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Research article

Reliability analysis of the Lindley distribution via unified hybrid censoring with applications in medical survival and biological lifetime data

Said G. Nassr[1], **T. S. Taher**[2], **Tmader Alballa**[3,*] **and Neema M. Elharoun**[1]

[1] Department of Statistics and Insurance, Faculty of Commerce, Arish University, Al-Arish 45511, Egypt
[2] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt
[3] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: tsalballa@pnu.edu.sa.

Abstract: Unified hybrid censored sampling results from the need to effectively evaluate survival data impacted by different censoring techniques, enabling researchers to use partial information and lower bias in parameter estimates. This study employs unified hybrid censored samples to examine the maximum likelihood and Bayesian estimation methods for unknown parameters of the Lindley distribution, encompassing the reliability function, hazard rate function, cumulative hazard function, and mean residual life. Initially, we estimate the model parameters via the maximum likelihood method. We also examine bootstrapped confidence intervals and asymptotic confidence intervals for unknown parameters of the Lindley distribution. We also derive the confidence interval for the mean residual life, the hazard rate function, the cumulative hazard function, and the reliability function. Moreover, using flexible gamma priors for parameters combined with a non-informative prior yields Bayes estimators based on the principles of squared error, linear exponential, and general entropy loss functions. Consequently, we identify the corresponding highest posterior density credible intervals for unknown parameters, the reliability function, the hazard rate function, the cumulative hazard function, and the mean residual lifetime. Using numerous criteria, Monte Carlo simulations help evaluate the accuracy of the given estimations. We also assess sampling techniques across different rival censoring systems, including bootstrap and censored samples. Ultimately, we investigated a medical data set to show the useful value of the suggested approaches. The numerical results confirm the efficiency of our proposed approaches.

Keywords: Lindley distribution; unified hybrid censored sampling; Bayesian estimation; maximum likelihood estimation; Monte Carlo simulation; Bootstrapping sample
Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

Unified hybrid censored sampling (UHCS) has arisen as a remedy for numerous obstacles faced in data collecting and analysis, especially in the context of incomplete or censored data. In numerous practical situations, acquiring comprehensive datasets is either unrealistic or unattainable due to limitations such as time, expense, or external factors like equipment malfunctions or participant attrition. Conventional censored sampling methods, like Type I (time-based) and Type II (failure-based), although successful in specific scenarios, frequently prove inadequate when confronted with the intricacies of contemporary studies. UHCS mitigates these restrictions by incorporating many censoring techniques into a cohesive framework, enhancing both flexibility and efficiency. A significant issue contributing to the adoption of UHCS is the unpredictability of real-world data acquisition. In reliability testing, certain items may break prematurely, whilst others may endure significantly longer than anticipated, complicating the establishment of a definite time frame (Type I) or a predetermined number of failures (Type II). Hybrid censoring, integrating elements from both methodologies, offers limited alleviation but remains deficient in the adaptability necessary for highly changeable situations. UHCS enhances this by providing a more holistic approach that can be customized to the particular requirements of a study, reconciling the limitations of time and sample size. A further problem pertains to the inefficacy of resource usage unconventional methods. Type I censoring frequently necessitates the continuation of an experiment until a predetermined time, despite the collection of adequate data for analysis. Likewise, Type II censoring can extend a research needlessly to achieve a predetermined number of failures. UHCS mitigates these inefficiencies by permitting the cessation of trials upon the fulfillment of predetermined criteria—whether temporal, failure-related, or a combination thereof—thus preserving resources while maintaining result integrity.

The primary distinction between UHCS and alternative censored sampling techniques is its adaptability and breadth. Conventional censoring techniques are tailored for particular scenarios and may find it challenging to adjust to unforeseen alterations during data acquisition. UHCS offers a cohesive architecture capable of seamlessly integrating various types of censorship, enhancing its versatility for dynamic real-world applications. In medical research, UHCS can accommodate fluctuating dropout rates, premature terminations, and varied follow-up periods, hence assuring optimal utilization of all available data. Statistically, UHCS distinguishes itself in its methodology for parameter estimation and model fitting. Integrating data from several censoring systems diminishes bias and enhances the accuracy of estimates. Conventional approaches may overlook partially censored observations or neglect their influence on the research, resulting in diminished accuracy of conclusions. UHCS guarantees that even partial data significantly contributes to the whole analysis, hence augmenting the trustworthiness of statistical judgments.

Chandrasekar et al. [1] categorize hybrid censored approaches into Type-I HCS and Type-II HCS. Unlike Type-II HCS, which may necessitate an extended duration to ascertain the minimum number of failures, Type-I HCS, for example, may yield several failure instances prior to the termination time. Consequently, these censored sampling methodologies have been enhanced with the incorporation of two expansions referred to as generalized Type-I and Type-II HCS. Chandrasekar et al. [1] proposed two generalized HCSs to address these issues: ensuring the quantity of failures and establishing a temporal limit for the experimental phase. These aimed to mitigate the intrinsic limitations of Type-I

and Type-II HCSs. Balakrishnan et al. [2] integrated generalized Type-I and Type-II HCS approaches to develop a UHCS. It offers enhanced adaptability in comparison to Type I and II HCS.

Censored data are essential in medical research, as researchers are unable to evaluate each individual's entire lifespan in a longitudinal study due to temporal and financial limitations. Due to the persistent absence of comprehensive data, censoring algorithms exist that maximize both time and resources. Balakrishnan et al. [2] characterize UHCS as an amalgamation of generalized Type-I HCS and generalized Type-II HCS, which can be articulated as follows. Examine the assessment of n identical units, where the lifespan of each unit is governed by independent, identically distributed (i.i.d) variables. Let r and k be integers inside the set $\{1, 2, \ldots, n\}$, such that $r < k < n$, and let T_1 and T_2 represent times in the interval $(0, \infty)$. The experiment concludes at the moment $\min\{\max\{X_{k:n}, T_1\}, T_2\}$, provided that the r-th failure transpires prior to T_1. Should the r-th failure occur between T_1 and T_2, the experiment terminates at $\min\{X_{k:n}, T_2\}$. Ultimately, if the r-th failure transpires subsequent to T_2, the experiment finishes at $X_{r:n}$.

The UHCS addresses Type-I, Type-II, hybrid, and generalized hybrid censorship flaws:

- **Enhanced flexibility:** Termination is based on time limits (T_1, T_2) and failure counts (k, r), enabling adaptation to changing situations.
- **Resource efficiency:** Experiments end by T_2 after at least k failures, eliminating wasteful extension and costs without compromising inferential validity.
- **Improved estimation:** UHCS uses all partially filtered data to lessen parameter estimate bias. Kumar et al. [3] found that maximum likelihood estimates (MLEs) and Bayes estimators are more accurate than traditional or single-criterion techniques.
- **Robustness to complex scenarios:** UHCS combines time and failure-count criteria to maintain data integrity in medical or reliability research with unpredictable dropout or failure patterns.

UHCS thus guarantees that each experiment lasts no longer than T_2 while recording at least k failures, balancing logistical constraints and data needs. Its efficacy has been confirmed by Balakrishnan and Kundu [4], Huang and Yang [5], Rabie and Li [6], Jeon and Kang [7], and Dutta et al. [8].

The Lindley distribution was introduced by Lindley [9] within the framework of fiducial and Bayesian statistics to demonstrate the distinction between fiducial and posterior distributions. Additionally, Ghitany et al. [10] examined its statistical features and showed that, in many applications, Lindley provides a better fit than the exponential model. By construction. The probability density function (PDF) and the cumulative distribution function (CDF) are provided by

$$f(x;\lambda) = \frac{\lambda^2}{(\lambda+1)}(1+x)\exp(-\lambda x); \quad x \geq 0, \lambda, > 0. \tag{1.1}$$

and

$$F(x;\lambda) = 1 - \frac{(\lambda + \lambda x + 1)\ \exp(-\lambda x)}{(\lambda+1)}; \quad x \geq 0, \lambda, > 0, \tag{1.2}$$

where λ scale parameter. mixes an exponential(λ) and a Gamma$(2, \lambda)$ component in proportion $\frac{\lambda}{\lambda+1}$. Bakouch et al. [11] proposed an enhanced Lindley variant, Ghitany et al. [12] introduced a two-parameter extension, and Nadarajah et al. [13] developed further generalizations—all underscoring the family's flexibility. The Lindley model enjoys wide applicability in industry,

medicine, biology, and beyond. For example, Ghitany et al. [14] used it to assess reliability in strength systems, and Gómez–Déniz et al. [**?**] derived a bounded-domain density based on its generalized form. Despite these extensions, the one-parameter core remains underutilized. In our real-data analyses, Lindley consistently outperforms more flexible two- and three-parameter families—such as the Weibull and Q-Weibull—in both goodness-of-fit and information criteria.

Lindley's single-parameter form yields closed-form expressions for key reliability measures like the survival function, hazard rate, cumulative hazard, and mean residual life, avoiding numerical integration and ensuring robust estimation under MLE and Bayesian frameworks, especially with censored data. Multi-parameter variants have been suggested, but the original one-parameter form offers the best blend of parsimony, interpretability, and operational efficiency for upper-hybrid censored samples. Finally, the Lindley distribution is stable and interpretable since it follows Occam's rule by describing data with one parameter. This research uses its skills to improve UHCS data modelling.

The reliability function $R(t)$, hazard rate function $H(t)$, cumulative hazard function $CH(t)$, and mean residual lifetime $MR(t)$ of the Lindley distribution are provided, respectively,

$$R(t) = \frac{(\lambda + \lambda t + 1)\ \exp(-\lambda t)}{(\lambda + 1)}, \tag{1.3}$$

$$H(t) = \frac{\lambda^2(1 + t)}{(\lambda + \lambda t + 1)}, \tag{1.4}$$

$$CH(t) = \int_0^t H(u)du = \log(\lambda + 1) + \lambda t - \log(\lambda + \lambda t + 1). \tag{1.5}$$

and

$$MR(t) = E(T - t \mid T > t) = \frac{\int_t^\infty R(u)du}{R(t)} = \frac{\lambda + \lambda t + 2}{\lambda + \lambda^2 + \lambda^2 t}. \tag{1.6}$$

For further information regarding the residual lifetime of the Lindley distribution, see Ebrahimi [16] and Goel and Krishna [17].

System performance and lifetime are evaluated using key metrics: reliability, hazard rate, cumulative hazard function, and mean residual life. Reliability represents the probability of a system operating without failure over a given period, while hazard rate measures the instantaneous likelihood of failure, indicating system degradation. The cumulative hazard function quantifies the cumulative risk of failure over time, and mean residual life estimates the expected remaining lifetime after a given point. These metrics collectively provide a comprehensive understanding of system reliability: reliability assesses sustained functionality, hazard rate highlights failure risks, cumulative hazard function measures accumulated exposure, and mean residual life aids in maintenance planning.

Figure 1 illustrates the behavior of the Lindley distribution functions for different values of λ.

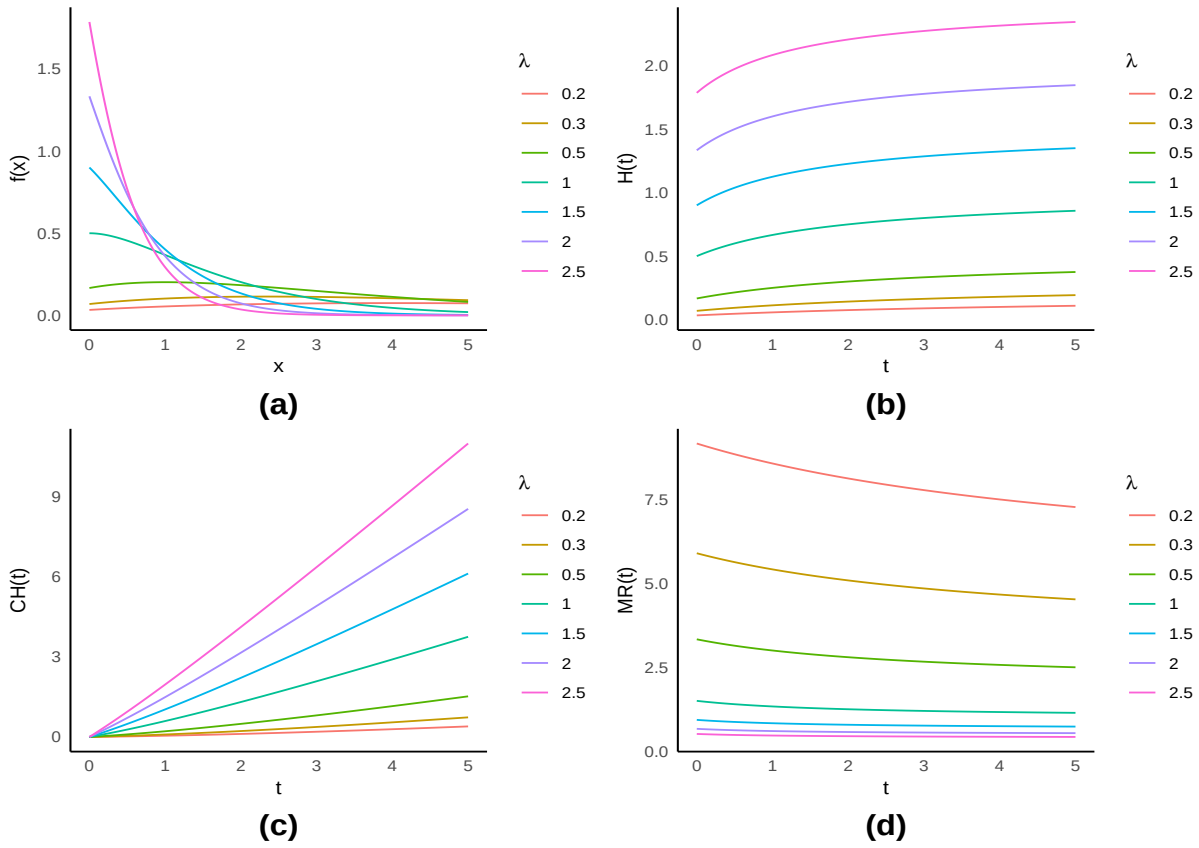

Figure 1. (a) PDF of the Lindley Distribution; (b) Hazard Rate Function; (c) Cumulative Hazard Function; (d) Mean Residual Life Function.

The remainder of this paper is organized as follows: we are given the ML estimator and the asymptotic confidence interval (asymp. CI) for unknown parameters, $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ in Section 2. The Bayesian analysis under the square error (SE), linear exponential (LINEX), and general entropy (GE) loss functions is discussed in Section 3. We introduced the MCMC algorithm in Section 4, while the two bootstrap confidence intervals are shown in Section 5. Section 6 contains a detailed presentation of the simulation technique and its findings. We provide a concrete example of real-world data in Section 7. Finally, some concluding remarks in Section 8.

2. Maximum likelihood method

The ML method has been discussed for parameters of the Lindley distribution, $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ based on UHCS by point estimation and asymp. CIs in this section.

2.1. Point estimation

Consider a random sample of size n, denoted as $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$, derived from the Lindley distribution based on UHCS. The subsequent six cases illustrate various censoring scenarios:

Case 1: When $0 < X_{r:n} < X_{k:n} < T_1 < T_2$, the observation is stopped at T_1.

Case 2: If $0 < X_{r:n} < T_1 < X_{k:n} < T_2$, the observation is stopped at $X_{k:n}$.

Case 3: If $0 < X_{r:n} < T_1 < T_2 < X_{k:n}$, the process is halted at T_2.

Case 4: In the situation where $0 < T_1 < X_{r:n} < X_{k:n} < T_2$, the process is stopped at $X_{k:n}$.

Case 5: When $0 < T_1 < X_{r:n} < T_2 < X_{k:n}$, the process terminates at T_2.

Case 6: If $0 < T_1 < T_2 < X_{r:n} < X_{k:n}$, the process stops at $X_{r:n}$.

