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Research article

General numerical radius for products of sectorial matrices

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Abstract: In this paper, we investigate the generalized numerical radius ω_N, associated with a matrix norm N defined by $\omega_N(X) = \sup_{\theta\in\mathbb{R}} N(\mathrm{Re}(e^{i\theta}X))$. We focus on matrices whose numerical ranges are contained in sectors of the complex plane (sectorial matrices) and derive upper bounds for $\omega_N(XY)$ and $\omega_N(X \circ Y)$ for such matrices X and Y. Our results generalize and refine well-known numerical radius inequalities. Several known inequalities for $\omega(X)$ are recovered as special cases.

Keywords: sectorial-matrices; accretive-matrices; dissipative-matrices; general numerical radius; Hadamard product
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1. Introduction

Let $\mathbb{M}_n$ denote the algebra of all $n \times n$ complex matrices. The Hadamard product of two matrices, $X = [x_{i,j}]$ and $Y = [y_{i,j}]$, in $\mathbb{M}_n$ is defined as the following matrix:

$$X \circ Y = [x_{i,j}y_{i,j}].$$

For a comprehensive study of the Hadamard product, see [12, Chapter 5]. For more recent developments in this area, refer to [10, 11, 14]. A matrix $X \in \mathbb{M}_n$ can be expressed as

$$X = \mathrm{Re}(X) + i\,\mathrm{Im}(X), \tag{1.1}$$

where $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ are Hermitian matrices defined as $\mathrm{Re}(X) = \frac{X+X^*}{2}$ and $\mathrm{Im}(X) = \frac{X-X^*}{2i}$, respectively. This decomposition is commonly referred to as the Cartesian decomposition of X.

A matrix X is called accretive (dissipative, respectively) if, in its Cartesian decomposition (1.1), the matrix $\mathrm{Re}(X)$ ($\mathrm{Im}(X)$, respectively) is positive definite. If both $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ are positive definite, then the matrix X is referred to as accretive-dissipative.

A norm $\|\cdot\|$ on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies the property $\|UXV\|$ for all $X \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. An example of such a norm is the operator norm, which is defined as follows:

$$\|X\| = \sup_{\|x\|=1} \|Xx\|.$$

For a matrix $X \in \mathbb{M}_n$, we define its numerical range as follows:

$$W(X) = \{\langle Xx, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{C}^n$, and $\|\cdot\|$ represents the Euclidean norm on $\mathbb{C}^n$. It is well known that $W(X)$ is a compact and convex subset of the complex plane.

Let $\alpha \in [0, \pi/2)$, and let S_α denote the sector of complex numbers defined as

$$S_\alpha = \{z = x + iy \in \mathbb{C} : x > 0, |y| \leq \tan(\alpha)x\}.$$

If the numerical range of a matrix X is subset of S_α, then X is called sectorial. The smallest α for which this condition holds is referred to as the index of sectoriality. Let $\mathbb{M}^s_{n,\alpha}, \alpha \in [0, \pi/2)$ represent the class of all $n \times n$ matrices X such that $W(zX) \subset S_\alpha$, where z is a complex number that satisfies $|z| = 1$. For more on sectorial matrices, see [2, 4, 7]. Let

$$\omega(X) = \sup\{|z| : z \in W(X)\}.$$

The function $\omega(X)$ is usually referred to as the numerical radius of X.

It is well established that $\omega(\cdot)$ constitutes a norm on $\mathbb{M}_n$, which satisfies the following inequalities:

$$\frac{1}{2}\|X\| \leq \omega(X) \leq \|X\|; \quad \forall X \in \mathbb{M}_n. \tag{1.2}$$

Furthermore, if $X \in \mathbb{M}_n$ is a normal matrix, then $\omega(X) = \|X\|$, thus demonstrating that the bounds in (1.2) are sharp.

