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Research article

The time-dependent attractor for beam equation with rotational inertia and structural damping

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Abstract: In this paper, we discussed the asymptotic behavior of the solutions to the beam equation with rotational inertia and structural damping:

$$\varepsilon(t)(1+(-\Delta)^{\alpha})\partial_t^2 u+\Delta^2 u+\gamma(-\Delta)^{\theta}\partial_t u+f(u)=g(x),$$

where $\varepsilon(t)$ was a decreasing bounded function. We found a more optimized subcritical exponent $p^*=\frac{N+2\theta}{N-4}$ depending on θ. Additionally, we showed that when the growth exponent p of nonlinear terms $f(u)$ was within the range $1\leqslant p<p^*$, the well-posedness and regularity were established. Finally, within the theory of process on time-dependent spaces, we investigated the existence of the time-dependent attractor by using the contraction function method and more detailed estimates in the time-dependent space $\mathcal{H}_t^{\alpha}$. The results refined and extended the model and the work in literature [Longtime behavior for an extensible beam equation with rotational inertia and structural nonlinear damping, *J. Math. Anal. Appl.*, **496** (2021), 124785.] from general energy space to time-dependent space in some sense.

Keywords: rotational inertia; regularity; time-dependent global attractors; structural damping
Mathematics Subject Classification: 35B40, 35B41, 37L15, 37L30

1. Introduction

In this article, we are concerned with the long-time dynamics of a beam equation with rotational inertia and structural damping

$$\varepsilon(t)(1+(-\Delta)^{\alpha})\partial_t^2 u+\Delta^2 u+\gamma(-\Delta)^{\theta}\partial_t u+f(u)=g(x),\quad (x,t)\in\Omega\times[\tau,+\infty), \tag{1.1}$$

$$u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0,\qquad u(x,\tau)=u_0(x),\qquad \partial_t u(x,\tau)=u_1(x),\qquad x\in\Omega, \tag{1.2}$$

where $\gamma>0$, $\theta\in(\frac{1}{2},\frac{2}{3})$ is a dissipation index, $\alpha\in[0,4\theta-2]$ is a rotational inertia index, and Ω is a bounded domain of $\mathbb{R}^N(N\geqslant 5)$ with smooth boundary $\partial\Omega$.

We presume that the time-dependent coefficient ε and the nonlinear functions f satisfy the following conditions.

Assumptions:

(i) $\varepsilon \in C^1(\mathbb{R})$ is a decreasing bounded function and satisfies

$$\lim_{t\to+\infty} \varepsilon(t) = 1. \tag{1.3}$$

In particular, there exists a constant $L > 0$, such that

$$\sup_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \leqslant L. \tag{1.4}$$

(ii) $f \in C^1(\mathbb{R}),\ f(0) = 0$, and for any $s \in \mathbb{R}$, f satisfies the dissipative condition

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} \geqslant -\lambda_1, \tag{1.5}$$

and the growth condition

$$|f'(s)| \leqslant C(1 + |s|^{p-1}), \qquad 1 \leqslant p < p^* = \frac{N + 2\theta}{N - 4}(N \geqslant 5), \tag{1.6}$$

where $C > 0$ and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$.

(iii) $g \in L^2(\Omega)$, $(u_0, u_1) \in \mathcal{H}_\tau^\alpha$ satisfies $\|(u_0, u_1)\|_{\mathcal{H}_\tau^\alpha} \leqslant R$, where R is a positive constant (the definition of function space $\mathcal{H}_\tau^\alpha$ can be found on page 5, line 7).

Remark 1.1. *Formula (1.5) implies that there is a positive constant* β_0, *for* $\frac{1}{2} < \beta_0 < 1$, *such that*

$$\langle F(u), 1\rangle \geqslant \frac{1 - \beta_0}{2}\|u\|^2 - C_{\beta_0},$$
$$\langle f(u), u\rangle \geqslant -(1 - \beta_0)\|u\|^2 - C_{\beta_0}, \quad \forall u \in L^2,$$

where $F(s) = \int_0^s f(r)\mathrm{d}r$ *and* C_{β_0} *is a positive constant (the definition of* $\langle ., .\rangle$ *can be found on page 4, line 24).*

When the coefficient ε in the Eq (1.1) is equivalent to a constant, there are many results for the beam equation with rotational inertia. Neglecting the rotational inertia and taking $\alpha = 0$ in Eq (1.1), Silva and Narciso [1] studied the well-posedness and longtime behavior of solutions for the extensible beam equation with structural nonlinear damping

$$\partial_t^2 u + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + N(\|\nabla u\|^2)(-\Delta)^\beta \partial_t u + f(u) = g(x), \quad x \in \Omega, t > 0, \tag{1.7}$$

with $0 \leq \beta \leq 1$. They obtained the well-posedness of the weak and strong solutions, respectively, and by using an approach on quasi-stable dynamical systems, they established the existence of global attractor and generalized exponential attractor provided that the nonlinearity $f(u)$ is of more optimised subcritical growth: $1 \leqslant p < \infty$ if $N \leqslant 4$; $1 \leqslant p \leqslant p^* := \frac{N}{N-4}$ if $N \geqslant 5$. For the research achievements of the related extensible beam models with $\alpha = 0$ in (1.1), one can refer to literatures [2–4].

When $\alpha > 0$, Chueshov and Lasiecka [5] proposed the following Berger extensible beam/plate model with rotational forces

$$(1-\omega\Delta)\partial_t^2 u + \Delta^2 u - \gamma\Delta\partial_t u + (Q + \|\nabla u\|^2)\Delta u = p(u, u_t), \tag{1.8}$$

where the parameter $\omega \geqslant 0$ represents rotational forces, the parameter Q describes in-plane forces applied to the plate, and the function p represents transverse loads which may depend on the displacement u and the velocity $\partial_t u$. They studied the well-posedness and longtime behavior of model (1.8).

Recently, Niimura [6] investigated the following model with nonlocal structural damping:

$$(1-\alpha\Delta)\partial_t^2 u + \Delta^2 u - \phi(\|\nabla u\|^2)\Delta u - M(\|\nabla u\|^2)\Delta\partial_t u + f(u) = h, \tag{1.9}$$

with the parameter $\alpha \in [0, 1]$. When the nonlinearity $f(u)$ is of subcritical growth:

$$|f'(s)| \leqslant C(1 + |s|^{p-1}),\ \textit{with}\ 1 \leqslant p < \frac{N}{(N-4)^+},$$

the author obtained the well-posedness of global solutions, as well as established the existence of a global attractor and an exponential attractor for each $\alpha \in [0, 1]$. At the same time, the family continuity of the global and exponential attractors at the perturbed parameter $\alpha = 0$ were also achieved.

Ding and Yang [7] proposed the following extensible beam model with structural nonlinear damping:

$$(1 + (-\Delta)^\theta)\partial_t^2 u + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + N(\|\nabla u\|^2)(-\Delta)^\alpha\partial_t u + f(u) = g(x), \tag{1.10}$$

where θ is called a rotational index and α is called a dissipative index, with $0 \leqslant \theta < \alpha \leqslant 1$. They found an optimal critical exponent $p_\theta^\alpha \equiv \frac{N+2(2\alpha-\theta)}{(N-4)^+}$ and showed that the model (1.10) has a strong global attractor and a strong exponential attractor for each $\theta \in [0, \alpha]$, when the nonlinearity $f(u)$ satisfies the optimal subcritical growth: $1 \leqslant p < p_\theta^\alpha$.

Sun and Yang [8] considered the more complex model with energy damping:

$$(1-\alpha\Delta)\partial_t^2 u + \Delta^2 u - \phi(\|\nabla u\|^2)\Delta u - M(\|\xi_u\|_H^2)\Delta\partial_t u + f(u) = h,\ \ \Omega\times(0,\infty), \tag{1.11}$$

where $\|\xi_u\|_H^2 = \|(u, \partial_t u)\|_H^2 = \|\Delta u\|^2 + \|\partial_t u\|^2$ is the energy norm. The author found the more optimised subcritical exponent $p^* = \frac{N+2}{N-4}$, with $N \geqslant 5$ and showed that the equation is well-posed when the growth exponent p of the nonlinearity $f(u)$ is up to the range: $1 \leqslant p \leqslant p^*$. When $1 \leqslant p < p^*$, the corresponding dynamical system possesses a global attractor and an exponential attractor for each $\alpha \in [0, 1]$. Meanwhile, the attractors are continuous with respect to $\alpha \in [0, 1]$.

If Eq (1.1) contains the time dependent coefficient $\varepsilon(t)$, and $\varepsilon(t)$ is a bounded monotone decreasing function when it approaches zero at infinity, then problems (1.1) and (1.2) will become more complex. The energy functional of the dynamical system dependent on the time t and the dissipation of $\varepsilon(\cdot)$ limits the existence of absorbing sets in the general sense. Some classical theories (global attractors or uniform attractors) and conventional methods are not suitable for solving such problems. To overcome this difficulty, Conti et al. [9–11] proposed the concept of time-dependent attractor and established the theory of time-dependent attractor. With the above theory, we found that the problems (1.1) and (1.2) can be studied.

However, as far as we know, when the parameter $\theta \in (\frac{1}{2}, \frac{2}{3})$ and $\alpha \in [0, 4\theta - 2]$, the long-term dynamic behavior of the solution for the beam equation with rotational inertia and structural damping in the time dependent spaces has not been studied. The structural damping term, nonlinear term and rotational inertia term in the equation will bring some essential difficulties to the dissipative estimation of the solution, the existence of the bounded absorbing set, and the verification of the asymptotic compactness of the solution process. We will deal with problems (1.1) and (1.2) by adopting the ideal of [12–14] and applying to the theory framework of [7,15,16]. By use of the asymptotic regular estimation technique, the contraction function method, and the time-dependent attractor theory, we prove the well-posedness and regularity of the solutions of the problems (1.1) and (1.2), and establish Lipschitz-continuity of the solution. Furthermore, the asymptotic compactness of the process is verified. Finally, when the growth exponent of the nonlinear terms satisfies $1 \leqslant p < p^* = \frac{N+2\theta}{N-4}$, with $N \geqslant 5$, we show the existence of the time dependent attractor of problems (1.1) and (1.2) in the time-dependent space $\mathcal{H}_t^\alpha$.

This paper is organized as follows. In Section 2, we recall some preliminaries and abstract results. In Section 3, we discuss the well-posedness and the regularity of weak solutions. Finally, in Section 4, we establish the existence of the time dependent attractor.