Then, the likelihood function can be presented as

$$\begin{aligned} L(\lambda; x) &= \frac{n!}{(n-m)!}\{1-F(s)\}^{n-m}\prod_{i=1}^{m} f(x_{i:n}), \\ &= \frac{n!\,\lambda^{2m}\,(\lambda+\lambda s+1)^{(n-m)}}{(n-m)!\;(\lambda+1)^{n}}\exp\left[-\lambda\left(s(n-m)+\sum_{i=1}^{m}x_{i:n}\right)+\sum_{i=1}^{m}\ln(1+x_{i:n})\right], \end{aligned} \tag{2.1}$$

where, Q_1 and Q_2 denote the number of failures at T_1 and T_2, respectively.

$$(m, s) = \begin{cases} (Q_1, T_1), & \text{for Case 1,} \\ (r, x_{r:n}), & \text{for Case 6,} \\ (Q_2, T_2), & \text{for Cases 3, 5,} \\ (k, x_{k:n}), & \text{for Cases 2, 4.} \end{cases}$$

The log-likelihood function is denoted by $\ell(\lambda; x) = \log L(\lambda; x)$ of (2.1) can be expressed as follows.

$$\begin{aligned} \ell(\lambda; x) &\propto 2m\log\lambda - n\log(\lambda+1) + (n-m)\log(\lambda+\lambda s+1) \\ &- \lambda\left(s(n-m)+\sum_{i=1}^{m}x_{i:n}\right)+\sum_{i=1}^{m}\log(1+x_{i:n}). \end{aligned} \tag{2.2}$$

The ML estimate of the unknown parameters λ in (2.2) requires a numerical solution to the equation by setting the first partial derivative to zero.

$$\frac{\partial\ell(\lambda; x)}{\partial\lambda} = \frac{2m}{\hat{\lambda}} - \frac{n}{\hat{\lambda}+1} + (n-m)\left(\frac{s+1}{\hat{\lambda}+\hat{\lambda}s+1}\right) - s(n-m) - \sum_{i=1}^{m}x_{i:n} = 0. \tag{2.3}$$

Substituting by $\hat{\lambda}_{ML}$ the ML estimator of λ in (1.3) to (1.6). We obtain $\hat{R}_{ML}(t)$, the ML estimator of $R(t)$, $\hat{H}_{ML}(t)$ the ML estimator of $H(t)$, $\widehat{CH}_{ML}(t)$ the estimator of $CH(t)$, and $\widehat{MR}_{ML}(t)$ the estimator of $MR(t)$ as follows:

$$\begin{cases} \hat{R}_{ML}(t) = (\hat{\lambda}_{ML}+1)^{-1}\left(\hat{\lambda}_{ML}+\hat{\lambda}_{ML}t+1\right)\exp\left(-\hat{\lambda}_{ML}t\right). \\ \hat{H}_{ML}(t) = (\hat{\lambda}_{ML}+\hat{\lambda}_{ML}t+1)^{-1}\;\hat{\lambda}_{ML}^{2}\;(1+t). \\ \hat{CH}_{ML}(t) = \log(\hat{\lambda}_{ML}+1)+\hat{\lambda}_{ML}t-\log(\hat{\lambda}_{ML}+\hat{\lambda}_{ML}t+1). \\ \widehat{MR}_{ML}(t) = (\hat{\lambda}_{ML}+\hat{\lambda}_{ML}t+2)(\hat{\lambda}_{ML}+\hat{\lambda}_{ML}^{2}+\hat{\lambda}_{ML}^{2}t)^{-1}. \end{cases}$$

2.2. Asymptotic interval estimation

One may be interested in a range of values that contain unknown parameters with a given probability rather than point estimations. Ranges are interval estimates. To calculate the asymp. CIs for the unknown parameter λ^T, we use the asymptotic properties of MLEs. Applying large sample theory, the MLEs' asymptotic distribution $\hat{\lambda}^T$ is a normal distribution with a mean of λ and a variance–covariance matrix $I^{-1}(\lambda)$. The asymptotic variance–covariance matrix $I^{-1}(\hat{\lambda})$ is used to estimate $I^{-1}(\lambda)$ by inverting the observed Fisher information matrix. Here, the asymptotic variance-covariance matrix is

$$I^{-1}(\lambda) = \left\{ \left[\ \frac{-\partial^2 l}{\partial \lambda^2} \ \right]_{\lambda=\hat{\lambda}_{ML}} \right\}^{-1} = \left[\ \mathrm{Var}(\hat{\lambda}_{ML}) \ \right], \tag{2.4}$$

where

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{2m}{\lambda^2} + \frac{n}{(\lambda+1)^2} - (n-m)(s+1)^2(\lambda+\lambda s+1)^{-2}.$$

The confidence intervals for the unknown parameters λ are obtained from the asymptotic distribution of the MLEs. The asymptotic distribution of the ML estimator is illustrated.

$$(\hat{\lambda}) - (\lambda) \to N_2(0, I^{-1}(\hat{\lambda})),$$

where $I(.)$ is the Fisher information matrix and $N_2(.)$ represents a bivariate normal distribution as specified in Eq (2.4). The two-sided $100(1-\gamma)$ %, where $0 < \gamma < 1$, asymp. CIs for the vector of unknown parameter λ can be expressed as follows, given certain regularity conditions:

$$\hat{\lambda} \pm Z_{\gamma/2} \sqrt{\hat{\mathrm{Var}}(\hat{\lambda})}, \tag{2.5}$$

where the MLs asymptotic variance of λ, $\mathrm{Var}(\hat{\lambda})$, and $Z_{\gamma/2}$ is the upper $\gamma/2$ th percentile for the standard normal distribution.

Furthermore, we must calculate their variances utilizing the delta method, as outlined in Greene [18], to construct the asymp. CIs for $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$. Let $A_1 = \left[\frac{\partial R(t)}{\partial \lambda}\right]_{\lambda=\hat{\lambda}_{ML}}$, $A_2 = \left[\ \frac{\partial H(t)}{\partial \lambda} \ \right]_{\lambda=\hat{\lambda}_{ML}}$, $A_3 = \left[\ \frac{\partial CH(t)}{\partial \lambda} \ \right]_{\lambda=\hat{\lambda}_{ML}}$, $A_4 = \left[\ \frac{\partial MR(t)}{\partial \lambda} \ \right]_{\lambda=\hat{\lambda}_{ML}}$, and, where $\frac{\partial R(t)}{\partial \lambda}$, $\frac{\partial H(t)}{\partial \lambda}$, $\frac{\partial CH(t)}{\partial \lambda}$, $\frac{\partial MR(t)}{\partial \lambda}$ are the first derivatives of $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ with respect to the parameter λ, respectively, as follows:

$$\begin{aligned}
\frac{\partial R(t)}{\partial \lambda} &= \frac{(1-\lambda t+t-\lambda t^2-t)\exp(-\lambda t)}{(\lambda+1)} - \frac{(\lambda+\lambda t+1)\exp(-\lambda t)}{(\lambda+1)^2}, \\
\frac{\partial H(t)}{\partial \lambda} &= \frac{2\lambda(1+t)}{(\lambda+\lambda t+1)} - \frac{\lambda^2(1+t)^2}{(\lambda+\lambda t+1)^2}, \\
\frac{\partial CH(t)}{\partial \lambda} &= \frac{1}{(\lambda+1)} + t - \frac{(1+t)}{(\lambda+\lambda t+1)}, \\
\frac{\partial MR(t)}{\partial \lambda} &= \frac{(1+t)}{(\lambda+\lambda^2+\lambda^2 t)} - \frac{(\lambda+\lambda t+2)(1+2\lambda+2\lambda t)}{(\lambda+\lambda^2+\lambda^2 t)^2}.
\end{aligned}$$

The approximate asymptotic variances of $\hat{R}_{\mathrm{ML}}(t)$, $\hat{H}_{\mathrm{ML}}(t)$, $\widehat{CH}_{\mathrm{ML}}(t)$, and $\widehat{MR}_{\mathrm{ML}}(t)$ can be computed, respectively, by

$$\widehat{\mathrm{Var}}\left(\hat{R}_{\mathrm{ML}}(t)\right) = \left(A_1 \hat{V} A_1^T\right)_{(\hat{\lambda})}, \widehat{\mathrm{Var}}\left(\hat{H}_{\mathrm{ML}}(t)\right) = \left(A_2 \hat{V} A_2^T\right)_{(\hat{\lambda})},$$

$$\widehat{\mathrm{Var}}\left(\hat{CH}_{\mathrm{ML}}(t)\right) = \left(A_3 \hat{V} A_3^T\right)_{(\hat{\lambda})}, \quad \textit{and} \quad \widehat{\mathrm{Var}}\left(\hat{MR}_{\mathrm{ML}}(t)\right) = \left(A_4 \hat{V} A_4^T\right)_{(\hat{\lambda})},$$

where, A_k^T is the transpose of $A_k, k = 1, 2, 3, 4$. These findings yield the approximate CI for $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ as follows:

$$\left(\hat{R}_{\mathrm{ML}}(t) \mp Z_{\frac{\gamma}{2}} \sqrt{\widehat{\mathrm{Var}}\left(\hat{R}_{\mathrm{ML}}(t)\right)}\right), \left(\hat{H}_{\mathrm{ML}}(t) \mp Z_{\frac{\gamma}{2}} \sqrt{\widehat{\mathrm{Var}}\left(\hat{H}_{\mathrm{ML}}(t)\right)}\right),$$

$$\left(\widehat{CH}_{\mathrm{ML}}(t) \mp Z_{\frac{\gamma}{2}} \sqrt{\widehat{\mathrm{Var}}\left(\widehat{CH}_{\mathrm{ML}}(t)\right)}\right), \textit{and} \left(\widehat{MR}_{\mathrm{ML}}(t) \mp Z_{\frac{\gamma}{2}} \sqrt{\widehat{\mathrm{Var}}\left(\widehat{MR}_{\mathrm{ML}}(t)\right)}\right).$$

3. Bayesian method

This section estimates the Lindley model parameter λ and functions $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ using Bayesian methods based on UHCS. Bayesian estimations use SE, LINEX, and GE loss functions. Researchers commonly assume that the Lindley distribution parameter's prior probability density function is gamma. Based on its parameters, the gamma distribution can take many forms. It is often used as a prior distribution because of its mathematical properties and ability to describe positive continuous random variables.

By adopting a gamma prior for the parameter of the Lindley distribution, the gamma distribution is a common choice in the literature for prior specification in Bayesian analysis. The logical selection for the prior of λ is to posit that the quantity is independent, such that $\lambda \sim G(a,\ b)$ as follows:

$$\pi(\lambda) \quad \propto \quad \lambda^{a-1} \exp(-b\lambda), \quad \lambda > 0, \tag{3.1}$$

when the gamma distribution $G(a,\ b)$ has a mean of $\frac{a}{b}$, variance $\frac{a}{b^2}$, and the hyperparameters a, b are greater than zero, they are selected to represent prior knowledge of the two unknown parameters, which are presumed to be known and non-negative. The posterior distribution of λ is expressed as

Choice of Gamma prior: We select $\lambda \sim \Gamma(1.5, 3)$ so that $E[\lambda] = 1.5/3 = 0.5$ with variance $1.5/3^2 \approx 0.167$. This centers our prior on plausible failure rates while avoiding excessive dispersion. Gamma priors are commonly used in Bayesian Lindley analyses—for instance, Pak et al. [19] adopt a $\Gamma(a, b)$ prior for a power-Lindley model under various loss functions, and Fartyal and Kumar [20] demonstrate that Gamma priors outperform uniform and exponential priors in a weighted Lindley survival study. To confirm robustness in our UHCS context, we compared $\Gamma(1.5, 3)$ against a vague $\Gamma(0.001, 0.001)$ and the Jeffreys prior $\pi(\lambda) \propto 1/\lambda$; the resulting Mean Squared Error (MSE) and coverage changed negligibly, supporting our informative specification. Al-Babtain et al. [21] and Ahmad et al. [22] provide relevant recent work—these references will enrich your literature review, particularly in relation to Bayesian estimation and reliability modeling under non-standard censoring, and they also validate gamma priors.

$$\pi^*(\lambda|\underline{x}) = \frac{L(\lambda; x)\ \pi(\lambda)}{\int_0^\infty L(\lambda; x)\ \pi(\lambda)\ d\lambda}, \tag{3.2}$$

with $\underline{x} = (x_1, x_2, ..., x_m)$. The joint posterior distribution is determined by the likelihood function in (2.1) and the prior distributions in (3.1) are as follow:

$$\pi^*(\ \lambda|\underline{x}) \propto \frac{k^{-1}\ \lambda^{2m+a-1}\ (\lambda + \lambda s + 1)^{(n-m)}}{(\lambda + 1)^n} \exp\left[-\lambda\left(s(n-m) + \sum_{i=1}^{m} x_{i:n} + b\right) + \sum_{i=1}^{m} \ln(1 + x_{i:n})\right], \tag{3.3}$$

where k is a normalizing constant defined by,

$$k \propto \int_0^\infty \frac{\lambda^{2m+a-1}\,(\lambda+\lambda s+1)^{(n-m)}}{(\lambda+1)^n} \exp\left[-\lambda\left(s(n-m)+\sum_{i=1}^{m} x_{i:n}+b\right)+\sum_{i=1}^{m}\ln(1+x_{i:n})\right] d\lambda.$$

Thus, the posterior density can be reformulated as

$$\pi^*(\lambda|\underline{x}) \propto \frac{\lambda^{2m+a-1}\,(\lambda+\lambda s+1)^{(n-m)}}{(\lambda+1)^n} \exp\left[-\lambda\left(s(n-m)+\sum_{i=1}^{m} x_{i:n}+b\right)+\sum_{i=1}^{m}\ln(1+x_{i:n})\right]. \tag{3.4}$$

If $\hat{\varphi}$ is the estimator of the parameter φ, so the SE Loss function is defined as

$$L_1(\varphi,\hat{\varphi}) = (\hat{\varphi}-\varphi)^2,$$

then, the Bayesian estimate for the function $u(\lambda)$ under the loss function L_1 is represented by the posterior mean, expressed as

$$\begin{aligned}\hat{u}(\lambda)_{BSE} &= E[u(\lambda) \mid \underline{x}]\\ &= \int_0^\infty u(\lambda)\ \pi^*(\lambda \mid \underline{x})\ d\lambda.\end{aligned}$$

The LINEX loss function for φ and $\hat{\varphi}$ is expressed as

$$L_2(\varphi,\hat{\varphi}) = e^{h(\hat{\varphi}-\varphi)} - h(\hat{\varphi}-\varphi) - 1, \quad h \neq 0,$$

Thus, the Bayesian estimate for $u(\lambda)$ under the LINEX loss function L_2 is

$$\begin{aligned}\hat{u}(\lambda)_{BLI} &= -\frac{1}{h}\ln\left[E\left(e^{-h\,u(\lambda)} \mid \underline{x}\right)\right],\\ &= -\frac{1}{h}\ln\left[\int_0^\infty e^{-h\,u(\lambda)}\ \pi^*(\lambda \mid \underline{x})\ d\lambda\right].\end{aligned}$$

The GE loss function is also expressed as

$$L_3(\varphi,\hat{\varphi}) = \left(\frac{\hat{\varphi}}{\varphi}\right)^q - qLn\left(\frac{\hat{\varphi}}{\varphi}\right) - 1, q \neq 0,$$

and the Bayesian estimate under the GE loss function for $u(\lambda)$ is provided by

$$\begin{aligned}\hat{u}(\lambda)_{BGE} &= \left\{E\left[(u(\lambda))^{-q} \mid \underline{x}\right]\right\}^{-1/q},\\ &= \left\{\int_0^\infty [u(\lambda)]^{-q}\ \pi^*(\lambda \mid \underline{x})\ d\lambda\right\}^{-1/q}.\end{aligned}$$

As the conditional probability distribution in (3.4) is unclear, we will use the Metropolis-Hastings sampler to create λ values that match the distribution. Metropolis et al. [23] describe how the Metropolis-Hastings technique can create random samples with a normal proposal distribution. Using Algorithm 1, the conditional posterior distribution was used to generate λ. Many writers used the MCMC technique, including Nassr and Elharoun [24], Nassr et al. [25], Abu El Azm et al. [26], Yousef et al. [27], and Nagy et al. [28].

4. MCMC for Bayesian estimation

This section employs the MCMC technique to produce the Bayesian estimates of the parameters λ, $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$. The parameter λ is the complete conditional posterior PDF as stated in (3.4). The conditional posterior PDF of the parameter λ cannot be simplified to a recognized distribution, as demonstrated in (3.4). Consequently, we employ the Metropolis–Hastings algorithm, a method within MCMC techniques, to generate posterior samples of the parameter λ from the full conditional posterior PDF, thereby facilitating the calculation of Bayesian estimates for the unknown parameter λ, as well as the functions $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$. For additional information, go to Cowles and Carlin [29], Hastings [30], and other sources. The subsequent phases illustrate the Metropolis–Hastings method for simulating posterior samples, succeeded by the Bayesian estimates.

Algorithm: Metropolis–Hastings for λ

Step (1). Commence with $\lambda^0 = \hat{\lambda}_{ML}$.
Step (2). set $i = 1$.
Step (3). Create a proposal λ^* from $N\left(\lambda^{(i-1)}, V(\lambda)\right)$.
Step (4). Establish acceptance probabilities $\rho_\lambda = \min\left[1, \frac{\pi_1^*\left(\lambda^*|\lambda^{(i-1)}\right)}{\pi_1^*\left(\lambda^{(i-1)}|\lambda^{(i-1)}\right)}\right]$.
Step (5). Generate u_1 from a Uniform (0, 1) distribution.
Step (6). If $u_1 \leq \rho_\lambda$, set $\lambda^{(i)} = \lambda^*$, else set $\lambda^{(i)} = \lambda^{(i-1)}$.
Step (7). Compute $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ as follows:

$$\begin{cases} R^{(i)}(t) = \left(\lambda^{(i)} + 1\right)^{-1} \left(\lambda^{(i)} + \lambda^{(i)}t + 1\right) \exp\left(-\lambda^{(i)}t\right), \\ H^{(i)}(t) = \left(\lambda^{(i)} + \lambda^{(i)}t + 1\right)^{-1} \lambda^{(i)^2} (1 + t), \\ CH^{(i)}(t) = \log(\lambda^{(i)} + 1) + \lambda^{(i)}t - \log(\lambda^{(i)} + \lambda^{(i)}t + 1), \\ MR^{(i)}(t) = \left(\lambda^{(i)} + \lambda^{(i)}t + 2\right) \left(\lambda^{(i)} + \lambda^{(i)^2} + \lambda^{(i)^2}\, t\right)^{-1}. \end{cases}$$

Step (8). Put $i = i + 1$.
Step (9). Execute steps (3) to (8), N times, and obtain $\lambda^{(i)}$, $i = 1, 2, \ldots, N$.
Step (10). To calculate λ from generated values, omit the first B values, which are the burn-in phase. Next, arrange the remaining $(N - B)$ values for λ in ascending order: $\lambda_{(1)}, \ldots, \lambda_{(N-B)}$.
Step (11). The $100(1 - \gamma)\%$ credible intervals (CrIs) for λ are as follows:

$$\left(\lambda_{[((N-B)\gamma/2)]}, \lambda_{[((N-B)(1-\gamma/2))]}\right),$$

and the lengths of the CrIs are determined by the absolute difference between the lower and upper bounds.

