation space defined by A^α and $T(t)(t \geqslant 0)$ be the analytic semigroup generated by $-A$. Then,*
(i) for each $\alpha > 0$, $T(t) : X \to X_\alpha$ is continuous by operator norm on $t > 0$;
(ii) for each $\alpha > 0$, $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in \mathcal{D}(A^\alpha)$ and $t \geqslant 0$;
(iii) for each $\alpha > 0$, there exist constant $M_\alpha > 0$ such that

$$\| A^\alpha T(t) \| \leqslant M_\alpha t^{-\alpha} e^{-\nu t} < M_\alpha t^{-\alpha}, \ t > 0.$$

Now, we recall some results for the abstract linear evolution equations. Let J denote the infinite interval $[0, \infty)$. We consider the initial value problem

$$\begin{cases} z'(t) + Az(t) = h(t), \ t \in J, \\ z(0) = x_0. \end{cases} \tag{2.2}$$

By [3], in the case that the $T(t)(t \geqslant 0)$ is a strongly continuous semigroup, when $x_0 \in X_1$ and $h \in C^1(J, X)$, (2.2) has only one classical solution $z \in C^1(J, X) \cap C(J, X_1)$ given by

$$z(t) = T(t)x_0 + \int_0^t T(t-s)h(s)ds, \tag{2.3}$$

where $X_1 = \mathcal{D}(A)$ is a Banach space with the graph norm $\|\cdot\|_1=\|\cdot\| + \| A\cdot \|$. Universally, when $x_0 \in X$ and $h \in C(J, X)$, the z given by (2.3) belongs to $C(J, X)$ and it is called a mild solution of (2.2). The z is called a strong solution of (2.2) if it is continuous on J, differentiable a.e. on $(0, \infty)$, $z' \in L^1_{loc}(J, X)$, and satisfies (2.2). Furthermore, if A is a sectorial operator, we give the following lemmas.

Lemma 2.4. *[3] Let A be a sectorial operator, $0 \leqslant \alpha < \beta \leqslant 1$, and $\gamma := \beta - \alpha$. Then, for each $x_0 \in X_\beta$, $h \in C(I, X)$, the mild solution for (2.2), z, belongs to $C^\gamma(I, X_\alpha)$.*

Lemma 2.5. *[3] Let A be a sectorial operator. Then, for each $x_0 \in X$, $h \in C^\mu(I, X)$, the mild solution for (2.2), z, is a classical solution, that is,*

$$z \in C^1((0, \omega], X) \cap C([0, \omega], X_1).$$

Let $(Y, \|\cdot\|_Y)$ be another Banach space, and let there be $0 \leqslant \alpha < 1$ such that $X_\alpha \hookrightarrow Y \hookrightarrow X$. So, there is constant $N > 0$ such that $\| x \|_Y \leqslant N \| x \|_\alpha$, $x \in X_\alpha$. Let $C_\omega(\mathbb{R}, X_\alpha)$ be the Banach space of all continuous ω-periodic functions from $\mathbb{R}$ to X_α with the maximum norm $\| z \|_{C_\alpha} = \max_{t\in I} \| z(t) \|_\alpha$, and $C_\omega(\mathbb{R}, Y)$ be the Banach space of all continuous ω-periodic functions from $\mathbb{R}$ to Y with the maximum norm $\| z \|_{C_Y} = \max_{t\in I} \| z(t) \|_Y$. Clearly, $C_\omega(\mathbb{R}, X_\alpha) \hookrightarrow C_\omega(\mathbb{R}, Y)$.

Define the operator $B : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$, expressed by

$$Bz(t) = z(t) - cz(t - \delta),\ t \in \mathbb{R},\ z \in C_\omega(\mathbb{R}, X). \tag{2.4}$$

Lemma 2.6. *[35] Let $| c |< 1$. Then, B has a bounded inverse operator which is expressed as*

$$B^{-1}y(t) = \sum_{j=0}^{\infty} c^j y(t - j\delta),\ y \in C_\omega(\mathbb{R}, X) \tag{2.5}$$

and $\| B^{-1} \| \leqslant \frac{1}{1-|c|}$.

Setting $y = Bz$, Eq (2.1) can be rewritten as

$$y'(t) + Ay(t) = h(t),\ t \in \mathbb{R}. \tag{2.6}$$

Now, the regularity results of the periodic problem for Eq (2.1) are presented.

Lemma 2.7. *Let $| c |< 1$ and $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator. Then, for each $\alpha \in [0, 1)$, $0 < \gamma < 1-\alpha$, and $h \in C_\omega(\mathbb{R}, X)$, Eq (2.1) has a unique ω-periodic mild solution $z \in C^\gamma_\omega(\mathbb{R}, X_\alpha)$ expressed by*

$$z(t) = B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)h(s)ds := Sh(t),\ t \in \mathbb{R}, \tag{2.7}$$

and the solution operator $S : C_\omega(\mathbb{R}, X) \to C^\gamma_\omega(\mathbb{R}, X_\alpha)$ is bounded linear operator.

Proof. Since A is a sectorial operator, $T(t)(t \geqslant 0)$ is exponentially stable. By [11], Eq (2.6) has only one ω-periodic mild solution y given by

$$y(t) = T(t)E(h) + \int_0^t T(t - s)h(s)ds := Ph(t),\ t \in \mathbb{R}, \tag{2.8}$$

where $E(h) = Bx_0 = (I - T(\omega))^{-1} \int_0^\omega T(\omega - s)h(s)ds$. By $y = Bz$ and Lemma 2.6, Eq (2.1) has only one periodic mild solution z expressed by

$$
\begin{aligned}
z(t) &= B^{-1}\left(T(t)E(h) + \int_0^t T(t-s)h(s)ds\right) \\
&= B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s)h(s)ds := (B^{-1} \circ P)h(t), \ t \in \mathbb{R}.
\end{aligned} \tag{2.9}
$$

We choose $\beta : 0 \leqslant \alpha < \beta < 1$ that satisfies $\gamma = \beta - \alpha$. If $h \in C(I, X)$, then $E(h) \in X_\beta$. For each $0 \leqslant t_1 \leqslant t_2 \leqslant \omega$, it follows from Lemma 2.3 that

$$
\begin{aligned}
\| B^{-1}(T(t_2)E(h) - T(t_1)E(h)) \|_\alpha &= \| A^\alpha B^{-1}(T(t_2) - T(t_1))E(h) \| \\
&\leqslant \| B^{-1} \| \cdot \| A^{-(\beta-\alpha)}(T(t_2) - T(t_1)) \cdot A^\beta E(h) \| \\
&\leqslant \frac{1}{1- | c |} \| A^{-\gamma}(T(t_2) - T(t_1)) \| \cdot \| E(h) \|_\beta \\
&\leqslant \frac{M}{1- | c |} \| A^{-\gamma}(T(t_2 - t_1) - I) \| \cdot \| E(h) \|_\beta \\
&= \frac{M}{1- | c |} \left\| \int_0^{t_2 - t_1} A^{1-\gamma}T(s)ds \right\| \cdot \| E(h) \|_\beta \\
&\leqslant \frac{M}{1- | c |} \int_0^{t_2 - t_1} \| A^{1-\gamma}T(s) \| ds \cdot \| E(h) \|_\beta \\
&\leqslant \frac{MM_{1-\gamma}}{1- | c |} \int_0^{t_2 - t_1} s^{\gamma - 1} ds \cdot \| E(h) \|_\beta \\
&= \frac{MM_{1-\gamma}}{(1- | c |)\gamma} | t_2 - t_1 |^\gamma \cdot \| E(h) \|_\beta \ .
\end{aligned}
$$