Clearly, $\omega(\cdot)$ represents a weakly unitarily invariant norm on $\mathbb{M}_n$, which satisfies the condition $\omega(UXU) = \omega(X)$ for all $X \in \mathbb{M}_n$ and all unitarily matrices $U \in \mathbb{M}_n$. However, $\omega(\cdot)$ is not necessarily a unitarily invariant norm. For a comprehensive study of the numerical radius and related topics, see [12, Chapter I]. For recent developments, also refer to [8, 9].

In [15], it was established that for every $X \in \mathbb{M}_n$, the following holds:

$$\omega(X) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}X)\|.$$

Motivated by this result, the authors of [5] defined the generalized numerical radius as follows.

Definition 1.1. *Let $X \in \mathbb{M}_n$, and let N be a norm on $\mathbb{M}_n$. The generalized numerical radius of X, induced by N, is defined as follows:*

$$\omega_N(X) = \sup_{\theta\in\mathbb{R}} N(Re(e^{i\theta}X)).$$

They established several properties similar to those of $\omega(\cdot)$. Notably, they demonstrated that $\omega_N(\cdot)$ defines a norm on $\mathbb{M}_n$. Moreover, they showed that if $N(\cdot)$ is weakly unitarily invariant, then $\omega_N(\cdot)$ retains this invariance. Refer to [5] for more comprehensive details.

We recall that a norm N on $\mathbb{M}_n$ is called multiplicative if it satisfies

$$N(XY) \le N(X)N(Y), \quad \text{for all } X, Y \in \mathbb{M}_n,$$

and it is called Hadamard multiplicative if it satisfies

$$N(X \circ Y) \le N(X)N(Y), \quad \text{for all } X, Y \in \mathbb{M}_n.$$

In general, the norm $\omega(\cdot)$ is neither multiplicative nor Hadamard multiplicative. However, it is relatively straightforward to show that

$$\omega(XY) \le 4\,\omega(X)\omega(Y), \quad \forall X, Y \in \mathbb{M}_n, \tag{1.3}$$

where the constant 4 is optimal in (1.3). Additionally, we have

$$\omega(X \circ Y) \le 2\,\omega(X)\,\omega(Y), \tag{1.4}$$

where the constant 2 is optimal in (1.4); see [12, p. 73].

By selecting specific forms of X and Y, it is possible to derive sharper estimates than those presented in (1.3) and (1.4). For example, if $A, X \in \mathbb{M}_n$ and $A = [a_{ij}]$, then the following inequality holds:

$$\omega(A \circ X) \le \left(\max_j a_{jj}\right)\,\omega(X). \tag{1.5}$$

For further details, refer to [6, Corollary 4] and [13, Proposition 4.1].

If $X = [x_{ij}] \in \mathbb{M}^s_{n,\alpha_1}$ and $Y = [y_{ij}] \in \mathbb{M}^s_{n,\alpha_2}$, then the following inequalities hold:

$$\omega(XY) \le \sec(\alpha_1)\sec(\alpha_2)\omega(X)\omega(Y), \tag{1.6}$$

$$\omega(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2)\omega(X)\omega(Y), \tag{1.7}$$

and

$$\omega(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2)\min\left\{\max_j |x_{jj}|\ \omega(Y),\ \max_j |y_{jj}|\ \omega(X)\right\}. \tag{1.8}$$

For further details, see [1].

In the sequel, unless stated otherwise, let N be a multiplicative, unitarily invariant, and self-adjoint norm, and let ω_N be defined as in Definition 1.1.

This note aims to investigate the general numerical radius ω_N when applied to the product and the Hadamard product of sectorial matrices. Several properties will be established, including Propositions 2.1–2.3. Furthermore, the inequalities (1.5)–(1.8) are extended to the generalized numerical radius ω_N.

2. Main results

We begin this section by presenting the following three lemmas. The proofs of these lemmas are available in [2–4].