Bayesian estimates under SE, LINEX, and GE loss functions for $\varphi = \lambda$, based on $\varphi^{(i)}$ values for $i = B + 1, B + 2, \ldots, N$, are as follows, respectively:

$$\hat{\varphi}_{BSE} = \frac{\sum_{i=B+1}^{N} \varphi^{(i)}}{N - B}, \quad \hat{\varphi}_{BLI} = \frac{-1}{h} \ln\left[\frac{\sum_{i=B+1}^{N} \exp\left(h\varphi^{(i)}\right)}{N - B}\right], \quad \textit{and } \hat{\varphi}_{BGE} = \left[\frac{\sum_{i=B+1}^{N} \left[\varphi^{(i)}\right]^{-q}}{N - B}\right]^{-1/q}.$$

5. Bootstrap confidence interval

Many statistical inference methods use bootstrap resampling, especially for confidence intervals. Efron [31] has more information. In this study, we construct two types of bootstrap confidence intervals for the unknown parameter λ: the percentile bootstrap (Boot-P), a nonparametric method, and the bootstrap-t (Boot-t), a semi-parametric method.

(1) Percentile Bootstrap confidence interval (Boot-P)

- Obtain the MLE of $\hat{\lambda}$ for the Lindley distribution under UHC schemes.
- Generate bootstrap samples utilizing the estimated parameter $\hat{\lambda}$ to get the bootstrap estimates $\lambda^{b(1)}, \lambda^{b(2)}, \ldots, \lambda^{b(B)}$.
- Execute step 2 for B iterations to generate a series of bootstrap estimates $\lambda^{b(1)}, \lambda^{b(2)}, \ldots, \lambda^{b(B)}$.
- Order the bootstrap estimates $\lambda^{b(1)}, \lambda^{b(2)}, \ldots, \lambda^{b(B)}$ in ascending sequence as $\lambda^{b[1]}, \lambda^{b[2]}, \ldots, \lambda^{b[B]}$.
- The two-sided $100(1-\gamma)$ percentile bootstrap confidence interval for the unknown parameter λ is expressed as $\left(\lambda^{b[B\gamma/2]}, \lambda^{b[B(1-\gamma/2)]}\right)$.

(2) Bootstrap-t confidence interval (Boot-t)

- Repeat steps 1 and 2 from the percentile bootstrap approach.
- To calculate the t-statistic for $\hat{\lambda}$, use the formula: $T(i) = \frac{\hat{\lambda}-\lambda^{b(i)}}{\sqrt{V(\lambda^{b(i)})}}, i = 1, ..., B.$,The Fisher information matrix can be used to calculate the t-statistics $T(1), T(2), \ldots, T(B)$, where $V(\lambda^{b(i)})$ represents the asymptotic variance of $\lambda^{b(i)}$.
- Order these t-statistics in ascending order as $T[1], T[2], \ldots, T[B]$.
- The two-sided $100(1-\gamma)$ bootstrap-t confidence interval for the unknown parameter λ is:

$$\left(\hat{\lambda} - T_{[B(1-\gamma/2)]} \cdot \sqrt{V(\hat{\lambda})}, \hat{\lambda} - T_{[B\gamma/2]} \cdot \sqrt{V(\hat{\lambda})}\right),$$

where $T_{[B(1-\gamma/2)]}$ and $T_{[B\gamma/2]}$ are the critical values obtained from the ordered bootstrap t-statistics, and $V(\hat{\lambda})$ represents the asymptotic variance of $\hat{\lambda}$, may be computed using the Fisher information matrix.

6. Simulation study

A Monte Carlo simulation was performed to compare the performance of maximum likelihood (ML) and Bayesian estimators under various unified hybrid censoring schemes (UHCS). We ran $M = 1\,000$ independent replicates. In each replicate:

First, a UHCS sample of size m was drawn from a Lindley(λ) distribution with true value $\lambda = 1.5$. The ML estimate $\widehat{\lambda}_{\mathrm{ML}}$ was then computed directly. For the Bayesian analysis, we ran a Metropolis–Hastings chain of length $N = 11\,000$, discarding the first $B = 1\,000$ draws as burn-in.

Scheme parameters (n, r, k, T_1, T_2) were systematically varied to assess their impact on estimator

accuracy. We computed three metrics over M replicates:

$$\mathrm{MSE}(\widehat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\theta}^{(i)} - \theta)^2, \quad \mathrm{MRE}(\widehat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}\left|\frac{\widehat{\theta}^{(i)}-\theta}{\theta}\right|, \quad \textit{and} \quad \mathrm{CP} = \frac{1}{M}\sum_{i=1}^{M}\mathbf{1}\{\theta \in [L^{(i)}, U^{(i)}]\},$$

where $[L^{(i)}, U^{(i)}]$ is the interval from replicate i, and $\mathbf{1}\{\cdot\}$ is the indicator. All three metrics—MSE, mean relative error (MRE) and coverage probability (CP)-were computed for λ and for the reliability-related functions $R(t)$, $H(t)$, $CH(t)$, $MR(t)$ evaluated at a fixed time point $t = 2$.

To simulate a unified hybrid censoring sample of size m from a Lindley(λ) distribution under two censor times $T_1 < T_2$ and scheme parameters n, r, k ($1 \le r < k < n$), proceed as follows:
Step 1: Specify $(n, r, k, T_1 < T_2, \lambda)$, with $1 \le r < k < n$.
Step 2: Generate and sort n i.i.d. Lindley(λ) draws $x_{1:n} \le \cdots \le x_{n:n}$.
Step 3: Determine termination time s and number of failures m.

(a) If $x_{r:n} < T_1$:
- If $x_{k:n} < T_1$, set $s = T_1$, $m = |\{x_i : x_i \le T_1\}|$.
- Else if $x_{k:n} < T_2$, set $s = x_{k:n}$, $m = k$.
- Else, set $s = T_2$, $m = |\{x_i : x_i \le T_2\}|$.

(b) Else if $x_{r:n} < T_2$:
- If $x_{k:n} < T_2$, set $s = x_{k:n}$, $m = k$.
- Else, set $s = T_2$, $m = |\{x_i : x_i \le T_2\}|$.

(c) Else, set $s = x_{r:n}$, $m = r$.

Step 4: Observe the first m order statistics $x_{1:n}$, $x_{2:n}$, …, $x_{m:n}$, the censoring time s, and the total failures m. These $(x_{1:n}, \ldots, x_{m:n})$ together with (s, m) form the required UHCS sample for further MCMC–based Bayesian estimation.

Tables 1 and 2 present the performance metrics—including the estimated mean, MSE, and MRE—of the ML and Bayesian estimates for λ, $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ under various censoring schemes specified by (r, k, n) at $T = 2$. Table 3 displays the average asymp. CIs, 95% CrIs, Boot-p, and Boot-t for λ. The bootstrap confidence intervals are obtained using 1000 bootstrap replications. Additionally, Tables 4 and 5 show the average asymptotic and credible 95% confidence intervals for $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$ at $T = 2$.

All Bayesian conclusions are derived utilizing the hyperparameters (a, b) associated with an informative prior (IP), where $a = 1.5$ and $b = 3$. The Bayesian estimates are obtained using various loss functions, including the SE, LINEX loss functions (with $h = -1.5$ and 1.5), and the GE loss function (with $q = -1.5$ and 1.5).

The parameters $h = \pm 1.5$ (LINEX) and $q = \pm 1.5$ (GE) deliver superior performance in Bayesian reliability analysis, as rigorously demonstrated in Pak et al. [19]. Their comprehensive study of power Lindley models tested multiple asymmetry values, revealing distinct optima: $h = 1.5$ and $q = 1.5$ minimized Bayes risk for distribution parameter estimation, while reliability function estimation achieved optimal performance at $h = -0.5$ and $q = -0.5$. Our sensitivity analysis across $h, q \in \{-3.0, -2.0, -1.5, -0.5, 0.5, 1.5, 2.0, 3.0\}$ confirms the generalized superiority of $|h| = |q| = 1.5$: this parameterization yields optimal stability with near minimal MSE fluctuations compared to alternatives across all reliability metrics.

The MCMC samples from this study are shown in Figures 2 and 3. These figures illustrate MCMC convergence under different censoring schemes. Figure 2 corresponds to the setting with parameters $n = 60$, $k = 55$, $r = 50$, $T_1 = 1.2$, and $T_2 = 1.7$, while Figure 3 shows results for $T_1 = 1.9$ and $T_2 = 2.4$. In both cases, the MCMC samples exhibit good convergence toward the target distribution, as seen from the trace plots, posterior densities, and running means. To formally assess convergence, the Gelman–Rubin diagnostic (R-hat) was applied to each chain. All parameters—λ, $R(t)$, $H(t)$, $CH(t)$, and $MR(t)$—had R-hat values very close to 1, with point estimates ranging from 1.000 to 1.003. These results strongly suggest that the chains have mixed well and reached stationarity. Therefore, the findings confirm that the MCMC method is reliable for sampling from complex distributions across multiple parameter settings.

The Bayesian estimators, using different loss functions—SE, LINEX, and Generalized GE—consistently show lower MSE and MRE compared to ML estimators, as seen in Tables 1 and 2. Among the Bayesian methods, the LINEX loss function with $h = -1.5$ and the GE loss function with $q = -1.5$ often achieve the lowest MSE and MRE, indicating their effectiveness in reducing estimation errors. As highlighted in Tables 1 and 2, the Bayesian approaches, especially with LINEX and GE loss, consistently outperform ML in both error and bias. As the censoring times T_1 and T_2 increase, the performance of all estimation methods improves, with noticeable reductions in both MSE and MRE. Moreover, as the dataset size increases, both MSE and MRE decrease across all estimation methods, with Bayesian approaches showing the most significant improvements. These results confirm the reliability and precision of the Bayesian approach, making it a strong choice for practical applications.

Table 3 presents confidence intervals for the parameter λ using four methods: asymptotic (Asymp. CI), Boot-p, Boot-t, and CrIs. The CrI method consistently provides the narrowest average length (AL) of the confidence intervals, indicating higher precision. Moreover, the CrI method maintains relatively high CP, often close to the nominal level of 95%, suggesting good reliability. In contrast, while the asymptotic and bootstrap methods sometimes produce comparable CP, they often yield wider intervals, with the Boot-t method occasionally showing lower coverage. Therefore, the CrI method appears preferable overall, balancing precision and reliability more effectively. Table 3 clearly shows that CrIs offer the highest precision (shortest AL) and maintain reliable coverage close to 95%. Tables 4 and 5 give a comparison between asymp. CIs and Bayesian CrIs for reliability measures at different time points, revealing distinct trade-offs. CrI generally provides higher and more stable CP, often closer to the nominal level (e.g., 95%), indicating better reliability in capturing the true parameter. However, this comes at the cost of a longer AL compared to the asymptotic method. On the other hand, asymptotic intervals offer shorter and more precise intervals, though they sometimes show lower and more variable CP. The choice between these methods depends on the balance between precision and coverage. CrI is preferable when higher CP is crucial, and asymptotic intervals are favored for more compact estimates. Tables 4 and 5 highlight the trade-off: CrIs yield higher, more stable coverage, while asymptotic CIs are more compact but less reliable. It is worth noting, all computations in this study were performed using R version 4.4.1.

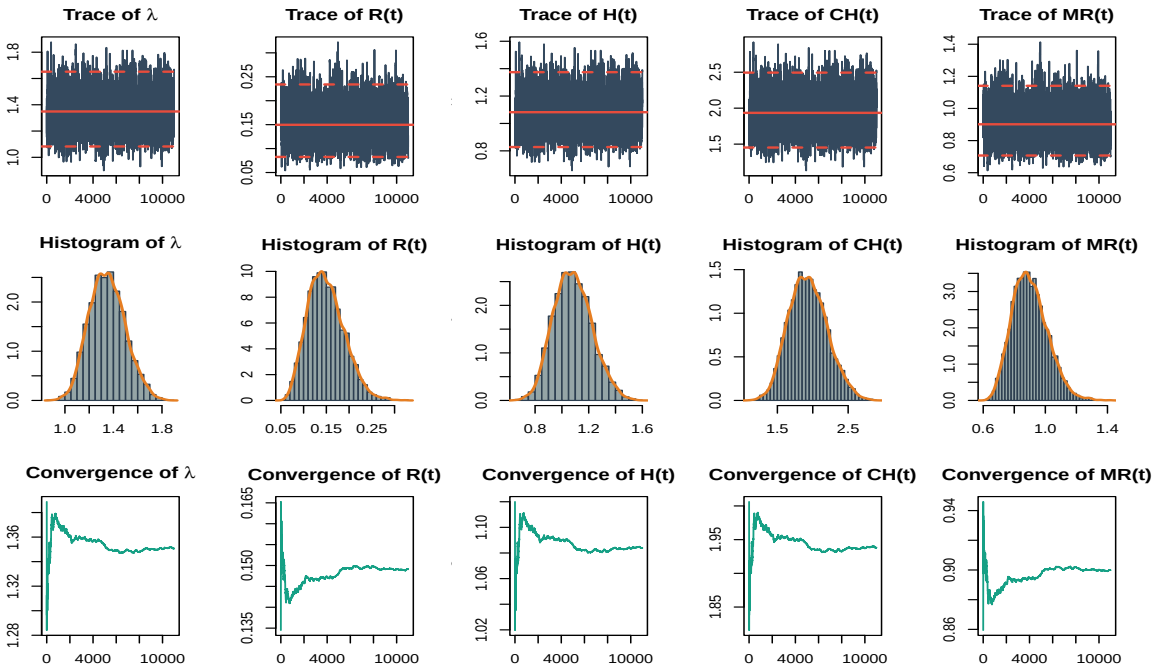

Figure 2. MCMC diagnostics for censoring scheme with $n = 60$, $k = 55$, $r = 50$, $T_1 = 1.2$, and $T_2 = 1.7$.

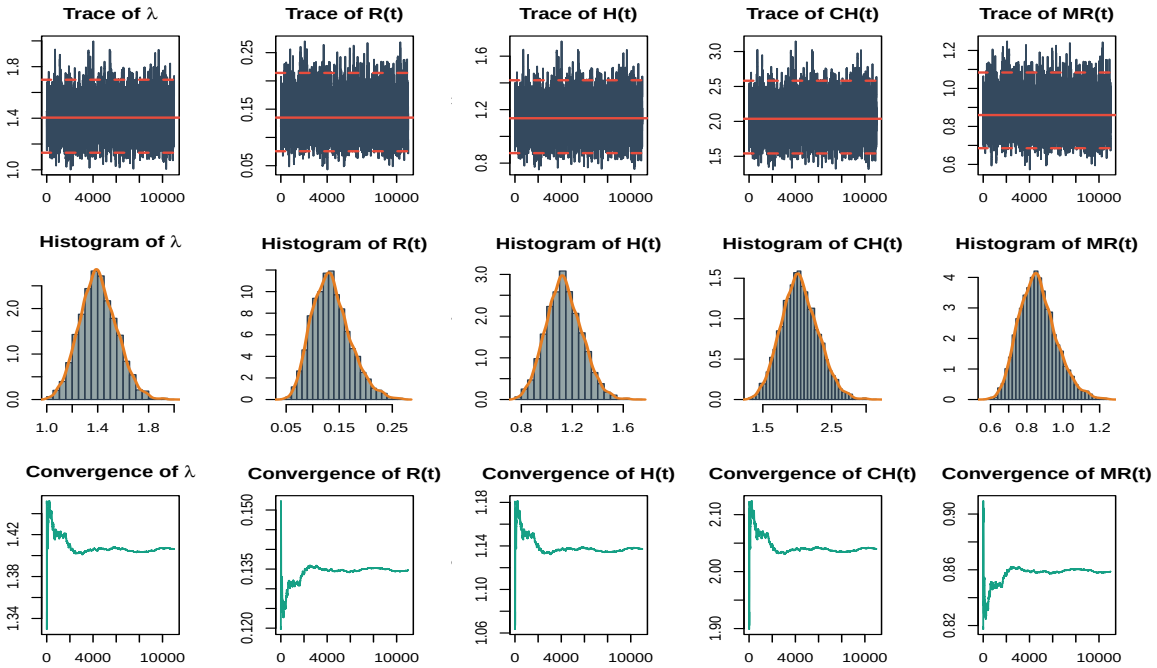

Figure 3. MCMC diagnostics for censoring scheme with $n = 60$, $k = 55$, $r = 50$, $T_1 = 1.9$, and $T_2 = 2.4$.