Thus, $B^{-1}T(t)E(h) \in C^\gamma(I, X_\alpha)$. Setting $V(t) = B^{-1} \int_0^t T(t-s)h(s)ds$, for $0 \leqslant t_1 \leqslant t_2 \leqslant \omega$, we have

$$
\begin{aligned}
\| V(t_2) - V(t_1) \|_\alpha &= \left\| B^{-1} \int_0^{t_2} T(t_2 - s)h(s)ds - B^{-1} \int_0^{t_1} T(t_1 - s)h(s)ds \right\|_\alpha \\
&= \left\| B^{-1} \left(\int_0^{t_1} (T(t_2 - s) - T(t_1 - s))h(s)ds + \int_{t_1}^{t_2} T(t_2 - s)h(s)ds \right) \right\|_\alpha \\
&= \left\| A^\alpha B^{-1} \left(\int_0^{t_1} (T(t_2 - s) - T(t_1 - s))h(s)ds + \int_{t_1}^{t_2} T(t_2 - s)h(s)ds \right) \right\| \\
&\leqslant \left\| A^\alpha B^{-1} \int_0^{t_1} (T(t_2 - s) - T(t_1 - s))h(s)ds \right\| + \left\| A^\alpha B^{-1} \int_{t_1}^{t_2} T(t_2 - s)h(s)ds \right\| \\
&\leqslant \frac{1}{1- | c |} \left\| \int_0^{t_1} A^\alpha (T(t_2 - s) - T(t_1 - s))h(s)ds \right\| + \frac{1}{1- | c |} \left\| \int_{t_1}^{t_2} A^\alpha T(t_2 - s)h(s)ds \right\| \\
&:= V_1 + V_2.
\end{aligned}
$$

We estimate V_1 and V_2 separately, obtaining

$$
V_1 \leqslant \frac{1}{1- | c |} \int_0^{t_1} \| A^\alpha (T(t_2 - s) - T(t_1 - s)) \| ds \cdot \| h \|_C
$$

$$
\begin{aligned}
&\leqslant \frac{M}{1-|c|}\int_0^{t_1}\| A^{\alpha}(T(t_2-t_1+s)-T(s))\| ds\cdot \| h\|_C\\
&= \frac{M}{1-|c|}\int_0^{t_1}\left\|\int_s^{t_2-t_1+s} A^{\alpha+1}T(r)dr\right\| ds\cdot \| h\|_C\\
&\leqslant \frac{MM_{\alpha+1}}{1-|c|}\int_0^{t_1}\int_s^{t_2-t_1+s} r^{-(\alpha+1)}drds\cdot \| h\|_C\\
&= \frac{MM_{\alpha+1}}{\alpha(1-|c|)}\int_0^{t_1} s^{-\alpha}-(t_2-t_1+s)^{-\alpha}ds\cdot \| h\|_C\\
&\leqslant \frac{MM_{\alpha+1}}{\alpha(1-|c|)(1-\alpha)}(t_2-t_1)^{1-\alpha}\cdot \| h\|_C,
\end{aligned}
$$

and

$$
\begin{aligned}
V_2 &\leqslant \frac{1}{1-|c|}\int_{t_1}^{t_2}\| A^{\alpha}T(t_2-s)h(s)\| ds\\
&\leqslant \frac{M_{\alpha}}{1-|c|}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds\cdot \| h\|_C\\
&\leqslant \frac{M_{\alpha}}{(1-|c|)(1-\alpha)}(t_2-t_1)^{1-\alpha}\cdot \| h\|_C\ .
\end{aligned}
$$

Setting $M(\alpha)=\max\left\{\frac{MM_{\alpha+1}}{\alpha(1-|c|)(1-\alpha)},\frac{M_{\alpha}}{(1-|c|)(1-\alpha)}\right\}$, then

$$
\| V(t_2)-V(t_1)\|_{\alpha}\leqslant 2M(\alpha)(t_2-t_1)^{1-\alpha}\cdot \| h\|_C\ .
$$

Since $\gamma<1-\alpha$, we get that $V\in C^{1-\alpha}(I,X_{\alpha})\hookrightarrow C^{\gamma}(I,X_{\alpha})$. Thus, from the periodicity and the definition of P, we define operator $S=B^{-1}\circ P: C_{\omega}(\mathbb{R},X)\to C_{\omega}^{\gamma}(\mathbb{R},X_{\alpha})$. Obviously, S is a bounded linear operator. We complete the proof. □

Lemma 2.8. *Let $|c|<1$ and $A:\mathcal{D}(A)\subset X\to X$ be a sectorial operator. Then, for $h\in C_{\omega}^{\mu}(\mathbb{R},X)(0<\mu<1)$, the periodic mild solution to Eq (2.1) $z\in C_{\omega}^{1}(\mathbb{R},X)\cap C_{\omega}(\mathbb{R},X_1)$ is a classical solution.*

Proof. Let $z(t)=B^{-1}T(t)E(h)+V(t),\ t\in\mathbb{R}$ be a periodic mild solution of Eq (2.1), where $V(t)=B^{-1}\int_0^t T(t-s)h(s)ds$. By the analyticity of $T(t)$ and (2.5), we find that $B^{-1}T(t)E(h)$ is a classical solution of Eq (2.1). Next, we need to prove that $V(t)\in\mathcal{D}(A),\ t\in\mathbb{R}$, and $AV(t)$ is continuous on $\mathbb{R}$. For each $t\in\mathbb{R}$, we have

$$
\begin{aligned}
V(t)&=B^{-1}\int_0^t T(t-s)h(s)ds\\
&=B^{-1}\int_0^t T(t-s)(h(s)-h(t))ds+B^{-1}\int_0^t T(t-s)h(t)ds\\
&=B^{-1}\int_0^t T(t-s)(h(s)-h(t))ds+B^{-1}\int_0^t T(s)h(t)ds\\
&:=B^{-1}\Lambda_1(t)+B^{-1}\Lambda_2(t).
\end{aligned}
$$

By the property of the C_0-semigroup, one has $\Lambda_2(t)\in\mathcal{D}(A)$ and $A\Lambda_2(t)=T(t)h(t)-h(t)$. So, $\Lambda_2\in C_{\omega}(\mathbb{R},X_1)$. By (2.5), we have $B^{-1}\Lambda_2\in C_{\omega}(\mathbb{R},X_1)$.

