Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 14472–14487.
DOI:10.3934/math.2025651
Received: 28 April 2025
Revised: 10 June 2025
Accepted: 17 June 2025
Published: 24 June 2025

Research article

Inversion transformations with respect to conics in hybrid number planes

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Abstract: This paper presents a comprehensive study of geometric inversion with respect to central conics in hybrid number planes, which unify complex, hyperbolic, and dual numbers within a single algebraic structure. By employing the hybrid scalar product and the associated pseudo-Euclidean metric, the hybridian planes were classified as elliptic, hyperbolic, or parabolic. Explicit inversion formulas were derived for points, lines, and conics in each plane type. It was shown that lines passing through the inversion center remain invariant, while others transform into conics. Homothetic conics preserve their type under inversion, whereas non-homothetic conics yield cubic or quartic curves depending on their relation to the inversion center. These results extend classical inversion geometry into a unified hybrid setting, providing a new framework for geometric transformations in generalized number systems.

Keywords: inversion of conics; hybrid numbers; pseudo-Euclidean geometry; hypercomplex systems; geometric transformations
Mathematics Subject Classification: 11R52, 30C20, 30G35, 51M15, 51N20

1. Introduction

Geometric transformations serve as a foundational theme in both classical and modern mathematics, providing a robust framework for analyzing structure, symmetry, and motion. Among these, inversion transformations are of particular interest due to their rich geometric, algebraic, and conformal properties. Traditionally defined in the Euclidean plane with respect to circles, inversion generalizes the concept of reflection and plays a fundamental role in the study of Möbius transformations and complex analysis [1, 2].

As a starting point, consider the classical notion of symmetry with respect to a line. In Figure 1, the point B is the reflection of point A across the line ℓ, such that the perpendicular distances from both points to the line are equal.

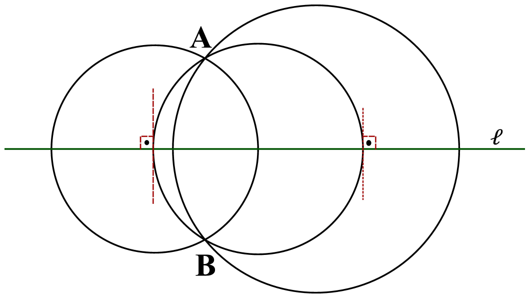

Figure 1. Point B, which is the reflection of point A with respect to the line ℓ.

Extending this idea leads to circular inversion, as illustrated in Figure 2, where for a fixed center O and radius R, the points A and B satisfy the relation

$$|OA| \cdot |OB| = R^2,$$

[3]. This transformation lies at the core of inversive geometry.

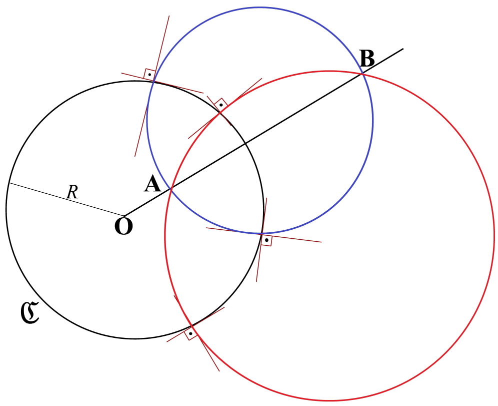

Figure 2. The point B is the circular inversion of the point A with respect to the circle $\mathfrak{C}$ (in black), centered at O with radius R.

As illustrated in Figure 2, the concept of circular inversion can also be characterized via orthogonality: all circles passing through a point A and orthogonal to a fixed reference circle $\mathfrak{C}$ intersect at a second point B. According to this geometric principle, B is defined as the circular inversion of A with respect to $\mathfrak{C}$. This interpretation offers an elegant and purely geometric understanding of inversion, complementing the traditional metric definition.

In classical settings, inversion is elegantly expressed using complex numbers, where inversion with respect to the unit circle is defined by $z \mapsto \frac{1}{z}$. This concept has been extended to general conic sections through analytical methods and Clifford algebra approaches [4–7]. However, these generalizations largely remain confined to Euclidean or complex frameworks [8].

Meanwhile, hypercomplex number systems—such as complex, hyperbolic, and dual numbers—have emerged as powerful algebraic tools for modeling distinct geometric behaviors [9–11]. These systems correspond, respectively, to elliptic, hyperbolic, and parabolic geometries and are closely linked to transformation groups like $SL_2(\mathbb{R})$ via Möbius-type actions [12, 13]. These ideas align with Klein's Erlangen Program, which advocates for the classification of geometries based on their invariance under transformation groups [14–17].

To unify these distinct number systems within a single algebraic and geometric framework, hybrid numbers were introduced [18]. Although recent studies have explored the algebraic properties and applications of hybrid numbers [19–22], a comprehensive theory of inversion with respect to conics in hybridian planes has yet to be developed. This gap presents an opportunity to extend classical inversive geometry into a generalized pseudo-Euclidean setting, offering potential applications in physics, computer graphics, and non-Euclidean modeling [23–26]. In a broader context, the techniques and insights developed here may also support advances in inverse problems involving scattering phenomena and conic singularities [27–29].

In this study, we propose a rigorous geometric formulation of inversion transformations with respect to conic sections in hybrid number planes. Utilizing the hybrid scalar product and the metric classification of planes, we derive explicit inversion formulas for points, lines, and conics. Furthermore, we analyze the nature of inverse curves, identifying conditions under which inversions yield conics, cubic curves, or quartic curves, motivated by the aim to generalize classical inversion concepts into hybrid number systems and provide a unified framework that captures new geometric behaviors not seen in traditional settings.

By bridging classical inversive geometry with hybrid number theory, this work establishes a unified framework for geometric transformations in pseudo-Euclidean spaces, laying the foundation for future research in generalized symmetry, metric transformations, and complex motion analysis.

The structure of the paper is as follows. Section 2 provides preliminary information about hybrid numbers and the geometric structure of hybridian planes. In Section 3, we construct inversion transformations with respect to central conics and derive explicit formulas for points, lines, and conics in elliptic, hyperbolic, and parabolic hybrid planes. Section 4 concludes the paper with a summary of results and directions for future work.