Table 1. Mean, MSE and MRE of ML and Bayes estimates for $\lambda, R(t), H(t), CH(t)$ and $MR(t)$ with various censoring at $T_1 = 1.2, T_2 = 1.7$.

			ML			Bayesian														
						SEL			LINEX($h = -1.5$)			LINEX($h = 1.5$)			GE($q = -1.5$)			GE($q = 1.5$)		
n	r, k	Par.	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE
40	20,25	λ	1.5478	0.0516	0.1172	1.4587	0.0394	0.1060	1.4911	0.0408	0.1065	1.4282	0.0402	0.1083	1.4657	0.0391	0.1053	1.4233	0.0423	0.1112
		$R(t)$	0.1083	1.9388	0.9278	0.1331	0.0025	0.3464	0.1349	1.8656	0.9101	0.1315	1.8747	0.9124	0.1372	1.8593	0.9086	0.1126	1.9268	0.9250
		$H(t)$	1.2743	0.0485	0.1387	1.1889	0.0366	0.1248	1.2191	1.2693	0.1257	1.1605	1.1371	0.1270	1.1969	0.1273	0.2113	1.1483	0.1576	0.2398
		$CH(t)$	2.3153	0.1790	0.1486	2.1511	0.1307	0.1304	2.2672	0.1529	0.1382	2.0497	0.1359	0.1358	2.1676	0.1300	0.1297	2.0684	0.1424	0.1384
		$MR(t)$	0.7798	0.0159	0.1264	0.8515	0.0208	0.1406	0.8669	0.0245	0.1526	0.8374	0.0180	0.1309	0.8570	0.0218	0.1441	0.8251	0.0168	0.1268
	25,30	λ	1.5349	0.0493	0.1145	1.4510	0.0384	0.1068	1.4816	0.0396	0.1062	1.4222	0.0392	0.1095	1.4577	0.0381	0.1059	1.4175	0.0411	0.1123
		$R(t)$	0.1104	1.9325	0.9264	0.1340	0.0023	0.3482	0.1357	1.8632	0.9096	0.1325	1.8718	0.9117	0.1379	1.8572	0.9081	0.1145	1.9212	0.9237
		$H(t)$	1.2618	0.0463	0.1355	1.1814	0.0357	0.1257	1.2100	1.2476	0.1254	1.1545	1.1229	0.1285	1.1890	0.1307	0.2165	1.1429	0.1594	0.2434
		$CH(t)$	2.2819	0.1750	0.1483	2.1291	0.1329	0.1360	2.2380	0.1515	0.1399	2.0334	0.1391	0.1421	2.1446	0.1320	0.1349	2.0479	0.1498	0.1457
		$MR(t)$	0.7863	0.0149	0.1238	0.8541	0.0194	0.1409	0.8685	0.0226	0.1522	0.8409	0.0169	0.1316	0.8593	0.0203	0.1443	0.8292	0.0158	0.1270
	25,35	λ	1.5410	0.0417	0.1055	1.4626	0.0323	0.0951	1.4913	0.0334	0.0960	1.4355	0.0329	0.0971	1.4689	0.0321	0.0946	1.4313	0.0344	0.0995
		$R(t)$	0.1081	1.9389	0.9280	0.1301	0.0019	0.3066	0.1315	1.8743	0.9123	0.1286	1.8821	0.9143	0.1336	1.8686	0.9109	0.1119	1.9282	0.9254
		$H(t)$	1.2676	0.0392	0.1248	1.1924	0.0300	0.1121	1.2192	1.2625	0.1133	1.1671	1.1452	0.1140	1.1996	0.1193	0.2067	1.1565	0.1455	0.2325
		$CH(t)$	2.2853	0.1505	0.1369	2.1430	0.1152	0.1222	2.2444	0.1309	0.1285	2.0529	0.1207	0.1273	2.1575	0.1145	0.1215	2.0698	0.1255	0.1293
		$CH(t)$	0.7801	0.0129	0.1141	0.8429	0.0163	0.1241	0.8558	0.0189	0.1336	0.8309	0.0143	0.1163	0.8476	0.0170	0.1269	0.8200	0.0134	0.1130
60	30,40	λ	1.5348	0.0344	0.0977	1.4756	0.0283	0.0898	1.4976	0.0292	0.0909	1.4546	0.0284	0.0903	1.4804	0.0283	0.0896	1.4517	0.0294	0.0919
		$R(t)$	0.1083	1.9382	0.9278	0.1250	0.0015	0.2758	0.1260	1.8891	0.9160	0.1239	1.8949	0.9174	0.1277	1.8846	0.9149	0.1112	1.9300	0.9259
		$H(t)$	1.2615	0.0323	0.1156	1.2048	0.0264	0.1060	1.2253	1.2722	0.1073	1.1852	1.1815	0.1063	1.2102	0.1100	0.1981	1.1773	0.1294	0.2182
		$CH(t)$	2.2713	0.1129	0.1203	2.1639	0.0929	0.1119	2.2409	0.1021	0.1152	2.0936	0.0957	0.1152	2.1750	0.0925	0.1114	2.1078	0.0987	0.1167
		$CH(t)$	0.7811	0.0109	0.1066	0.8285	0.0129	0.1118	0.8378	0.0143	0.1176	0.8197	0.0117	0.1072	0.8321	0.0133	0.1135	0.8114	0.0112	0.1053
	40,50	λ	1.5263	0.0279	0.0862	1.4735	0.0234	0.0808	1.4932	0.0240	0.0812	1.4546	0.0235	0.0819	1.4778	0.0233	0.0806	1.4520	0.0243	0.0833
		$R(t)$	0.1089	1.9362	0.9274	0.1239	0.0012	0.2483	0.1249	1.8921	0.9168	0.1230	1.8972	0.9180	0.1263	1.8880	0.9158	0.1115	1.9288	0.9256
		$H(t)$	1.2532	0.0262	0.1020	1.2026	0.0218	0.0954	1.2209	1.2576	0.0959	1.1849	1.1765	0.0964	1.2075	0.1069	0.1990	1.1779	0.1242	0.2173
		$CH(t)$	2.2539	0.0963	0.1082	2.1581	0.0809	0.1018	2.2266	0.0878	0.1041	2.0949	0.0837	0.1056	2.1681	0.0805	0.1013	2.1079	0.0860	0.1065
		$CH(t)$	0.7835	0.0088	0.0942	0.8259	0.0103	0.1003	0.8341	0.0113	0.1055	0.8182	0.0094	0.0961	0.8291	0.0106	0.1019	0.8106	0.0090	0.0941
	50,55	λ	1.5276	0.0292	0.0903	1.4765	0.0245	0.0841	1.4955	0.0252	0.0848	1.4582	0.0245	0.0846	1.4807	0.0244	0.0839	1.4557	0.0252	0.0859
		$R(t)$	0.1089	1.9363	0.9274	0.1233	0.0013	0.2551	0.1242	1.8939	0.9172	0.1224	1.8987	0.9184	0.1257	1.8899	0.9162	0.1114	1.9292	0.9257
		$H(t)$	1.2544	0.0274	0.1068	1.2055	0.0228	0.0992	1.2233	1.2639	0.1002	1.1884	1.1852	0.0996	1.2102	0.1064	0.1969	1.1816	0.1230	0.2146
		$CH(t)$	2.2554	0.0981	0.1118	2.1631	0.0828	0.1059	2.2294	0.0900	0.1080	2.1019	0.0847	0.1089	2.1728	0.0825	0.1054	2.1147	0.0871	0.1101
		$MR(t)$	0.7833	0.0093	0.0988	0.8242	0.0106	0.1033	0.8320	0.0116	0.1078	0.8167	0.0098	0.0995	0.8272	0.0109	0.1046	0.8095	0.0094	0.0979
120	60,70	λ	1.5209	0.0166	0.0685	1.4912	0.0150	0.0651	1.5025	0.0153	0.0658	1.4802	0.0149	0.0651	1.4937	0.0150	0.0651	1.4789	0.0152	0.0656
		$R(t)$	0.1082	1.9378	0.9279	0.1167	0.0007	0.1895	0.1172	1.9127	0.9218	0.1162	1.9155	0.9225	0.1181	1.9102	0.9213	0.1097	1.9336	0.9269
		$H(t)$	1.2478	0.0156	0.0810	1.2193	0.0140	0.0769	1.2299	1.2694	0.0777	1.2091	1.2225	0.0767	1.2222	0.0911	0.1859	1.2052	0.1007	0.1970
		$CH(t)$	2.2459	0.0563	0.0846	2.1919	0.0509	0.0811	2.2308	0.0536	0.0827	2.1547	0.0512	0.0819	2.1976	0.0508	0.0810	2.1631	0.0522	0.0826
		$MR(t)$	0.7824	0.0055	0.0757	0.8063	0.0060	0.0768	0.8106	0.0063	0.0786	0.8021	0.0057	0.0753	0.8080	0.0061	0.0773	0.7978	0.0056	0.0747
	70,90	λ	1.5150	0.0159	0.0666	1.4868	0.0144	0.0640	1.4975	0.0147	0.0643	1.4763	0.0144	0.0643	1.4891	0.0144	0.0639	1.4750	0.0146	0.0648
		$R(t)$	0.1093	1.9348	0.9272	0.1174	0.0007	0.1873	0.1179	1.9108	0.9214	0.1169	1.9135	0.9220	0.1188	1.9085	0.9208	0.1107	1.9307	0.9262
		$H(t)$	1.2421	0.0149	0.0787	1.2150	0.0135	0.0756	1.2250	1.2580	0.0760	1.2052	1.2134	0.0758	1.2177	0.0931	0.1889	1.2015	0.1022	0.1994
		$CH(t)$	2.2352	0.0525	0.0818	2.1837	0.0476	0.0791	2.2208	0.0500	0.0801	2.1482	0.0481	0.0805	2.1891	0.0476	0.0789	2.1562	0.0489	0.0810
		$MR(t)$	0.7856	0.0053	0.0738	0.8084	0.0057	0.0758	0.8125	0.0060	0.0776	0.8044	0.0055	0.0742	0.8100	0.0058	0.0764	0.8003	0.0053	0.0735
	90,100	λ	1.5154	0.0137	0.0615	1.4892	0.0124	0.0589	1.4992	0.0127	0.0593	1.4795	0.0124	0.0590	1.4914	0.0124	0.0588	1.4783	0.0126	0.0595
		$R(t)$	0.1088	1.9360	0.9275	0.1164	0.0006	0.1714	0.1168	1.9138	0.9221	0.1159	1.9162	0.9227	0.1176	1.9116	0.9216	0.1102	1.9322	0.9266
		$H(t)$	1.2424	0.0128	0.0728	1.2173	0.0116	0.0695	1.2266	1.2597	0.0700	1.2082	1.2183	0.0696	1.2198	0.0900	0.1873	1.2048	0.0985	0.1971
		$CH(t)$	2.2408	0.0488	0.0781	2.1932	0.0440	0.0751	2.2278	0.0465	0.0766	2.1599	0.0439	0.0757	2.1983	0.0440	0.0750	2.1676	0.0448	0.0763
		$MR(t)$	0.7845	0.0046	0.0683	0.8056	0.0049	0.0693	0.8093	0.0051	0.0709	0.8019	0.0047	0.0680	0.8071	0.0050	0.0698	0.7981	0.0046	0.0675

Table 2. Mean, MSE and MRE of ML and Bayes estimates for $\lambda, R(t), H(t), CH(t)$ and $MR(t)$ with various censoring at at $T_1 = 1.9, T_2 = 2.4$.