2. Notations and preliminary results

In this section, we introduce some function spaces which will be used throughout this paper:

$$L^p = L^p(\Omega), \quad W^{m,p} = W^{m,p}(\Omega), \quad H^m = W^{m,2},$$

$$V_1 = H_0^1, \quad V_2 = H^2 \cap H_0^1, \quad \|\cdot\|_{V_2} = \|\cdot\|_2,$$

with $p \geqslant 1$. We denote the norm and the inner product by $\|\cdot\|_{L^2}$ and $\langle\cdot,\cdot\rangle$ in $L^2(\Omega)$, and C_i or $C(\cdot,\cdot)$ are positive constants depending on the quantities appearing in the parenthesis. Let $A = -\Delta$ with domain $D(A) = H_0^1 \cap H^2$. For $s \in \mathbb{R}$, we define the Hilbert spaces

$$V_s = D(A^{\frac{s}{2}}), \quad \langle u, v\rangle_s = \langle A^{\frac{s}{2}}u,\ A^{\frac{s}{2}}v\rangle, \quad \|u\|_s = \|A^{\frac{s}{2}}u\|, \quad u, v \in V_s.$$

Applying Sobolev embedding theorem, we can obtain the compact embedding

$$V_{s_1} \hookrightarrow\hookrightarrow V_{s_2}, \text{ as } s_1 > s_2, \tag{2.1}$$

and the continuous embedding

$$V_s \hookrightarrow L^{\frac{2N}{N-2s}}, \quad s > 0. \tag{2.2}$$

At the same time, the Poincaré inequality that we will widely use

$$\lambda_1^s \int_\Omega |v|^2 \mathrm{d}x \leqslant \int_\Omega |A^{\frac{s}{2}}v|^2 \mathrm{d}x, \quad \forall\, v \in V_s \tag{2.3}$$

also holds. Therefore, the problems (1.1) and (1.2) can be written in the following form:

$$\varepsilon(t)(1 + A^\alpha)\partial_t^2 u + A^2 u + \gamma A^\theta \partial_t u + f(u) = g, \quad t > \tau, \tag{2.4}$$

$$u(\tau) = u_0,\ \partial_t u(\tau) = u_1. \tag{2.5}$$

Define a family of Hilbert spaces $\mathcal{H}_t^{2+\alpha}$ as follows:

$$\mathcal{H}_t^{2+\alpha} = V_4 \times V_{2+\alpha},$$

and the norm in this family of spaces is defined by the formula

$$\|(u, \partial_t u)\|^2_{\mathcal{H}_t^{2+\alpha}} = \|u\|_4^2 + \varepsilon(t)\|\partial_t u\|^2_{2+\alpha}. \tag{2.6}$$

Especially for the family of Hilbert spaces, $\mathcal{H}_t^{\alpha}$ can be defined as

$$\mathcal{H}_t^{\alpha} = V_2 \times V_\alpha, \tag{2.7}$$

and the norm in this space is defined by

$$\|(u, \partial_t u)\|^2_{\mathcal{H}_t^{\alpha}} = \|u\|_2^2 + \varepsilon(t)\|\partial_t u\|^2_{\alpha}. \tag{2.8}$$

Moreover, when $\alpha > 0$,

$$\mathcal{H}_t^{2+\alpha} \hookrightarrow\hookrightarrow \mathcal{H}_t^{\alpha}.$$

We introduce some abstract results to study the long-time dynamical behavior of solutions in time-dependent spaces.

Definition 2.1. [11] *Let X_t be a family of normed spaces. A two-parameter family of operators $\{U(t,\tau) : X_\tau \to X_t, \tau \leqslant t, \tau \in \mathbb{R}\}$ is said to be a process, if for any $\tau \in \mathbb{R}$,*

(i) $U(\tau,\tau) = \mathrm{Id}$ *is the identity operator on* X_τ;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall \tau \leqslant s \leqslant t$.

Definition 2.2. [11] *A family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded, if there exists a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$, $\forall\, t \in \mathbb{R}$.*

For every $t \in \mathbb{R}$, the R-ball of X_t is defined by:

$$\mathbb{B}_t(R) = \{z \in X_t \,|\, \|z\|_{X_t} \leqslant R\}.$$

Definition 2.3. [11] *A uniformly bounded family $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ is called a time-dependent absorbing set for the process $U(t,\tau)$, if for any $R > 0$, there exist a $t_0 = t_0(R) \leqslant t$ and $R_0 > 0$ such that*

$$\tau \leqslant t - t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset \mathbb{B}_t(R_0).$$

The process $U(t,\tau)$ is said to be dissipative if it possesses a time-dependent absorbing set.

Lemma 2.1. [17] *Let x_n be a bounded sequence, and also let $\psi \in C(\mathbb{R})$ be a monotone function. Then,*

$$\psi(\liminf_{n\to\infty} x_n) \leqslant \liminf_{n\to\infty} \psi(x_n).$$

Lemma 2.2. [18] *Let X, B and Y be three Banach spaces. For any $T > 0$, if $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$, and*

$$\mathrm{W} = \{u \in L^p([0,T];X) | \partial_t u \in L^r([0,T];Y)\},\ with\ \ r > 1,\ 1 \leqslant p < \infty,$$

$$\mathrm{W}_1 = \{u \in L^\infty([0,T];X) | \partial_t u \in L^r([0,T];Y)\},\ with\ \ r > 1,$$

then,

$$W \hookrightarrow\hookrightarrow L^p([0,T];B),\ W_1 \hookrightarrow\hookrightarrow C([0,T];B).$$

Theorem 2.1. [11] *If $U(t,\tau)$ is asymptotically compact, that is, the set*

$$\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} |\ Each\ K_t\ is\ compact\ in\ X_t,\ \mathfrak{K}\ is\ attracting\}$$

is not empty, then the time-dependent attractor $\mathfrak{A}$ exists and coincides with $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. In particular, it is unique.

Definition 2.4. [19] *Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\{C_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subsets of $\{X_t\}_{t\in\mathbb{R}}$. We call a function $\Phi_\tau^t(\cdot,\cdot)$ defined on $X_t \times X_t$ a contractive function on $C_\tau \times C_\tau$, if for any fixed $t \in \mathbb{R}$ and any sequence $\{x_n\}_{n=1}^\infty \subset C_\tau$, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\Phi_\tau^t(x_{n_k}, x_{n_l}) = 0,$$

where $\tau \leqslant t$.

Theorem 2.2. [19] *Let $U(\cdot,\cdot)$ be a process on $\{X_t\}_{t\in\mathbb{R}}$. Assume that $U(\cdot,\cdot)$ possesses a time-dependent absorbing set $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$. If for any $\varepsilon > 0$ there exists a subsequence $T(\varepsilon) \leqslant t$, $\Phi_T^t \in \mathfrak{C}(\mathbb{B}_T(R))$ such that*

$$\|U(t,T)x - U(t,T)y\| \leqslant \varepsilon + \Phi_T^t(x,\ y),\ \forall x,\ y \in \mathbb{B}_T(R),$$

for any fixed $t \in \mathbb{R}$, then $U(\cdot,\cdot)$ is asymptotically compact.

Theorem 2.3. [17] *Let $U(\cdot,\cdot)$ be a process in a family of Banach spaces. Then, $U(\cdot,\cdot)$ has a time-dependent global attractor $\mathfrak{A}^* = \{A_t^*\}_{t\in\mathbb{R}}$ satisfying*

$$A_t^* = \bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s} U(t,\tau)B_\tau(R)},$$

if, and only if,

(i) *$U(\cdot,\cdot)$ has a time-dependent absorbing set family $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$;*

(ii) *$U(\cdot,\cdot)$ is asymptotically compact.*

Definition 2.5. [17, 20, 21] *A time-dependent attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant, if for all $\tau \leqslant t$,*

$$U(t,\tau)A_\tau = A_t.$$

3. Well-posedness and regularity of solutions

First, we define the solution of the problems (2.4) and (2.5) as follows.

Definition 3.1. *A binary* $y = (u, \partial_t u)$ *is said to be a weak solution of the problems* (2.4) *and* (2.5) *on an interval* $[\tau, T]$, *for* $\tau \in \mathbb{R}$, *if*

$$u \in L^\infty([\tau, T]; V_2) \cap L^2([\tau, T]; V_{4-3\theta}), \quad \partial_t u \in L^\infty([\tau, T]; V_\alpha) \cap L^2([\tau, T]; V_\theta),$$

and satisfies

$$\begin{aligned}&\langle \varepsilon(t)(1 + A^\alpha)\partial_t^2 u, \omega\rangle + \langle u, \omega\rangle_2 + \gamma\langle \partial_t u, \omega\rangle_\theta + \langle f(u), \omega\rangle = \langle g, \omega\rangle, \quad t > \tau,\\ &u(\tau) = u_0, \ \partial_t u(\tau) = u_1,\end{aligned}$$

for all $\tau \leqslant t$ *and any* $\omega \in V_2$.

Theorem 3.1. *If assumptions* (i)-(iii) *hold, then for every* $T > \tau$, $\theta \in (\frac{1}{2}, \frac{2}{3})$, $\alpha \in [0, 4\theta - 2]$, *there exists a unique weak solution* $y = (u, \partial_t u)$ *of the problems* (2.4) *and* (2.5) *with* $(u, \partial_t u) \in C([\tau, T]; \mathcal{H}_t^\alpha) \cap L^2([\tau, T]; V_{4-3\theta} \times V_\theta)$, $\partial_t^2 u \in L^\infty([\tau, T]; V_{\alpha-4\theta}) \cap L^2([\tau, T]; V_{\alpha-3\theta})$, *and*

$$\begin{aligned}&\|u(t)\|_2^2 + \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 + \varepsilon^2(t)\|\partial_t^2 u(t)\|_{2\alpha-4\theta}^2\\ &+ \int_t^{t+1} (\|u(s)\|_{4-3\theta}^2 + \varepsilon^2(t)\|\partial_t^2 u(s)\|_{2\alpha-3\theta}^2)\mathrm{d}s + \int_\tau^t \|\partial_t u(s)\|_\theta^2 \mathrm{d}s\\ \leqslant& C(R, \|g\|, L, \gamma), \ t \geqslant \tau. \end{aligned} \tag{3.1}$$

Moreover, the solution satisfies the following properties:

(i) *Energy equation: For every* $\tau \leqslant s \leqslant t$, *the following energy identity*

$$\begin{aligned}E(u(t), \partial_t u(t)) + 2\gamma \int_s^t \|\partial_t u(r)\|_\theta^2 \mathrm{d}r =& \int_s^t [\varepsilon'(r)\|\partial_t u(r)\|^2 + \varepsilon'(r)\|\partial_t u(r)\|_\alpha^2]\mathrm{d}r\\ &+ E(u(s), \partial_t u(s)),\end{aligned} \tag{3.2}$$

holds, where

$$E(u(t), \partial_t u(t)) = \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 + \|u(t)\|_2^2 + 2\langle F(u(t)), 1\rangle - 2\langle g, u(t)\rangle.$$

(ii) *Lipschitz stability in weak topological space: The solution* $y = (u, \partial_t u)$ *is Lipschitz continuous on* $V_{2-\theta+\alpha} \times V_{2\alpha-\theta}$, *that is,*

$$\|(\tilde{u}(t), \partial_t \tilde{u}(t)\|_{\mathcal{H}_t^{2\alpha-\theta}}^2 \leqslant C(R, \|g\|, L, T)\|(\tilde{u}, \partial_t \tilde{u})(\tau)\|_{\mathcal{H}_\tau^{2\alpha-\theta}}^2, \ as \ \tau \leqslant t \leqslant T < \infty, \tag{3.3}$$

where $\tilde{y} = (\tilde{u}, \partial_t \tilde{u}) = y_1 - y_2$ *and* $y_i = (u_i, \partial_t u_i)$ $(i = 1, 2)$ *are two weak solutions of the problems* (2.4) *and* (2.5) *corresponding to the initial data* (u_{i_0}, u_{i_1}) $(i = 1, 2)$, *respectively.*

(iii) *Dissipativity: There exists a positive constant* R_0 *independent of* α *and* θ *such that*

$$\|(u, \partial_t u)\|_{\mathcal{H}_t^\alpha} \leqslant R_0, \quad \forall \ t \geqslant t(R), \tag{3.4}$$

where, $\tau \leqslant t - t(R)$ and $t(R)$ is a moment that is dependent on R.