Lemma 2.1. *Let $X \in \mathbb{M}_n$. If the numerical range of X is a subset of S_α, then*

$$N(X) \leq \sec(\alpha) N(Re(X)).$$

Lemma 2.2. *Let $X \in \mathbb{M}_n$. If the numerical range of X is a subset of S_α, then*

$$\begin{pmatrix} \tan(\alpha)Re(X) & Im(X) \\ Im(X) & \tan(\alpha)Re(X) \end{pmatrix} \geq 0.$$

Lemma 2.3. *Let $X \in \mathbb{M}_n$. If the numerical range of X is a subset of S_α, then*

$$\begin{pmatrix} \sec(\alpha)Re(X) & X \\ X^* & \sec(\alpha)Re(X) \end{pmatrix} \geq 0.$$

In the following propositions, we present some basic properties of $\omega_N(\cdot)$ that will be used in the proofs.

Proposition 2.1. *Let $X \in \mathbb{M}_n$. Then,*

$$\omega_N(Re(X)) \leq \omega_N(X).$$

Proof. First, we remark that the fact that $Re(X^*) = Re(X)$ implies $\omega_N(X^*) = \omega_N(X)$. Now, observe that

$$\begin{aligned} \omega_N(Re(X)) &= \omega_N\left(\frac{X + X^*}{2}\right) \\ &= \frac{1}{2}\left(\omega_N(X + X^*)\right) \\ &\leq \frac{1}{2}\left(\omega_N(X) + \omega_N(X^*)\right) \\ &= \frac{1}{2}\left(\omega_N(X) + \omega_N(X)\right) \\ &= \omega_N(X). \end{aligned}$$

□

Proposition 2.2. *Let $X \in \mathbb{M}_n$. If the numerical range of X is a subset of S_α, then*

$$\omega_N(Im(X)) \leq \tan(\alpha)\omega_N(Re(X)).$$

Proof. Let $X = A + iB$ be the Cartesian decomposition of X. Since $W(X) \subset S_\alpha$, by Lemma 2.2, we have the following

$$\begin{pmatrix} \tan(\alpha)A & B \\ B & \tan(\alpha)A \end{pmatrix} \geq 0.$$

Thus,

$$N(B) \le \tan(\alpha)N(A). \tag{2.1}$$

Since B is Hermitian, we have $\mathrm{Re}(e^{i\theta}B) = \cos(\theta)B$. Therefore,

$$\begin{aligned}
\omega_N(B) &= \sup_{\theta\in\mathbb{R}} N(\mathrm{Re}(e^{i\theta}B)) \\
&= \sup_{\theta\in\mathbb{R}} |\cos(\theta)|N(B) \\
&= N(B) \\
&\le \tan(\alpha)N(A)\,(\text{by (2.1)}) \\
&= \tan(\alpha)\omega_N(A)\,(\text{since } A \text{ is Hermitian}) \\
&= \tan(\alpha)\omega_N(\mathrm{Re}(X)).
\end{aligned}$$

□

Proposition 2.3. *Let $X \in \mathbb{M}_n$. If the numerical range of X is a subset of S_α, then*

$$\omega_N(X) \le \sec(\alpha)\ \omega_N(\mathrm{Re}(X)).$$

Proof. First, by Lemma 2.1, observe that we have the following

$$N(X) \le \sec(\alpha)N(\mathrm{Re}(X)). \tag{2.2}$$

Now,

$$\begin{aligned}
\omega_N(X) &= \sup_{\theta\in\mathbb{R}} N(Re(e^{i\theta}X)) \\
&= \sup_{\theta\in\mathbb{R}} \frac{1}{2}N(e^{i\theta}X + e^{-i\theta}X^*) \\
&\le \sup_{\theta\in\mathbb{R}} \frac{1}{2}\left(N(e^{i\theta}X) + N(e^{-i\theta}X^*)\right) \\
&= \sup_{\theta\in\mathbb{R}} \frac{1}{2}\left(N(e^{i\theta}X) + N(e^{i\theta}X)\right) = N(X) \qquad (2.3) \\
&\le \sec(\alpha)N(\mathrm{Re}(X))\,(\text{by (2.2) }) \\
&= \sec(\alpha)\omega_N(\mathrm{Re}(X)).
\end{aligned}$$

□

2.1. Inequalities for products of sectorial matrices

The main result of this subsection is as follows.