2. Preliminaries on hybrid geometry

To establish a comprehensive framework for inversion transformations, we begin by outlining the algebraic structure of hybrid numbers and their associated geometric properties [18, 19].

A hybrid number is expressed as:

$$\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h},$$

where the units satisfy the relations $\mathbf{i}^2 = -1$, $\varepsilon^2 = 0$, $\mathbf{h}^2 = 1$, and $\mathbf{ih} = \mathbf{hi} = \mathbf{i} + \varepsilon$ [18]. Each hybrid number can be decomposed into its scalar part $S_{\mathbf{p}} = p_1$ and vector part $\mathbf{V_p} = p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$. The conjugate of $\mathbf{p}$ is defined by $\overline{\mathbf{p}} = S_{\mathbf{p}} - \mathbf{V_p}$. The multiplication of hybrid numbers adheres to the algebraic rules established for these units [19, 21].

The character of a hybrid number is given by

$$C(\mathbf{p}) = \mathbf{p}\overline{\mathbf{p}} = p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2.$$

Accordingly, a hybrid number is classified as spacelike if $C(\mathbf{p}) < 0$, timelike if $C(\mathbf{p}) > 0$, and lightlike if $C(\mathbf{p}) = 0$. The norm is defined by $\|\mathbf{p}\| = \sqrt{|C(\mathbf{p})|}$, and the inverse exists when $\|\mathbf{p}\| \neq 0$, given by $\mathbf{p}^{-1} = \dfrac{\overline{\mathbf{p}}}{C(\mathbf{p})}$. Lightlike hybrid numbers have no inverse.

The hybridian scalar product for two hybrid numbers $\mathbf{q}$ and $\mathbf{p}$ is defined as:

$$\langle \mathbf{q}, \mathbf{p} \rangle_{\mathbb{K}} = \frac{\mathbf{q}\overline{\mathbf{p}} + \mathbf{p}\overline{\mathbf{q}}}{2} = q_1 p_1 + (q_2 - q_3)(p_2 - p_3) - q_3 p_3 - q_4 p_4,$$

where $\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\varepsilon + q_4\mathbf{h}$ and $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ [18]. This scalar product induces a pseudo-metric structure on $\mathbb{R}^3$ [30]. In this context, pure hybrid numbers are treated as vectors in the hybridian 3-space $\mathbb{H}^3$. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be hybrid vectors in $\mathbb{H}^3$, equipped with the bilinear form

$$\langle \mathbf{u}, \mathbf{v} \rangle = -(u_1 - u_2)(v_1 - v_2) + u_2 v_2 + u_3 v_3.$$

Vectors in $\mathbb{H}^3$ are classified according to the discriminant

$$\Delta(\mathbf{u}) = -(u_1 - u_2)^2 + u_2^2 + u_3^2$$

as elliptic if $\Delta(\mathbf{u}) < 0$, hyperbolic if $\Delta(\mathbf{u}) > 0$, or parabolic if $\Delta(\mathbf{u}) = 0$. The discriminant Δ determines the geometric type of the hybridian plane associated with a vector, analogous to how the sign of the discriminant in conic sections characterizes ellipses, hyperbolas, or parabolas. Note that for any hybrid vector $\mathbf{v} = (v_1, v_2, v_3)$, the discriminant satisfies $\Delta(\mathbf{v}) = \mathbf{v}^2$, where the square is taken under the hybrid multiplication rules. Two vectors are pseudo-orthogonal when their scalar product is zero [22].

For any non-zero pure hybrid vector $\mathbf{v}$, the associated plane is defined as $\mathbb{P}_{\mathbf{v}} = \{O, \mathbf{v}\} \subset \mathbb{R}^4$. The nature of $\mathbb{P}_{\mathbf{v}}$ depends on the type of $\mathbf{v}$: if $\mathbf{v}$ is elliptic, then $\mathbb{P}_{\mathbf{v}}$ is isomorphic to the complex plane since $\mathbf{v}^2 = -1$ [31]; if $\mathbf{v}$ is hyperbolic, $\mathbb{P}_{\mathbf{v}}$ corresponds to a hyperbolic plane with $\mathbf{v}^2 = 1$; and if $\mathbf{v}$ is parabolic, $\mathbb{P}_{\mathbf{v}}$ represents a dual plane where $\mathbf{v}^2 = 0$. This classification provides the geometric foundation necessary for extending classical inversion transformations into hybridian geometry [20, 32].

3. Inversion map in hybrid number planes

We begin by analyzing the general equations of conics within the hybrid plane $\mathbb{P}_{\mathbf{v}}$. In hybrid geometry, each point on a conic maintains a constant hybridian distance from its center. Utilizing this property, the standard form of a conic in the plane $\mathbb{P}_{\mathbf{v}}$, for a non-zero pure hybrid number $\mathbf{v} = (v_1, v_2, v_3)$, is given by

$$-A^2(x-h)^2 + \Delta B^2(y-s)^2 = r, \quad r \in \{-1, 1\}, \tag{3.1}$$

where $\Delta = -v_1^2 + 2v_1v_2 + v_3^2$. The parameter r takes the value -1 for parabolic and elliptic planes, and either $+1$ or -1 for hyperbolic planes. Based on the type of hybridian plane, conics centered at (h, s) are classified as follows:

- **Elliptic Plane** $\mathbb{P}_{\mathbf{v}}$: If $\mathbf{v}$ is unit elliptic, then

$$-A^2(x-h)^2 - B^2(y-s)^2 = -1$$

  with parametric equation $\left(\dfrac{\cos\theta}{A} + h, \dfrac{\sin\theta}{B} + s\right)$.

- **Hyperbolic Plane** $\mathbb{P}_{\mathbf{v}}$: If $\mathbf{v}$ is unit hyperbolic, then

$$-A^2(x-h)^2 + B^2(y-s)^2 = \pm 1$$

with parametric equation $\left(\dfrac{\cosh\theta}{A} + h, \dfrac{\sinh\theta}{B} + s\right)$.