Since $h \in C_\omega^\mu(\mathbb{R}, X)(0 < \mu < 1)$, one gets

$$\begin{aligned}\int_0^t \| AT(t-s)(h(s)-h(t)) \| ds &\leqslant \int_0^t \| AT(t-s) \| \cdot \| h(s)-h(t) \| ds \\ &\leqslant M_1 \| h \|_{C^\mu} \int_0^t (t-s)^{-(1-\mu)} ds \\ &= \frac{M_1 \| h \|_{C^\mu}}{\mu}(t-s)^\mu < +\infty.\end{aligned}$$

Hence, $\Lambda_1(t) \in \mathcal{D}(A)$ and $A\Lambda_1(t) = \int_0^t AT(t-s)(h(s)-h(t))ds \in C_\omega(\mathbb{R}, X)$. Furthermore, we obtain $\Lambda_1 \in C_\omega(\mathbb{R}, X_1)$. So, it follows from (2.5) that $B^{-1}\Lambda_1 \in C_\omega(\mathbb{R}, X_1)$. Consequently, we have $V = B^{-1}\Lambda_1 + B^{-1}\Lambda_2 \in C_\omega(\mathbb{R}, X_1)$. Therefore, $z \in C_\omega^1(\mathbb{R}, X) \cap C_\omega(\mathbb{R}, X_1)$ is a classical solution of Eq (2.1). We complete the proof. □

3. Existence of periodic mild solution

In this section, we study the existence of ω-periodic mild solutions to Eq (1.1) in a subspace $Y \subset X$ on the premise that A is a sectorial operator.

Theorem 3.1. *Let $| c |< 1$, $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator, and the C_0-semigroup $T(t)(t \geqslant 0)$ generated by $-A$ be compact. If the conditions*

(H1) $f : \mathbb{R} \times Y \times Y \to X$ is ω-periodic in t and satisfies

(i) $f(t, \cdot, \cdot) : Y \times Y \to X$ is continuous for each $t \in \mathbb{R}$;

(ii) $f(\cdot, v, w) : \mathbb{R} \to X$ is strongly measurable for each $(v, w) \in Y \times Y$,

(H2) for every $r > 0$, there exists positive continuous function $h_r : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup_{\|v\|_Y, \|w\|_Y \leqslant r} \| f(t, v, w) \| \leqslant h_r(t)$$

hold, the function $s \mapsto \frac{h_r(s)}{(t-s)^\alpha}$ is Lebesgue integrable, and there is constant $0 < \rho < \frac{1-|c|}{NM_T M_\alpha}$ such that

$$\liminf_{r\to\infty} \frac{1}{r} \int_{t-\omega}^t \frac{h_r(s)}{(t-s)^\alpha} ds < \rho < \infty,$$

then Eq (1.1) has at least one ω-periodic mild solution in $C_\omega(\mathbb{R}, Y)$.

Proof. Let $M_T = \| (I - T(\omega))^{-1} \|$. Defining an operator Q in $C_\omega(\mathbb{R}, Y)$ expressed by

$$Qz = (S \circ F)z, \tag{3.1}$$

where S is periodic solution operator defined by (2.7), $F : C_\omega(\mathbb{R}, Y) \to C_\omega(\mathbb{R}, X)$ is defined as

$$F(z)(t) = f(t, z(t), z(t-\tau)), \ t \in \mathbb{R}, \ z \in C_\omega(\mathbb{R}, Y). \tag{3.2}$$

Thus,

$$Qz(t) = B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s) f(s, z(s), z(s-\tau)) ds, \ t \in \mathbb{R}, \ z \in C_\omega(\mathbb{R}, Y). \tag{3.3}$$

It is easy to verify that $Q : C_\omega(\mathbb{R}, Y) \to C^\gamma_\omega(\mathbb{R}, X_\alpha) \hookrightarrow C^\gamma_\omega(\mathbb{R}, Y) \hookrightarrow C_\omega(\mathbb{R}, Y)$. Since the fixed point of Q is equivalent to periodic mild solution to Eq (1.1), we need show that Q has a fixed point.

For each $r > 0$, we set

$$B_r = \{z \in C_\omega(\mathbb{R}, Y) |\ \| z(t) \|_Y \leqslant r,\ t \in \mathbb{R}\}. \tag{3.4}$$

Now, we testify that $\exists\, r > 0$ such that $QB_r \subset B_r$. In reality, for all $r > 0$, there exist $z_r \in B_r$ and $t \in \mathbb{R}$ such that $\| Qz_r(t) \|_Y > r$. By (H3), one has

$$\begin{aligned}
r &< \| Qz_r(t) \|_Y \\
&= \left\| B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s) f(s,\ z_r(s),\ z_r(s-\tau)) ds \right\|_Y \\
&\leqslant N \left\| B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s) f(s,\ z_r(s),\ z_r(s-\tau)) ds \right\|_\alpha \\
&\leqslant \frac{NM_T}{1-|c|} \int_{t-\omega}^{t} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z_r(s),\ z_r(s-\tau)) \| ds \\
&\leqslant \frac{NM_T M_\alpha}{1-|c|} \int_{t-\omega}^{t} \frac{h_r(s)}{(t-s)^\alpha} ds.
\end{aligned}$$

Multiplying both sides of the above inequality by $\frac{1}{r}$ and calculating the lower limit as $r \to \infty$, we obtain

$$\rho \geqslant \frac{1-|c|}{NM_T M_\alpha}, \tag{3.5}$$

which is a contradiction to $0 < \rho < \frac{1-|c|}{NM_T M_\alpha}$ of (H2). Thus, there is a constant $r > 0$ such that $QB_r \subset B_r$.

Next, we show that set $\{Qz |\ z \in B_r\}$ is relatively compact in $C_\omega(\mathbb{R}, Y)$.

Step 1. We prove that $Q : B_r \to B_r$ is continuous.

Let $\{z_n\} \subset B_r$ and $z_n \to z$ as $n \to \infty$. It follows from (H1) that

$$f(t,\ z_n(t),\ z_n(t-\tau)) \to f(t,\ z(t),\ z(t-\tau)),\ (n \to \infty). \tag{3.6}$$

Since

$$\| f(t,\ z_n(t),\ z_n(t-\tau)) - f(t,\ z(t),\ z(t-\tau)) \| \leqslant 2h_r(t),\ t \in \mathbb{R},$$

by Lebesgue's bounded convergence theorem, one has

$$\begin{aligned}
\| Qz_n(t) - Qz(t) \|_Y &\leqslant N \| Qz_n(t) - Qz(t) \|_\alpha \\
&\leqslant \frac{NM_T}{1-|c|} \int_{t-\omega}^{t} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z_n(s),\ z_n(s-\tau)) - f(s,\ z(s),\ z(s-\tau)) \| ds \\
&\leqslant \frac{NM_T M_\alpha}{1-|c|} \int_{t-\omega}^{t} (t-s)^{-\alpha} \cdot \| f(s,\ z_n(s),\ z_n(s-\tau)) - f(s,\ z(s),\ z(s-\tau)) \| ds \\
&\leqslant \frac{NM_T M_\alpha \omega^{1-\alpha}}{(1-|c|)(1-\alpha)} \| f(s,\ z_n(s),\ z_n(s-\tau)) - f(s,\ z(s),\ z(s-\tau)) \| \to 0,\ (n \to \infty).
\end{aligned}$$