(iv) *Regularity when $t > \tau$: For any $\tau < qa < a \leqslant t \leqslant T$ (with $0 < q < 1$, as $a > 0$ or $q > 1$, as $a < 0$),*

$$(u, \partial_t u, \partial_t^2 u) \in L^\infty([a,T]; V_{2+\theta-\alpha} \times V_{2-\theta+\alpha} \times V_{2\alpha-\theta}) \cap L^2([a,T]; V_{4-\alpha} \times V_2 \times V_\alpha), \tag{3.5}$$

satisfying

$$\begin{aligned}&\|u\|^2_{2+\theta-\alpha} + \|\partial_t u\|^2_{2+\alpha-\theta} + \varepsilon(t)\|\partial_t^2 u\|^2_{2\alpha-\theta} + \int_t^{t+1} (\|u(s)\|^2_{4-\alpha} + \|\partial_t^2 u(s)\|^2_\alpha + \|\partial_t u(s)\|^2_2)\mathrm{d}s \\ &\leqslant \Big(\frac{1}{(t-qa)^{\frac{\theta}{1-\theta}}} + 1\Big)\Big(\frac{1}{(qa-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}} + 1\Big)C_1,\end{aligned} \tag{3.6}$$

where,

$$C_1 = (\frac{3\theta}{k_1(\theta-\alpha)})^{\frac{3\theta}{\theta-\alpha}} C(R, \|g\|, L, \gamma)^{\frac{3\theta}{\theta-\alpha}}.$$

Proof. (i) Existence of the weak solution: Taking the scalar product of (2.4) with $\partial_t u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t), \partial_t u(t)) + 2\gamma\|\partial_t u\|^2_\theta = \varepsilon'(t)\|\partial_t u\|^2 + \varepsilon'(t)\|\partial_t u\|^2_\alpha, \tag{3.7}$$

where

$$E(u(t), \partial_t u(t)) = \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|^2_\alpha + \|u(t)\|^2_2 + 2\langle F(u(t)), 1\rangle - 2\langle g, u(t)\rangle. \tag{3.8}$$

Taking into account $\varepsilon'(t) < 0$ and integrating (3.7) over $[s,t]$, we have

$$\begin{aligned}E(u(t), \partial_t u(t)) + 2\gamma\int_\tau^t \|\partial_t u(s)\|^2_\theta \mathrm{d}s &= \int_s^t [\varepsilon'(r)\|\partial_t u(r)\|^2 + \varepsilon'(r)\|\partial_t u(r)\|^2_\alpha]\mathrm{d}r + E(u(s), \partial_t u(s)) \\ &\leqslant E(u_0, u_1), \text{ as } t \geqslant \tau.\end{aligned} \tag{3.9}$$

From Remark 1.1 and the embedding $V_2 \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we can get

$$\kappa\|(u(t), \partial_t u(t))\|^2_{\mathcal{H}_t^\alpha} - C(R, \|g\|) \leqslant E(u(t), \partial_t u(t)) \leqslant C(R, L)\|(u(t), \partial_t u(t))\|^2_{\mathcal{H}_t^\alpha} + C(\|g\|), \tag{3.10}$$

where $\kappa > 0$ suitably small.

From (3.9) and (3.10), we obtain

$$\|(u, \partial_t u(t)\|^2_{\mathcal{H}_t^\alpha} + \int_\tau^t \|\partial_t u(s)\|^2_\theta \mathrm{d}s \leqslant C(R, \|g\|, L, \gamma), \text{ as } t \geqslant \tau. \tag{3.11}$$

By use of the embedding $L^{1+\frac{1}{p}}(\Omega) \hookrightarrow\hookrightarrow V_{-2} \hookrightarrow\hookrightarrow V_{\alpha-4\theta}$ and Eq (2.4), we can obtain that

$$\begin{aligned}&\varepsilon^2(t)\|\partial_t^2 u(t)\|^2_{2\alpha-4\theta} \\ &\leqslant \|u(t)\|^2_{4-4\theta} + \gamma^2\|\partial_t u(t)\|^2_{-2\theta} + \|f(u)\|^2_{-4\theta} + \|g\|^2_{-4\theta} \\ &\leqslant C(R, \gamma)(\|u(t)\|^2_{4-4\theta} + \|\partial_t u(t)\|^2_{-2\theta} + \|f(u)\|^2_{L^{1+\frac{1}{p}}} + \|g\|^2)\end{aligned}$$

$$\leqslant C(R,\gamma)(\|u(t)\|_2^2+\|u(t)\|_2^{2p}+\|\partial_t u(t)\|_{-2\theta}^2+\|g\|^2)$$
$$\leqslant C(R,\|g\|,L,\gamma). \tag{3.12}$$

Taking the scalar product in $L^2(\Omega)$ of (2.4) with $A^{2-3\theta}u$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon(t)\langle(1+A^{\alpha})\partial_t u, A^{2-3\theta}u\rangle)+\|u\|_{4-3\theta}^2+\gamma\langle A^{\theta}\partial_t u, A^{2-3\theta}u\rangle+\langle f(u), A^{2-3\theta}u\rangle$$
$$=\varepsilon(t)\|\partial_t u\|_{2-3\theta}^2+\varepsilon(t)\|\partial_t u\|_{2-3\theta+\alpha}^2+\varepsilon'(t)\langle(1+A^{\alpha})\partial_t u, A^{2-3\theta}u\rangle+\langle g, A^{2-3\theta}u\rangle. \tag{3.13}$$

Next, we will handle each item of (3.13),

$$|\langle g, A^{2-3\theta}u\rangle|\leqslant\|g\|_{-3\theta}\|u\|_{4-3\theta}\leqslant\frac{1}{4}\|u\|_{4-3\theta}^2+C(\|g\|,\lambda_1),$$
$$|\gamma\langle A^{\theta}\partial_t u, A^{2-3\theta}u\rangle|\leqslant\gamma\|\partial_t u\|_{-\theta}\|u\|_{4-3\theta}\leqslant\frac{1}{4}\|u\|_{4-3\theta}^2+C(\gamma,\lambda_1)\|\partial_t u\|_{\theta}^2,$$
$$|\varepsilon(t)\langle(1+A^{\alpha})\partial_t u, A^{2-3\theta}u\rangle|\leqslant L\|u\|_{4-6\theta}\|\partial_t u\|+L\|u\|_{4-6\theta+\alpha}\|\partial_t u\|_{\alpha}\leqslant C(R,\|g\|,L,\gamma),$$
$$|\varepsilon'(t)\langle(1+A^{\alpha})\partial_t u, A^{2-3\theta}u\rangle|\leqslant L\|u\|_{4-6\theta}\|\partial_t u\|+L\|u\|_{4-6\theta+\alpha}\|\partial_t u\|_{\alpha}\leqslant C(R,\|g\|,L,\gamma).$$

By use of the Sobolev embedding based on the facts $4-6\theta+\alpha<1+\alpha<2$, $2-3\theta<\theta$, and $2-3\theta+\alpha\leqslant\theta$, we have

$$|\langle f(u), A^{2-3\theta}u\rangle|\leqslant C\int_{\Omega}(|u|+|u|^p)|A^{2-3\theta}u|\mathrm{d}x$$
$$\leqslant C\Big(\int_{\Omega}(|u|+|u|^p)^{\frac{2N}{N+6\theta}}\mathrm{d}x\Big)^{\frac{N+6\theta}{2N}}\Big(\int_{\Omega}|A^{2-3\theta}u|^{\frac{2N}{N-6\theta}}\mathrm{d}x\Big)^{\frac{N-6\theta}{2N}}$$
$$\leqslant C(\|u\|_2^2+\|u\|_2^p)(\|u\|_{4-3\theta}^2)$$
$$\leqslant\frac{1}{4}\|u\|_{4-3\theta}^2+C(R,\|g\|,L,\gamma). \tag{3.14}$$

Inserting the above estimates into (3.13), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon(t)\langle(1+A^{\alpha})\partial_t u, A^{2-3\theta}u\rangle)+\frac{1}{4}\|u\|_{4-3\theta}^2\leqslant C(R,\|g\|,L,\gamma)(1+\|\partial_t u\|_{\theta}^2).$$

Therefore,

$$\int_t^{t+1}\|u(s)\|_{4-3\theta}^2\mathrm{d}s\leqslant C(R,\|g\|,L,\gamma). \tag{3.15}$$

Using the embedding $L^{\frac{2N}{N+6\theta}}\hookrightarrow V_{-3\theta}$ and (3.14), we obtain

$$\|f(u)\|_{-3\theta}^2\leqslant C\|f(u)\|_{L^{\frac{2N}{N+6\theta}}}^2$$
$$\leqslant C(\|u\|_{L^{\frac{2N}{N+6\theta}}}^4+\|u\|_{L^{\frac{2Np}{N+6\theta}}}^{2p})$$
$$\leqslant C(\|u\|_2^4+\|u\|_2^{2p})$$
$$\leqslant C(R,\|g\|,L,\gamma).$$

Hence, $f(u) \in L^2([\tau,T];V_{-3\theta})$. By use of the embedding $L^{\frac{2N}{N+3}}(\Omega) \hookrightarrow\hookrightarrow V_{\alpha-3\theta} \hookrightarrow\hookrightarrow V_{-3\theta}$ and Eq (2.4), we can obtain that

$$\begin{aligned}\varepsilon^2(t)\|\partial_t^2 u(t)\|_{2\alpha-3\theta}^2 &\leqslant \|u(t)\|_{4-3\theta}^2 + \gamma^2\|\partial_t u(t)\|_{-\theta}^2 + \|f(u)\|_{-3\theta}^2 + \|g\|_{-3\theta}^2 \\ &\leqslant C(R,\gamma)(\|u(t)\|_{4-3\theta}^2 + \|\partial_t u(t)\|_\theta^2 + +\|u\|_2^2 + \|u\|_2^{2p} + \|g\|^2) \\ &\leqslant C(R,\|g\|,L,\gamma)(\|u(t)\|_{4-3\theta}^2 + \|\partial_t u(t)\|_\theta^2). \end{aligned} \tag{3.16}$$

Combining (3.11) with (3.15), we have

$$\partial_t^2 u \in L^2([\tau,T];V_{2\alpha-3\theta}). \tag{3.17}$$

Finally, from (3.11), (3.12), (3.15), and (3.17), we gain the estimate (3.1).

Next, we will prove the existence of solutions for the problems (2.4) and (2.5) in the space $C([\tau,T];\mathcal{H}_t^\alpha) \cap L^2([\tau,T];V_{4-3\theta}\times V_\theta)$.