- **Parabolic Plane** $\mathbb{P}_{\mathbf{v}}$: If $\mathbf{v}$ is parabolic, then

$$-A^2(x-h)^2 = -1$$

with parametric equation $\left(\dfrac{1}{A}(Ah \pm 1), t\right)$.

Let M be a point in the plane $\mathbb{P}_{\mathbf{v}}$, and let $k > 0$ be a given scalar. The homothetic transformation F with center M and ratio k is defined as the mapping that fixes M and sends any other point P to a point Q such that

$$\overrightarrow{MQ} = k \cdot \overrightarrow{MP},$$

where M, P, and Q are collinear and lie on the same side relative to M [33].

This transformation can be formally expressed as:

$$F : \mathbb{R}^2 \cup \{1\} \longrightarrow \mathbb{R}^2 \cup \{1\}, \quad P \mapsto F(P) = Q$$

with the matrix representation given by

$$F(P) = \begin{bmatrix} q_1 \\ q_2 \\ 1 \end{bmatrix} = \begin{bmatrix} k & 0 & (1-k)h \\ 0 & k & (1-k)s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}, \tag{3.2}$$

where $P = (p_1, p_2)$, $Q = (q_1, q_2)$, and $M = (h, s)$ [33].

Under the homothety transformation defined by

$$X = kx + (1-k)h, \quad Y = ky + (1-k)s,$$

where x and y denote the original coordinate variables in the conic $\mathfrak{J}$, we observe that

$$X - h = k(x-h), \quad Y - s = k(y-s).$$

Substituting these expressions into the general conic equation

$$-A^2(X-h)^2 + \Delta B^2(Y-s)^2 = r$$

yields

$$-A^2[k(x-h)]^2 + \Delta B^2[k(y-s)]^2 = r,$$

which simplifies to

$$-A^2k^2(x-h)^2 + \Delta B^2k^2(y-s)^2 = r.$$

Thus, the conic is transformed into an equivalent equation under the given homothety.

It is straightforward to verify that the eccentricity e of the conic remains invariant under homothety

$$e = \sqrt{1 + \Delta\left(\frac{Ak}{Bk}\right)^2} = \sqrt{1 + \Delta\frac{A^2}{B^2}}.$$

Definition 3.1. [3, 7, 34] Two conics are said to be of the same semi-form if they have parallel axes and equal eccentricities. Furthermore, if their principal axes are also parallel, then the conics are homothetic.

Definition 3.2. [3, 7, 34] Let $\mathfrak{J}$ be a conic and $\mathfrak{C}$ a central conic (i.e., a conic centered at the origin). The conic $\mathfrak{J}'$ is called the image of $\mathfrak{J}$ under inversion with respect to $\mathfrak{C}$ if $\mathfrak{J}'$ consists of all points that are the images of the points of $\mathfrak{J}$.

Therefore, the inversion map in the hybridian plane with respect to a conic is defined as follows:

Definition 3.3. Let $\mathfrak{C}$ be a conic with center $M = (h, s)$ in the plane $\mathbb{P}_{\mathbf{v}} = \text{span}\{1, \mathbf{v}\}$. The inversion map with respect to $\mathfrak{C}$ is defined as:

$$\phi : \mathbb{P}_{\mathbf{v}} \setminus \{M\} \longrightarrow \mathbb{P}_{\mathbf{v}} \setminus \{M\},$$

such that $\phi(P) = P'$, where the points satisfy $|MP| \cdot |MP'| = |MQ|^2$. Here, P' lies on the ray $\overrightarrow{MP}$, and Q is the intersection point of the conic $\mathfrak{C}$ with the ray $\overrightarrow{MP}$.

Theorem 3.1. *Let P' be the inverse of a point $P = (x_0, y_0)$ with respect to the inversion conic*

$$\mathfrak{C} : -A^2(x-h)^2 + B^2(y-s)^2\Delta = r,$$

centered at $M = (h, s)$, where $r \in \{-1, 1\}$ depending on the type of the conic. The point P' is the intersection of the line MP and the polar line of P. Then,

$$P' = \left(\frac{r(x_0 - h)}{\rho_P} + h, \frac{r(y_0 - s)}{\rho_P} + s \right), \tag{3.3}$$

where $\Delta = -v_1^2 + 2v_1v_2 + v_3^2$ and $\rho_P = -A^2(x_0 - h)^2 + B^2(y_0 - s)^2\Delta$.

Proof. The ray $\overrightarrow{MP} = (x_0 - h, y_0 - s)$ intersects $\mathfrak{C}$ at

$$Q = (\eta(x_0 - h) + h, \eta(y_0 - s) + s)$$

for $\eta > 0$. Since $Q \in \mathfrak{C}$, we have

$$\begin{aligned} -A^2\left(\eta(x_0 - h)\right)^2 + B^2\left(\eta(y_0 - s)\right)^2\Delta &= r, \\ \eta^2\left(-A^2(x_0 - h)^2 + B^2(y_0 - s)^2\Delta\right) &= r. \end{aligned} \tag{3.4}$$

The polar of P is given by the line

$$l_P : -A^2(x-h)(x_0 - h) + B^2(y-s)(y_0 - s)\Delta = r.$$

This line intersects the ray $\overrightarrow{MP}$ at the point

$$P' = (\xi(x_0 - h) + h, \xi(y_0 - s) + s)$$

for $\xi > 0$. Since $P' \in l_P$, it follows that

$$-A^2\left(\xi(x_0 - h)\right)(x_0 - h) + B^2\left(\xi(y_0 - s)\right)(y_0 - s)\Delta = r,$$

$$\xi\left(-A^2(x_0-h)^2+B^2(y_0-s)^2\Delta\right) = r. \tag{3.5}$$

From (3.4) and (3.5), we obtain $\xi = \eta^2$. Therefore, we have

$$\begin{aligned} |MP| &= \sqrt{|-(x_0-h)^2-(y_0-s)^2\Delta|}, \\ |MP'| &= \xi \cdot \sqrt{|-(x_0-h)^2-(y_0-s)^2\Delta|}, \\ |MQ| &= \eta \cdot \sqrt{|-(x_0-h)^2-(y_0-s)^2\Delta|}. \end{aligned}$$