Namely,

$$\| Qz_n - Qz \|_{C_Y} \to 0,\ (n \to \infty), \tag{3.7}$$

and hence $Q : B_r \to B_r$ is continuous.
Step 2. We show that the set $\{Qz|\ z \in B_r\}$ is equicontinuous in $C_\omega(\mathbb{R}, Y)$.
For any $z \in B_r$ and $0 \leqslant t_1 < t_2 \leqslant \omega$, it follows by (3.3) that

$$\begin{aligned} Qz(t_2) - (Qz)(t_1) &= B^{-1}(I - T(\omega))^{-1} \int_{t_2-\omega}^{t_2} T(t_2 - s)f(s,\ z(s),\ z(s-\tau))ds \\ &\quad - B^{-1}(I - T(\omega))^{-1} \int_{t_1-\omega}^{t_1} T(t_1 - s)f(s,\ z(s),\ z(s-\tau))ds \\ &= B^{-1}(I - T(\omega))^{-1} \int_{t_1}^{t_2} T(t_2 - s)F(z)(s)ds \\ &\quad - B^{-1}(I - T(\omega))^{-1} \int_{t_1-\omega}^{t_2-\omega} T(t_1 - s)F(z)(s)ds \\ &\quad + B^{-1}(I - T(\omega))^{-1} \int_{t_2-\omega}^{t_1} (T(t_2 - s) - T(t_1 - s))F(z)(s)ds \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Obviously, one has

$$\| Qz_n(t) - Qz(t) \|_Y \leqslant \| I_1 \|_Y + \| I_2 \|_Y + \| I_3 \|_Y . \tag{3.8}$$

Let us separately estimate $\| I_1 \|_Y$, $\| I_2 \|_Y$, and $\| I_3 \|_Y$. For $\| I_1 \|_Y$ and $\| I_2 \|_Y$, it follows from (H2) that

$$\begin{aligned} \| I_1 \|_Y &\leqslant N \| I_1 \|_\alpha \\ &\leqslant \frac{NM_T}{1-|c|} \int_{t_1}^{t_2} \| A^\alpha T(t_2 - s) \| \cdot \| F(z)(s) \| ds \\ &\leqslant \frac{NM_T M_\alpha}{1-|c|} \int_{t_1}^{t_2} \frac{h_r(s)}{(t_2 - s)^\alpha} ds, \end{aligned}$$

$$\begin{aligned} \| I_2 \|_Y &\leqslant N \| I_2 \|_\alpha \\ &\leqslant \frac{NM_T}{1-|c|} \int_{t_1-\omega}^{t_2-\omega} \| A^\alpha T(t_1 - s) \| \cdot \| F(z)(s) \| ds \\ &\leqslant \frac{NM_T M_\alpha}{1-|c|} \int_{t_1-\omega}^{t_2-\omega} \frac{h_r(s)}{(t_1 - s)^\alpha} ds. \end{aligned}$$

Obviously, $\| I_1 \|_Y$, $\| I_2 \|_Y \to 0$ as $t_2 - t_1 \to 0$.

Since $T_\alpha(t)(t \geqslant 0)$ is the part of $T(t)(t \geqslant 0)$ in X_α, namely, $T(t)(t \geqslant 0)$ is a C_0-semigroup in X_α and $\| T(t) \|_\alpha \leqslant \| T(t) \|$, $t \geqslant 0$. Thus, for $\| I_3 \|_Y$, we get

$$\begin{aligned} \| I_3 \|_Y &\leqslant N \| I_3 \|_\alpha \\ &\leqslant \frac{NM_T}{1-|c|} \int_{t_2-\omega}^{t_1} \| (T(t_2 - s) - T(t_1 - s)) \cdot F(z)(s) \|_\alpha ds \\ &\leqslant \frac{NM_T}{1-|c|} \int_{t_2-\omega}^{t_1} \| T(t_1 - s)(T(t_2 - t_1) - I) \cdot F(z)(s) \|_\alpha ds \\ &\leqslant \frac{NMM_T}{1-|c|} \int_{t_2-\omega}^{t_1} \| (T(t_2 - t_1) - I) \cdot F(z)(s) \|_\alpha ds \to 0,\ (t_2 - t_1 \to 0), \end{aligned}$$

so $\| Qz(t_2) - (Qz)(t_1) \|_Y \to 0$ as $t_2 - t_1 \to 0$. Thus, the set $\{Qz|\, z \in B_r\}$ is equicontinuous in $C_\omega(\mathbb{R}, Y)$.

Step 3. We verify that the set $\{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\}$ is relatively compact in Y.

For the convenience of proof, we define the set

$$(Q_\xi B_r)(t) := \{(Q_\xi z)(t)|\, z \in B_r,\ 0 < \xi < \omega,\ t \in \mathbb{R}\} \tag{3.9}$$

expressed by

$$\begin{aligned}(Q_\xi z)(t) &= B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t-\xi} T(t-s) f(s,\ z(s),\ z(s-\tau)) ds \\ &= T(\xi) B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t-\xi} T(t-s-\xi) f(s,\ z(s),\ z(s-\tau)) ds.\end{aligned}$$

Obviously, $(Q_\xi B_r)(t) \subset \{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\}$, and we obtain

$$\begin{aligned}\| (Q_\xi z)(t) \|_\alpha &= \left\| A^\alpha B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t-\xi} T(t-s) f(s,\ z(s),\ z(s-\tau)) ds \right\| \\ &\leqslant \frac{M_T}{1-|c|} \int_{t-\omega}^{t-\xi} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z(s),\ z(s-\tau)) \| ds \\ &\leqslant \frac{M_T M_\alpha}{1-|c|} \int_{t-\omega}^{t-\xi} \frac{h_r(s)}{(t-s)^\alpha} ds := \widetilde{M} < \infty.\end{aligned}$$

Since $T_\alpha(\xi)$ is a compact operator in X_α, for each $t \in \mathbb{R}$, one has that $(Q_\xi B_r)(t)$ is relatively compact in X_α, that is, any open coverage of $(Q_\xi B_r)(t)$ has finite sub coverage. Thus, it follows from (3.2), (3.3), and (3.9) for every $z_i \in B_r$ and $t \in \mathbb{R}$ that

$$\begin{aligned}\| Qz_i(t) - Q_\xi z_i(t) \|_\alpha &= \left\| B^{-1}(I - T(\omega))^{-1} \left(\int_{t-\omega}^{t} T(t-s) F(z_i)(s) - \int_{t-\omega}^{t-\xi} T(t-s) F(z_i)(s) \right) ds \right\|_\alpha \\ &\leqslant \| B^{-1} \| \cdot \| (I - T(\omega))^{-1} \| \int_{t-\xi}^{t} \| A^\alpha T(t-s) \| \cdot \| F(z_i)(s) \| ds \\ &\leqslant \frac{M_T M_\alpha}{1-|c|} \int_{t-\xi}^{t} \frac{h_r(s)}{(t-s)^\alpha} ds (\xi \to 0).\end{aligned}$$

Therefore, for any $\epsilon > 0$, if ξ is small enough, then $\| Q_\xi z_i(t) - Qz_i(t) \|_\alpha < \epsilon$. So, by the compactness of $(Q_\xi B_r)(t)$, we find that $\{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\} \subset \bigcup_{i=1}^{n} B(Q_\xi z_i(t), \epsilon)$. Namely, the $\{Q_\xi z_1(t),\ Q_\xi z_2(t), \ldots,\ Q_\xi z_n(t)\}$ is a finite ϵ-net of $\{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\}$. Thus, $\{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\}$ is totally bounded in Y, and we find that $\{Qz(t)|\, z \in B_r,\ t \in \mathbb{R}\}$ is relatively compact in Y.