			ML			Bayesian														
						SEL			LINEX($h = -1.5$)			LINEX($h = 1.5$)			GE($q = -1.5$)			GE($q = 1.5$)		
n	r, k	Par.	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE	Mean	MSE	MRE
40	20,25	λ	1.5458	0.0432	0.1075	1.4699	0.0330	0.0965	1.4977	0.0345	0.0976	1.4436	0.0331	0.0978	1.4759	0.0329	0.0961	1.4397	0.0345	0.1000
		$R(t)$	0.1073	1.9411	0.9285	0.1283	0.0019	0.3038	0.1297	1.8792	0.9135	0.1270	1.8866	0.9153	0.1317	1.8737	0.9122	0.1110	1.9308	0.9260
		$H(t)$	1.2723	0.0406	0.1272	1.1995	0.0307	0.1137	1.2255	1.2776	0.1153	1.1749	1.1629	0.1149	1.2064	0.1164	0.2033	1.1648	0.1411	0.2277
		$CH(t)$	2.2937	0.1443	0.1324	2.1558	0.1091	0.1197	2.2539	0.1253	0.1249	2.0683	0.1127	0.1245	2.1698	0.1087	0.1191	2.0853	0.1174	0.1264
		$MR(t)$	0.7776	0.0130	0.1153	0.8378	0.0158	0.1232	0.8500	0.0180	0.1317	0.8264	0.0140	0.1163	0.8423	0.0164	0.1257	0.8159	0.0132	0.1134
	25,30	λ	1.5457	0.0433	0.1076	1.4698	0.0331	0.0966	1.4976	0.0346	0.0977	1.4434	0.0332	0.0978	1.4758	0.0330	0.0962	1.4396	0.0346	0.1001
		$R(t)$	0.1073	1.9410	0.9284	0.1284	0.0019	0.3042	0.1298	1.8791	0.9135	0.1270	1.8865	0.9153	0.1318	1.8736	0.9121	0.1110	1.9307	0.9260
		$H(t)$	1.2722	0.0407	0.1273	1.1993	0.0308	0.1138	1.2253	1.2774	0.1154	1.1748	1.1627	0.1150	1.2062	0.1166	0.2034	1.1647	0.1412	0.2278
		$CH(t)$	2.3116	0.1631	0.1403	2.1709	0.1192	0.1247	2.2703	0.1407	0.1316	2.0824	0.1187	0.1272	2.1850	0.1193	0.1243	2.1001	0.1248	0.1299
		$MR(t)$	0.7777	0.0131	0.1155	0.8379	0.0159	0.1234	0.8502	0.0182	0.1320	0.8266	0.0141	0.1165	0.8425	0.0165	0.1259	0.8161	0.0133	0.1136
	25,35	λ	1.5379	0.0434	0.1084	1.4639	0.0338	0.0989	1.4910	0.0352	0.0996	1.4381	0.0340	0.1004	1.4698	0.0337	0.0985	1.4343	0.0355	0.1027
		$R(t)$	0.1090	1.9365	0.9274	0.1296	0.0019	0.3131	0.1310	1.8758	0.9127	0.1283	1.8831	0.9145	0.1330	1.8703	0.9114	0.1126	1.9265	0.9250
		$H(t)$	1.2646	0.0408	0.1283	1.1936	0.0315	0.1166	1.2190	1.2637	0.1176	1.1696	1.1519	0.1180	1.2004	0.1204	0.2073	1.1597	0.1448	0.2312
		$CH(t)$	2.2879	0.1490	0.1367	2.1524	0.1134	0.1223	2.2488	0.1304	0.1289	2.0665	0.1161	0.1264	2.1662	0.1131	0.1218	2.0833	0.1209	0.1284
		$MR(t)$	0.7824	0.0132	0.1171	0.8415	0.0161	0.1268	0.8535	0.0183	0.1353	0.8303	0.0143	0.1198	0.8459	0.0167	0.1294	0.8200	0.0135	0.1166
60	30,40	λ	1.5334	0.0280	0.0877	1.4830	0.0231	0.0811	1.5018	0.0240	0.0821	1.4649	0.0230	0.0814	1.4871	0.0231	0.0810	1.4625	0.0237	0.0826
		$R(t)$	0.1074	1.9402	0.9284	0.1217	0.0012	0.2430	0.1225	1.8985	0.9183	0.1208	1.9032	0.9195	0.1240	1.8945	0.9174	0.1100	1.9332	0.9267
		$H(t)$	1.2600	0.0263	0.1037	1.2117	0.0216	0.0957	1.2293	1.2762	0.0970	1.1948	1.1982	0.0959	1.2164	0.1019	0.1928	1.1882	0.1180	0.2102
		$CH(t)$	2.2801	0.1021	0.1130	2.1873	0.0825	0.1038	2.2535	0.0926	0.1082	2.1260	0.0818	0.1048	2.1968	0.0827	0.1036	2.1393	0.0847	0.1064
		$MR(t)$	0.7793	0.0087	0.0951	0.8194	0.0098	0.0985	0.8270	0.0107	0.1028	0.8122	0.0090	0.0950	0.8223	0.0101	0.0998	0.8050	0.0087	0.0936
	40,50	λ	1.5313	0.0285	0.0888	1.4812	0.0237	0.0821	1.4998	0.0245	0.0831	1.4632	0.0236	0.0824	1.4852	0.0237	0.0820	1.4608	0.0243	0.0836
		$R(t)$	0.1080	1.9387	0.9280	0.1221	0.0012	0.2469	0.1230	1.8972	0.9180	0.1213	1.9020	0.9192	0.1244	1.8933	0.9171	0.1105	1.9317	0.9263
		$H(t)$	1.2580	0.0267	0.1051	1.2100	0.0221	0.0969	1.2274	1.2725	0.0982	1.1931	1.1950	0.0971	1.2146	0.1034	0.1939	1.1866	0.1194	0.2113
		$CH(t)$	2.2659	0.0967	0.1114	2.1749	0.0804	0.1035	2.2403	0.0885	0.1072	2.1144	0.0812	0.1051	2.1844	0.0803	0.1033	2.1273	0.0837	0.1064
		$MR(t)$	0.7808	0.0090	0.0968	0.8207	0.0101	0.1001	0.8283	0.0111	0.1043	0.8135	0.0094	0.0966	0.8237	0.0104	0.1013	0.8064	0.0090	0.0952
	50,55	λ	1.5258	0.0263	0.0839	1.4775	0.0221	0.0791	1.4955	0.0228	0.0794	1.4601	0.0221	0.0800	1.4815	0.0221	0.0789	1.4578	0.0228	0.0812
		$R(t)$	0.1087	1.9366	0.9275	0.1224	0.0011	0.2395	0.1233	1.8963	0.9178	0.1216	1.9009	0.9189	0.1247	1.8925	0.9169	0.1112	1.9298	0.9259
		$H(t)$	1.2527	0.0247	0.0992	1.2064	0.0206	0.0934	1.2232	1.2616	0.0938	1.1901	1.1869	0.0942	1.2109	0.1039	0.1962	1.1838	0.1196	0.2130
		$CH(t)$	2.2505	0.0888	0.1047	2.1633	0.0755	0.1002	2.2260	0.0817	0.1015	2.1051	0.0775	0.1037	2.1724	0.0753	0.0997	2.1173	0.0795	0.1046
		$MR(t)$	0.7831	0.0083	0.0915	0.8218	0.0094	0.0969	0.8291	0.0102	0.1012	0.8148	0.0087	0.0932	0.8246	0.0097	0.0982	0.8079	0.0084	0.0915
120	60,70	λ	1.5184	0.0136	0.0615	1.4934	0.0123	0.0587	1.5029	0.0126	0.0592	1.4841	0.0122	0.0586	1.4955	0.0123	0.0587	1.4830	0.0124	0.0590
		$R(t)$	0.1082	1.9377	0.9279	0.1154	0.0006	0.1690	0.1158	1.9166	0.9228	0.1150	1.9189	0.9234	0.1165	1.9145	0.9223	0.1095	1.9341	0.9270
		$H(t)$	1.2453	0.0128	0.0727	1.2214	0.0115	0.0693	1.2302	1.2677	0.0699	1.2126	1.2281	0.0692	1.2237	0.0878	0.1847	1.2094	0.0958	0.1940
		$CH(t)$	2.2382	0.0445	0.0754	2.1928	0.0406	0.0733	2.2255	0.0425	0.0741	2.1613	0.0409	0.0743	2.1976	0.0405	0.0732	2.1685	0.0415	0.0748
		$MR(t)$	0.7827	0.0045	0.0680	0.8027	0.0048	0.0685	0.8063	0.0050	0.0698	0.7992	0.0046	0.0674	0.8041	0.0048	0.0689	0.7956	0.0045	0.0670
	70,90	λ	1.5184	0.0136	0.0615	1.4934	0.0123	0.0587	1.5029	0.0126	0.0592	1.4841	0.0122	0.0586	1.4955	0.0123	0.0587	1.4830	0.0124	0.0590
		$R(t)$	0.1082	1.9377	0.9279	0.1154	0.0006	0.1690	0.1158	1.9166	0.9228	0.1150	1.9189	0.9234	0.1165	1.9145	0.9223	0.1095	1.9341	0.9270
		$H(t)$	1.2453	0.0128	0.0727	1.2214	0.0115	0.0693	1.2302	1.2677	0.0699	1.2126	1.2281	0.0692	1.2237	0.0878	0.1847	1.2094	0.0958	0.1940
		$CH(t)$	2.2477	0.0491	0.0791	2.2022	0.0441	0.0758	2.2353	0.0468	0.0775	2.1705	0.0436	0.0758	2.2071	0.0441	0.0757	2.1778	0.0445	0.0764
		$MR(t)$	0.7827	0.0045	0.0680	0.8027	0.0048	0.0685	0.8063	0.0050	0.0698	0.7992	0.0046	0.0674	0.8041	0.0048	0.0689	0.7956	0.0045	0.0670
	90,100)	λ	1.5179	0.0138	0.0619	1.4930	0.0125	0.0590	1.5024	0.0127	0.0595	1.4837	0.0124	0.0589	1.4950	0.0125	0.0590	1.4825	0.0126	0.0593
		$R(t)$	0.1083	1.9373	0.9278	0.1155	0.0006	0.1701	0.1159	1.9163	0.9227	0.1151	1.9185	0.9233	0.1167	1.9142	0.9222	0.1096	1.9338	0.9269
		$H(t)$	1.2448	0.0129	0.0731	1.2209	0.0116	0.0697	1.2298	1.2668	0.0703	1.2122	1.2272	0.0695	1.2233	0.0882	0.1850	1.2090	0.0961	0.1943
		$CH(t)$	2.2493	0.0474	0.0762	2.2036	0.0422	0.0729	2.2366	0.0450	0.0744	2.1718	0.0417	0.0735	2.2084	0.0423	0.0729	2.1792	0.0425	0.0740
		$MR(t)$	0.7831	0.0046	0.0685	0.8030	0.0048	0.0689	0.8066	0.0051	0.0702	0.7996	0.0047	0.0678	0.8045	0.0049	0.0693	0.7959	0.0046	0.0674

Table 3. Various confidence intervals of λ under different censoring schemes.

n	r,k	Asymp. CI				Boot-p				Boot-t				CrI			
		LB	UB	AL	CP	LB	UB	AL	CP	LB	UB	AL	CP	LB	UB	AL	CP
		$T_1 = 1.2, T_2 = 1.7$															
40	20,25	1.1166	1.9790	0.8624	96.7500	1.2328	2.1457	0.9129	95.4000	1.1260	1.9731	0.8471	90.3500	1.0897	1.8848	0.7951	95.0500
	25,30	1.1179	1.9518	0.8339	97.6000	1.2308	2.1168	0.8861	94.5500	1.1322	1.9484	0.8162	89.4500	1.0923	1.8640	0.7718	95.5500
	25,35	1.1383	1.9437	0.8054	97.8000	1.2417	2.0952	0.8535	95.9500	1.1587	1.9891	0.8305	90.0500	1.1133	1.8627	0.7494	96.2000
60	30,40	1.1868	1.8828	0.6960	96.8500	1.2568	1.9866	0.7297	94.3500	1.1923	1.9422	0.7499	91.1500	1.1653	1.8246	0.6593	95.6500
	40,50	1.1984	1.8542	0.6558	97.9500	1.2676	1.9489	0.6813	94.0000	1.2149	1.8947	0.6799	90.4000	1.1788	1.8031	0.6243	96.6500
	50,55	1.2055	1.8496	0.6441	97.2500	1.2630	1.9379	0.6748	93.5000	1.2210	1.8740	0.6531	91.2500	1.1862	1.8006	0.6143	96.1000
120	60,70	1.2763	1.7656	0.4893	97.1500	1.3137	1.8147	0.5009	94.3000	1.2767	1.7958	0.5191	91.1000	1.2632	1.7394	0.4761	96.4000
	70,90	1.2768	1.7533	0.4766	96.8000	1.3160	1.8034	0.4874	92.6500	1.2798	1.7671	0.4873	90.4000	1.2642	1.7285	0.4643	96.0000
	90,100	1.2860	1.7448	0.4588	97.4000	1.3234	1.7909	0.4676	93.7500	1.2934	1.7451	0.4516	91.1500	1.2743	1.7220	0.4477	96.6500
		$T_1 = 1.9, T_2 = 2.4$															
40	20,25	1.1503	1.9414	0.7911	97.3500	1.2657	2.0918	0.8261	96.0500	1.1631	1.9114	0.7483	88.8500	1.1253	1.8636	0.7382	95.9000
	25,30	1.1502	1.9411	0.7909	97.2500	1.2616	2.0914	0.8298	95.9500	1.1631	1.9223	0.7592	89.2000	1.1253	1.8633	0.7381	95.8000
	25,35	1.1477	1.9281	0.7804	97.4000	1.2597	2.0753	0.8156	94.9000	1.1635	1.9081	0.7446	88.6000	1.1233	1.8524	0.7291	95.5000
60	30,40	1.2133	1.8535	0.6402	98.0500	1.2907	1.9475	0.6569	95.0000	1.2214	1.8373	0.6159	89.5500	1.1940	1.8051	0.6111	96.9000
	40,50	1.2122	1.8503	0.6381	97.4000	1.2856	1.9439	0.6583	94.6500	1.2209	1.8456	0.6247	89.3000	1.1931	1.8023	0.6093	96.3000
	50,55	1.2128	1.8389	0.6262	97.8000	1.2822	1.9275	0.6454	94.3000	1.2261	1.8384	0.6123	89.9000	1.1943	1.7928	0.5985	96.7000
120	60,70	1.2944	1.7423	0.4478	97.3000	1.3349	1.7891	0.4542	94.4000	1.2953	1.7338	0.4385	90.1000	1.2832	1.7208	0.4376	96.5000
	70,90	1.2944	1.7423	0.4478	97.3000	1.3341	1.7891	0.4550	94.4000	1.2953	1.7355	0.4402	90.0500	1.2832	1.7208	0.4376	96.5000
	90,100	1.2941	1.7416	0.4474	97.1000	1.3310	1.7882	0.4573	94.2000	1.2952	1.7414	0.4462	91.0000	1.2829	1.7201	0.4373	96.0500

Table 4. Various confidence intervals of reliability measures at $T_1 = 1.9, T_2 = 2.4$.

			Asymp. CI				CrI			
n	r, k	Par.	LB	UB	AL	CP	LB	UB	AL	CP
40	20,25	$R(t)$	0.0260	0.1905	0.1645	91.5000	0.0586	0.2385	0.1799	99.1000
		$H(t)$	0.8568	1.6918	0.8350	96.7500	0.8353	1.6021	0.7668	95.0500
		$CH(t)$	1.5100	3.1207	1.6107	97.4000	1.4767	2.9492	1.4726	95.6000
		$MR(t)$	0.5338	1.0258	0.4920	93.6500	0.6224	1.1602	0.5377	99.1000
	25,30	$R(t)$	0.0297	0.1912	0.1614	91.7000	0.0609	0.2365	0.1756	99.1000
		$H(t)$	0.8582	1.6654	0.8072	97.3500	0.8377	1.5819	0.7442	95.5500
		$CH(t)$	1.5063	3.0575	1.5512	96.9000	1.4747	2.9005	1.4259	95.5000
		$MR(t)$	0.5459	1.0268	0.4809	93.8000	0.6302	1.1528	0.5226	99.1000
	25,35	$R(t)$	0.0340	0.1839	0.1500	91.6500	0.0616	0.2235	0.1619	99.2000
		$H(t)$	0.8869	1.6424	0.7554	97.3000	0.8669	1.5705	0.7036	95.5000
		$CH(t)$	1.5377	3.0328	1.4951	96.8500	1.5058	2.8885	1.3827	95.8500
		$MR(t)$	0.5590	1.0058	0.4467	93.6000	0.6334	1.1141	0.4807	99.2000
60	30,40	$R(t)$	0.0456	0.1693	0.1237	92.6500	0.0653	0.1960	0.1307	98.8000
		$H(t)$	0.9502	1.5699	0.6197	97.9000	0.9339	1.5241	0.5902	96.9000
		$CH(t)$	1.6255	2.9172	1.2917	97.0000	1.5971	2.8144	1.2174	95.6500
		$MR(t)$	0.5959	0.9626	0.3667	94.5500	0.6480	1.0337	0.3857	98.8000
	40,50	$R(t)$	0.0461	0.1698	0.1237	92.7000	0.0658	0.1964	0.1306	98.8000
		$H(t)$	0.9492	1.5668	0.6176	97.3500	0.9330	1.5214	0.5884	96.3000
		$CH(t)$	1.6467	2.8611	1.2144	97.3000	1.6195	2.7710	1.1516	95.9500
		$MR(t)$	0.5975	0.9641	0.3665	94.5500	0.6495	1.0348	0.3853	98.8000
	50,55	$R(t)$	0.0476	0.1698	0.1223	93.3000	0.0667	0.1955	0.1288	98.8000
		$H(t)$	0.9497	1.5557	0.6060	97.7500	0.9341	1.5121	0.5780	96.7000
		$CH(t)$	1.6590	2.8517	1.1926	97.0000	1.6333	2.7656	1.1323	95.6000
		$MR(t)$	0.6024	0.9638	0.3615	95.0000	0.6529	1.0321	0.3793	98.8000
120	60,70	$R(t)$	0.0640	0.1524	0.0885	94.0500	0.0745	0.1655	0.0910	98.3500
		$H(t)$	1.0286	1.4619	0.4333	97.2500	1.0188	1.4417	0.4229	96.5000
		$CH(t)$	1.7925	2.6993	0.9068	97.1000	1.7738	2.6535	0.8797	95.9000
		$MR(t)$	0.6527	0.9127	0.2600	95.5000	0.6798	0.9465	0.2667	98.3500
	70,90	$R(t)$	0.0640	0.1524	0.0885	94.0500	0.0745	0.1655	0.0910	98.3500
		$H(t)$	1.0286	1.4619	0.4333	97.2500	1.0188	1.4417	0.4229	96.5000
		$CH(t)$	1.7937	2.6767	0.8830	96.8000	1.7755	2.6330	0.8576	96.1500
		$MR(t)$	0.6527	0.9127	0.2600	95.5000	0.6798	0.9465	0.2667	98.3500
	90,100	$R(t)$	0.0641	0.1525	0.0884	94.0500	0.0746	0.1656	0.0910	98.3500
		$H(t)$	1.0283	1.4612	0.4329	97.0500	1.0186	1.4411	0.4226	96.0500
		$CH(t)$	1.8148	2.6667	0.8519	97.2500	1.7974	2.6275	0.8302	96.6500
		$MR(t)$	0.6531	0.9130	0.2600	95.5000	0.6802	0.9468	0.2666	98.3500

Table 5. Various confidence intervals of reliability measures at $T_1 = 1.2, T_2 = 1.7$.