Let $y_n = (u_n, \partial_t u_n)$ be the solutions of approximation equation corresponding to the problems (2.4) and (2.5). It is easy to see that estimate (3.1) holds for the Galerkin approximation subsequence $\{y_n\}$. Thus, there exist a binary $y = (u,\partial_t u) \in L^\infty([\tau,T];\mathcal{H}_t^\alpha) \cap L^2([\tau,T];V_{4-3\theta}\times V_\theta)$, $\partial_t^2 u \in L^\infty([\tau,T];V_{\alpha-4\theta}) \cap L^2([\tau,T];V_{\alpha-3\theta})$, such that

$$\begin{aligned}&(u_n,\partial_t u_n) \to (u,\partial_t u) \text{ weakly}^* \text{ in } L^\infty([\tau,T];\mathcal{H}_t^\alpha),\\ &(u_n,\partial_t u_n) \to (u,\partial_t u) \text{ weakly in } L^2([\tau,T];V_{4-3\theta}\times V_\theta),\\ &\partial_t^2 u_n \to \partial_t^2 u \text{ weakly}^* \text{ in } L^\infty([\tau,T];V_{2\alpha-4\theta}),\\ &\partial_t^2 u_n \to \partial_t^2 u \text{ weakly in } L^2([\tau,T];V_{2\alpha-3\theta}).\end{aligned}$$

Applying Lemma 2.2, we can deduce that

$$(u_n,\partial_t u_n) \to (u,\partial_t u) \text{ in } C([\tau,T];V_{2-\eta}\times V_{\alpha-\eta}) \text{ with } \eta: 0<\eta\ll 1, \tag{3.18}$$

$$u_n \to u \text{ in } L^2([\tau,T];V_2) \text{ and } u_n(x,t)\to u(x,t), \ \text{ a.e. } (x,t)\in\Omega\times[\tau,T], \tag{3.19}$$

$$\partial_t u_n \to \partial_t u \ \text{ in } L^2([\tau,T];V_\alpha), \tag{3.20}$$

$$f(u_n) \to f(u) \text{ weakly in } L^{1+\frac{1}{p}}([\tau,T];L^{1+\frac{1}{p}}(\Omega)). \tag{3.21}$$

Hence, for any $\xi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\tau^T \langle A^2u_n - A^2u,\xi\rangle \mathrm{d}t &\leqslant \int_\tau^T \|A(u_n(t)-u(t))\|\|A\xi\|\mathrm{d}t\\ &= \int_\tau^T \|(u_n(t)-u(t))\|_2\|\xi\|_2\mathrm{d}t \to 0.\end{aligned}$$

Besides, for any $\xi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\tau^T \langle f(u_n)-f(u),\xi\rangle \mathrm{d}t &\leqslant C\int_\tau^T (\|u_n\|_2 + \|u\|_2 + \|u_n\|_2^{p-1} + \|u\|_2^{p-1})\|u_n-u\|_2\|\xi\|_2\mathrm{d}t\\ &\leqslant C(R,\|g\|,L,\gamma)\|u_n-u\|_{L^2([\tau,T],V_2)} \to 0.\end{aligned}$$

As a result, we gain that the limiting function $y = (u, \partial_t u)$ is a weak solution of the problems (2.4) and (2.5) satisfying estimate (3.1).

Recall that $(u(t), \partial_t u(t)) \in C([\tau, T]; V_{2-\eta} \times V_{\alpha-\eta}) \cap L^\infty(\tau, T; \mathcal{H}_t^\alpha)$. Therefore, $(u, \partial_t u) \in C_w([\tau, T]; \mathcal{H}_t^\alpha)$ and $\|(u, \partial_t u)\|_{\mathcal{H}_t^\alpha} \leqslant \liminf_{s\to t} \|(u(s), \partial_t u(s))\|_{\mathcal{H}_s^\alpha}$.

For any $t \in [\tau, T]$, it follows from (3.2) that

$$\lim_{s\to t} E(u(s), \partial_t u(s)) = E(u(t), \partial_t u(t)). \tag{3.22}$$

Additionally from (3.19), we deduce that $u(x, s) \to u(x, t)$ *a.e.* $x \in \Omega$ *as* $s \to t$, and by Lemma 2.1, Remark 1.1, and the Fatou lemma, we have

$$\begin{aligned}
&\|(u(t), \partial_t u(t))\|^2_{\mathcal{H}_t^\alpha} \leqslant \liminf_{s\to t} \|(u(s), \partial_t u(s))\|^2_{\mathcal{H}_t^\alpha},\\
&\lim_{s\to t} 2\langle g, u(s)\rangle = 2\langle g, u(t)\rangle,\\
&\quad\int_\Omega (2F(u(t)) + (1-\beta_0)\lambda_1^2 |u(t)|^2 + C(\beta_0))\mathrm{d}x\\
&\leqslant \liminf_{s\to t} \int_\Omega (2F(u(s)) + (1-\beta_0)\lambda_1^2 |u(s)|^2 + C(\beta_0))\mathrm{d}x\\
&\leqslant \liminf_{s\to t} \int_\Omega 2F(u(s))\mathrm{d}x + (1-\beta_0)\lambda_1^2 \|u\|^2 + C(\beta_0)|\Omega|.
\end{aligned}$$

That is,

$$\int_\Omega 2F(u(t))\mathrm{d}x \leqslant \liminf_{s\to t} \int_\Omega 2F(u(s))\mathrm{d}x.$$

From (3.21) and the above estimates, we get

$$\begin{aligned}
&\liminf_{s\to t} [\varepsilon(s)\|\partial_t u(s)\|^2 + \varepsilon(s)\|\partial_t u(s)\|_\alpha^2] + \liminf_{s\to t} [\|u(s)\|_2^2] + 2\langle F(u(s)), 1\rangle]\\
&\leqslant \lim_{s\to t} [\varepsilon(s)\|\partial_t u(s)\|^2 + \varepsilon(s)\|\partial_t u(s)\|_\alpha^2 + \|u(s)\|_2^2 + 2\langle F(u(s)), 1\rangle]\\
&= \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 + \|u(t)\|_2^2 + 2\langle F(u(t)), 1\rangle\\
&\leqslant \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 + \liminf_{s\to t} [\|u(s)\|_2^2] + \liminf_{s\to t} 2\langle F(u(s)), 1\rangle\\
&\leqslant \varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 + \liminf_{s\to t} [\|u(s)\|_2^2 + 2\langle F(u(s)), 1\rangle]\\
&\leqslant \liminf_{s\to t} [\varepsilon(s)\|\partial_t u(s)\|^2 + \varepsilon(s)\|\partial_t u(s)\|_\alpha^2] + \liminf_{s\to t} [\|u(s)\|_2^2] + 2\langle F(u(s)), 1\rangle].
\end{aligned}$$

Then,

$$\varepsilon(t)\|\partial_t u(t)\|^2 + \varepsilon(t)\|\partial_t u(t)\|_\alpha^2 = \lim_{s\to t} [\varepsilon(s)\|\partial_t u(s)\|^2 + \varepsilon(s)\|\partial_t u(s)\|_\alpha^2].$$

Similar to the above estimate, we have

$$\|u(t)\|_2^2 = \lim_{s\to t} \|u(s)\|_2^2. \tag{3.23}$$

In the light of the uniform convexity of the space $\mathcal{H}_t^\alpha$, (3.22), (3.23), and $(u, \partial_t u) \in C_w([\tau, T];\ \mathcal{H}_t^\alpha)$, we conclude that $(u, \partial_t u) \in C([\tau, T]; \mathcal{H}_t^\alpha)$. Up to now, the proof of the existence of the solutions have been completed. □

(ii) Lipschitz stability: Let $y_i(t)$, $i = 1,2$ be the corresponding solutions of the problems (2.4) and (2.5) with $y_i(\tau) \in \mathcal{H}_\tau^\alpha$ satisfying $\|y_i(\tau)\|_{\mathcal{H}_\tau^\alpha} \leqslant R$, $i = 1,2$. Let $\tilde{y} = (\tilde{u}, \partial_t\tilde{u}) = y_1 - y_2$. Substituting $\tilde{y}$ into (2.4) and (2.5), we have

$$\varepsilon(t)(1 + A^\alpha)\partial_t^2\tilde{u} + A^2\tilde{u} + \gamma A^\theta\partial_t\tilde{u} + f_1 - f_2 = 0,\ t \in [\tau, \infty), \tag{3.24}$$

$$\tilde{u}(\tau) = u_{1_0} - u_{2_0}, \quad \partial_t\tilde{u}(\tau) = u_{1_1} - u_{2_1}, \tag{3.25}$$

where $f_i = f(u_i)$, $i = 1,2$. In the following, let δ be sufficiently small. Multiplying (3.24) by $2A^{\alpha-\theta}\partial_t\tilde{u} + 2\delta\tilde{u}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}K(\tilde{u}, \partial_t\tilde{u}) + 2\delta\|\tilde{u}\|_2^2 + 2\gamma\|\partial_t\tilde{u}\|_\alpha^2 - 2\delta\varepsilon(t)\|\partial_t\tilde{u}\|_\alpha^2 - 2\delta\varepsilon(t)\|\partial_t\tilde{u}\|^2\\ =&\sum_{j=1}^4 \Pi_j + \varepsilon'(t)\|\partial_t\tilde{u}\|_{\alpha-\theta}^2 + \varepsilon'(t)\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2,\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}K(\tilde{u}, \partial_t\tilde{u}) =&2\delta\varepsilon(t)\langle A^\alpha\partial_t\tilde{u}, \tilde{u}\rangle + 2\delta\varepsilon(t)\langle\tilde{u}, \partial_t\tilde{u}\rangle + \varepsilon(t)\|\partial_t\tilde{u}\|_{\alpha-\theta}^2\\ &+ \varepsilon(t)\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2 + \|\tilde{u}\|_{2+\alpha-\theta}^2 + \gamma\delta\|\tilde{u}\|_\theta^2,\\ \Pi_1 =& 2\delta\varepsilon'(t)\langle\partial_t\tilde{u}, \tilde{u}\rangle,\ \ \Pi_2 = 2\delta\varepsilon'(t)\langle A^\alpha\partial_t\tilde{u}, \tilde{u}\rangle,\\ \Pi_3 =& -2\langle f(u_1) - f(u_2), A^{\alpha-\theta}\partial_t\tilde{u}\rangle,\ \ \Pi_4 = -2\langle f(u_1) - f(u_2), \delta\tilde{u}\rangle.\end{aligned}$$

By using the Young inequality, Poincaré inequality, and assumption (1.4), we obtain

$$\begin{aligned}|2\delta\varepsilon(t)\langle\tilde{u}, \partial_t\tilde{u}\rangle| &\leqslant \frac{4\delta^2L}{\lambda_1^{2+3\alpha-2\theta}}\|\tilde{u}\|_{2+\alpha-\theta}^2 + \frac{\varepsilon(t)}{4}\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2,\\ |2\delta\varepsilon(t)\langle A^\alpha\partial_t\tilde{u}, \tilde{u}\rangle| &\leqslant \frac{4\delta^2L}{\lambda_1^{2+\alpha-2\theta}}\|\tilde{u}\|_{2+\alpha-\theta}^2 + \frac{\varepsilon(t)}{4}\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2.\end{aligned}$$