From the above discussion, using the Arzela-Ascoli theorem, $\{Qz|\, z \in B_r\}$ is relatively compact in $C_\omega(\mathbb{R}, Y)$. So, $Q : B_r \to B_r$ is a completely continuous operator. By the Schauder fixed point theorem, Q has at least one fixed point $z \in B_r$ which is a ω-periodic mild solutions to Eq (1.1). We complete the proof. □

In (H2), if $h_r(t)$ does not depend on time t, we easily get $0 < \rho < \frac{1-|c|}{N M_T M_\alpha}$. Thus, (H2) can be replaced by the following condition.

(H2*) There exist nonnegative constants $l_1,\ l_2$ and l_0 satisfying $l_1 + l_2 < \frac{(1-|c|)(1-\alpha)}{NM_TM_\alpha\omega^{1-\alpha}}$ such that for any $t \in \mathbb{R}$ and $v,\ w \in Y$,

$$\| f(t,\ v,\ w) \| \leqslant l_1 \| v \|_Y + l_2 \| w \|_Y + l_0.$$

Obviously, (H2*)⇒ (H2). In fact, for every $r > 0$, if $\| v \|_Y,\ \| w \|_Y \leqslant r$, one obtains

$$\| f(t,\ v,\ w) \| \leqslant r(l_1 + l_2) + l_0 := h_r(t),\ t \in \mathbb{R}, \tag{3.10}$$

and hence

$$\liminf_{r\to\infty} \frac{1}{r} \int_{t-\omega}^{t} \frac{h_r(s)}{(t-s)^\alpha} ds = (l_1 + l_2)\frac{\omega^{1-\alpha}}{1-\alpha} := \rho > 0. \tag{3.11}$$

Thus, we get the following corollary.

Corollary 3.1. *Let $| c |< 1$, $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator and the C_0-semigroup $T(t)(t \geqslant 0)$ generated by $-A$ be compact. If (H1) and (H2 *) hold, then Eq (1.1) has at least one ω-periodic mild solution in $C_\omega(\mathbb{R}, Y)$.*

Furthermore, if $T(t)(t \geqslant 0)$ is a non-compact semigroup and the f satisfies the Lipschitz condition, we can get the following result.

Theorem 3.2. *Let $| c |< 1$ and $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator. If the condition (H3) $f : \mathbb{R} \times Y \times Y \to X$ is ω-periodic in t and there exist constants $C_1,\ C_2 > 0$ satisfying $C_1 + C_2 < \frac{(1-|c|)(1-\alpha)}{NM_TM_\alpha\omega^{1-\alpha}}$ such that for $v_i,\ w_i \in Y (i = 1, 2)$,*

$$\| f(t,\ v_2,\ w_2) - f(t,\ v_1,\ w_1) \| \leqslant C_1 \| v_2 - v_1 \|_Y + C_2 \| w_2 - w_1 \|_Y,\ t \in \mathbb{R}$$

holds, then Eq (1.1) has only one ω-periodic mild solution in $C_\omega(\mathbb{R}, Y)$.

Proof. Let Q be the operator defined by (3.1), that is, $Q = S \circ F : C_\omega(\mathbb{R}, Y) \to C_\omega(\mathbb{R}, Y)$. For each $z_1,\ z_2 \in C_\omega(\mathbb{R}, Y)$, it follows by (H4) and (3.3) that

$$\begin{aligned}
&\| Qz_2(t) - Qz_1(t) \|_Y \leqslant N \| Qz_2(t) - Qz_1(t) \|_\alpha \\
= & N \left\| B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)\big(f(s,\ z_2(s),\ z_2(s-\tau)) - f(s,\ z_1(s),\ z_1(s-\tau))\big)ds \right\|_\alpha \\
\leqslant & \frac{NM_T}{1-|c|} \int_{t-\omega}^{t} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z_2(s),\ z_2(s-\tau)) - f(s,\ z_1(s),\ z_1(s-\tau)) \| ds \\
\leqslant & \frac{NM_TM_\alpha}{1-|c|} \int_{t-\omega}^{t} (t-s)^{-\alpha}(C_1 \| z_2(s) - z_1(s) \|_Y + C_2 \| z_2(s-\tau) - z_1(s-\tau) \|_Y)ds \\
\leqslant & \frac{NM_TM_\alpha(C_1 + C_2)}{1-|c|} \int_{t-\omega}^{t} (t-s)^{-\alpha} ds \| z_2 - z_1 \|_{C_Y} \\
= & \frac{NM_TM_\alpha\omega^{1-\alpha}(C_1 + C_2)}{(1-|c|)(1-\alpha)} \| z_2 - z_1 \|_{C_Y},
\end{aligned}$$

and hence, by (H4), one has

$$\| Qz_2 - Qz_1 \|_{C_Y} \leqslant \frac{NM_TM_\alpha\omega^{1-\alpha}(C_1 + C_2)}{(1-|c|)(1-\alpha)} \| z_2 - z_1 \|_{C_Y} < \| z_2 - z_1 \|_{C_Y}. \tag{3.12}$$

Thus, $Q : C_\omega(\mathbb{R}, Y) \to C_\omega(\mathbb{R}, Y)$ is a contractive mapping. Applying the Banach contraction mapping principle, Q has a unique fixed point $z^* \in C_\omega(\mathbb{R}, Y)$ which is an ω-periodic mild solution to Eq (1.1). We complete the proof. □

4. Regularity of periodic mild solution

In this section, based on the regularity conclusion of the neutral linear evolution equation Eq (2.1), we further discuss the regularity of periodic mild solutions in the interpolation space X_α for Eq (1.1).

Theorem 4.1. *Let* $|\, c \,| < 1$ *and* $A : \mathcal{D}(A) \subset X \to X$ *be sectorial operator. If* $f : \mathbb{R} \times X_\alpha \times X_\alpha \to X$ *is* ω*-periodic in t and satisfies*
(H4) there exist constants $0 < \mu_1 < 1$ *and* $0 < L < \frac{(1-|c|)(1-\alpha)}{2M_T M_\alpha \omega^{1-\alpha}}$ *such that for* $\forall\ t_i \in \mathbb{R}$ *and* $v_i,\ w_i \in X_\alpha (i = 1, 2)$

$$\| f(t_2,\ v_2,\ w_2) - f(t_1,\ v_1,\ w_1) \| \leqslant L(|\, t_2 - t_1 \,|^{\mu_1} + \| v_2 - v_1 \|_\alpha + \| w_2 - w_1 \|_\alpha),$$

then Eq (1.1) has only one ω*-periodic classical solution* $z^* \in C^1_\omega(\mathbb{R}, X) \cap C_\omega(\mathbb{R}, X_1)$.

Proof. Define an operator Q in $C_\omega(\mathbb{R}, X_\alpha)$ expressed by

$$Qz(t) = B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s) f(s,\ z(s),\ z(s-\tau)) ds,\ t \in \mathbb{R},\ z \in C_\omega(\mathbb{R}, X_\alpha). \tag{4.1}$$

It is easily seen that $Q : C_\omega(\mathbb{R}, X_\alpha) \to C^\gamma_\omega(\mathbb{R}, X_\alpha) \hookrightarrow C_\omega(\mathbb{R}, X_\alpha)$ is continuous.