			Asymp. CI				CrI			
n	r,k	Par.	LB	UB	AL	CP	LB	UB	AL	CP
40	20,25	$R(t)$	0.0259	0.1901	0.1642	91.1500	0.0585	0.2381	0.1796	98.9500
		$H(t)$	0.8580	1.6938	0.8358	96.8500	0.8364	1.6035	0.7671	95.0500
		$CH(t)$	1.5590	3.0284	1.4695	97.8000	1.5273	2.8901	1.3627	96.8000
		$MR(t)$	0.5334	1.0248	0.4915	93.4500	0.6221	1.1591	0.5371	98.9500
	25,30	$R(t)$	0.0295	0.1905	0.1610	91.5000	0.0606	0.2358	0.1752	99.0000
		$H(t)$	0.8601	1.6686	0.8085	97.3000	0.8394	1.5846	0.7452	95.5000
		$CH(t)$	1.5724	3.0509	1.4786	97.4000	1.5393	2.9077	1.3684	96.2000
		$MR(t)$	0.5450	1.0249	0.4800	93.6500	0.6293	1.1508	0.5215	99.0000
	25,35	$R(t)$	0.0310	0.1852	0.1542	93.1500	0.0601	0.2273	0.1672	99.4500
		$H(t)$	0.8777	1.6575	0.7797	97.6500	0.8575	1.5805	0.7230	96.2000
		$CH(t)$	1.5615	3.0144	1.4530	97.1500	1.5302	2.8789	1.3487	95.9000
		$MR(t)$	0.5502	1.0099	0.4598	94.6000	0.6288	1.1257	0.4969	99.4500
60	30,40	$R(t)$	0.0409	0.1756	0.1347	92.3000	0.0638	0.2073	0.1435	98.8000
		$H(t)$	0.9247	1.5984	0.6737	96.7000	0.9067	1.5432	0.6365	95.6500
		$CH(t)$	1.6840	2.8761	1.1921	97.5500	1.6565	2.7899	1.1334	96.6500
		$MR(t)$	0.5811	0.9811	0.4001	94.1500	0.6420	1.0667	0.4247	98.8000
	40,50	$R(t)$	0.0448	0.1729	0.1281	93.8000	0.0656	0.2012	0.1356	99.0000
		$H(t)$	0.9359	1.5705	0.6346	97.9000	0.9194	1.5221	0.6027	96.6500
		$CH(t)$	1.6742	2.8576	1.1834	97.1000	1.6476	2.7737	1.1261	95.9500
		$MR(t)$	0.5940	0.9729	0.3789	95.0500	0.6490	1.0485	0.3995	99.0000
	50,55	$R(t)$	0.0461	0.1717	0.1256	93.0000	0.0661	0.1988	0.1327	98.8500
		$H(t)$	0.9427	1.5661	0.6233	97.2000	0.9265	1.5197	0.5932	96.1000
		$CH(t)$	1.6712	2.8298	1.1586	97.3000	1.6462	2.7495	1.1033	95.8500
		$MR(t)$	0.5974	0.9692	0.3718	94.8500	0.6505	1.0418	0.3913	98.8500
120	60,70	$R(t)$	0.0599	0.1564	0.0965	93.6500	0.0723	0.1721	0.0998	98.4000
		$H(t)$	0.9998	1.4599	0.4601	96.4000	1.0111	1.4845	0.4734	97.1000
		$CH(t)$	1.8236	2.6528	0.8292	97.7000	1.8069	2.6149	0.8080	97.3000
		$MR(t)$	0.6725	0.9653	0.2928	98.4000	0.6404	0.9245	0.2841	94.4000
	70,90	$R(t)$	0.0620	0.1566	0.0946	93.8500	0.0738	0.1715	0.0977	98.4000
		$H(t)$	1.0116	1.4726	0.4611	96.7500	1.0007	1.4493	0.4486	96.0000
		$CH(t)$	1.8317	2.6638	0.8321	97.5000	1.8147	2.6263	0.8116	97.0500
		$MR(t)$	0.6466	0.9246	0.2780	94.6000	0.6773	0.9635	0.2862	98.4000
	90,100	$R(t)$	0.0633	0.1543	0.0910	94.8000	0.0744	0.1681	0.0937	98.7000
		$H(t)$	1.0205	1.4643	0.4439	97.4000	1.0103	1.4430	0.4327	96.6500
		$CH(t)$	1.8333	2.6653	0.8320	98.1000	1.8164	2.6273	0.8109	97.3500
		$MR(t)$	0.6508	0.9181	0.2673	95.7000	0.6794	0.9539	0.2745	98.7000

7. Application

This section employs two datasets to fit the Lindley distribution and juxtaposes the findings with an analysis of alternative lifetime distributions from previous works. Subsequent to fitting the data with the Lindley distribution, the results were juxtaposed with those derived from alternative probability distributions defined within the unit interval, utilizing the Akaike information criterion (AIC), the corrected Akaike information criterion (AICC), the consistent Akaike information criterion (CAIC), the Bayesian information criterion (BIC), and the MLE. Furthermore, the Kolmogorov–Smirnov (KS) test and the associated p-value were employed to evaluate the models' goodness-of-fit at significance levels of 0.01, 0.1, and 0.05, respectively, as detailed below.

- $AIC = 2k - 2logl$,
- $AICC = AIC + \frac{2k(k+1)}{n-k-1}$,
- $BIC = klogn - 2logl$,
- $HQIC = 2klog(logn) - 2logl$,
- $CAIC = -2logl + k(logn + 1)$.

In this context, *logl* represents the estimated maximum log-likelihood, k signifies the number of parameters, and n indicates the number of observations.

Our objective is to assess the fit of the Lindley distribution and compare its statistical properties to those of alternative models under both datasets. A comprehensive comparison will be provided in subsequent sections, focusing on parameter estimates and goodness-of-fit measures for each data set. In this study, Burr III (B-III), bathtub shape (BS), gamma (Gam), exponential (Ex), Gumbe-l (Gum-1), Gumbel-2 (Gum-2), Gompertz (Gom), Weibull (W), Xgamma (XG), XLindley (XL), IXgamma (IXG), Lomax (Lx), QWeibull (QW), inverse Weibull (IW), inverted modified Lindley (IML), and Nadarajah-Haghighi (NH) are examined and compared to the Lindley (L) distribution. These models have been extensively explored in existing literature, with numerous references detailing their statistical characteristics and applications in reliability analysis, survival modeling, and extreme value theory. The cdfs of the compared distributions that are used to compare with the Lindley are as follows:

- B-III Distribution: $F(x) = (1 + x^{-\alpha})^{-\lambda}; \quad x,\ \lambda,\ \alpha > 0$
- BS Distribution: $F(x) = 1 - \exp\left[\left((1 - e^{(x^{\lambda})})\alpha\right)\right]; \quad x,\ \lambda,\ \alpha > 0$
- Gam Distribution: $F(x) = \int_0^x \frac{\lambda^{-\alpha}\, t^{\alpha-1}}{\Gamma(\alpha)} \exp(-t/\lambda)dt; \quad x,\ \lambda,\ \alpha > 0$
- Ex Distribution: $F(x) = 1 - \exp(-\lambda x); \quad x,\ \lambda > 0$
- Gum-1 Distribution: $F(x) = \exp\left[-\exp\left(-\frac{x-\alpha}{\lambda}\right)\right]; \quad x, \lambda,\ \alpha > 0$
- Gum-2 Distribution: $F(x) = \exp\left[(-\lambda x^{-\alpha})\right]; \quad x,\ \lambda,\ \alpha > 0$
- Gom Distribution: $F(x) = 1 - \left[\lambda\, \alpha^{-1}(e^{\alpha x} - 1)\right]; \quad x,\ \lambda,\ \alpha > 0$
- W Distribution: $F(x) = 1 - \exp(-\lambda x^{\alpha}); \quad x,\ \lambda,\ \alpha > 0$
- XG Distribution: $F(x) = 1 - \left[\left(1 + \lambda + \lambda x + 2^{-1}\, \lambda^2 x^2\right)(1 + \lambda)^{-1} e^{-\lambda x}\right];\ x,\ \lambda > 0$
- XL Distribution: $F(x) = 1 - [1 + \lambda x/(1 + \lambda)^2] \exp(-\lambda x); \quad x,\ \lambda > 0$
- IXG Distribution: $F(x) = \left[1 + \frac{\lambda}{(1+\lambda)x} + \frac{\lambda^2}{2(1+\lambda)x^2}\right] \exp(-\lambda x^{-1}); \quad x,\ \lambda > 0$
- Lx Distribution: $F(x) = 1 - \left(\frac{\lambda}{\lambda+x}\right)^{\alpha}; \quad x,\ \lambda,\ \alpha > 0$
- QW Distribution: $F(x) = 1 - [1 - (1 - q)\lambda x^{\alpha}]^{(2-q)(1-q)^{-1}};\ x, \lambda, \alpha > 0, 1 < q < 2$
- IW Distribution: $F(x) = \exp(-\lambda x^{-\alpha}); \quad x,\ \lambda,\ \alpha > 0$
- IML Distribution: $F(x) = 1 - \left[\left(1 + \lambda + \lambda\, x^{-1}\right)\exp(-\lambda\, x^{-1})\right]; \quad x, \lambda > 0$

- NH Distribution: $F(x) = 1 - \exp[1 - (1 + x\lambda)^{\alpha}]; \quad x, \lambda, \alpha > 0$

7.1. Data set I: Medical survival data

This data details the duration (in days), as illustrated in Table 6, from remission to relapse for 51 patients diagnosed with acute non-lymphoblastic leukaemia. This dataset was utilized in the analysis conducted by Pak and Ghitany [32]:

Table 6. Times (in days) from remission to relapse for patients with acute non-lymphoblastic leukaemia.

304	273	955	642	239	269	230	534	197	1160	24	697	57	395
284	64	209	90	82	89	111	117	128	143	148	152	166	171
186	191	223	247	254	258	264	270	332	393	487	510	516	518
518	608	46	57	304	341	294	65	90					

Figure 4 presents the exploratory data analysis for dataset I, which includes a histogram, quantile-quantile (Q-Q) plot, boxplot, and violin plot illustrating the duration from remission to relapse for patients diagnosed with acute non-lymphoblastic leukaemia. Table 7 presents the model's goodness of fit for data set I. In contrast, Figure 5 presents the estimated pdf, cdf, cumulative hazard plots, and Profile Log-Likelihood Plot of lambda. Meanwhile, Figure 6 displays the Q-Q graphs of the calculated densities. Figures 7 and 8 show MCMC diagnostics under two censoring schemes for dataset I. At the same time, Table 8 shows evaluating ML and Bayesian estimates for $\lambda, R(t), H(t), CH(t)$, and $MR(t)$ with censoring data set I. LINEX1/2 denote LINEX loss function with $h = \mp 1.5$; GE1/2 denote GE loss function with $q = \mp 1.5$.

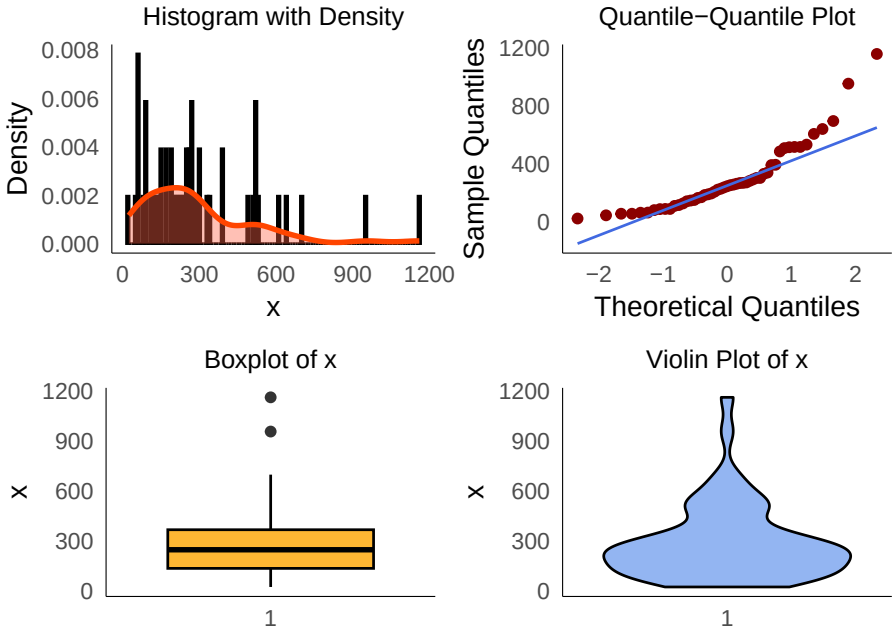

Figure 4. Exploratory data analysis: Histogram, Q-Q plot, boxplot, and violin plot for data set I.

Table 7. Model fit statistics and p-values for additional distributions data set I.

Models	AIC	AICC	BIC	HQIC	CAIC	KS Stat	p-value
L	673.9641	674.0457	675.8959	674.7023	676.8959	0.08669	0.83800
XL	673.9939	674.0756	675.9258	674.7321	676.9258	0.08673	0.83764
Gam	675.7071	675.9571	679.5707	677.1835	681.5707	0.08746	0.83017
Gum-1	681.8777	682.1277	685.7441	683.3542	687.7441	0.09328	0.76657
XG	681.4427	681.5243	683.3745	682.1809	684.3745	0.12464	0.40652
B-III	689.0841	689.3341	692.9477	690.5605	694.9477	0.12567	0.39620
IW	689.1587	689.4337	693.0474	690.6601	695.0474	0.12645	0.38861
Gum-2	689.1587	689.4337	693.0474	690.6601	695.0474	0.12645	0.38861
W	682.9083	682.9083	688.5219	684.1347	688.5219	0.13102	0.36296
QW	684.7192	685.2299	690.5147	686.9338	690.5147	0.12982	0.35640
BS	683.2922	683.2922	688.9058	685.5186	690.9058	0.13191	0.35650
NH	683.4671	683.7171	687.3308	684.9436	689.3308	0.13442	0.31542
Ex	679.0292	679.0292	683.6429	680.2557	686.0209	0.13817	0.28453
Lx	685.1126	685.3626	688.9763	686.5891	690.9763	0.13817	0.28453
IML	689.2911	689.2911	691.2229	690.0293	692.2229	0.15602	0.16691
IXG	689.3187	689.4004	691.2505	690.0569	692.2505	0.15764	0.15847

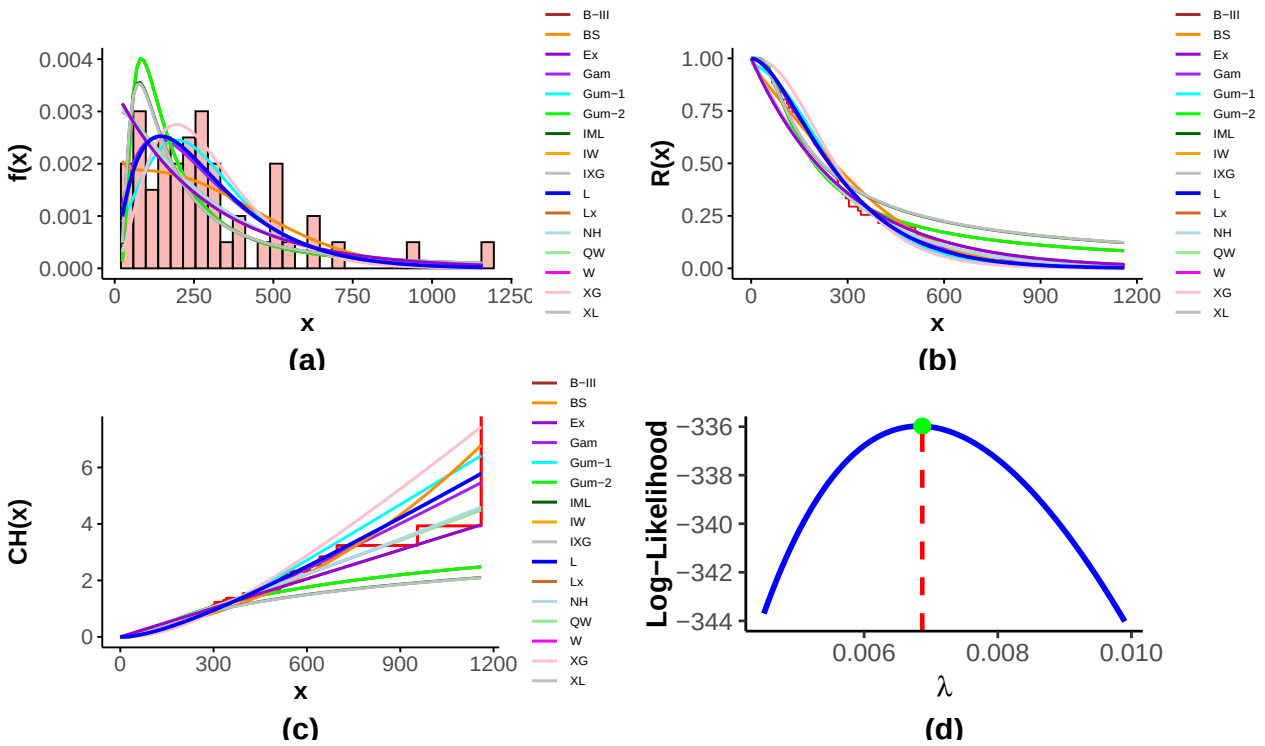

Figure 5. Plots for the fit of dataset I: (a) Estimated PDFs; (b) Reliability plots; (c) Cumulative Hazard plots; (d) Profile Log-Likelihood plot of λ.

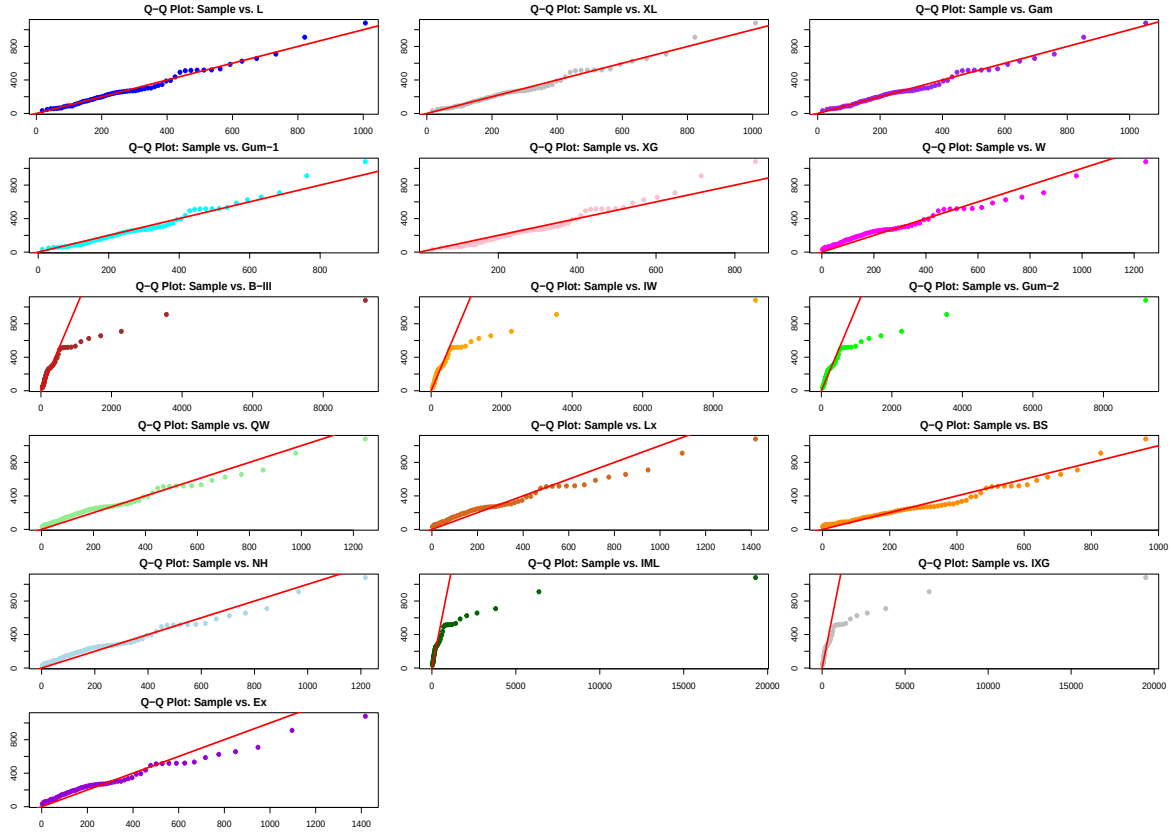

Figure 6. The Q-Q plots of the different models for the data set I.