Hence, there are some proper constants μ_2, μ_3 such that

$$\mu_2(\|\tilde{u}(t)\|_{2+\alpha-\theta}^2 + \varepsilon(t)\|\partial_t\tilde{u}(t)\|_{2\alpha-\theta}^2) \leqslant K(\tilde{u}, \partial_t\tilde{u}) \leqslant \mu_3(\|\tilde{u}(t)\|_{2+\alpha-\theta}^2 + \varepsilon(t)\|\partial_t\tilde{u}(t)\|_{2\alpha-\theta}^2), \tag{3.27}$$

where $\mu_2 = \min\{\frac{1}{2}, 1 - \frac{4\delta^2L}{\lambda_1^{2+3\alpha-2\theta}} - \frac{4\delta^2L}{\lambda_1^{2+\alpha-2\theta}}\}$, $\mu_3 = \max\{\frac{3}{2} + \frac{1}{\lambda_1^\alpha}, 1 + \frac{\gamma\delta}{\lambda_1^{2+\alpha-2\theta}} + \frac{4\delta^2L}{\lambda_1^{2+3\alpha-2\theta}} + \frac{4\delta^2L}{\lambda_1^{2+\alpha-2\theta}}\}$. Note (3.1) and the interpolation inequality. Then,

$$\begin{aligned}|\Pi_1| \leqslant&\frac{2\delta^2L}{\lambda_1^{2+3\alpha-2\theta}}\|\tilde{u}\|_{2+\alpha-\theta}^2 + \frac{L}{2}\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2,\\ |\Pi_2| \leqslant&\frac{2\delta^2L}{\lambda_1^{2+\alpha-2\theta}}\|\tilde{u}\|_{2+\alpha-\theta}^2 + \frac{L}{2}\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2,\\ |\Pi_3| \leqslant&2\int_\Omega |f(u_1) - f(u_2)| \cdot |A^{\alpha-\theta}\partial_t\tilde{u}|\mathrm{d}x\\ \leqslant&C(\int_\Omega (|u_1| + |u_2| + |u_1|^{p-1} + |u_2|^{p-1})^{\frac{2N}{6+2\alpha-2\eta}}\mathrm{d}x)^{\frac{6+2\alpha-2\eta}{2N}}(\int_\Omega |\tilde{u}|^{\frac{2N}{N-4+2\eta}}\mathrm{d}x)^{\frac{N-4+2\eta}{2N}}\end{aligned}$$

$$
\begin{aligned}
&(\int_\Omega |A^{\alpha-\theta}\partial_t\tilde{u}|^{\frac{2N}{N-2\alpha-2}}\mathrm{d}x)^{\frac{N-2\alpha-2}{2N}}\\
&\leqslant C(\|u_1\|_2+\|u_2\|_2+\|u_1\|_2^{p-1}+\|u_2\|_2^{p-1})(\|\tilde{u}\|_{2-\eta}^2+\|A^{\alpha-\theta}\partial_t\tilde{u}\|_{\alpha+1}^2)\\
&\leqslant\frac{\delta}{4}(\|\tilde{u}\|_2^2+\|\partial_t\tilde{u}\|_\alpha^2)+C(R,\|g\|,L,\gamma)(\|\tilde{u}\|_{2+\alpha-\theta}^2+\|\partial_t\tilde{u}\|_{2\alpha-\theta}^2),\\
|\Pi_4|&\leqslant 2\int_\Omega |f(u_1)-f(u_2)|\cdot|\delta\tilde{u}|\mathrm{d}x\\
&\leqslant C\delta(\int_\Omega(|u_1|+|u_2|+|u_1|^{p-1}+|u_2|^{p-1})^{\frac{p+1}{p-1}}\mathrm{d}x)^{\frac{p-1}{p+1}}(\int_\Omega|\tilde{u}|^{p+1}\mathrm{d}x)^{\frac{2}{p+1}}\\
&\leqslant C\delta(\|u_1\|_{L^{p+1}}+\|u_2\|_{L^{p+1}}+\|u_1\|_{L^{p+1}}^{p-1}+\|u_2\|_{L^{p+1}}^{p-1})(\|\tilde{u}\|_{L^{p+1}}^2)\\
&\leqslant C(R,\|g\|,L,\gamma)\|\tilde{u}\|_{2-\eta}^2\\
&\leqslant\frac{\delta}{4}\|\tilde{u}\|_2^2+C(R,\|g\|,L,\gamma)\|\tilde{u}\|_{2-\theta+\alpha}^2,
\end{aligned}
$$

where Sobolev embedding: $V_{2-\eta}\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$ with $0<\eta\ll 1$ has been used.

Inserting the above estimates into (3.26), we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}K(\tilde{u},\partial_t\tilde{u})+k(\|\tilde{u}\|_2^2+\|\partial_t\tilde{u}\|_\alpha^2)\leqslant C(R,\|g\|,L,\gamma)K(\tilde{u},\partial_t\tilde{u}),\tag{3.28}
$$

where $k=\min\{\frac{3\delta}{2},2\gamma-2\delta L-\delta-2L\delta\frac{1}{\lambda_1^\alpha}\}$.

Applying the Gronwall inequality, we obtain

$$
\|(\tilde{u}(t),\partial_t\tilde{u}(t)\|_{\mathcal{H}_t^{2\alpha-\theta}}^2\leqslant C(R,\|g\|,L,T)\|(\tilde{u},\partial_t\tilde{u})(\tau)\|_{\mathcal{H}_\tau^{2\alpha-\theta}}^2.\tag{3.29}
$$

Consequently, this implies (3.3).

(iii) Dissipativity: Taking the scalar product of (2.4) with $\partial_t u+\delta u$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}K_1(u,\partial_t u)+2\Lambda(u,\partial_t u)=0,\tag{3.30}
$$

where $\delta>0$ is sufficiently small,

$$
\begin{aligned}
K_1(u,\partial_t u)&=E(u,\partial_t u)+\delta\varepsilon(t)\langle(1+A^\alpha)\partial_t u,u\rangle,\\
\Lambda(u,\partial_t u)&=\|\partial_t u\|_\theta^2+\delta\|u\|_2^2+\delta\langle f(u),u\rangle+\delta\langle A^\theta\partial_t u,u\rangle-\delta\langle g,u\rangle-\delta\varepsilon(t)(\|\partial_t u\|^2+\|\partial_t u\|_\alpha^2)\\
&\quad-\varepsilon'(t)(\|\partial_t u\|^2+\|\partial_t u\|_\alpha^2)-\delta\varepsilon'(t)\langle(1+A^\alpha)\partial_t u,u\rangle.
\end{aligned}\tag{3.31}
$$

From (3.9) and (3.10), we get

$$
\kappa\|(u,\partial_t u)\|_{\mathcal{H}_t^\alpha}^2-C(R,\|g\|)\leqslant K_1(u,\partial_t u)\leqslant C(R,L)\|(u,\partial_t u)\|_{\mathcal{H}_t^\alpha}^2+C(\|g\|).\tag{3.32}
$$

Remark 1.1 and (3.1) implies

$$
\begin{aligned}
\|u\|_2^2+(f(u),u)&\geqslant\|u\|_2^2+(\beta_0-1)\|u\|_2^2-C_{\beta_0}\\
&\geqslant\beta_0\|u\|_2^2-C_{\beta_0},
\end{aligned}
$$

$$|\langle A^{\theta}\partial_t u, u\rangle| \leqslant \frac{\beta_0}{4}\|u\|_2^2 + \frac{1}{\beta_0\lambda_1^{\alpha+2-2\theta}}\|u_t\|_{\alpha}^2,$$

$$|\langle g, u\rangle| \leqslant \frac{\beta_0}{4}\|u\|_2^2 + C(R, \|g\|, \beta_0), \tag{3.33}$$

hence, we get

$$\begin{aligned}\Lambda(u, u_t) &\geqslant (\lambda_1^{\theta-\alpha} - \frac{\delta}{\beta_0\lambda_1^{\alpha+2-2\theta}} - \delta\lambda_1^{\alpha}L - \delta L)\|u_t\|_{\alpha}^2 + \frac{\delta\beta_0}{2}\|u\|_2^2 - \delta C(R, \|g\|, \beta_0)\\ &\geqslant \varsigma\|(u(t), u_t(t))\|_{\mathcal{H}_t^{\alpha}}^2 - \varsigma C(R, \|g\|, \beta_0, L),\end{aligned}$$

where $\varsigma > 0$ and $\varsigma = \min\left\{\lambda_1^{\theta-\alpha} - \frac{\delta}{\beta_0\lambda_1^{\alpha+2-2\theta}} - \delta\lambda_1^{\alpha}L - \delta L, \frac{\delta\beta_0}{2}, \delta\right\}$.

Based on the above estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}K_1(u, \partial_t u) + \varsigma K_1(u, \partial_t u) \leqslant \varsigma C(R, \|g\|, \beta_0, L). \tag{3.34}$$

Applying the Gronwall inequality to (3.34), we get

$$K_1(u, \partial_t u) \leqslant K_1(u(\tau), \partial_t u(\tau))\mathrm{e}^{-\varsigma(t-\tau)} + C(R, \|g\|, \beta_0, L). \tag{3.35}$$

From (3.35), the dissipativity of solutions of the problems (2.4) and (2.5) can be achieved.

(iv) Regularity when $t > \tau$: We differentiate (2.4) with respect to t and substitute into $v = \partial_t u$. Then

$$\varepsilon(t)(1 + A^{\alpha})\partial_t^2 v + \varepsilon'(t)(1 + A^{\alpha})\partial_t v + A^2 v + \gamma A^{\theta}\partial_t v + f'(u)v = 0. \tag{3.36}$$

Multiplying (3.36) by $A^{\alpha-\theta}\partial_t v + \delta v$ and integrating over Ω , we have

$$\frac{\mathrm{d}}{\mathrm{d}t}K_2(v(t), \partial_t v(t)) + 2\delta\|v\|_2^2 + 2(\gamma - \delta\varepsilon(t))\|\partial_t v\|_{\alpha}^2 - 2\delta\varepsilon(t)\|\partial_t v\|^2 = \sum_{i=1}^{2}\mathcal{M}_i, \tag{3.37}$$

where

$$\begin{aligned}K_2(v, \partial_t v) =&2\delta\varepsilon(t)\langle\partial_t v, v\rangle + 2\delta\varepsilon(t)\langle A^{\alpha}\partial_t v, v\rangle + \varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2\\ &+ \varepsilon(t)\|\partial_t v\|_{\alpha-\theta}^2 + \gamma\delta\|v\|_{\theta}^2 + \|v\|_{2+\alpha-\theta}^2,\\ \mathcal{M}_1 =& - 2\varepsilon'(t)\langle A^{\alpha}\partial_t v + \partial_t v, A^{\alpha-\theta}\partial_t v\rangle,\\ \mathcal{M}_2 =& - 2\langle f'(u)v, A^{\alpha-\theta}\partial_t v + \delta v\rangle.\end{aligned}$$

We can obtain from the Hölder inequality

$$2|\delta\varepsilon(t)\langle\partial_t v, v\rangle| \leqslant \frac{4\delta^2 L}{\lambda_1^{2+3\alpha-2\theta}}\|v\|_{2+\alpha-\theta}^2 + \frac{1}{4}\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2,$$

$$2|\delta\varepsilon(t)\langle A^{\alpha}\partial_t v, v\rangle| \leqslant \frac{4\delta^2 L}{\lambda_1^{2+\alpha-2\theta}}\|v\|_{2+\alpha-\theta}^2 + \frac{1}{4}\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2.$$

Then, there are the appropriate constants μ_6 and μ_7, such that

$$\mu_6(\|v\|_{2+\alpha-\theta}^2 + \varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2) \leqslant K_2(v, \partial_t v) \leqslant \mu_7(\|v\|_{2+\alpha-\theta}^2 + \varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2), \tag{3.38}$$

where $\mu_6 = \min\{\frac{1}{2}, 1 - \frac{4\delta^2 L}{\lambda_1^{2+3\alpha-2\theta}} - \frac{4\delta^2 L}{\lambda_1^{2+\alpha-2\theta}}\}$, $\mu_7 = \max\{\frac{3}{2} + \frac{1}{\lambda_1^{\alpha}}, 1 + \frac{4\delta^2 L}{\lambda_1^{2+3\alpha-2\theta}} + \frac{4\delta^2 L}{\lambda_1^{2+\alpha-2\theta}} + \frac{\gamma\delta}{\lambda_1^{2+\alpha-2\theta}}\}$. From (1.4), we get

$$|\mathcal{M}_1| \leqslant 2L\|\partial_t v\|_{2\alpha-\theta}^2 + 2L\|\partial_t v\|_{\alpha-\theta}^2.$$

Having in mind the growth exponent $1 \leqslant p < p^* = \frac{N+2\theta}{N-4}$, $N \geqslant 5$, we obtain

$$\begin{aligned}
|\mathcal{M}_2| &\leqslant C\int_\Omega (|u| + |u|^{p-1})|v||\delta v|\mathrm{d}x + C\int_\Omega (|u| + |u|^{p-1})|v||A^{\alpha-\theta}\partial_t v|\mathrm{d}x \\
&\leqslant C\delta(\|u\|_{L^{p+1}} + \|u\|_{L^{p+1}}^{p-1})\|v\|_{L^{p+1}}^2 + C(\|u\|_2 + \|u\|_2^{p-1})(\|v\|_{2-\eta}^2 + \|A^{\alpha-\theta}\partial_t v\|_{\alpha+1}^2) \\
&\leqslant \delta\|\partial_t v\|_\alpha^2 + \delta\|v\|_2^2 + C(R, \|g\|, L, \gamma)(\|v\|_{2-\theta+\alpha}^2 + \|\partial_t v\|_{2\alpha-\theta}^2),
\end{aligned} \tag{3.39}$$

where we have used interpolation inequality and (1.6).