Theorem 2.1. *Let $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$, with $\alpha_1, \alpha_2 \in [0, \pi/2)$. Then,*

$$\omega_N(XY) \le \sec(\alpha_1)\sec(\alpha_2)\omega_N(X)\omega_N(Y).$$

Proof. Since $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$, we can find two complex numbers, z and w, such that $|z| = |w| = 1$ where the numerical ranges of zX and wY are subsets of $S_{\alpha_1}, S_{\alpha_2}$, respectively. Furthermore, we have

$$\begin{aligned}
\omega_N(XY) &\leq N(XY) \quad \text{(by (2.3))} \\
&\leq N(X)\ N(Y) \\
&= N(zX)\ N(wY) \\
&\leq \sec(\alpha_1)\sec(\alpha_2) N(Re(zX))\, N(Re(wY)) \quad \text{(by Lemma 2.1)} \\
&= \sec(\alpha_1)\sec(\alpha_2)\omega_N(Re(zX))\ \omega_N(Re(wY)) \\
&\text{(since both } Re(zX)) \text{ and } Re(wY) \text{ are Hermitian)} \\
&\leq \sec(\alpha_1)\sec(\alpha_2)\omega_N(zX)\ \omega_N(wY) \quad \text{(by Proposition 2.1)} \\
&= \sec(\alpha_1)\sec(\alpha_2)\omega_N(X)\ \omega_N(Y).
\end{aligned}$$

□

As a direct result of Theorem 2.1, we derive the following corollaries.

Corollary 2.1. *If $X, Y \in \mathbb{M}^s_{n,\alpha}$, then*

$$\omega_N(XY) \leq \sec^2(\alpha)\omega_N(X)\omega_N(Y). \tag{2.4}$$

Corollary 2.2. *If $X, Y \in \mathbb{M}_n$ are accretive-dissipative, then*

$$\omega_N(XY) \leq 2\ \omega_N(X)\omega_N(Y).$$

Proof. Since $X, Y \in \mathbb{M}_n$ are accretive-dissipative, we have $X, Y \in \mathbb{M}^s_{n,\alpha}$ with $\alpha = \pi/4$. Now, the result follows from Theorem 2.1. Note that when X is accretive-dissipative, both its real and imaginary parts are positive definite. Consequently, the numerical range $W(X)$ lies entirely within the first quadrant of the complex plane. As a result, the rotated numerical range satisfies $W(e^{-\pi i/4}X) \subset S_{\pi/4}$. □

Corollary 2.3. *Let $X_j \in \mathbb{M}^s_{n,\alpha_j}$, $j = 1, 2, 3, ..., m$. Then,*

$$\omega_N(X_1 X_2 ... X_m) \leq \left(\prod_{j=1}^{m} \sec(\alpha_j)\right) \omega_N(X_1)\omega_N(X_2)...\omega_N(X_m).$$

Corollary 2.4. *If $X_1, X_2, ..., X_m \in \mathbb{M}^s_{n,\alpha}$, then*

$$\omega_N(X_1 X_2 ... X_m) \leq \sec^m(\alpha)\omega_N(X_1)\omega_N(X_2)...\omega_N(X_m).$$

Corollary 2.5. *Suppose that $X_1, X_2, ..., X_m \in \mathbb{M}_n$ such that each one of them is accretive and dissipative; then,*

$$\omega_N(X_1 X_2 ... X_m) \leq 2^{m/2}\ \omega_N(X_1)\omega_N(X_2)...\omega_N(X_m).$$

2.2. Inequalities for Hadamard products of sectorial matrices

We begin this subsection with the following auxiliary lemma.

Lemma 2.4. *[12, Theorem 5.5.7]. If N is a unitarily invariant norm on $\mathbb{M}_n$, then it is Hadamard submultiplicative if and only if it is submultiplicative.*

In general, the generalized numerical radius ω_N is not Hadamard multiplicative. However, in a special case, we can prove the following result.