Thus, $|MP| \cdot |MP'| = |MQ|^2$. Using the equations of the polar line of $\mathfrak{C}$ and the line MP,

$$\frac{y-s}{x-h} = \frac{y_0-s}{x_0-h},$$

the point P' given in (3.3) is obtained. □

Remark 3.1. *The quantity ρ_P represents the square of the distance between the inversion center M and the point P, according to the hybridian metric. Its sign and magnitude depend on the value of the scalar parameter Δ, which functions as a discriminant classifying the type of the hybrid plane. Specifically, $\Delta = -1$ corresponds to elliptic, $\Delta = 0$ to parabolic, and $\Delta = 1$ to hyperbolic geometries. These distinctions directly affect the behavior of the inversion, as illustrated in Figure 3.*

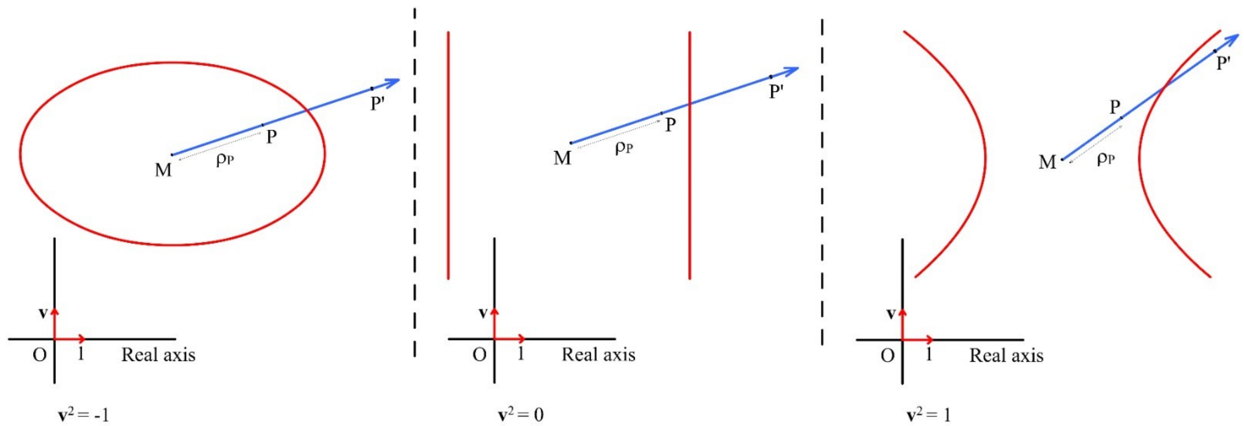

Figure 3. Visualization of the square of the hybridian distance between the inversion center and a point P in elliptic, parabolic, and hyperbolic planes.

Remark 3.2. *Let $P' = (x_1, y_1)$ be the inverse of the point $P = (x_0, y_0)$ with respect to the conic*

$$\mathfrak{C} : -A^2(x-h)^2 + B^2(y-s)^2\Delta = r.$$

Theorem 3.1 can be expressed using the homothety matrix (3.2) as follows:

$$F(P) = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{r}{\rho_P} & 0 & \left(1-\dfrac{r}{\rho_P}\right)h \\ 0 & \dfrac{r}{\rho_P} & \left(1-\dfrac{r}{\rho_P}\right)s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}, \tag{3.6}$$

where $\rho_P = -A^2(x_0-h)^2 + B^2(y_0-s)^2\Delta$.

Theorem 3.2. *An inversion maps a line ℓ passing through the center of inversion M to itself.*

Proof. The result is obvious from the definition of inversion, since any line passing through the center of inversion remains invariant. □

Theorem 3.3. *Let $\mathfrak{C}$ be a central conic of inversion in the plane $\mathbb{P}_{\mathbf{v}}$. The inverse of a line ℓ that does not pass through the center of inversion O (the origin) with respect to $\mathfrak{C}$ is a conic $\mathfrak{J}$ passing through the center of inversion of the same form as the central conic of inversion, and conversely. The type of the conic $\mathfrak{J}$ is given in the table below according to the type of the plane $\mathbb{P}_{\mathbf{v}}$.*

$\mathbb{P}_{\mathbf{v}}$	$\mathfrak{C}$	$\mathfrak{J}$
Elliptic	Ellipse	Ellipse
Hyperbolic	Hyperbola	Hyperbola
Parabolic	Dual circle	For $x = c$, line
Parabolic	Dual circle	For $y = ax + b$, parabola

Proof. Consider the line $y = ax + b$. The image of a point $P = (x, y) = (x, ax + b)$ under inversion is

$$P' = r\left(\frac{x}{\rho_P}, \frac{y}{\rho_P}\right) = r\left(\frac{x}{\rho_P}, \frac{ax+b}{\rho_P}\right),$$

where $r = -1$ for parabolic and elliptic planes, and $r = \pm 1$ for hyperbolic planes.

It can be verified that the point P' satisfies the following equation:

$$-A^2x^2 + \Delta B^2y^2 = r\left(\frac{ax-y}{b}\right).$$

Rearranging this equation using P', we obtain the equation of the conic, which is the inverse of the line $y = ax + b$:

$$\begin{aligned} -A^2\left(x + \frac{ra}{2A^2b}\right)^2 + \Delta B^2\left(y + \frac{\Delta r}{2B^2b}\right)^2 &= -\left(\frac{a}{2Ab}\right)^2 + \Delta\left(\frac{1}{2Bb}\right)^2 && \text{for } \Delta \neq 0,\\ -A^2\left(x - \frac{a}{2A^2b}\right)^2 &= -\left(\frac{a}{2Ab}\right)^2 + \frac{y}{b} && \text{for } \Delta = 0. \end{aligned}$$

It is clear that these conics are of the same form. If the line is of the form $x = c$, then

$$-A^2x^2 + \Delta B^2y^2 = \frac{rx}{c}.$$

Since $\Delta = 0$ in the dual plane, the inverse of $x = c$ is

$$x = \frac{1}{A^2c}.$$

Conversely, under the same conditions, the inverse of these conics is a line. □

Example 3.1. The conic $\mathfrak{J}'$ (in red) shown in the table below represents the inversion of the line $y = 3x + 1$ (in blue), which does not pass through the origin, with respect to the conic $\mathfrak{C}$ (in black) (Figure 4).