We substitute the above estimate into (3.37), then

$$\frac{\mathrm{d}}{\mathrm{d}t}K_2(v(t), \partial_t v(t)) + k_1(\|v\|_2^2 + \|\partial_t v\|_\alpha^2) \leqslant C(R, \|g\|, L, \gamma)K_2(v(t), \partial_t v(t)), \tag{3.40}$$

where $k_1 = \min\{\delta, 2\gamma - 2\delta L - 2\delta L\lambda_1^{-\alpha} - \delta\}$.

For any $t > \tau$, multiplying (3.40) by $(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}} K_2(v(t), \partial_t v(t))] + k_1(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}(\|v\|_2^2 + \|\partial_t v\|_\alpha^2) \\
\leqslant &C(R, \|g\|, L, \gamma)(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}} K_2(v(t), \partial_t v(t)) \\
&+ \frac{4\theta-\alpha}{\theta-\alpha}(t-\tau)^{\frac{3\theta}{\theta-\alpha}}\mu_7(\|v\|_{2+\alpha-\theta}^2 + \varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2) \\
\leqslant &C(R, \|g\|, L, \gamma)(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}} K_2(v(t), \partial_t v(t)) \\
&+ \frac{k_1}{2}(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}(\|v\|_2^2 + \|\partial_t v\|_\alpha^2) + C_1,
\end{aligned}$$

where we have used

$$\begin{aligned}
\frac{4\theta-\alpha}{\theta-\alpha}(t-\tau)^{\frac{3\theta}{\theta-\alpha}}\mu_7\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2 &\leqslant \frac{4\theta-\alpha}{\theta-\alpha}(t-\tau)^{\frac{3\theta}{\theta-\alpha}}\mu_7\varepsilon(t)\|\partial_t v\|_\alpha^{2\varrho}\|\partial_t v\|_{2\alpha-4\theta}^{2(1-\varrho)} \\
&\leqslant \frac{k_1}{2}(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}\|\partial_t v\|_\alpha^2 + C_1
\end{aligned}$$

and

$$\begin{aligned}
\frac{4\theta-\alpha}{\theta-\alpha}(t-\tau)^{\frac{3\theta}{\theta-\alpha}}\mu_7\|v\|_{2+\alpha-\theta}^2 &\leqslant \frac{4\theta-\alpha}{\theta-\alpha}(t-\tau)^{\frac{3\theta}{\theta-\alpha}}\mu_7\varepsilon(t)\|v\|_2^{2\varrho}\|v\|_\alpha^{2(1-\varrho)} \\
&\leqslant \frac{k_1}{2}(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}\|v\|_2^2 + C_1,
\end{aligned}$$

with $\varrho = \frac{3\theta}{4\theta-\alpha} < 1$ and

$$C_1 = (\frac{3\theta}{k_1(\theta-\alpha)})^{\frac{3\theta}{\theta-\alpha}} C(R, \|g\|, L, \gamma)^{\frac{3\theta}{\theta-\alpha}}.$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}((t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}K_2(v(t),\partial_t v(t)))+\frac{k_1}{2}(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}(\|v\|_2^2+\|\partial_t v\|_\alpha^2)$$
$$\leqslant C(R,\|g\|,L,\gamma)(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}K_2(v(t),\partial_t v(t)). \tag{3.41}$$

Integrating (3.41) over $[\tau, t]$, we obtain

$$\|v\|_{2+\alpha-\theta}^2+\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2\leqslant\frac{1}{(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}}C_1,\ \text{as } \tau<t\leqslant 1. \tag{3.42}$$

From (3.42), and integrating (3.40) over $[1, t]$, we get

$$\|v\|_{2+\alpha-\theta}^2+\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2\leqslant C_1,\ \text{as } 1<t. \tag{3.43}$$

Integrating (3.40) over $[t, t+1]$, and taking account of (3.42)–(3.43), we have

$$\int_t^{t+1}(\|v\|_2^2+\|\partial_t v\|_\alpha^2)\mathrm{d}s\leqslant\Big(\frac{1}{(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}}+1\Big)C_1,\ \text{as } \tau<t. \tag{3.44}$$

Combining with (3.42)–(3.44), we get

$$\|v\|_{2+\alpha-\theta}^2+\varepsilon(t)\|\partial_t v\|_{2\alpha-\theta}^2+\int_t^{t+1}(\|v\|_2^2+\|\partial_t v\|_\alpha^2)\mathrm{d}s\leqslant C_1,\ \forall\tau<t,$$

where $C_2=\Big(\frac{1}{(t-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}}+1\Big)C_1$.

Taking the scalar product in L^2 of (2.4) and $A^{2-\alpha}u$, we can obtain

$$\gamma\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{2-\alpha+\theta}^2+2\|u\|_{4-\alpha}^2=2\langle\varepsilon(t)(1+A^\alpha)\partial_t^2u,A^{2-\alpha}u\rangle+2\langle g,A^{2-\alpha}u\rangle-2\langle f(u),A^{2-\alpha}u\rangle. \tag{3.45}$$

It is easy to get

$$|2\langle g,A^{2-\alpha}u\rangle|\leqslant\frac{4}{\lambda_1^\alpha}\|g\|^2+\frac{1}{4}\|u\|_{4-\alpha}^2, \tag{3.46}$$

$$|2\varepsilon(t)\langle(1+A^\alpha)\partial_t^2u,A^{2-\alpha}u\rangle|\leqslant(\frac{2L^2}{\lambda_1^{2\alpha}}+2L^2)\|\partial_t^2u\|_\alpha^2+\frac{1}{2}\|u\|_{4-\alpha}^2. \tag{3.47}$$

When the growth exponent is $1\leqslant p<p^*=\frac{N+2\theta}{N-4},\ N\geqslant 5$, we can know that

$$|-2\langle f(u),A^{2-\alpha}u\rangle|\leqslant C(\int_\Omega(1+|u|+|u|^p)^{\frac{2N}{N+2\alpha}}\mathrm{d}x)^{\frac{N+2\alpha}{2N}}(\int_\Omega|A^{2-\alpha}u|^{\frac{2N}{N-2\alpha}}\mathrm{d}x)^{\frac{N-2\alpha}{2N}}$$
$$\leqslant C(1+\|u\|_2+\|u\|_2^p)\|A^{2-\alpha}u\|_\alpha$$
$$\leqslant C(R,\|g\|,L,\gamma)+\frac{1}{4}\|u\|_{4-\alpha}^2. \tag{3.48}$$

Based on the above estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{2-\alpha+\theta}^2+\|u\|_{4-\alpha}^2\leqslant C(R,\|g\|,L,\gamma)(\|\partial_t^2u\|_\alpha^2+1). \tag{3.49}$$

For any $\tau < qa < a \leqslant t$, with $0 < q < 1$, as $a > 0$ or $q > 1$, as $a < 0$, we multiply (3.49) by $(t-qa)^{\frac{1}{1-\theta}}$ and use the interpolation inequality. Then,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}((t-qa)^{\frac{1}{1-\theta}}\|u\|^2_{2-\alpha+\theta}) + (t-qa)^{\frac{1}{1-\theta}}\|u\|^2_{4-\alpha}\\
\leqslant&\frac{1}{1-\theta}(t-qa)^{\frac{\theta}{1-\theta}}\|u\|^2_{2-\alpha+\theta} + C(R,\|g\|,L,\gamma)(t-qa)^{\frac{1}{1-\theta}}(\|\partial_t^2 u\|^2_\alpha + 1)\\
\leqslant&C(R,\|g\|,L,\gamma)(t-qa)^{\frac{1}{1-\theta}}(\|\partial_t^2 u\|^2_\alpha + 1) + (t-qa)^{\frac{1}{1-\theta}}\|u\|^2_{4-\alpha} + C(R)\frac{(t-qa)^{\frac{\theta+\alpha-\alpha\theta}{(1-\theta)(2-\theta)}}}{(1-\theta)^{\frac{2-\alpha}{2-\theta}}}\|u\|^2_2\\
\leqslant&C(R,\|g\|,L,\gamma)(t-qa)^{\frac{1}{1-\theta}}(\|\partial_t^2 u\|^2_\alpha + 1) + (t-qa)^{\frac{1}{1-\theta}}\|u\|^2_{4-\alpha},
\end{aligned} \tag{3.50}$$

where

$$\frac{1}{1-\theta}(t-qa)^{\frac{\theta}{1-\theta}}\|u\|^2_{2-\alpha+\theta} \leqslant \frac{1}{2}(t-qa)^{\frac{1}{1-\theta}}\|u\|^2_{4-\alpha} + C(R)\frac{(t-qa)^{\frac{\theta+\alpha-\alpha\theta}{(1-\theta)(2-\theta)}}}{(1-\theta)^{\frac{2-\alpha}{2-\theta}}}\|u\|^2_2.$$

Integrating (3.50) over $[qa, t]$ and making use of (3.44), we obtain

$$\|u(t)\|^2_{2-\alpha+\theta} \leqslant \frac{1}{(t-qa)^{\frac{\theta}{1-\theta}}}\Big(\frac{1}{(qa-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}} + 1\Big)C_1, \text{ as } qa < t \leqslant qa+1. \tag{3.51}$$

From (3.51), and integrating (3.49) over $[qa+1, t]$, we get

$$\|u(t)\|^2_{2-\frac{2\alpha}{3}+\theta} \leqslant \Big(\frac{1}{(qa-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}} + 1\Big)C_1, \text{ as } qa+1 \leqslant t. \tag{3.52}$$

Integrating (3.49) over $[t, t+1]$, and taking account of (3.51)–(3.52), we have

$$\int_t^{t+1}\|u(s)\|^2_{4-\alpha}\mathrm{d}s \leqslant \Big(\frac{1}{(t-qa)^{\frac{\theta}{1-\theta}}} + 1\Big)\Big(\frac{1}{(qa-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}} + 1\Big)C_1, \text{ as } qa < t. \tag{3.53}$$

Combining with (3.51)–(3.53), we get

$$\|u(t)\|^2_{2-\alpha+\theta} + \int_t^{t+1}\|u(s)\|^2_{4-\alpha}\mathrm{d}s \leqslant \Big(\frac{1}{(t-qa)^{\frac{\theta}{1-\theta}}} + 1\Big)\Big(\frac{1}{(qa-\tau)^{\frac{4\theta-\alpha}{\theta-\alpha}}} + 1\Big)C_1, \ \forall \tau < t.$$

Finally, (3.6) can be obtained.

According to Theorem 3.1, a process $U(t,\tau)$ corresponding to the problems (2.4) and (2.5) can be defined as follows:

$$y(t) = U(t,\tau)y(\tau) : \mathcal{H}^\alpha_\tau \to \mathcal{H}^\alpha_t,$$

which is continuous from $\mathcal{H}^\alpha_\tau$ to $\mathcal{H}^\alpha_t$.

4. The existence of time-dependent global attractor in $\mathcal{H}^\alpha_t$

4.1. Time-dependent absorbing set in $\mathcal{H}^\alpha_t$

By Theorem 3.1 (iii), we can obtain the following result.