Theorem 2.2. *Let $X, Y \in \mathbb{M}_n$ such that at least one of X or Y is Hermitian. Then,*

$$\omega_N(X \circ Y) \leq \omega_N(X)\omega_N(Y).$$

Proof. Suppose $X, Y \in \mathbb{M}_n$ such that Y is Hermitian. Then,

$$\begin{aligned}
\omega_N(X \circ Y)) &= \sup_\theta N(Re(e^{i\theta}(X \circ Y))) \\
&= \frac{1}{2}\sup_\theta N(e^{i\theta}(X \circ Y) + e^{-i\theta}(X \circ Y)^*) \\
&= \frac{1}{2}\sup_\theta N(e^{i\theta}X \circ Y + e^{-i\theta}X^* \circ Y) \\
&= \frac{1}{2}\sup_\theta N(\left(e^{i\theta}X + e^{-i\theta}X^*\right) \circ Y) \\
&= \sup_\theta N(Re\left(e^{i\theta}X\right) \circ Y) \\
&\leq \sup_\theta N(Re\left(e^{i\theta}X\right))N(Y) \\
&= \omega_N(X)\omega_N(Y).
\end{aligned} \tag{2.5}$$

□

The main theorem of this subsection is stated as follows.

Theorem 2.3. *Let $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$. Then, the following inequality holds:*

$$\omega_N(X \circ Y) \leq \sec(\alpha_1)\sec(\alpha_2)\omega_N(X)\omega_N(Y). \tag{2.6}$$

Proof. Since $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$, there exist two complex numbers, z and $w \in \mathbb{C}$, such that $|z| = |w| = 1$, where the numerical ranges of zX and wY are subsets of S_{α_1} and S_{α_2}, respectively. Therefore, by Lemma 2.3, the following two blocks are positive semidefinite:

$$\begin{pmatrix} \sec(\alpha_1)Re(zX) & zX \\ \bar{z}X^* & \sec(\alpha_1)Re(zX) \end{pmatrix},$$

$$\begin{pmatrix} \sec(\alpha_2)Re(wY) & wY \\ \overline{w}Y^* & \sec(\alpha_2)Re(wY) \end{pmatrix}.$$

Hence, their Hadamard product,

$$\begin{pmatrix} \sec(\alpha_1)\sec(\alpha_2)Re(zX) \circ Re(wY) & zw\,(X \circ Y) \\ \overline{zw}\,(X \circ Y)^* & \sec(\alpha_1)\sec(\alpha_2)Re(zX) \circ Re(wY) \end{pmatrix},$$

is also positive semidefinite. This implies that

$$\begin{aligned} N(X\circ Y) &= N(zw(X\circ Y)) \\ &\leq \sec(\alpha_1)\sec(\alpha_2)N(Re(zX)\circ Re(wY)). \end{aligned}$$

Therefore, we have the following

$$\begin{aligned} \omega_N(X\circ Y) &\leq N(X\circ Y) \\ &\leq \sec(\alpha_1)\sec(\alpha_2)N(Re(zX)\circ Re(wY)) \qquad (2.7) \\ &\leq \sec(\alpha_1)\sec(\alpha_2)N(Re(zX))\,N(Re(wY)) \quad \text{(by Lemma 2.4)} \\ &= \sec(\alpha_1)\sec(\alpha_2)\omega_N(Re(zX)\omega_N(Re(wY)) \\ &\text{(since both } Re(zX) \text{ and } Re(wY) \text{ are Hermitian)} \\ &\leq \sec(\alpha_1)\sec(\alpha_2)\omega_N(zX)\omega_N(wY) \quad \text{(by Proposition 2.1)} \\ &= \sec(\alpha_1)\sec(\alpha_2)|z|\omega_N(X)|w|\omega_N(Y) \\ &= \sec(\alpha_1)\sec(\alpha_2)\omega_N(X)\omega_N(Y). \end{aligned}$$

□

As a consequence, we have the following corollary.