$\mathfrak{C}$	$\mathfrak{I}'$
$2x^2 + 3y^2 = 1$	$2x^2 + 3x + 3y^2 - y = 0$
$-2x^2 + 3y^2 = 1$	$2x^2 - 3x - 3y^2 + y = 0$
$-2x^2 = -1$	$(4x + 3)^2 = 8y + 9$

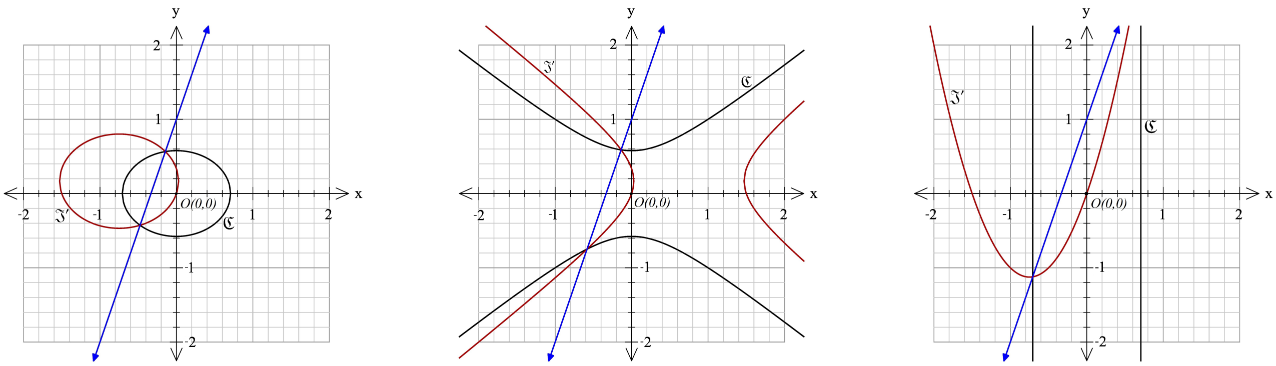

Figure 4. Inversion of a line with respect to the elliptic, hyperbolic, and dual circles, respectively.

Theorem 3.4. *Let $\mathfrak{C}$ and $\mathfrak{I}$ be homothetic conics, where $\mathfrak{C}$ is a central conic in the plane $\mathbb{P}_{\mathbf{v}}$. If $\mathfrak{I}$ does not pass through the center of inversion O, then the inverse of $\mathfrak{I}$ with respect to $\mathfrak{C}$ is a conic $\mathfrak{I}'$, which is homothetic to $\mathfrak{I}$.*

Proof. Let

$$\mathfrak{C} \;:\; -A^2x^2 + \Delta B^2y^2 = r,$$

$$\mathfrak{I} \;:\; -A_0^2(x-h)^2 + \Delta B_0^2(y-s)^2 = r, \tag{3.7}$$

$$A_0 = kA, \quad B_0 = kB, \quad k \in \mathbb{R}, \tag{3.8}$$

$$\rho_P = -A^2x^2 + B^2y^2\Delta. \tag{3.9}$$

Consider the point

$$P' = \left(\frac{rx}{\rho_P}, \frac{ry}{\rho_P}\right).$$

Substituting P', along with (3.8) and (3.9), into equation (3.7), we get

$$\begin{aligned}
& -(Ak)^2\left(\frac{rx}{\rho_P} - h\right)^2 + \Delta(Bk)^2\left(\frac{ry}{\rho_P} - s\right)^2 = r \\
\Longrightarrow & -A^2\left(\frac{rx}{\rho_P} - h\right)^2 + \Delta B^2\left(\frac{ry}{\rho_P} - s\right)^2 = \frac{r}{k^2} \\
\Longrightarrow & \frac{1}{\rho_P^2}\left(-A^2r^2x^2 + \Delta B^2r^2y^2\right) + \frac{2r}{\rho_P}\left(A^2hx - \Delta B^2sy\right) - A^2h^2 + \Delta B^2s^2 = \frac{r}{k^2} \\
\Longrightarrow & \frac{1}{\rho_P} + \frac{-2r(-A^2hx + \Delta B^2sy)}{\rho_P} - A^2h^2 + \Delta B^2s^2 = \frac{r}{k^2}
\end{aligned}$$

$$\Longrightarrow -A^2x^2 + \Delta B^2y^2 = \frac{1 - 2r(-A^2hx + \Delta B^2sy)}{\frac{r}{k^2} - (-A^2h^2 + \Delta B^2s^2)}.$$

Thus, the equation represents a conic that is homothetic to $\mathfrak{I}$. □

Example 3.2. The conics $\mathfrak{I}'$ (in red) shown in the table below represent the inversion of the conics $\mathfrak{I}$ (in blue), which do not pass through the origin, with respect to the central conics $\mathfrak{C}$ (in black) (Figure 5).

$\mathfrak{C}$	$\mathfrak{I}$	$\mathfrak{I}'$
$\frac{x^2}{15} + \frac{y^2}{4} = 1$	$\frac{12(x-1)^2}{15} + 3(y-1)^2 = 1$	$-\frac{7x^2}{15} - \frac{7y^2}{4} = 15(2-y) - 4x$
$-\frac{x^2}{15} + \frac{y^2}{4} = 1$	$-\frac{12(x-1)^2}{15} + 3(y-1)^2 = 1$	$\frac{3y^2}{4} - \frac{x^2}{5} = 15(y-2) - 4x$
$2x^2 = 1$	$-24(x-1)^2 = -1$	$-2x^2 = \frac{12}{23} - \frac{48x}{23}$
$\left(\frac{1}{3}\right)^2 x^2 = 1$	$4y = 4 - (x-1)^2$	$12y = x^2 + 6x - 27$