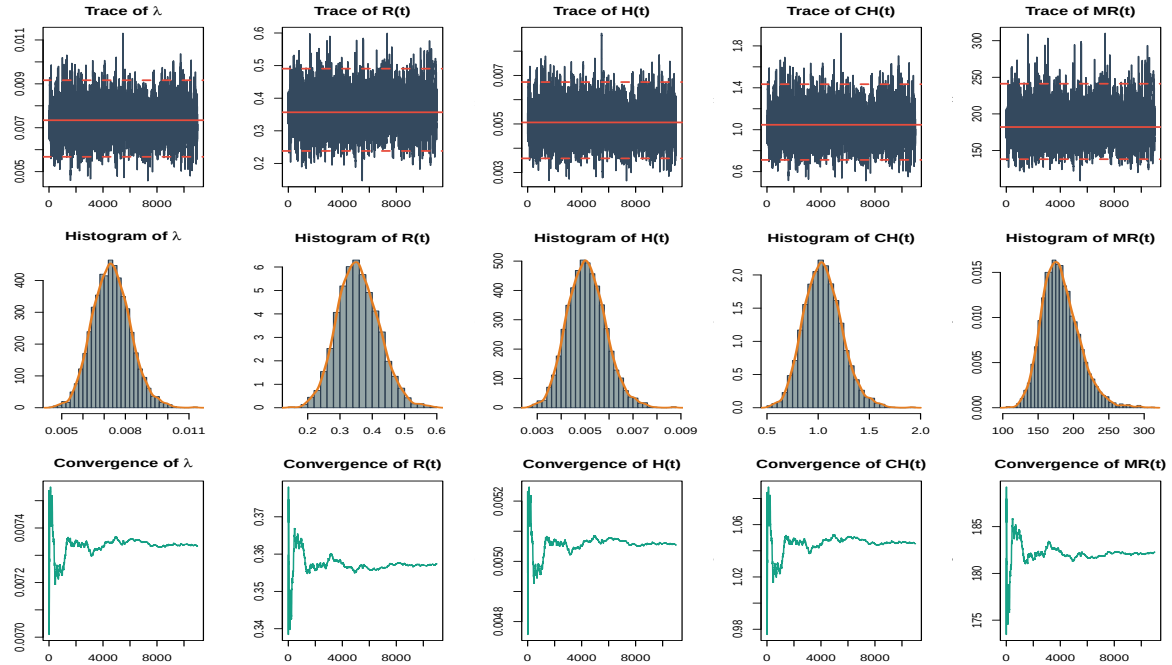

Figure 7. MCMC diagnostics: Trace, density, and convergence for dataset I ($r = 30$, $k = 40$, $T_1 = 150$, $T_2 = 250$).

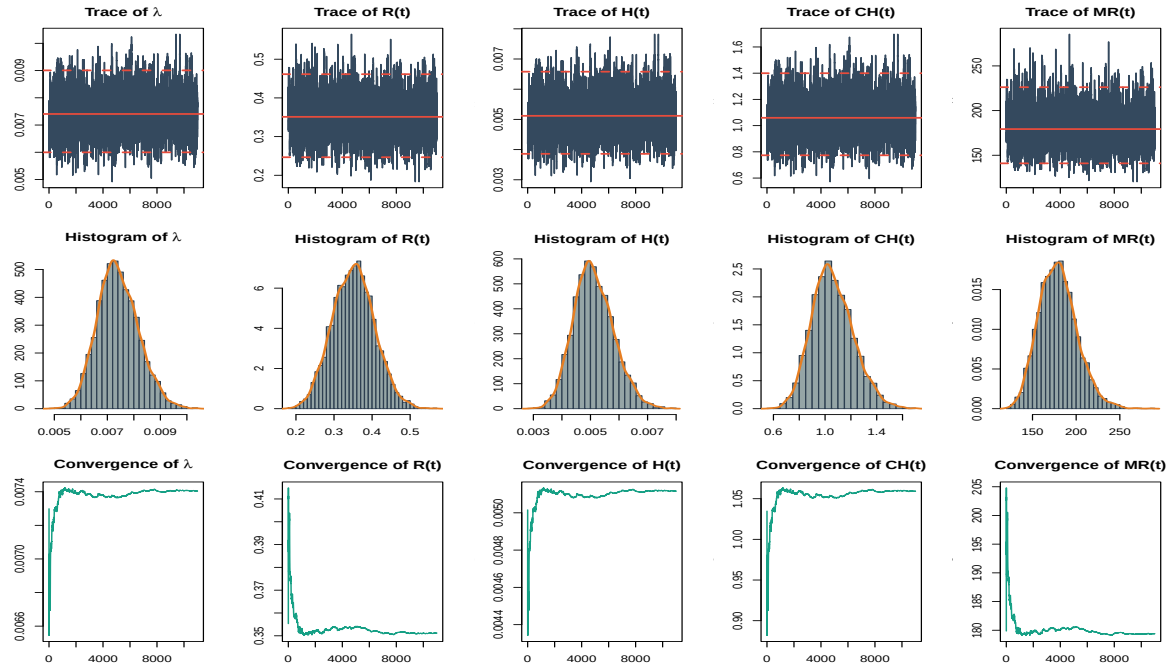

Figure 8. MCMC diagnostics: Trace, density, and convergence for dataset I ($r = 30,\ k = 40,\ T_1 = 300,\ T_2 = 600$).

Table 8. Evaluating ML and Bayesian performance for $\lambda,\ R(t),\ H(t),\ CH(t)$ and $MR(t)$ with censoring data I.

			ML	Bayesian					Asymp. CI	Boot-p	Boot-t	CrI
$r,\ k$	T_1, T_2	Par.		SEL	LINEX1	LINEX2	GE1	GE2	AL	AL	AL	AL
20,30	150 ,250	λ	0.0070	0.0071	0.0071	0.0071	0.0072	0.0070	0.0034	0.0035	0.0033	0.0036
		$R(t)$	0.3776	0.3733	0.3768	0.3698	0.3764	0.3569	0.2635	-	-	0.2657
		$H(t)$	0.0048	0.0049	0.0049	0.0049	0.0049	0.0047	0.0031	-	-	0.0032
		$CH(t)$	0.9740	1.0027	1.0302	0.9770	1.0114	0.9581	0.6979	-	-	0.7334
		$MR(t)$	188.7380	188.9850	329.5878	115.4508	190.0877	183.7283	107.3486	-	-	112.0889
	300 ,600	λ	0.0074	0.0076	0.0076	0.0076	0.0076	0.0075	0.0033	0.0038	0.0031	0.0033
		$R(t)$	0.3454	0.3382	0.3408	0.3356	0.3408	0.3249	0.2360	-	-	0.2304
		$H(t)$	0.0051	0.0053	0.0053	0.0053	0.0053	0.0052	0.0030	-	-	0.0030
		$CH(t)$	1.0630	1.0996	1.1243	1.0765	1.1068	1.0637	0.6833	-	-	0.6989
		$MR(t)$	176.0349	174.4705	287.5407	115.2206	175.2177	170.8242	90.4448	-	-	89.9263
30,40	150 ,250	λ	0.0072	0.0073	0.0073	0.0073	0.0074	0.0072	0.0034	0.0030	0.0031	0.0034
		$R(t)$	0.3598	0.3567	0.3597	0.3537	0.3595	0.3421	0.2486	-	-	0.2430
		$H(t)$	0.0050	0.0051	0.0051	0.0051	0.0051	0.0049	0.0030	-	-	0.0031
		$CH(t)$	1.0222	1.0471	1.0727	1.0231	1.0549	1.0076	0.6910	-	-	0.7006
		$MR(t)$	181.6148	181.9256	318.9812	112.2760	182.8243	177.5811	97.8263	-	-	98.3764
	300 ,600	λ	0.0073	0.0074	0.0074	0.0074	0.0074	0.0072	0.0030	0.0027	0.0032	0.0032
		$R(t)$	0.3553	0.3551	0.3578	0.3525	0.3576	0.3426	0.2224	-	-	0.2287
		$H(t)$	0.0050	0.0051	0.0051	0.0051	0.0051	0.0049	0.0027	-	-	0.0028
		$CH(t)$	1.0348	1.0494	1.0712	1.0284	1.0562	1.0147	0.6261	-	-	0.6515
		$MR(t)$	179.8582	181.1165	292.1560	129.9057	181.9038	177.3405	86.7939	-	-	92.7469
40,50	150 ,250	λ	0.0073	0.0074	0.0074	0.0074	0.0074	0.0073	0.0030	0.0026	0.0028	0.0031
		$R(t)$	0.3553	0.3550	0.3575	0.3526	0.3573	0.3433	0.2224	-	-	0.2228
		$H(t)$	0.0050	0.0051	0.0051	0.0051	0.0051	0.0049	0.0027	-	-	0.0028
		$CH(t)$	1.0348	1.0488	1.0692	1.0291	1.0551	1.0158	0.6261	-	-	0.6355
		$MR(t)$	179.8582	181.0147	349.1234	125.0204	181.7683	177.4600	86.7939	-	-	89.9369
	300 ,600	λ	0.0071	0.0071	0.0071	0.0071	0.0071	0.0070	0.0028	0.0024	0.0026	0.0029
		$R(t)$	0.3685	0.3708	0.3731	0.3686	0.3728	0.3605	0.2122	-	-	0.2152
		$H(t)$	0.0049	0.0049	0.0049	0.0049	0.0049	0.0048	0.0025	-	-	0.0026
		$CH(t)$	0.9984	1.0032	1.0203	0.9867	1.0088	0.9751	0.5758	-	-	0.5890
		$MR(t)$	185.0611	187.2099	290.5867	136.4770	187.8874	183.9091	84.8867	-	-	89.1814

7.2. *Data set II: Biological lifetime data*

This data was presented by Okash et al. [33], which comprises the survival times (in days) of guinea pigs subjected to different dosages of virulent tubercle bacilli. This data set, as illustrated in Table 9, contains 64 observations and has been examined in the research conducted by Haj Ahmad et al. [34].

Figure 9 displays the exploratory data analysis for dataset II, which includes a histogram, Q-Q plot, boxplot, and violin plot of the survival times of guinea pigs exposed to virulent tubercle bacilli. Table 10 shows the model's goodness of fit for dataset I. But, Figure 10 presents the estimated pdf, cdf, cumulative hazard plots, and profile log-likelihood plot of lambda. While Figure 11 displays the Q-Q graphs of the calculated densities. Figures 12 and 13 show MCMC diagnostics under two censoring schemes for dataset II. At the same time, Table 11 shows evaluating ML and Bayesian estimates for $\lambda, R(t), H(t), CH(t)$, and $MR(t)$ with censoring data II.

Table 9. Survival times (in days) of guinea pigs exposed to virulent tubercle bacilli.

12	15	22	24	24	32	32	33	38	38	43	44	48	52	53	54
55	56	57	58	58	59	60	60	60	61	62	63	65	65	67	68
70	72	73	75	76	76	81	83	85	87	91	95	96	98	99	109
121	127	129	131	143	146	146	175	211	233	258	258	263	297	341	341

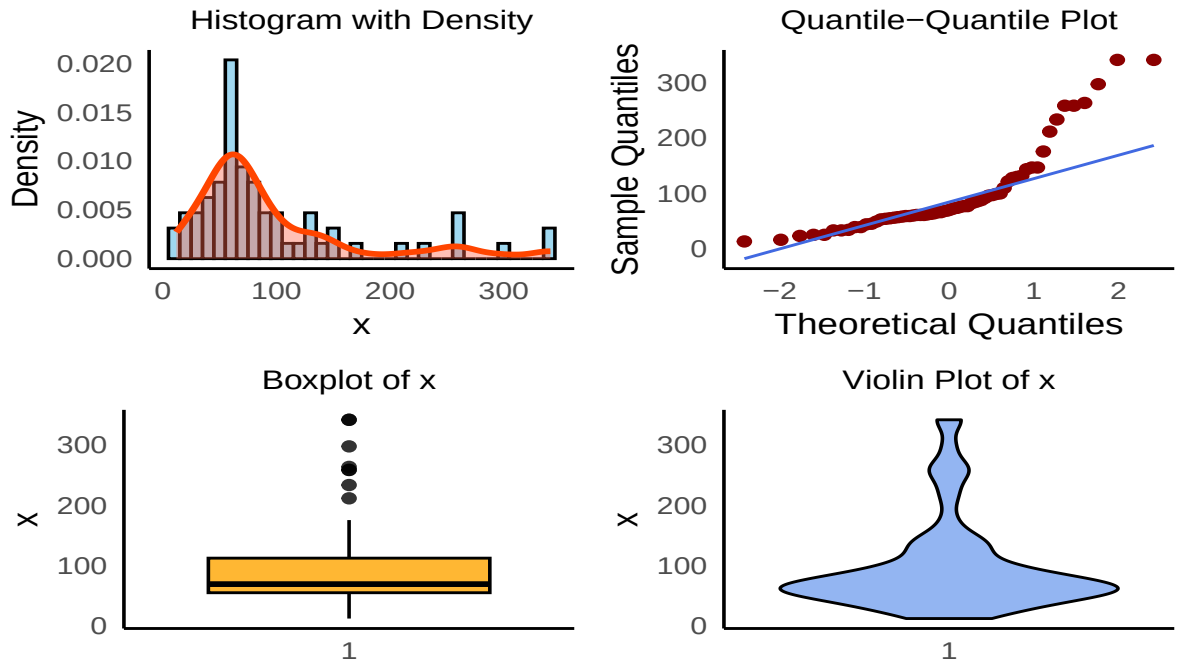

Figure 9. Exploratory data analysis: Histogram, Q-Q plot, boxplot, and violin plot for data set II.

Table 10. Model fit statistics and p-values for all distributions based on data set II.

Models	AIC	AICC	BIC	HQIC	CAIC	KS Stat	p-value
L	699.3687	699.4333	701.5276	700.2129	702.5276	0.13041	0.22650
XL	699.8017	699.8662	701.9606	700.6522	702.9606	0.13042	0.22635
Gam	700.7982	700.9949	705.1160	702.4992	707.1160	0.13755	0.17740
Gum-1	707.1215	707.3120	711.4393	708.8223	713.4393	0.15008	0.11192
W	707.6603	707.8570	711.9738	709.3613	713.9780	0.15071	0.10920
B-III	705.3836	705.5803	709.7014	707.0846	711.7014	0.15405	0.09588
IW	705.5414	705.7381	709.8592	707.2424	711.8592	0.15495	0.09255
Gum-2	705.5414	705.7381	709.8592	707.2424	711.8592	0.15495	0.09255
Gom	714.2821	714.4788	718.3934	715.9334	720.5998	0.16041	0.07506
QW	708.3382	708.7382	714.8149	710.8897	717.8149	0.16477	0.06192
BS	720.0061	720.2029	724.3239	721.7077	726.3239	0.17281	0.04375
XG	703.8040	703.8685	705.9629	704.6545	706.9629	0.17502	0.03964
NH	713.7876	713.7876	718.1054	715.4868	720.1054	0.17590	0.03797
IML	715.1596	715.2241	717.3184	716.0100	718.3184	0.18840	0.02128
IXG	715.8512	715.9157	718.0100	716.7016	719.0100	0.19316	0.01687

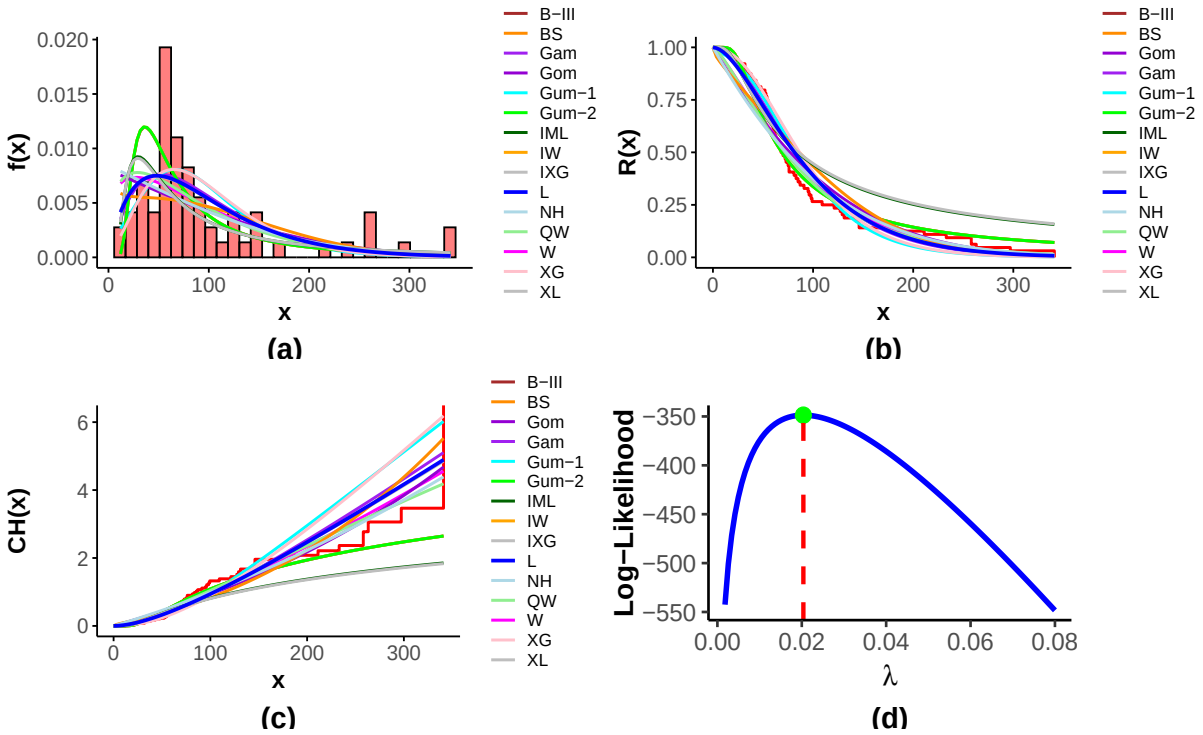

Figure 10. Plots for the fit of dataset II: (a) Estimated PDFs; (b) Reliability plots; (c) Cumulative hazard plots; (d) Profile log-likelihood plot of λ.