Theorem 4.1. *Assume that the assumption of Theorem* 3.1 *holds. For any initial data $y(\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}^\alpha_\tau$, there exists a positive constant R_0, such that the process $U(t,\tau)$ corresponding to the problems* (2.4) *and* (2.5) *possesses a time-dependent absorbing set, namely, the family $\mathfrak{B}_t = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$.*

4.2. A priori estimates

The following estimations will be used to establish the asymptotic compactness of the process $U(t,\tau)$.

Let $y_i(t) = (u_i(t), \partial_t u_i(t))$ $(i = 1,2)$ be the solutions for the problems (2.4) and (2.5) with the initial data $(u_i(\tau), \partial_t u_i(\tau)) \in \{\mathbb{B}_\tau(R)\}_{\tau\in\mathbb{R}}$. The difference $\tilde{y}(t) = y_1(t) - y_2(t) = (\omega(t), \partial_t\omega(t))$ satisfies

$$\varepsilon(t)(1 + A^{\alpha})\partial_t^2\omega + A^2\omega + \gamma A^{\theta}\partial_t\omega + f_1 - f_2 = 0, \ t \in [\tau, \infty), \tag{4.1}$$

$$\omega(\tau) = u_{1_0} - u_{2_0}, \ \partial_t\omega(\tau) = u_{1_1} - u_{2_1}, \tag{4.2}$$

where, $f_i = f(u_i), \ i = 1,2$.

Define

$$H(t) = \varepsilon(t)\|\partial_t\omega(t)\|^2 + \varepsilon(t)\|\partial_t\omega(t)\|_\alpha^2 + \|\omega(t)\|_2^2.$$

We will make a priori estimation in the following four steps.

Step 1. We multiply (4.1) by $2\partial_t\omega$ and integrate over $[s,t]\times\Omega$. Then,

$$\begin{aligned}
&H(t) - H(s) + 2\gamma\int_s^t\int_\Omega |A^{\frac{\theta}{2}}\partial_t\omega(r)|^2\mathrm{d}x\mathrm{d}r + 2\int_s^t\int_\Omega (f(u_1) - f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r \\
=&\int_s^t\int_\Omega \varepsilon'(r)|\partial_t\omega(r)|^2\mathrm{d}x\mathrm{d}r + \int_s^t\int_\Omega \varepsilon'(r)|A^{\frac{\alpha}{2}}\partial_t\omega(r)|^2\mathrm{d}x\mathrm{d}r,
\end{aligned} \tag{4.3}$$

where $T \leqslant s \leqslant t$.

By assumption (1.4), we obtain

$$\varepsilon(t)|\partial_t\omega|^2 + \varepsilon(t)|A^{\frac{\alpha}{2}}\partial_t\omega|^2 \leqslant L(|\partial_t\omega|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega|^2) - \varepsilon'(t)(|\partial_t\omega|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega|^2),$$

then

$$\begin{aligned}
&\int_T^t\int_\Omega (\varepsilon(r)|\partial_t\omega(r)|^2 + \varepsilon(r)|A^{\frac{\alpha}{2}}\partial_t\omega|^2)\mathrm{d}x\mathrm{d}r \\
\leqslant& L\int_T^t\int_\Omega (|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega|^2)\mathrm{d}x\mathrm{d}r - \int_T^t\int_\Omega \varepsilon'(r)(|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega|^2)\mathrm{d}x\mathrm{d}s \\
\leqslant& H(T) + L\int_T^t\int_\Omega (|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega|^2)\mathrm{d}x\mathrm{d}r - 2\int_T^t\int_\Omega (f(u_1) - f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r.
\end{aligned} \tag{4.4}$$

Step 2. Multiplying (4.1) by ω and integrating over $[T,t]\times\Omega$, we get

$$\begin{aligned}
&\int_\Omega \varepsilon(t)[\partial_t\omega(t)\omega(t) + (A^\alpha\partial_t\omega(t))\omega(t)]\mathrm{d}x + \frac{\gamma}{2}\|\omega(t)\|_\theta^2 \\
&- \int_\Omega \varepsilon(T)[\partial_t\omega(T)\omega(T) + (A^\alpha\partial_t\omega(T))\omega(T)]\mathrm{d}x + \int_T^t\int_\Omega |A\omega(r)|^2\mathrm{d}x\mathrm{d}r \\
&+ \int_T^t\int_\Omega (f(u_1) - f(u_2))\omega(r)\mathrm{d}x\mathrm{d}r - \int_T^t \varepsilon(r)(\|\partial_t\omega(r)\|^2 + \|\partial_t\omega(r)\|_\alpha^2)\mathrm{d}r \\
=&\frac{\gamma}{2}\|\omega(T)\|_\theta^2 + \int_T^t\int_\Omega \varepsilon'(r)[\partial_t\omega(r)\omega(r) + (A^\alpha\partial_t\omega(r))\omega(r)]\mathrm{d}x\mathrm{d}r.
\end{aligned} \tag{4.5}$$

Estimates (4.4) and (4.5) imply that

$$\begin{aligned}\int_T^t H(r)\mathrm{d}r =& \int_T^t (\varepsilon(r)\|\partial_t\omega(r)\|^2 + \varepsilon(r)\|\partial_t\omega(r)\|_\alpha^2 + \|\omega(r)\|_2^2)\mathrm{d}r \\ \leqslant& H(T) + L\int_T^t\int_\Omega (|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega(r)|^2)\mathrm{d}x\mathrm{d}r - 2\int_T^t\int_\Omega (f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r \\ &- \int_T^t\int_\Omega (f(u_1)-f(u_2))\omega(r)\mathrm{d}x\mathrm{d}r - \int_\Omega \varepsilon(t)[\partial_t\omega(t)\omega(t) + (A^\alpha\partial_t\omega(t))\omega(t)]\mathrm{d}x \\ &+ \int_T^t \varepsilon(r)(\|\partial_t\omega(r)\|^2 + \|\partial_t\omega(r)\|_\alpha^2)\mathrm{d}r + \int_\Omega \varepsilon(T)[\partial_t\omega(T)\omega(T) + (A^\alpha\partial_t\omega(T))\omega(T)]\mathrm{d}x \\ &+ \int_T^t\int_\Omega \varepsilon'(r)[\partial_t\omega(r)\omega(r) + (A^\alpha\partial_t\omega(r))\omega(r)]\mathrm{d}x\mathrm{d}r + \frac{\gamma}{2}\|\omega(T)\|_\theta^2.\end{aligned}$$

Step 3. Integrating (4.3) over $[T,t]$ with respect to s, we have

$$\begin{aligned}H(t)(t-T) \leqslant& \int_T^t H(s)\mathrm{d}s - 2\int_T^t\int_s^t\int_\Omega (f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s \\ \leqslant& H(T) + L\int_T^t\int_\Omega (|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega(r)|^2)\mathrm{d}x\mathrm{d}r - 2\int_T^t\int_\Omega (f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r \\ &- \int_T^t\int_\Omega (f(u_1)-f(u_2))\omega(r)\mathrm{d}x\mathrm{d}r - \int_\Omega \varepsilon(t)[\partial_t\omega(t)\omega(t) + (A^\alpha\partial_t\omega(t))\omega(t)]\mathrm{d}x \\ &+ \int_T^t \varepsilon(r)(\|\partial_t\omega(r)\|^2 + \|\partial_t\omega(r)\|_\alpha^2)\mathrm{d}r + \int_\Omega \varepsilon(T)[\partial_t\omega(T)\omega(T) + (A^\alpha\partial_t\omega(T))\omega(T)]\mathrm{d}x \\ &+ \int_T^t\int_\Omega \varepsilon'(r)[\partial_t\omega(r)\omega(r) + (A^\alpha\partial_t\omega(r))\omega(r)]\mathrm{d}x\mathrm{d}r + \frac{\gamma}{2}\|\omega(T)\|_\theta^2 \\ &- 2\int_T^t\int_s^t\int_\Omega (f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Step 4. Set

$$C(M) = H(T) + \frac{\gamma}{2}\|\omega(T)\|_\theta^2 + \int_\Omega \varepsilon(T)[\partial_t\omega(T)\omega(T) + (A^\alpha\partial_t\omega(T))\omega(T)]\mathrm{d}x, \tag{4.6}$$

and

$$\varphi_T^t((u_1(T),\partial_t u_1(T)),(u_2(T),\partial_t u_2(T))) = \Psi_1 + \Psi_2, \tag{4.7}$$

where

$$\begin{aligned}\Psi_1 =& \frac{1}{(t-T)}\Big[L\int_T^t\int_\Omega (|\partial_t\omega(r)|^2 + |A^{\frac{\alpha}{2}}\partial_t\omega(r)|^2)\mathrm{d}x\mathrm{d}r \\ &- \int_\Omega \varepsilon(t)[\partial_t\omega(t)\omega(t) + (A^\alpha\partial_t\omega(t))\omega(t)]\mathrm{d}x \\ &+ \int_T^t \varepsilon(r)(\|\partial_t\omega(r)\|^2 + \|\partial_t\omega(r)\|_\alpha^2)\mathrm{d}r\end{aligned}$$

$$+\int_T^t\int_\Omega \varepsilon'(r)[\partial_t\omega(r)\omega(r)+(A^\alpha\partial_t\omega(r))\omega(r)]\mathrm{d}x\mathrm{d}r],$$

$$\Psi_2=-\frac{1}{(t-T)}[2\int_T^t\int_\Omega(f(u_1)-f(u_2))\partial_t\omega(s)\mathrm{d}x\mathrm{d}s$$
$$+\int_T^t\int_\Omega(f(u_1)-f(u_2))\omega(s)\mathrm{d}x\mathrm{d}s$$
$$+2\int_T^t\int_s^t\int_\Omega(f(u_1)-f(u_2))\partial_t\omega(r)\mathrm{d}x\mathrm{d}r\mathrm{d}s].$$

Therefore,

$$H(t)\leqslant\frac{1}{t-T}C_M+\varphi_T^t((u_1(T),\partial_t u_1(T)),(u_2(T),\partial_t u_2(T))). \tag{4.8}$$

4.3. Asymptotic compactness

In this subsection, we will verify that the process $U(t,\tau)$ corresponding to the problems (2.4) and (2.5) is asymptotically compact by using method of the contraction function.

Theorem 4.2. *If the assumptions hold, for any fixed $t\in\mathbb{R}$ and any bounded $\{\tau_n\}_{n=1}^\infty\subset(-\infty,t]$ with $\tau_n\to-\infty$ as $n\to\infty$, and for any bounded sequence $\{x_n\}_{n=1}^\infty\subset\mathbb{B}_{\tau_n}(R)\subset\mathcal{H}_{\tau_n}^\alpha$, then sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ possesses a convergent subsequence.*

Proof. For any $\varepsilon>0$ and fixed t, there exists $T<t$ such that $\frac{C_M}{t-T}<\varepsilon$. Thanks to Theorem 2.2, we also need to show that $\Phi_T^t\in\mathfrak{C}(\mathbb{B}_T(R))$, for every fixed t.

Let $(u_n,\partial_t u_n)$ be the solution for the problems (2.4) and (2.5) with initial data $(u_{n_0},u_{n_1})\in\mathbb{B}_T(R)$. According to Theorem 3.1, we can know that $\|u_n\|_2^2+\varepsilon(\xi)\|\partial_t u_n\|_\alpha^2$ is bounded, and $\|u_n\|_2^2$ is also bounded. For any fixed t and $\forall\xi\in[T,t]$, in view of (1.4) and the boundedness of $\frac{1}{\varepsilon(\xi)}$, we gain the boundedness of $\|\partial_t u_n\|_\alpha^2$.