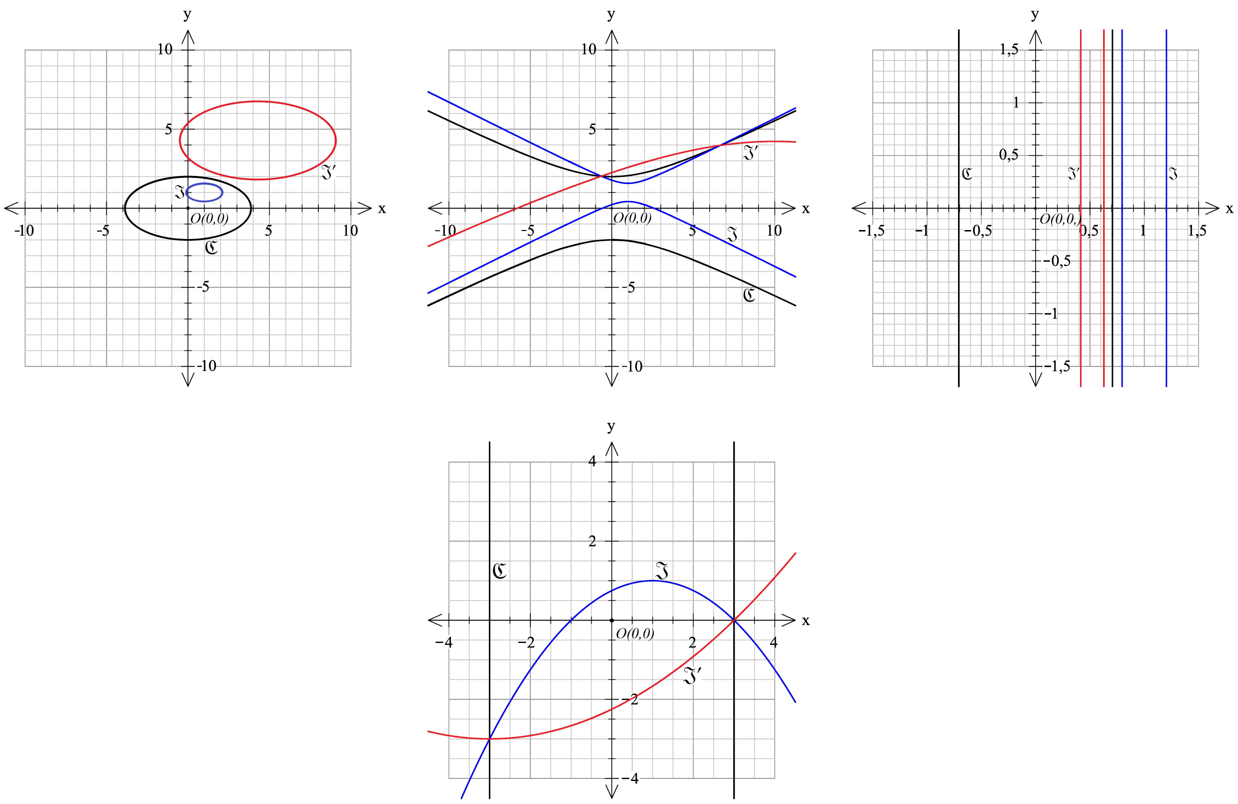

Figure 5. Inversion of a conic with respect to the elliptic, hyperbolic, and dual circle, respectively.

Theorem 3.5. *Let $\mathcal{P} : -A_0^2(x-h)^2 = a + y$ be a parabola and $\mathfrak{C} : -A^2x^2 = r$ be a dual circle. The inversion of the parabola $\mathcal{P}$ with respect to the dual circle $\mathfrak{C}$ is*

- a line, if $\mathcal{P}$ passes through the center of inversion;
- a parabola, if $\mathcal{P}$ does not pass through the center of inversion.

Proof. Let

$$\begin{aligned} \mathfrak{C} &: -A^2x^2 = r, \\ \mathcal{P} &: -A_0^2(x-h)^2 = a + y, \quad (3.10) \\ \rho_P &= -A^2x^2. \quad (3.11) \end{aligned}$$

The inversion point is

$$P' = \left(\frac{rx}{\rho_P}, \frac{ry}{\rho_P}\right).$$

Substituting P' and (3.11) into (3.10), we obtain

$$-A_0^2\left(\frac{-rx}{A^2x^2} - h\right)^2 = a + \frac{ry}{-A^2x^2}.$$

This simplifies to

$$A^4\left(A_0^2h^2 + a\right)x^2 - 2A^2A_0^2hx + A^2y + A_0^2 = 0.$$

If $\mathcal{P}$ passes through the center of inversion, then $-A_0^2h^2 = a$, yielding

$$y = 2A_0^2hx - \left(\frac{A_0}{A}\right)^2.$$

If $\mathcal{P}$ does not pass through the center, then

$$y = A^2\left(A_0^2h^2 + a\right)x^2 + 2A_0^2hx - \left(\frac{A_0}{A}\right)^2.$$

□

Theorem 3.6. *Let $\mathfrak{I}$ and $\mathfrak{C} : -A^2x^2 + \Delta B^2y^2 = r$ be two non-homothetic conics, where $\mathfrak{I}$ passes through the center of inversion. If the plane $\mathbb{P}_v$ is non-parabolic, then the inverse of the conic $\mathfrak{I}$ with respect to $\mathfrak{C}$ is a third-degree curve. If the plane $\mathbb{P}_v$ is parabolic, the inverse of $\mathfrak{I}$ is a line.*

Proof. Let us consider

$$\begin{aligned} \mathfrak{C} &: -A^2x^2 + \Delta B^2y^2 = r, \\ \mathfrak{I} &: -A_1^2(x-h)^2 + \Delta B_1^2(y-s)^2 = r, \quad (3.12) \\ A_1 &\neq kA, \quad B_1 \neq kB, \quad k \in \mathbb{R}, \\ \rho_P &= -A^2x^2 + \Delta B^2y^2. \quad (3.13) \end{aligned}$$

Since $\mathfrak{I}$ passes through the center of inversion, we have

$$-A_1^2h^2 + \Delta B_1^2s^2 = r.$$

The inversion of a point $P = (x, y)$ is given by

$$P' = \left(\frac{rx}{\rho_P}, \frac{ry}{\rho_P}\right).$$

Substituting P' and (3.13) into equation (3.12), we get

$$-A_1^2\left(\frac{rx}{\rho_P} - h\right)^2 + \Delta B_1^2\left(\frac{ry}{\rho_P} - s\right)^2 = r.$$

This can be rewritten as:

$$-A_1^2\left(\frac{rx}{-A^2x^2 + \Delta B^2y^2} - h\right)^2 + \Delta B_1^2\left(\frac{ry}{-A^2x^2 + \Delta B^2y^2} - s\right)^2 = -A_1^2h^2 + \Delta B_1^2s^2.$$

Clearing denominators and simplifying leads to the following cubic equation:

$$2A^2A_1^2hx^3 - 2A^2B_1^2s\Delta x^2y - 2B^2A_1^2h\Delta xy^2 + 2B^2B_1^2s\Delta^2y^3 + rA_1^2x^2 - rB_1^2\Delta y^2 = 0.$$

Therefore, the inverse of $\mathfrak{J}$ with respect to $\mathfrak{C}$ is a third-degree curve in non-parabolic planes.

