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Letter

An example showing that Schrijver's ϑ-function need not upper bound the Shannon capacity of a graph

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Abstract: This letter addresses an open question concerning a variant of the Lovász ϑ function, which was introduced by Schrijver and independently by McEliece et al. (1978). The question of whether this variant provides an upper bound on the Shannon capacity of a graph was explicitly stated by Bi and Tang (2019). This letter presents an explicit example of a graph on 32 vertices, which shows that, in contrast to the Lovász ϑ function, this variant does not necessarily upper bound the Shannon capacity of a graph. The example, previously outlined by the author in a recent paper (2024), is presented here in full detail, making it easy to follow and verify. By resolving this question, the note clarifies a subtle but significant distinction between these two closely related graph invariants.

Keywords: graph invariants; Lovász ϑ-function; Schrijver's ϑ-function; Shannon capacity of graphs; independence number; semidefinite programming
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1. Introduction

The concept of the Shannon capacity of a graph, as introduced in Shannon's seminal paper (1956) on zero-error communication [1], plays a key role in understanding the synergy and interaction between zero-error information theory and graph theory. The zero-error capacity of a discrete memoryless channel (DMC) equals the Shannon capacity of the corresponding confusability graph, whose vertices represent the input symbols of the channel, and any pair of its vertices are adjacent if they represent input symbols that can be confused by the channel (i.e., conditioned on each of these two input symbols, an identical output symbol can be received with some positive probabilities). The significance of the Shannon capacity of graphs, and the hardness of its computability in general, are highlighted in various survey papers [2–5].

The aim of this note is to present an example demonstrating that, in contrast to the Lovász ϑ-function, which upper bounds the Shannon capacity of a graph [6], the variant introduced by Schrijver does not [7, Example 5.24]. This resolves a query concerning the variant proposed by Schrijver, which is identical to the function independently presented by McEliece *et al.* (1978) [8,9], and that was posed as an open question by Bi and Tang [10]. They wrote in [10]: “In fact, it is quite difficult to disprove that the Schrijver ϑ-function is an upper bound, because at least for graphs G of moderate size the two values $\vartheta(\mathsf{G})$ (the Lovász ϑ-function of G) and $\vartheta'(\mathsf{G})$ (the Schrijver ϑ-function of G) are very close to each other”. Such an example is presented in Section 3, following a brief overview of preliminary material in Section 2.

2. Preliminaries

In the following, $\alpha(\mathsf{G})$, $\vartheta(\mathsf{G})$, $\vartheta'(\mathsf{G})$, and $\Theta(\mathsf{G})$ respectively denote the independence number, Lovász ϑ-function, Schrijver's ϑ-function, and the Shannon capacity of a simple, finite, and undirected graph G. We refer the reader to our recent paper (see [7, Section 2]) for preliminaries, definitions, and an account of the properties of Lovász and Schrijver's ϑ-functions of a graph, as well as the Shannon capacity of a graph. In order to introduce the Shannon capacity of a graph, we need the notion of a *strong product* of graphs.

Definition 1. (Strong products and strong powers of graphs) Let G and H be graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with a vertex set $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$ (the Cartesian product), and distinct vertices (g, h) and (g', h') are adjacent in $\mathsf{G} \boxtimes \mathsf{H}$ if and only if one of the following three conditions holds:
(1) $g = g'$ and $\{h, h'\} \in \mathsf{E}(\mathsf{H})$, (2) $\{g, g'\} \in \mathsf{E}(\mathsf{G})$ and $h = h'$, or (3) $\{g, g'\} \in \mathsf{E}(\mathsf{G})$ and $\{h, h'\} \in \mathsf{E}(\mathsf{H})$.
Strong products are therefore commutative and associative up to graph isomorphisms. The *k-fold strong power* of G is given by $\mathsf{G}^{\boxtimes k} \triangleq \mathsf{G} \boxtimes \cdots \boxtimes \mathsf{G}$, where G is multiplied by itself k times.

The following three results are used in this note.