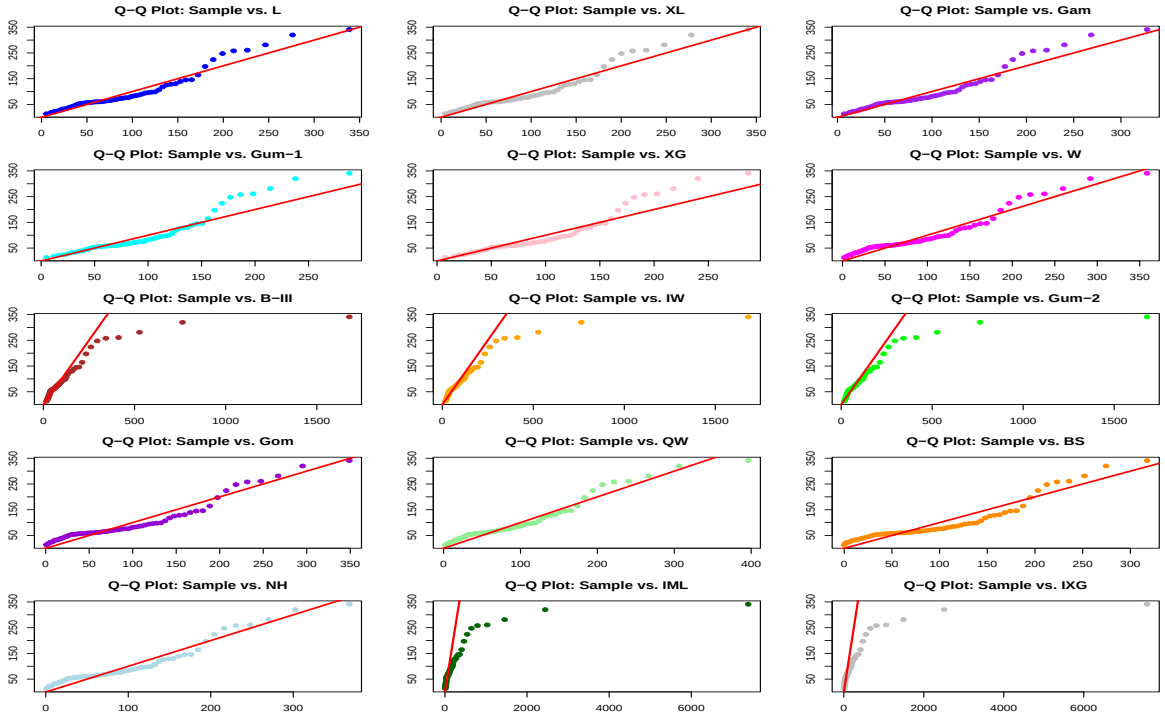

Figure 11. The Q-Q plots of the different models for the data set II.

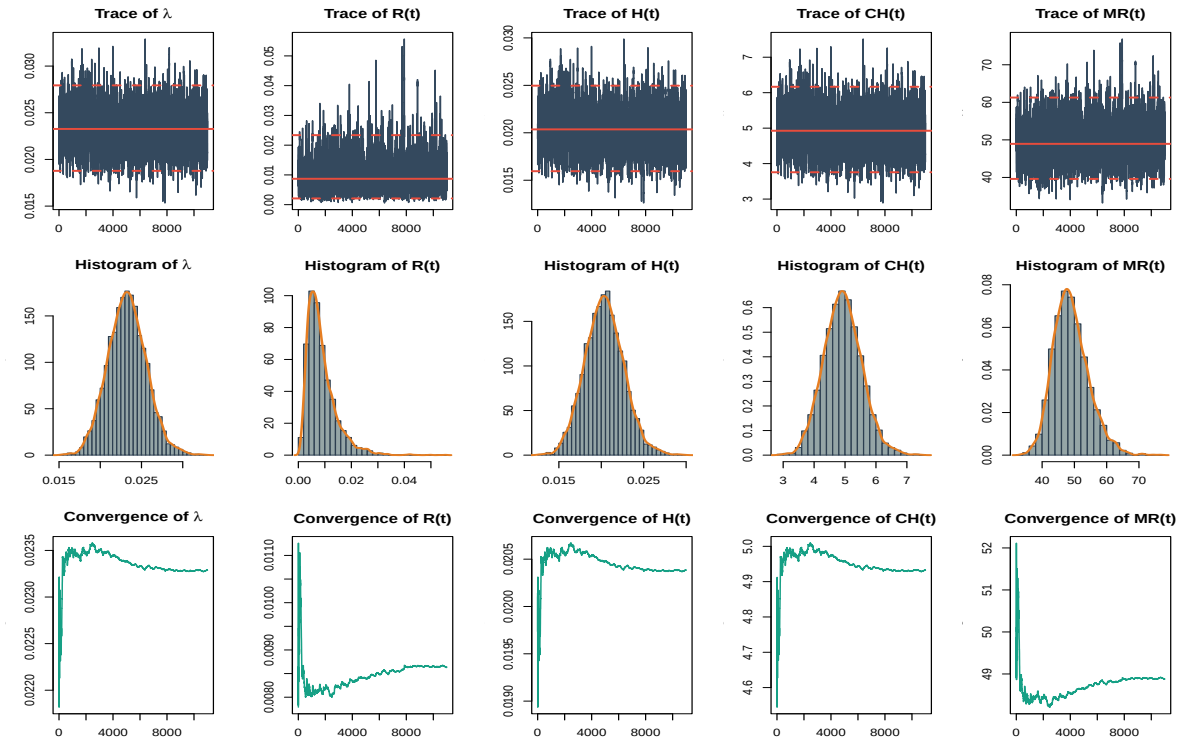

Figure 12. MCMC diagnostics: Trace, density, and convergence for dataset II ($r = 35,\ k = 45,\ T_1 = 50,\ T_2 = 100$).

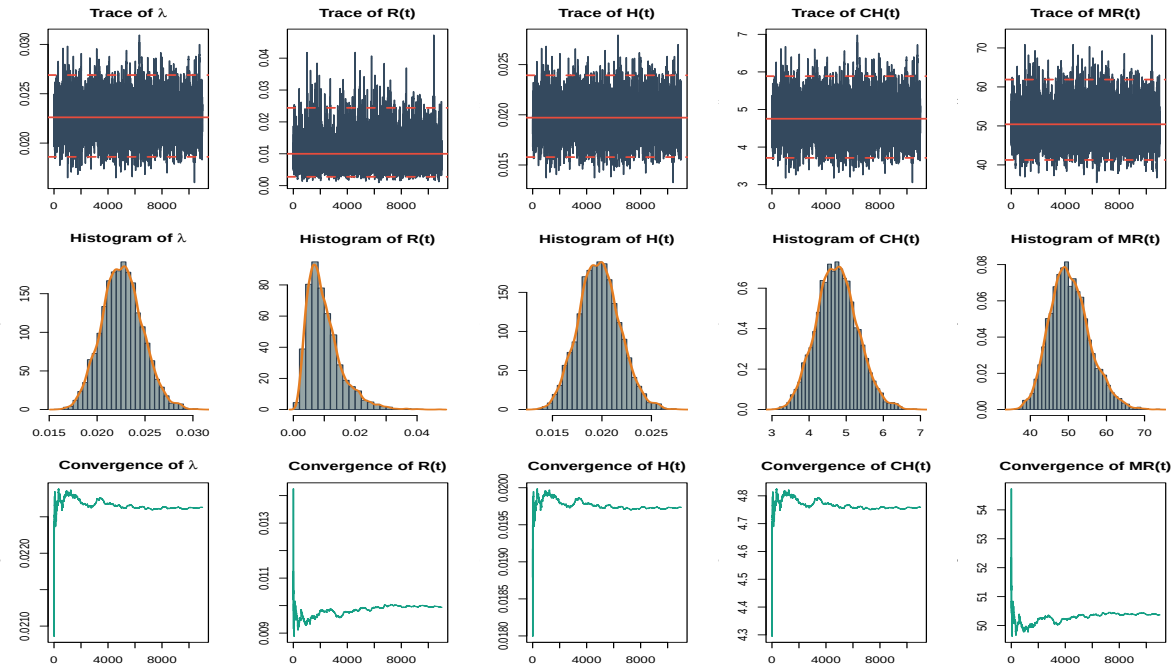

Figure 13. MCMC diagnostics: Trace, density, and convergence for dataset II ($r = 35,\ k = 45,\ T_1 = 150,\ T_2 = 300$).

Table 11. Evaluating ML and Bayesian performance for $\lambda,\ R(t),\ H(t),\ CH(t)$ and $MR(t)$ with censoring data II.

			ML	Bayesian					Asymp. CI	Boot-p	Boot-t	CrI
$r,\ k$	T_1, T_2	Par.		SEL	LINEX1	LINEX2	GE1	GE2	AL	AL	AL	AL
25,35	50, 100	λ	0.0224	0.0228	0.0228	0.0228	0.0229	0.0225	0.0097	0.0100	0.0099	0.0098
		$R(t)$	0.0091	0.0099	0.0100	0.0099	0.0109	0.0057	0.0232	-	-	0.0251
		$H(t)$	0.0195	0.0199	0.0199	0.0199	0.0200	0.0196	0.0096	-	-	0.0096
		$CH(t)$	4.6990	4.8122	5.1651	4.5188	4.8342	4.7018	2.5502	-	-	2.5623
		$MR(t)$	50.3756	50.0703	68.7021	37.4463	50.2521	49.1840	24.0436	-	-	23.9105
	150, 300	λ	0.0224	0.0226	0.0226	0.0226	0.0227	0.0224	0.0082	0.0082	0.0080	0.0081
		$R(t)$	0.0092	0.0099	0.0099	0.0099	0.0106	0.0068	0.0196	-	-	0.0206
		$H(t)$	0.0195	0.0197	0.0197	0.0197	0.0198	0.0195	0.0080	-	-	0.0080
		$CH(t)$	4.6929	4.7591	4.9978	4.5507	4.7747	4.6813	2.1384	-	-	2.1315
		$MR(t)$	50.4330	50.3451	66.1244	40.8823	50.4742	49.7090	20.2090	-	-	20.0369
35,45	50, 100	λ	0.0230	0.0234	0.0234	0.0234	0.0234	0.0231	0.0091	0.0127	0.0066	0.0090
		$R(t)$	0.0078	0.0084	0.0084	0.0084	0.0091	0.0053	0.0185	-	-	0.0195
		$H(t)$	0.0201	0.0205	0.0205	0.0205	0.0205	0.0202	0.0089	-	-	0.0088
		$CH(t)$	4.8551	4.9545	5.2485	4.6995	4.9729	4.8625	2.3793	-	-	2.3650
		$MR(t)$	48.9467	48.6638	69.8521	38.7486	48.8067	47.9636	21.1301	-	-	20.7312
	150, 300	λ	0.0224	0.0226	0.0226	0.0226	0.0227	0.0224	0.0082	0.0095	0.0077	0.0083
		$R(t)$	0.0092	0.0100	0.0100	0.0099	0.0107	0.0068	0.0196	-	-	0.0217
		$H(t)$	0.0195	0.0197	0.0197	0.0197	0.0198	0.0195	0.0080	-	-	0.0081
		$CH(t)$	4.6929	4.7585	4.9953	4.5367	4.7746	4.6766	2.1384	-	-	2.1684
		$MR(t)$	50.4330	50.3776	71.5722	41.3365	50.5147	49.7113	20.2090	-	-	20.6182
45,55	50, 100	λ	0.0233	0.0236	0.0236	0.0236	0.0237	0.0233	0.0090	0.0128	0.0099	0.0091
		$R(t)$	0.0071	0.0079	0.0079	0.0079	0.0086	0.0049	0.0170	-	-	0.0189
		$H(t)$	0.0204	0.0207	0.0207	0.0207	0.0208	0.0204	0.0089	-	-	0.0090
		$CH(t)$	4.9416	5.0166	5.3149	4.7572	5.0350	4.9239	2.3787	-	-	2.4034
		$MR(t)$	48.1909	48.1297	66.2535	38.6430	48.2703	47.4402	20.4527	-	-	20.6487
	150, 300	λ	0.0224	0.0226	0.0226	0.0226	0.0227	0.0224	0.0082	0.0074	0.0079	0.0086
		$R(t)$	0.0092	0.0100	0.0100	0.0099	0.0107	0.0066	0.0196	-	-	0.0220
		$H(t)$	0.0195	0.0198	0.0198	0.0198	0.0198	0.0195	0.0080	-	-	0.0084
		$CH(t)$	4.6929	4.7627	5.0186	4.5383	4.7794	4.6790	2.1384	-	-	2.2447
		$MR(t)$	50.4330	50.3514	66.7191	39.1555	50.4906	49.6686	20.2090	-	-	21.1253

8. Concluding remarks

This study analyzes UHCS statistical conclusions for the one-parameter Lindley distribution. We calculated point and interval estimates for model parameters, reliability, hazard rate, cumulative hazard, and mean residual life functions using classical and Bayesian methods. In addition to maximum likelihood point estimates, the approximation and bootstrap confidence intervals for the parameters are obtained using their asymptotic properties. The delta technique estimates reliability, hazard rate, cumulative hazard, and mean residual life function variances to provide intervals. Squared error, LINEX, and general entropy loss functions are used to generate Bayesian estimates utilizing the gamma prior distribution and previous knowledge of the unknown parameter. One cannot directly calculate the posterior distribution. Thus, Markov Chain Monte Carlo is used to obtain point estimates and maximum posterior density credible ranges. A simulation and application test the approaches' efficacy and adaptability.

The UHCS plan increases traditional by extending an experiment beyond a defined inspection interval if a few failures are observed. According to simulation results, Lindley parameters should be estimated using Bayesian methods with UHCS data. The Lindley model fit acute non-lymphoblastic leukemia data and guinea pig survival durations (in days) subjected to varied dosages of virulent tubercle bacilli better than several established models. Future research may use the maximum product of spacing approach with Bayesian estimation to estimate the Lindley distribution using the proposed censoring strategy and compare the findings to current work. It can also examine Bayesian estimating results with different loss functions. Future work could estimate Lindley distribution information measures under the suggested filtering technique using established information measure methodologies. For further information on these methods and estimation, see Nassr et al. [35], El-Saeed et al. [36], Alawady et al. [37], Barakat et al. [38], and Husseiny et al. [39].

Author contributions

Said G. Nassr: Methodology (equal); Writing–original draft (equal); Writing–review & editing (equal); T. S. Taher: Methodology (equal); Software (equal); Writing–original draft (equal); Formal analysis (equal); Tmader Alballa: Funding acquisition (equal); Methodology (equal); Project administration; Writing–review & editing (equal); Neema M. Elharoun: Methodology (equal); Data curation (equal); Writing–review & editing (equal). All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Aartificial intelligence(AI) tools in the creation of this article.

Data Availability

The data is available in the paper.

Acknowledgments

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Conflict of interest

The authors declare that they have no conflicts of interest.

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