For any $z_1,\ z_2 \in C_\omega(\mathbb{R}, X_\alpha)$, it follows from (H4) and (4.1) that

$$\begin{aligned}
&\| Qz_2(t) - Qz_1(t) \|_\alpha \\
=& \left\| B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)(f(s,\ z_2(s),\ z_2(s-\tau)) - f(s,\ z_1(s),\ z_1(s-\tau))) ds \right\|_\alpha \\
\leqslant& \frac{M_T}{1 - |\, c \,|} \int_{t-\omega}^{t} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z_2(s),\ z_2(s-\tau)) - f(s,\ z_1(s),\ z_1(s-\tau)) \| ds \\
\leqslant& \frac{L M_T M_\alpha}{1 - |\, c \,|} \int_{t-\omega}^{t} (t-s)^{-\alpha} (\| z_2(s) - z_1(s) \|_\alpha + \| z_2(s-\tau) - z_1(s-\tau) \|_\alpha) ds \\
\leqslant& \frac{2 L M_T M_\alpha}{1 - |\, c \,|} \int_{t-\omega}^{t} (t-s)^{-\alpha} ds \, \| z_2 - z_1 \|_{C_\alpha} \\
=& \frac{2 L M_T M_\alpha \omega^{1-\alpha}}{(1 - |\, c \,|)(1-\alpha)} \| z_2 - z_1 \|_{C_\alpha},
\end{aligned}$$

and hence, by (H5), we have

$$\| Qz_2 - Qz_1 \|_{C_\alpha} \leqslant \frac{2 L M_T M_\alpha \omega^{1-\alpha}}{(1 - |\, c \,|)(1-\alpha)} \| z_2 - z_1 \|_{C_\alpha} < \| z_2 - z_1 \|_{C_\alpha} . \tag{4.2}$$

Thus, $Q : C_\omega(\mathbb{R}, X_\alpha) \to C_\omega(\mathbb{R}, X_\alpha)$ is a contractive mapping, and by the applying Banach contraction mapping principle, we see that Q has a unique fixed point $z^* \in C_\omega(\mathbb{R}, X_\alpha)$, which is an ω-periodic mild solution of Eq (1.1).

Next, we show that z^* is a classical solution. We know that $z^* \in C_\omega(\mathbb{R}, X_\alpha)$ is a mild solution to Eq (1.1). Setting $h(t) = f(t,\ z(t),\ z(t-\tau))$ for $t \in \mathbb{R}$, obviously, $h \in C_\omega(\mathbb{R}, X)$, and hence z^* is also a mild solution of the linear equation

$$\frac{d}{dt}(z(t) - cz(t-\delta)) + A(z(t) - cz(t-\delta)) = h(t),\ t \in \mathbb{R}. \tag{4.3}$$

By Lemma 2.7, one obtains

$$z^* \in C^{\gamma}_{\omega}(\mathbb{R}, X_\alpha) \hookrightarrow C^{\gamma}_{\omega}(\mathbb{R}, Y) \hookrightarrow C^{\gamma}_{\omega}(\mathbb{R}, X),\ \gamma \in (0, 1-\alpha).$$

By means of (H4), we choose $\mu = \min\{\mu_1, \gamma\}$, and deduce that $h \in C^{\mu}_{\omega}(\mathbb{R}, X)$. Therefore, by virtue of Lemma 2.8, $z^* \in C^1_{\omega}(\mathbb{R}, X) \cap C_{\omega}(\mathbb{R}, X_1)$. We complete the proof. □

To obtain the existence of strong solution for Eq (1.1), we provide the definition of the strong solution for Eq (1.1).

Definition 4.1. *[3] If z is a periodic mild solution for Eq (1.1), is almost everywhere differentiable on $\mathbb{R}$, $z' \in L^1_{loc}(\mathbb{R}, X_\alpha)$, and satisfies Eq (1.1), then it is called a strong solution of Eq (1.1).*

Theorem 4.2. *Let X be reflexive Banach space and $|\ c\ |< 1$. Let $A : \mathcal{D}(A) \subset X \to X$ be sectorial operator. If $f : \mathbb{R} \times X_\alpha \times X_\alpha \to X$ is ω-periodic in t and satisfies*
(H5) there exists constant $0 < L < \frac{(1-|c|)(1-\alpha)}{2M_T M_\alpha \omega^{1-\alpha}}$ such that for $\forall\ t_i \in \mathbb{R}$ and $v_i,\ w_i \in X_\alpha (i = 1, 2)$

$$\| f(t_2,\ v_2,\ w_2) - f(t_1,\ v_1,\ w_1) \| \leqslant L(|\ t_2 - t_1\ | + \| v_2 - v_1 \|_\alpha + \| w_2 - w_1 \|_\alpha),$$

then Eq (1.1) has only one ω-periodic strong solution.

Proof. Let Q be the operator defined by (4.1). For all $r > 0$, we set

$$B_r = \{z \in C_\omega(\mathbb{R}, X_\alpha) |\ \| z(t) \|_\alpha \leqslant r\}. \tag{4.4}$$

We choose $\overline{C} = \max_{t\in[0,\pi]} \| f(t,\ \theta,\ \theta) \|$ and $r_0 \geqslant \frac{\overline{C} M_T M_\alpha \omega^{1-\alpha}}{(1-|c|)(1-\alpha) - 2LM_T M_\alpha \omega^{1-\alpha}}$. Then, for every $z \in B_{r_0}$, it follows by (4.1) and (H5) that

$$\begin{aligned}
\| Qz(t) \|_\alpha &= \left\| B^{-1}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s) f(s,\ z(s),\ z(s-\tau)) ds \right\|_\alpha \\
&\leqslant \frac{M_T}{1-|\ c\ |} \int_{t-\omega}^{t} \| A^\alpha T(t-s) \| \cdot \| f(s,\ z(s),\ z(s-\tau)) \| ds \\
&\leqslant \frac{M_T M_\alpha}{1-|\ c\ |} \int_{t-\omega}^{t} (t-s)^{-\alpha} (L(\| z(s) \|_\alpha + \| z(s-\tau) \|_\alpha) + \| f(t,\ \theta,\ \theta) \|) ds \\
&\leqslant \frac{M_T M_\alpha \omega^{1-\alpha}}{(1-|\ c\ |)(1-\alpha)} (2Lr_0 + \overline{C}) \leqslant r_0.
\end{aligned}$$

Thus, we deduce that there is constant $r_0 \geqslant \frac{\overline{C} M_T M_\alpha \omega^{1-\alpha}}{(1-|c|)(1-\alpha) - 2LM_T M_\alpha \omega^{1-\alpha}}$ such that $QB_{r_0} \subset B_{r_0}$.