Theorem 1. [1, 6] For every simple graph G,

$$\Theta(\mathsf{G}) \triangleq \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} \tag{2.1}$$

$$= \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}, \tag{2.2}$$

and

$$\alpha(\mathsf{G}) \leq \Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.3}$$

Theorem 2. [6, 9] For every graph G,

$$\alpha(\mathsf{G}) \leq \vartheta'(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.4}$$

Theorem 3. [6] Let G be d-regular of order n, and let $\lambda_n(\mathsf{G})$ be the smallest eigenvalue of its adjacency matrix. Then,

$$\vartheta(\mathsf{G}) \leq -\frac{n\,\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}, \tag{2.5}$$

with equality if G is an edge-transitive graph.

The interested reader is also referred to a recent survey paper [11] on upper bounds on the independence number of a graph, including the Lovász and Schrijver's ϑ functions (see (2.4)).

In light of Theorems 1 and 2, the following question was left open since [9], and recently resolved in [7, Example 5.24].

Question 1. Could the bound on the Shannon capacity, $\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G})$, be improved by the bound $\Theta(\mathsf{G}) \leq \vartheta'(\mathsf{G})$?

We give a negative answer to Question 1 by providing a complete and detailed presentation of the example in [7, Example 5.24]. The main result in this note is thus an explicit and detailed example demonstrating that

$$\Theta(\mathsf{G}) \nleq \vartheta'(\mathsf{G}). \tag{2.6}$$

3. Example

Let G be the Gilbert graph on 32 vertices, where

$$\mathsf{V}(\mathsf{G}) = \{0,1\}^5, \qquad \mathsf{E}(\mathsf{G}) = \Big\{\underline{u}, \underline{v} \in \{0,1\}^5 : \; 1 \leq d_{\mathrm{H}}(\underline{u}, \underline{v}) \leq 2\Big\}, \tag{3.1}$$

so, every two vertices are adjacent if and only if the Hamming distance between their corresponding binary 5-length tuples is either 1 or 2. Label each vertex in $\{0,1\}^5$ by its corresponding decimal value in $\{0, 1, \ldots, 31\}$, and let the i-th row and column of the adjacency matrix $\mathbf{A}$ correspond to the vertex labeled $i-1$, for each $i \in [32]$. Then, the following holds:

- The graph G is 15-regular, since $\binom{5}{1} + \binom{5}{2} = 5 + 10 = 15$.
- To motivate why this graph is a suitable candidate for such an example, it is instructive to observe the following structural properties:
  1. G is vertex-transitive, edge-transitive, and distance-regular, but not strongly regular.
  2. The complement graph $\overline{\mathsf{G}}$ is 16-regular and vertex-transitive, but it is not edge-transitive nor distance-regular (and thus not strongly regular).

  These properties, which can be observed using the SageMath software [12], are not essential for verifying the subsequent steps of the example. However, if the complement graph $\overline{\mathsf{G}}$ were edge-transitive in addition to being vertex-transitive, or if it were strongly regular, then by combining Theorem 5.9 and Eq (2.43) of [7] with Proposition 1 of [13], it would follow that $\vartheta(\mathsf{G}) = \vartheta'(\mathsf{G})$, and hence G could not serve as a counterexample to Question 1 (since $\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G})$ by Theorem 1 in [6]).
- The independence number of G is $\alpha(\mathsf{G}) = 4$. An example of such a maximal independent set of G:

$$\{(1,0,0,1,0),\ (0,1,1,1,0),\ (0,0,0,0,1),\ (1,1,1,0,1)\}. \tag{3.2}$$

- Solving the following minimization problem for $\vartheta'(\mathsf{G})$ [9] (see also [11]):

$$\boxed{\begin{array}{l} \text{minimize} \;\; \lambda_{\max}(\mathbf{X}) \\ \text{subject to} \\ \begin{cases} \mathbf{X} \in \mathcal{S}_{32} \\ A_{i,j} = 0 \;\Rightarrow\; X_{i,j} \geq 1, \quad \forall\, i,j \in \{1,\ldots,32\} \end{cases} \end{array}} \tag{3.3}$$

where $\mathcal{S}_{32}$ denotes the set of all the 32×32 real symmetric matrices, is a dual semidefinite programming (SDP) problem with strong duality. We aim to show that $\vartheta'(\mathsf{G}) = 4 = \alpha(\mathsf{G})$, relying on the CVX software [14].