Corollary 2.6. *If* $X, Y \in \mathbb{M}^s_{n,\alpha}$, *then*

$$\omega_N(X\circ Y) \leq \sec^2(\alpha)\omega_N(X)\omega_N(Y). \tag{2.8}$$

A straightforward modification of the arguments used in the proof of Theorem 2.3 yields the following Theorem.

Theorem 2.4. *Let* $X_j \in \mathbb{M}^s_{n,\alpha_j}$, $j = 1, 2, ..., m$. *Then,*

$$\omega_N(X_1\circ ...\circ X_m) \leq \left(\prod_{j=1}^{m}\sec(\alpha_j)\right)\omega_N(X_1)\omega_N(X_2)...\omega_N(X_m).$$

Consequently,

Corollary 2.7. *If* $X_1, ..., X_m \in \mathbb{M}_n$ *are accretive-dissipative, then*

$$\omega_N(X_1\circ ...\circ X_m) \leq 2^{m/2}\omega_N(X_1)\omega_N(X_2)...\omega_N(X_m).$$

2.3. Inequalities involving diagonal entries

We need the following lemma, which can be found in [12].

Lemma 2.5. *[12, Theorem 5.5.19]*
Let $X, Y \in \mathbb{M}_n$ *such that* $Y = [y_{ij}] > 0$. *Then,*

$$N(X\circ Y) \leq \max_i y_{ii}\, N(X).$$

Applying Lemma 2.5, we can establish the following result.

Lemma 2.6. *Let $X, Y \in \mathbb{M}_n$ such that $Y > 0$. Then,*

$$w_N(X \circ Y) \le \max_i y_{ii}\, w_N(X).$$

Proof. Observe the following:

$$\begin{aligned}
w_N(X \circ Y) &= \sup_\theta N(Re[e^{i\theta}(X \circ Y)]) \\
&= \sup_\theta N(Re[(e^{i\theta}X) \circ Y]) \\
&= \sup_\theta N\left(\frac{(e^{i\theta}X) \circ Y + ((e^{i\theta}X) \circ Y)^*}{2}\right) \\
&= \sup_\theta N\left(\frac{(e^{i\theta}X) \circ Y + (e^{i\theta}X)^* \circ Y^*}{2}\right) \\
&= \sup_\theta N\left(\frac{\left((e^{i\theta}X) + (e^{i\theta}X)^*\right) \circ Y}{2}\right) \\
&= \sup_\theta N\left(Re[e^{i\theta}X] \circ Y\right) \\
&\le \sup_\theta N(Re[e^{i\theta}X] \max_i y_{ii}) \text{ (by Lemma 2.5)} \\
&= \max_i y_{ii} \sup_\theta N(Re[e^{i\theta}X]) \\
&= \max_i y_{ii}\, w_N(X).
\end{aligned}$$

□

The result below estimates $\omega(X \circ Y)$ using the diagonal entries of the sectorial matrices X and Y.

Theorem 2.5. *Suppose $X = [x_{ij}] \in \mathbb{M}^s_{n,\alpha_1}$ and $Y = [y_{ij}] \in \mathbb{M}^s_{n,\alpha_2}$. Then,*

$$\omega_N(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2) \max_j |x_{jj}|\ \omega_N(Y)$$

and

$$\omega_N(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2) \max_j |y_{jj}|\ \omega_N(X).$$

Proof. By inequality (2.7), we have

$$\omega_N(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2)\omega_N(Re(zX) \circ Re(wY)),$$

where z and w are two complex numbers such that $|z| = |w| = 1$. Since $\mathrm{Re}(zX)$ is positive semidefinite, Lemma 2.6 implies that

$$\omega_N(\mathrm{Re}(zX) \circ Re(wY)) \le \max_j\ \mathrm{Re}(zx_{jj})\, \omega_N\left(\mathrm{Re}(wY)\right).$$