Furthermore, we choose a large enough constant $\widetilde{L} \geqslant \frac{LM_T M_\alpha \omega^{1-\alpha}}{(1-|c|)(1-\alpha) - 2LM_T M_\alpha \omega^{1-\alpha}}$ and define the set

$$\overline{\Omega} = \{z \in B_{r_0} |\ \| z(t_2) - z(t_1) \|_\alpha \leqslant \widetilde{L}\ |\ t_2 - t_1\ |,\ t_1,\ t_2 \in \mathbb{R}\}. \tag{4.5}$$

Clearly, $\overline{\Omega} \neq \emptyset$ is a bounded convex closed set. To prove that Q has a fixed point in $\overline{\Omega}$, we first need show that for each $z \in \overline{\Omega}$,

$$\| Qz(t_2) - Qz(t_1) \|_\alpha \leqslant \widetilde{L}\ |\ t_2 - t_1\ |,\ t_1,\ t_2 \in \mathbb{R}. \tag{4.6}$$

By virtue of (4.1) and (H6), we have

$$
\begin{aligned}
\| Qz(t_2) - Qz(t_1) \|_\alpha &= \Big\| B^{-1}(I - T(\omega))^{-1} \int_{t_2-\omega}^{t_2} T(t_2 - s)f(s,\ z(s),\ z(s-\tau))ds \\
&\quad - B^{-1}(I - T(\omega))^{-1} \int_{t_1-\omega}^{t_1} T(t_1 - s)f(s,\ z(s),\ z(s-\tau))ds \Big\|_\alpha \\
&\leqslant \frac{M_T}{1- \mid c \mid} \Big\| \int_{t_2-\omega}^{t_2} A^\alpha T(t_2 - s)f(s,\ z(s),\ z(s-\tau))ds \\
&\quad - \int_{t_1-\omega}^{t_1} A^\alpha T(t_1 - s)f(s,\ z(s),\ z(s-\tau))ds \Big\| \\
&= \frac{M_T}{1- \mid c \mid} \Big\| \int_0^\omega A^\alpha T(s)f(t_2 - s,\ z(t_2 - s),\ z(t_2 - s - \tau))ds \\
&\quad - \int_0^\omega A^\alpha T(s)f(t_1 - s,\ z(t_1 - s),\ z(t_1 - s - \tau))ds \Big\| \\
&\leqslant \frac{M_T}{1- \mid c \mid} \int_0^\omega \| A^\alpha T(s) \| \cdot \| f(t_2 - s,\ z(t_2 - s),\ z(t_2 - s - \tau)) \\
&\quad - f(t_1 - s,\ z(t_1 - s),\ z(t_1 - s - \tau)) \| ds \\
&\leqslant \frac{M_T M_\alpha}{1- \mid c \mid} \int_0^\omega s^{-\alpha}(L(\mid t_2 - t_1 \mid + \| z(t_2 - s) - z(t_1 - s) \|_\alpha \\
&\quad + \| z(t_2 - s - \tau) - z(t_1 - s - \tau) \|_\alpha))ds \\
&\leqslant \frac{LM_T M_\alpha \omega^{1-\alpha}(2\widetilde{L} + 1)}{(1- \mid c \mid)(1-\alpha)} \mid t_2 - t_1 \mid \leqslant \widetilde{L} \mid t_2 - t_1 \mid,
\end{aligned}
$$

and hence (4.6) holds, that is, $Q\overline{\Omega} \subset \overline{\Omega}$. According to the proof process of Theorem 4.1, by (H5) we easily infer that $Q : \overline{\Omega} \to \overline{\Omega}$ is a contractive mapping, and hence has unique fixed point $z^* \in \overline{\Omega}$, which is an ω-periodic mild solution to Eq (1.1).

Finally, we need prove that z^* is a strong solution of Eq (1.1).

Since $z^* \in \overline{\Omega}$, then the function

$$
z^*(t) - cz^*(t-\delta) = (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)f(s,\ z^*(s),\ z^*(s-\tau))ds,\ t \in \mathbb{R} \tag{4.7}
$$

is Lipschitz continuous. By the reflexivity of X, we deduce that X_α also is reflexive. By [36], we can obtain that $z^*(\cdot)$ is almost everywhere differentiable on $\mathbb{R}$ and $(z^*(t) - cz^*(t-\delta))' \in L^1_{loc}(\mathbb{R}, X_\alpha)$. Then, it follows from [3] that

$$
\begin{aligned}
\frac{d}{dt}(z^*(t) - cz^*(t-\delta)) &= \frac{d}{dt}(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)f(s,\ z^*(s),\ z^*(s-\tau))ds \\
&= (I - T(\omega))^{-1}\Big((I - T(\omega))f(t,\ z^*(t),\ z^*(t-\tau)) \\
&\quad - A \int_{t-\omega}^{t} T(t - s)f(s,\ z^*(s),\ z^*(s-\tau))ds\Big) \\
&= f(t,\ z^*(t),\ z^*(t-\tau)) - A(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)f(s,\ z^*(s),\ z^*(s-\tau))ds \\
&= f(t,\ z^*(t),\ z^*(t-\tau)) - A(z^*(t) - cz^*(t-\delta)).
\end{aligned}
$$

Namely,

$$\frac{d}{dt}(z^*(t) - cz^*(t-\delta)) + A(z^*(t) - cz^*(t-\delta)) = f(t,\ z^*(t),\ z^*(t-\tau)),\ a.e\ t \in \mathbb{R}, \tag{4.8}$$

Thus, from Definition 4.1, we conclude that z^* is a strong solution of Eq (1.1). We complete the proof.

5. Application

Example 5.1. Consider the time periodic problem of the neutral delayed parabolic equation

$$\begin{cases} \frac{\partial}{\partial t}y + A(x,D)y = g(x,\ t,\ z(x,\ t),\ \nabla z(x,\ t),\ z(x,\ t-\pi),\ \nabla z(x,\ t-\pi)),\ (x,\ t) \in \Omega \times \mathbb{R}, \\ z|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where $y(x,\ t) = Bz(x,\ t) = z(x,\ t) - \frac{2}{3}z(x,\ t-\frac{\pi}{2})$, and $\nabla z(x,\ t)$ represents gradient. Let $\Omega \subset \mathbb{R}^3$ be a bounded open area with a sufficiently smooth boundary $\partial\Omega$. Let

$$A(x,D)y = -\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) + a_0(x)y \tag{5.2}$$

be a strong elliptical operator in $\overline{\Omega}$, where the weight function $a_{ij} \in C^{1+\mu}(\overline{\Omega})(i,j = 1,2,3),\ a_0 \in C^{\mu}(\overline{\Omega})$, and $a_0(x) \geqslant 0$.

Theorem 5.1. *Let $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and ω-periodic in t. If g satisfies the assumption*
(F1) there exist nonnegative constant $l_1,\ l_2,\ l_0$ satisfying $l_1 + l_2 < \frac{1-\alpha}{12N^2 M_T M_\alpha \omega^{1-\alpha}}$ such that for each $(x,\ t,\ \varsigma,\ \phi,\ \psi,\ \zeta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$

$$|\ g(x,\ t,\ \varsigma,\ \phi,\ \psi,\ \zeta)\ | \leqslant l_1(|\ \varsigma\ | + |\ \phi\ |) + l_2(|\ \psi\ | + |\ \zeta\ |) + l_0,$$

then Eq (5.1) has at least one time ω-periodic mild solution $z \in C_\omega(\overline{\Omega} \times \mathbb{R})$.