If the plane $\mathbb{P}_\mathbf{v}$ is parabolic (i.e., $\Delta = 0$), this equation reduces to a linear equation, representing a line. □

The following table summarizes the information stated in Theorem 3.6.

Conic $\mathfrak{C}$	Inverse of $\mathfrak{J}$
Ellipse	$2A^2A_1^2hx^3 + 2A^2B_1^2sx^2y + 2B^2A_1^2hxy^2 + 2B^2B_1^2sy^3 - A_1^2x^2 - B_1^2y^2 = 0$
Hyperbola	$2A^2A_1^2hx^3 - 2A^2B_1^2sx^2y - 2B^2A_1^2hxy^2 + 2B^2B_1^2sy^3 + rA_1^2x^2 - rB_1^2y^2 = 0$
Dual circle	$2A^2hx = 1$

Theorem 3.7. *Let $\mathfrak{J}$ and $\mathfrak{C} : -A^2x^2 + \Delta B^2y^2 = r$ be two non-homothetic conics, where $\mathfrak{J}$ does not pass through the center of inversion. Then the inverse of $\mathfrak{J}$ with respect to $\mathfrak{C}$ is a fourth-degree curve with a singular point at the center of inversion.*

Proof. Consider the following conics:

$$\begin{aligned} \mathfrak{C} &: -A^2x^2 + \Delta B^2y^2 = r, \\ \mathfrak{J} &: -A_1^2(x-h)^2 + \Delta B_1^2(y-s)^2 = r, \quad (3.14) \\ A_1 &\neq kA, \quad B_1 \neq kB, \quad k \in \mathbb{R}, \\ \rho_P &= -A^2x^2 + \Delta B^2y^2. \quad (3.15) \end{aligned}$$

Since $\mathfrak{J}$ does not pass through the center of inversion, we have

$$-A_1^2h^2 + \Delta B_1^2s^2 \neq r.$$

The inversion of a point $P = (x, y)$ with respect to $\mathfrak{C}$ is given by

$$P' = \left(\frac{rx}{\rho_P}, \frac{ry}{\rho_P}\right).$$

Substituting P' into equation (3.14) yields

$$-A_1^2\left(\frac{rx}{\rho_P}-h\right)^2+\Delta B_1^2\left(\frac{ry}{\rho_P}-s\right)^2=r.$$

By replacing ρ_P from (3.15), we get

$$-A_1^2\left(\frac{rx}{-A^2x^2+\Delta B^2y^2}-h\right)^2+\Delta B_1^2\left(\frac{ry}{-A^2x^2+\Delta B^2y^2}-s\right)^2=r.$$

Clearing denominators and simplifying, we obtain the fourth-degree equation

$$\begin{aligned}&A^4(h^2A_1^2-B_1^2s^2\Delta+r)x^4+2rA^2A_1^2hx^3+B^4(A_1^2h^2-B_1^2s^2\Delta+r)\Delta^2y^4\\&+2(AB)^2(B_1^2s^2\Delta-A_1^2h^2\Delta-r\Delta)x^2y^2-2r\Delta A^2B_1^2sx^2y+2r\Delta^2B^2B_1^2sy^3\\&-2r\Delta B^2A_1^2hxy^2+A_1^2x^2-B_1^2\Delta y^2=0.\end{aligned}$$

Therefore, the inverse of $\mathfrak{J}$ with respect to $\mathfrak{C}$ is a fourth-degree curve featuring a singular point at the center of inversion. □

The following table summarizes the results stated in Theorem 3.7. These findings suggest a classification of fourth-degree equations as inverse images of conics. However, it should be noted that not all fourth-degree curves can be categorized in this manner.

Conic $\mathfrak{C}$	Inverse of $\mathfrak{J}$
Ellipse	$\left(A_1^2h^2+B_1^2s^2-1\right)\left(A^4x^4+B^4y^4\right)-2\left((AA_1)^2hx^3+(BB_1)^2sy^3\right)$ $+2(AB)^2\left(A_1^2h^2+B_1^2s^2-1\right)x^2y^2-2(AB_1)^2sx^2y$ $-2(BA_1)^2hxy^2+B_1^2y^2+A_1^2x^2=0$
Hyperbola	$\left(A_1^2h^2-B_1^2s^2+r\right)\left(A^4x^4+B^4y^4\right)+2r\left((AA_1)^2hx^3+(BB_1)^2sy^3\right)$ $+2(AB)^2\left(A_1^2h^2-B_1^2s^2+r\right)x^2y^2-2r(AB_1)^2sx^2y+$ $-2r(BA_1)^2hxy^2+A_1^2x^2-B_1^2y^2=0$
Dual circle	$A^4\left(A_1^2h^2-1\right)x^2-2(AA_1)^2hx+A_1^2=0$

4. Conclusions

In this paper, we developed a comprehensive framework for inversion transformations with respect to central conics in hybrid number planes. By leveraging the algebraic structure of hybrid numbers and the associated pseudo-Euclidean geometry, explicit inversion formulas were derived for points, lines, and conics across elliptic, hyperbolic, and parabolic planes. The results demonstrate that lines passing through the center of inversion remain invariant, while other lines transform into conic sections. Furthermore, homothetic conics preserve their type under inversion, whereas non-homothetic conics generate higher-order curves, including cubics and quartics, depending on their geometric configuration and the nature of the hybridian plane.