- To that end, consider the feasible solution of (3.3)

$$\mathbf{X} = [\mathbf{X}_\mathrm{l},\ \mathbf{X}_\mathrm{r}],$$

where $\mathbf{X}_\mathrm{l}$ and $\mathbf{X}_\mathrm{r}$ are given by the following 32×16 submatrices:

$$\mathbf{X}_\mathrm{l} = \begin{pmatrix}
+1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 \\
-1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 \\
-1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
-1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 \\
-1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\
-1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \\
-1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
-1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
+1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 \\
+1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \\
-1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +\mathbf{3} \\
-1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & +1 \\
-1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 \\
+1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +\mathbf{3} & +1 & +1 & +1 \\
-1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & -1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & +1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +\mathbf{3} & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
-1 & +1 & +1 & +1 & +1 & +1 & +1 & +\mathbf{3} & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & +1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 \\
+1 & +1 & +1 & -1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
+1 & +1 & +1 & +\mathbf{3} & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 \\
+1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\
+1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\
+\mathbf{3} & +1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1
\end{pmatrix}, \tag{3.4}$$

and

$$
\mathbf{X}_{\mathrm{r}} = \begin{pmatrix}
-1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +\mathbf{3} \\
-1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & +1 \\
-1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 \\
+1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +\mathbf{3} & +1 & +1 & +1 \\
-1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & -1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & +1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +\mathbf{3} & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
-1 & +1 & +1 & +1 & +1 & +1 & +1 & +\mathbf{3} & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & +1 & +\mathbf{3} & +1 & -1 & -1 & 1 & -1 & +1 & -1 & +1 & +1 \\
+1 & +1 & -1 & +1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & 1 \\
+1 & +1 & +1 & -1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
+1 & +1 & +1 & +\mathbf{3} & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & 1 \\
+1 & +1 & +\mathbf{3} & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\
+1 & +\mathbf{3} & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\
+\mathbf{3} & +1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 \\
-1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 \\
-1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\
-1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 \\
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-1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
-1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 \\
+1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & -1 & -1 \\
+1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}, \tag{3.5}
$$

which implies that the largest eigenvalue of the 32×32 matrix $\mathbf{X} = [\mathbf{X}_{\mathrm{l}},\ \mathbf{X}_{\mathrm{r}}]$ is equal to

$$\lambda_{\max}(\mathbf{X}) = 4. \tag{3.6}$$

- Combining the left-hand side of (2.4) with (3.2), (3.3), and (3.6) yields

$$4 = \alpha(\mathsf{G}) \leq \vartheta'(\mathsf{G}) \leq \lambda_{\max}(\mathbf{X}) = 4,$$

which gives

$$\vartheta'(\mathsf{G}) = 4 = \alpha(\mathsf{G}). \tag{3.7}$$

Note that all the entries of the matrix $\mathbf{X} = [\mathbf{X}_\mathrm{l},\ \mathbf{X}_\mathrm{r}]$, as given in (3.4) and (3.5), take the values $3, 1, -1$. Furthermore, $X_{i,j} = 3$ if and only if $i + j = 33$ and $i, j \in [32]$ (i.e., the bolded entries in (3.4) and (3.5) take the value 3, and they form the antidiagonal of $\mathbf{X}$).

- Although not needed directly for establishing our example, the graph G is 15-regular and edge-transitive on 32 vertices, whose smallest eigenvalue of its adjacency matrix equals $\lambda_{\min}(\mathsf{G}) = -3$. Hence, by Theorem 3, the Lovász ϑ-function of G is given by

$$\vartheta(\mathsf{G}) = -\frac{n\lambda_{\min}(\mathsf{G})}{d(\mathsf{G}) - \lambda_{\min}(\mathsf{G})} = \frac{32 \cdot 3}{15 + 3} = 5\frac{1}{3}. \tag{3.8}$$

This can also be verified numerically by solving the dual SDP problem

$$\boxed{\begin{array}{l} \text{minimize } \ \lambda_{\max}(\mathbf{X}) \\ \text{subject to} \\ \begin{cases} \mathbf{X} \in \mathcal{S}_{32} \\ A_{i,j} = 0 \ \Rightarrow \ X_{i,j} = 1, \quad \forall\, i, j \in \{1, \dots, 32\}, \end{cases} \end{array}} \tag{3.9}$$

differing from the dual SDP problem in (3.3) in the stronger requirement that, whenever $A_{i,j} = 0$, the entry $X_{i,j}$ is restricted to be equal to 1, rather than imposing the weaker condition that $X_{i,j} \geq 1$. It is interesting to note the following relation: let $\mathbf{X}$ and $\hat{\mathbf{X}}$ be the matrices that solve the SDP problems in (3.3) and (3.9), respectively. Then, the following holds:

(1) If $X_{i,j} = -1$, then $\hat{X}_{i,j} = -\frac{7}{9}$;

(2) If $X_{i,j} \in \{1, 3\}$, then $\hat{X}_{i,j} = 1$;

(3) All entries in the antidiagonal of $\mathbf{X}$ are equal to 3, and no other entry in $\mathbf{X}$ is equal to 3.

This transforms the optimized matrix $\mathbf{X}$ in (3.3) to the one in (3.9) and vice versa. Note that combining with (3.7), we have for the graph G,

$$4 = \alpha(\mathsf{G}) = \vartheta'(\mathsf{G}) < \vartheta(\mathsf{G}) = 5\frac{1}{3}, \tag{3.10}$$

so $\vartheta'(\mathsf{G})$ coincides with the independence number of G, and it is strictly smaller than $\vartheta(\mathsf{G})$.

- By the SageMath software, we have $\alpha(\mathsf{G} \boxtimes \mathsf{G}) = 20$, and the strong product graph $\mathsf{G} \boxtimes \mathsf{G}$ has 368,640 such maximal independent sets of size 20. In fact, it is enough for our purpose to establish the weaker result where

$$\alpha(\mathsf{G} \boxtimes \mathsf{G}) \geq 20. \tag{3.11}$$

To that end, an example of a (maximal) independent set of size 20 for the strong product of G by itself, $\mathsf{G} \boxtimes \mathsf{G}$, is given by

$$\Big\{((1,1,0,0,0),(1,1,1,1,1)), \quad ((1,0,1,0,0),(1,1,0,0,0)), \quad ((0,1,1,0,0),(0,0,1,1,0)),$$
$$((1,1,1,0,0),(0,0,0,0,1)), \quad ((1,0,0,1,0),(0,0,1,0,1)), \quad ((0,1,0,1,0),(1,0,0,0,0)),$$
$$((1,1,0,1,0),(0,1,0,1,0)), \quad ((0,0,1,1,0),(0,1,0,1,1)), \quad ((1,0,1,1,0),(1,0,1,1,0)),$$
$$((0,1,1,1,0),(1,1,1,0,1)), \quad ((1,0,0,0,1),(0,0,0,1,0)), \quad ((0,1,0,0,1),(0,1,0,0,1)),$$
$$((1,1,0,0,1),(1,0,1,0,0)), \quad ((0,0,1,0,1),(1,0,1,0,1)), \quad ((1,0,1,0,1),(0,1,1,1,1)),$$
$$((0,1,1,0,1),(1,1,0,1,0)), \quad ((0,0,0,1,1),(1,1,1,1,0)), \quad ((1,0,0,1,1),(1,1,0,0,1)),$$
$$((0,1,0,1,1),(0,0,1,1,1)), \quad ((0,0,1,1,1),(0,0,0,0,0))\Big\}. \tag{3.12}$$

- Consequently, it follows from (2.1), (3.7), and (3.11) that

$$\Theta(\mathsf{G}) \geq \sqrt{\alpha(\mathsf{G} \boxtimes \mathsf{G})} \geq \sqrt{20} > 4 = \vartheta'(\mathsf{G}), \tag{3.13}$$

demonstrating that $\vartheta'(\mathsf{G})$ does not serve as a universal upper bound on the Shannon capacity $\Theta(\mathsf{G})$, in contrast to the Lovász ϑ-function. On the other hand, $\vartheta'(\mathsf{G})$ yields an improved upper bound on the independence number $\alpha(\mathsf{G})$ than $\vartheta(\mathsf{G})$, and in fact, $\vartheta'(\mathsf{G})$ provides a tight upper bound for the considered graph G in (3.1) (see (3.10)).

Use of AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were utilized in creating this article.

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Conflict of interest

The author declares no conflicts of interest. He serves as the Guest Editor of the special issue "Mathematical Foundations of Information Theory" for *AIMS Mathematics*, but was not involved in the editorial review and the decision to publish this article.

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