Proof. Choosing the work space $X = L^2(\Omega)$ with norm $\|\cdot\|_2$, we easily find that X is a reflexive Banach space. We define the operator A in X given by

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),\ Ay = A(x,D)y. \tag{5.3}$$

By [3], we see that A is a sectorial operator, namely, the semigroup $T(t)(t \geqslant 0)$ generated by $-A$ is exponentially stable and analytic. And, because $A(x,D)$ has compact resolvent in $L^2(\Omega)$, then $T(t)(t \geqslant 0)$ is compact. Set $X_\alpha = \mathcal{D}(A^\alpha)$, where $\alpha \in [0,1]$, and assume that $\frac{3}{4} < \alpha < 1,\ 0 \leqslant \mu < \frac{4\alpha-3}{2}$. Then, $X_\alpha \hookrightarrow W^{1,2}(\Omega) \cap C^\mu(\overline{\Omega})$. Namely, there is a constant $N > 0$ such that $\|\ z\ \|_{1,2} \leqslant N\ \|\ z\ \|_\alpha,\ z \in X_\alpha$. Setting $z(t)(x) = (z(x,t),\ \nabla z(x,t))$ and

$$f(t,\ z(t),\ z(t-\pi))(x) = g(x,\ t,\ z(x,t),\ \nabla z(x,t),\ z(x,t-\pi),\ \nabla z(x,t-\pi)) \in C(\overline{\Omega}), \tag{5.4}$$

we deduce that $f : \mathbb{R} \times X_\alpha \times X_\alpha \to X$ is continuous and ω-periodic in t. Thus, we will convert Eq (5.1) into the abstract Eq (1.1).

For any $\varsigma,\ \psi \in X_\alpha$, by (F1) one has

$$\begin{aligned}\| f(t,\ \varsigma,\ \psi) \|_2 &= \Big(\int_\Omega \mid g(x,\ t,\ \varsigma,\ \nabla\varsigma,\ \psi,\ \nabla\psi) \mid^2 dx\Big)^{1/2}\\ &\leqslant \Big(\int_\Omega (l_1(\mid \varsigma \mid + \mid \nabla\varsigma \mid) + l_2(\mid \psi \mid + \mid \nabla\psi \mid) + l_0)^2 dx\Big)^{1/2}\\ &\leqslant 4l_1 \parallel \varsigma \parallel_{1,2} + 4l_2 \parallel \psi \parallel_{1,2} + 2l_0 \mid \Omega \mid\\ &\leqslant 4Nl_1 \parallel \varsigma \parallel_\alpha + 4Nl_2 \parallel \psi \parallel_\alpha + 2l_0 \mid \Omega \mid .\end{aligned}$$

For all $r > 0$ and $\| \varsigma \|_\alpha,\ \| \psi \|_\alpha \leqslant r$, we have

$$\| f(t,\ \varsigma,\ \psi) \|_2 \leqslant 4Nl_1 r + 4Nl_2 r + 2l_0 \mid \Omega \mid := h_r(t). \tag{5.5}$$

Obviously, f satisfies (H1) and (H2)*. Therefore, by Corollary 3.1, Eq (5.1) has at least one time ω-periodic mild solution $z \in C_\omega(\overline{\Omega} \times \mathbb{R})$. We complete the proof. □

Theorem 5.2. *Let $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and ω-periodic in t. If g satisfies the assumption*
(F2) there exist positive constant $l < \frac{1-\alpha}{6\widetilde{N} M_T M_\alpha \omega^{1-\alpha}}$ and $\mu \in (0, 1]$ such that for each $(x,\ t_i,\ \varsigma_i,\ \phi_i,\ \psi_i,\ \zeta_i) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 (i = 1, 2)$

$$\begin{aligned}&\mid g(x,\ t_2,\ \varsigma_2,\ \phi_2,\ \psi_2,\ \zeta_2) - g(x,\ t_1,\ \varsigma_1,\ \phi_1,\ \psi_1,\ \zeta_1) \mid\\ \leqslant& l(\mid t_2 - t_1 \mid^\mu + \mid \varsigma_2 - \varsigma_1 \mid + \mid \phi_2 - \phi_1 \mid + \mid \psi_2 - \psi_1 \mid + \mid \zeta_2 - \zeta_1 \mid),\end{aligned}$$

then Eq (5.1) has only one time ω-periodic classical solution or strong solution.

Proof. For every $t_i \in \mathbb{R}$ and $\varsigma_i,\ \psi_i \in X_\alpha (i = 1, 2)$, it follows from (5.4) and (F2) that

$$\begin{aligned}&\| f(t_2,\ \varsigma_2,\ \psi_2) - f(t_1,\ \varsigma_1,\ \psi_1) \|_2\\ =&\Big(\int_\Omega (g(x,\ t_2,\ \varsigma_2,\ \nabla\varsigma_2,\ \psi_2,\ \nabla\psi_2) - g(x,\ t_1,\ \varsigma_1,\ \nabla\varsigma_1,\ \psi_1,\ \nabla\psi_1))^2 dx\Big)^{1/2}\\ \leqslant&\Big(\int_\Omega l^2(\mid t_2 - t_1 \mid^\mu + \mid \varsigma_2 - \varsigma_1 \mid + \mid \nabla\varsigma_2 - \nabla\varsigma_1 \mid + \mid \psi_2 - \psi_1 \mid + \mid \nabla\psi_2 - \nabla\psi_1 \mid)^2 dx\Big)^{1/2}\\ \leqslant& l(2 \mid \Omega \mid \cdot \mid t_2 - t_1 \mid^\mu + 4 \parallel \varsigma_2 - \varsigma_1 \parallel_{1,2} + 4 \parallel \psi_2 - \psi_1 \parallel_{1,2})\\ \leqslant& l(2 \mid \Omega \mid \cdot \mid t_2 - t_1 \mid^\mu + 4N \parallel \varsigma_2 - \varsigma_1 \parallel_\alpha + 4N \parallel \psi_2 - \psi_1 \parallel_\alpha)\\ \leqslant& l\widetilde{N}(\mid t_2 - t_1 \mid^\mu + \parallel \varsigma_2 - \varsigma_1 \parallel_\alpha + \parallel \psi_2 - \psi_1 \parallel_\alpha),\end{aligned}$$

where $\widetilde{N} = \max\{2 \mid \Omega \mid,\ 4N\}$, and we choose $L = l\widetilde{N} < \frac{1-\alpha}{6M_T M_\alpha \omega^{1-\alpha}}$. If $\mu \in (0, 1)$, we see that condition (H4) is established, or if $\mu = 1$, we find that condition (H5) is established. Consequently, by Theorem (4.1) or Theorem (4.2), Eq (5.1) has only one time ω-periodic classical solution or strong solution. We complete the proof. □

6. Conclusions

The neutral delayed evolution Eq (1.1) has practical applications and it can be regarded as more general abstract form of the resistance-coupled transmission lines model [4, 5], thus its research has important theoretical significance and value. In this article, based on the analytic operator semigroup theory, fixed point theorems, and the fractional power of the sectorial operator, the existence and regularity conclusions of ω-periodic mild solution to Eq (1.1) are obtained under some suitable growth conditions of the nonlinear terms f. This article expands upon and supplements the existing literature, thus it is valuable and meaningful.

Author contributions

Shengbin Yang carried out the first draft of this manuscript; Shengbin Yang prepared the final version of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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