Now, we have the following:

$$
\begin{aligned}
\omega_N(X \circ Y) &\le \sec(\alpha_1)\sec(\alpha_2)\omega_N(\mathrm{Re}(zX) \circ \mathrm{Re}(wY)) \\
&\le \sec(\alpha_1)\sec(\alpha_2)\max_j\ \mathrm{Re}(zx_{jj})\,\omega_N\,(\mathrm{Re}(wY)) \\
&\le \sec(\alpha_1)\sec(\alpha_2)\max_j\ |zx_{jj}|\,\omega_N\,(wY) \\
&= \sec(\alpha_1)\sec(\alpha_2)\max_j\ |x_{jj}|\,\omega_N\,(Y)\,.
\end{aligned}
$$

Hence,

$$\omega_N(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2)\max_j\ |x_{jj}|\,\omega_N\,(Y)\,.$$

A similar argument implies the second inequality. □

Corollary 2.8. *Let $X = [x_{ij}] \in \mathbb{M}^s_{n,\alpha_1}$ and $Y = [y_{ij}] \in \mathbb{M}^s_{n,\alpha_2}$. Then,*

$$\omega_N(X \circ Y) \le \sec(\alpha_1)\sec(\alpha_2)\min\left\{\max_j |x_{jj}|\ \omega_N(Y), \max_j |y_{jj}|\ \omega_N(X)\right\}.$$

Corollary 2.9. *If $X = [x_{ij}], Y = [y_{ij}] \in \mathbb{M}_n$ are accretive-dissipative, then*

$$\omega_N(X \circ Y) \le 2\ \min\{\max_j |x_{jj}|\ \omega_N(Y), \max_j |y_{jj}|\ \omega(X)\}.$$

An alternative upper bound for $\omega(X \circ Y)$, where X and Y are sectorial, can be derived as follows.

Theorem 2.6. *Let $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$. Then,*

$$\omega_N(X \circ Y) \le \min\left\{(1 + \tan\alpha_1)\omega_N(Re\,X)\omega_N(Y), (1 + \tan\alpha_2)\omega_N(X)\omega_N(ReY)\right\}.$$

Consequently,

$$\omega_N(X \circ Y) \le (1 + \tan\alpha)\omega_N(X)\omega_N(Y),$$

where $\alpha = \max\{\alpha_1, \alpha_1\}$.

Proof. Since the second inequality follows from the first one, it is enough to prove the first inequality. Let $X = A + iB$ be the cartesian decomposition of X. Then,

$$
\begin{aligned}
\omega_N(X \circ Y) &= \omega_N((A + iB) \circ Y) \\
&= \omega_N(A \circ Y + iB \circ Y) \\
&\le \omega_N(A \circ Y) + \omega_N(B \circ Y) \\
&\le \omega_N(A)\omega(Y) + \omega_N(B)\omega_N(Y) \quad \text{(by Theorem 2.2)} \\
&\le \omega_N(A)\omega(Y) + \tan(\alpha_1)\,\omega_N(A)\omega(Y) \quad \text{(by Proposition 2.2)} \\
&= (1 + \tan\alpha_1)\omega_N(A)\omega_N(Y) \\
&= (1 + \tan\alpha_1)\omega_N(Re(X))\omega_N(Y).
\end{aligned}
$$

Hence,

$$\omega_N(X \circ Y) \le (1 + \tan\alpha_1)\omega_N(ReX)\omega(Y). \tag{2.9}$$

Similarly, one can show that

$$\omega_N(X \circ Y) \le (1 + \tan\alpha_2)\omega_N(ReY)\omega(X). \tag{2.10}$$

The result follows by combining (2.9) and (2.10). □

Corollary 2.10. *Let* $X, Y \in \mathbb{M}^s_{n,\alpha}$. *Then,*

$$\omega_N(X \circ Y) \leq (1 + \tan\alpha) \min\{(\omega_N(Re\, X)\omega_N(Y), \omega_N(X)\omega_N(ReY)\}.$$

Consequently,

$$\omega_N(X \circ Y) \leq (1 + \tan\alpha)\omega_N(X)\omega_N(Y). \tag{2.11}$$

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that he has no conflict of interest.

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