By the Alaoglu theorem, Lemma 2.2, and Theorem 3.1, for any $\tau\leqslant T\leqslant t$, without loss of generality (at most by passing subsequence), we presume that

$$u_n\to u \text{ weakly}^* \text{ in } L^\infty([T,t];V_2), \tag{4.9}$$
$$\partial_t u_n\to\partial_t u \text{ weakly}^* \text{ in } L^\infty([T,t];V_\alpha), \tag{4.10}$$
$$\partial_t^2 u_n\to\partial_t^2 u \text{ weakly}^* \text{ in } L^\infty([T,t];V_{2\alpha-4\theta}), \tag{4.11}$$
$$u_n\to u \text{ weakly in } L^2([T,t];V_{4-3\theta}), \tag{4.12}$$
$$\partial_t u_n\to\partial_t u \text{ weakly in } L^2([T,t];V_\theta), \tag{4.13}$$
$$\partial_t^2 u_n\to\partial_t^2 u \text{ weakly in } L^2([T,t];V_{2\alpha-3\theta}), \tag{4.14}$$
$$u_n\to u \text{ in } L^{p+1}([T,t];L^{p+1}(\Omega)), \tag{4.15}$$
$$u_n\to u \text{ in } L^2([T,t];V_2), \tag{4.16}$$
$$u_n(t)\to u(t) \text{ and } u_n(T)\to u(T) \text{ in } L^{p+1}(\Omega), \tag{4.17}$$
$$\partial_t u_n\to\partial_t u \text{ in } L^2([T,t];V_\alpha), \tag{4.18}$$

where we have used the Sobolev embedding $V_2\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$.

By virtue of (3.54), we know that

$$\{(u_n(s), \partial_t u_n(s))\} \subset C([T,t]; \mathcal{H}_s^\alpha) \text{ is a Cauchy sequence,} \tag{4.19}$$

and there exists a binary $(u(s), \partial_t u(s)) \in C([T,t]; \mathcal{H}_s^\alpha)$, such that

$$(u_n(s), \partial_t u_n(s)) \to (u(s), \partial_t u(s)) \text{ in } C([T,t]; \mathcal{H}_s^\alpha). \tag{4.20}$$

Next, we will handle each term of (4.7). First, we obtain from (4.16), (4.17), and (4.18),

$$\lim_{n\to\infty}\lim_{m\to\infty} L\int_T^t [\|\partial_t u_n - \partial_t u_m\|^2 + \|\partial_t u_n - \partial_t u_m\|_\alpha^2]\mathrm{d}s = 0, \tag{4.21}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \varepsilon(t)(\partial_t u_n - \partial_t u_m)(u_n - u_m)\mathrm{d}x\\
\leqslant &\lim_{n\to\infty}\lim_{m\to\infty} L\|\partial_t u_n - \partial_t u_m\|\|u_n - u_m\|\\
\leqslant &\lim_{n\to\infty}\lim_{m\to\infty} L(\|\partial_t u_n\| + \|\partial_t u_m\|)\|u_n - u_m\| = 0,
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega \varepsilon(t)(A^{\frac{\alpha}{2}}\partial_t u_n - A^{\frac{\alpha}{2}}\partial_t u_m)(A^{\frac{\alpha}{2}}u_n - A^{\frac{\alpha}{2}}u_m)\mathrm{d}x\\
\leqslant &\lim_{n\to\infty}\lim_{m\to\infty} L\|\partial_t u_n - \partial_t u_m\|_\alpha\|u_n - u_m\|_\alpha\\
\leqslant &\lim_{n\to\infty}\lim_{m\to\infty} L(\|\partial_t u_n\|_\alpha + \|\partial_t u_m\|_\alpha)\|u_n - u_m\|_\alpha = 0,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \varepsilon(s)[\|\partial_t u_n - \partial_t u_m\|^2 + \|\partial_t u_n - \partial_t u_m\|_\alpha^2]\mathrm{d}s\\
\leqslant &\lim_{n\to\infty}\lim_{m\to\infty} L\int_T^t [\|\partial_t u_n - \partial_t u_m\|^2 + \|\partial_t u_n - \partial_t u_m\|_\alpha^2]\mathrm{d}s = 0,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \varepsilon'(s)\langle\partial_t u_n - \partial_t u_m, u_n - u_m\rangle\mathrm{d}s\\
\leqslant &L\lim_{n\to\infty}\lim_{m\to\infty}(\int_T^t \|\partial_t u_n - \partial_t u_m\|^2 ds)^{\frac{1}{2}} \cdot (\int_T^t \|u_n - u_m\|^2 ds)^{\frac{1}{2}} = 0,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t \varepsilon'(s)\langle A^{\frac{\alpha}{2}}\partial_t u_n - A^{\frac{\alpha}{2}}\partial_t u_m, A^{\frac{\alpha}{2}}u_n - A^{\frac{\alpha}{2}}u_m\rangle\mathrm{d}s\\
\leqslant &L\lim_{n\to\infty}\lim_{m\to\infty}(\int_T^t \|\partial_t u_n - \partial_t u_m\|_\alpha^2 ds)^{\frac{1}{2}} \cdot (\int_T^t \|u_n - u_m\|_\alpha^2 ds)^{\frac{1}{2}} = 0.
\end{aligned} \tag{4.26}$$

Synthesizing (4.21)–(4.26), we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_1 = 0. \tag{4.27}$$

Second, from (4.16), we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega (f(u_n) - f(u_m))(u_n - u_m)\mathrm{d}x\mathrm{d}s\\
\leqslant &C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega (|u_n| + |u_m| + |u_n|^{p-1} + |u_m|^{p-1})|u_n - u_m|^2\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t(\|u_n\|_{L^{p+1}}+\|u_m\|_{L^{p+1}}+\|u_n\|_{L^{p+1}}^{p-1}+\|u_m\|_{L^{p+1}}^{p-1})\cdot\|u_n-u_m\|_{L^{p+1}}^2\mathrm{d}s\\
&\leqslant C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t(\|u_n\|_2+\|u_m\|_2+\|u_n\|_2^{p-1}+\|u_m\|_2^{p-1})\cdot\|u_n-u_m\|_2^2\mathrm{d}s\\
&\leqslant C\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\|u_n-u_m\|_2^2\mathrm{d}s\\
&=0. \qquad (4.28)
\end{aligned}
$$

It is easy to know

$$
\begin{aligned}
&\int_T^t\int_\Omega(f(u_n)-f(u_m))(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}s\\
&=\int_T^t\int_\Omega f(u_n)\partial_t u_n\mathrm{d}x\mathrm{d}s+\int_T^t\int_\Omega f(u_m)\partial_t u_m\mathrm{d}x\mathrm{d}s-\int_T^t\int_\Omega f(u_m)\partial_t u_n\mathrm{d}x\mathrm{d}s\\
&\quad-\int_T^t\int_\Omega f(u_n)\partial_t u_m\mathrm{d}x\mathrm{d}s\\
&=\int_\Omega F(u_n(t))\mathrm{d}x-\int_\Omega F(u_n(T))\mathrm{d}x+\int_\Omega F(u_m(t))\mathrm{d}x-\int_\Omega F(u_m(T))\mathrm{d}x\\
&\quad-\int_T^t\int_\Omega f(u_m)\partial_t u_n\mathrm{d}x\mathrm{d}s-\int_T^t\int_\Omega f(u_n)\partial_t u_m\mathrm{d}x\mathrm{d}s. \qquad (4.29)
\end{aligned}
$$

By (1.6) and the embedding $V_2\hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, we get

$$
\begin{aligned}
&|\int_\Omega(F(u_n(t))-F(u(t)))\mathrm{d}x|\\
&\leqslant\int_\Omega|f(u(t)+\vartheta(u_n(t)-u(t))||u_n(t)-u(t)|\mathrm{d}x\\
&\leqslant C\int_\Omega(|u_n(t)|^2+|(u(t)|^2+|u_n(t)|^p+|(u(t)|^p)|u_n(t)-u(t)|\mathrm{d}x\\
&\leqslant C(\|u_n(t)\|_{L^{p+1}}^2+\|u(t)\|_{L^{p+1}}^2+\|u_n(t)\|_{L^{p+1}}^p+\|u(t)\|_{L^{p+1}}^p)\|u_n(t)-u(t)\|_{L^{p+1}}\\
&\leqslant C\epsilon. \qquad (4.30)
\end{aligned}
$$

Because $f(u_n)\in L^2([\tau,T];V_{-3\theta})$ and $\partial_t u_m\in L^2([\tau,T];V_\theta)$ as $n\to\infty$, $m\to\infty$, we have

$$
\begin{aligned}
\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\langle f(u_n),\partial_t u_m\rangle\mathrm{d}s&=\lim_{n\to\infty}\int_T^t\langle f(u_n),\partial_t u\rangle\mathrm{d}s\\
&=\int_T^t\langle f(u),\partial_t u\rangle\mathrm{d}s\\
&=\int_\Omega F(u(t))\mathrm{d}x-\int_\Omega F(u(T))\mathrm{d}x.
\end{aligned}
$$

Similarly,

$$
\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\langle f(u_m),\partial_t u_n\rangle\mathrm{d}s=\int_\Omega F(u(t))\mathrm{d}x-\int_\Omega F(u(T))\mathrm{d}x.
$$

Therefore,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_\Omega (f(u_n)-f(u_m))(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}s=0. \tag{4.31}$$

For each fixed t, $|\int_s^t\int_\Omega (f(u_n) - f(u_m))(\partial_t u_n - \partial_t u_m)\mathrm{d}x\mathrm{d}r|$ is bounded. Then, thanks to the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_s^t\int_\Omega (f(u_n)-f(u_m))(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
=&\int_T^t\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_\Omega (f(u_n)-f(u_m))(\partial_t u_n-\partial_t u_m)\mathrm{d}x\mathrm{d}r\mathrm{d}s\\
=&\int_T^t 0\,\mathrm{d}s=0.
\end{aligned} \tag{4.32}$$

Consequently, we obtain from (4.28)–(4.32) that

$$\lim_{n\to\infty}\lim_{m\to\infty}\Psi_2=0. \tag{4.33}$$

Therefore, we conclude that $\Phi_T^t((u_1(T),\partial_t u_1(T)),(u_2(T),\partial_t u_2(T)))\in\mathfrak{C}(\mathbb{B}_T(R))$. □

4.4. The time-dependent global attractors

Theorem 4.3. *If the assumptions of Theorem 4.2 hold, then the dynamical system* $(U(t,\tau),\mathcal{H}_t^\alpha)$ *corresponding to the problems* (2.4) *and* (2.5) *possesses an invariant time-dependent global attractor* $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$.

Proof. According to Theorem 3.2, Theorem 3.3, and Theorems 4.1 and 4.2, there exists a time-dependent global attractor $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$.

From Remark 5.5 and Theorem 5.6 in [11], we know that if the process $U(t,\tau)$ is continuous, then the corresponding time-dependent global attractor is invariant. Hence, the time-dependent global attractor $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$ is invariant. □

5. Conclusions

This paper investigated the long-time dynamical behavior of solutions for the beam equation with rotational inertia and structural damping. By using the Faedo-Galerkin approximation method, the contraction function method, and the time-dependent attractor theory, the well-posedness and the existence of a time-dependent global attractor were obtained in the time-dependent space $\mathcal{H}_t^\alpha$.

Author contributions

Xuan Wang: Conceptualization, methodology, visualization, supervision, writing-review; Wei Wang: Conceptualization, methodology, writing-original draft, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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