This paper bridges classical inversive geometry with hypercomplex algebra, providing a unified approach to geometric transformations in generalized metric spaces. The findings not only fill a gap in the literature but also offer potential applications in fields such as non-Euclidean modeling, computer graphics, and theoretical physics. Future research may explore dynamic transformations, mappings in higher-dimensional hybrid spaces, and applications of hybrid inversions in solving geometric problems within applied sciences.

Author contributions

İskender Öztürk: Methodology, conceptualization, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing; Hasan Çakır: Methodology, conceptualization, validation, formal analysis, investigation, resources, writing-original draft preparation, writing-review and editing, funding acquisition; Mustafa Özdemir: Methodology, conceptualization, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank the reviewers for their helpful suggestions on how to improve the work. This study has been supported by the Recep Tayyip Erdoğan University Development Foundation (Grant Number: 02025004024447).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. I. M. Yaglom, *Geometric transformations IV: circular transformations*, Washington: Mathematical Association of America, 2014.

2. S. Ponnusamy, H. Silverman, *Complex variables with applications*, Boston: Birkhäuser, 2006. https://doi.org/10.1007/978-0-8176-4513-7

3. N. A. Childress, Inversion with respect to the central conics, *Mathematics Magazine*, **38** (1965), 147–149. https://doi.org/10.1080/0025570X.1965.11975615

4. J. Hrdina, A. Návrat, P. Vašík, Geometric algebra for conics, *Adv. Appl. Clifford Algebras*, **28** (2018), 66. https://doi.org/10.1007/s00006-018-0879-2

5. D. Klawitter, Reflections in conics, quadrics and hyperquadrics via Clifford algebra, *Beitr. Algebra Geom.*, **57** (2016), 221–242. https://doi.org/10.1007/s13366-014-0218-2

6. G. Casanova, Complement on elliptic and hyperbolic inversions, *Adv. Appl. Clifford Algebras*, **11** (2001), 293–295. https://doi.org/10.1007/BF03042319

7. J. L. Ramírez, G. N. Rubiano, Elliptic inversion of two-dimensional objects: a graphical point of view with Mathematica, *J. Geom.*, **3** (2014), 12–27.

8. G. Casanova, Conjugated of conics, *Adv. Appl. Clifford Algebras*, **18** (2008), 143–146. https://doi.org/10.1007/s00006-008-0069-8

9. G. Sobczyk, The hyperbolic number plane, *The College Mathematics Journal*, **26** (1995), 268–280. https://doi.org/10.1080/07468342.1995.11973712

10. W. Miller, R. Boehning, Gaussian, parabolic, and hyperbolic numbers, *The Mathematics Teacher*, **61** (1968), 377–382. https://doi.org/10.5951/MT.61.4.0377

11. A. A. Harkin, J. B. Harkin, Geometry of generalized complex numbers, *Mathematics Magazine*, **77** (2004), 118–129. https://doi.org/10.1080/0025570X.2004.11953236

12. V. V. Kisil, *Geometry of Mobius transformations: Elliptic, parabolic and hyperbolic actions of sl2(r)*, Singapore: World Scientific, 2012. https://doi.org/10.1142/p835

13. V. V. Kisil, Erlangen programme at large 3.2 ladder operators in hypercomplex mechanics, *Acta Polytech.*, **51** (2011), 44–53. https://doi.org/10.14311/1402

14. V. V. Kisil, Erlangen program at large: An overview, In: *Advances in applied analysis*, Basel: Birkhäuser, 2012, 1–94. https://doi.org/10.1007/978-3-0348-0417-2_1

15. V. V. Kisil, Induced representations and hypercomplex numbers, *Adv. Appl. Clifford Algebras*, **23** (2013), 417–440. https://doi.org/10.1007/s00006-012-0373-1

16. J. Rooney, On the three types of complex number and planar transformations, *Environment and Planning B: Planning and Design*, **5** (1978), 89–99. https://doi.org/10.1068/b050089

17. J. Rooney, Generalised complex numbers in mechanics, In: *Advances on theory and practice of robots and manipulators*, Cham: Springer, 2014, 55–62. https://doi.org/10.1007/978-3-319-07058-2_7

18. M. Özdemir, Introduction to hybrid numbers, *Adv. Appl. Clifford Algebras*, **28** (2018), 11. https://doi.org/10.1007/s00006-018-0833-3

19. İ. Öztürk, M. Özdemir, Similarity of hybrid numbers, *Math. Method. Appl. Sci.*, **43** (2020), 8867–8881. https://doi.org/10.1002/mma.6580

20. İ. Öztürk, M. Özdemir, Elliptical rotations with hybrid numbers, *Indian J. Pure Appl. Math.*, **55** (2024), 23–39. https://doi.org/10.1007/s13226-022-00343-5

21. H. Çakır, M. Özdemir, Hybrid number matrices, *Filomat*, **37** (2023), 9215–9227. https://doi.org/10.2298/FIL2327215C

22. H. Çakır, Consimilarity of hybrid number matrices and hybrid number matrix equations $\mathbf{A}\widetilde{\mathbf{X}}-\mathbf{XB}=\mathbf{C}$, *AIMS Mathematics*, **10** (2025), 8220–8234. https://doi.org/10.3934/math.2025378

23. S. Ulrych, Relativistic quantum physics with hyperbolic numbers, *Phys. Lett. B*, **625** (2005), 313–323. https://doi.org/10.1016/j.physletb.2005.08.072

24. R. C. Nunes, Erlangen's program for space-time through space-time geometric algebra induced by the R vector characteristic of the ring of hybrid numbers Z, (2021), arXiv: 2106.11106. https://doi.org/10.48550/arXiv.2106.11106

25. F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, P. Zampetti, *The mathematics of Minkowski space-time: with an introduction to commutative hypercomplex numbers*, Basel: Birkhäuser, 2008. https://doi.org/10.1007/978-3-7643-8614-6

26. F. Catoni, R. Cannata, V. Catoni, P. Zampetti, Hyperbolic trigonometry in two-dimensional space-time geometry, *Italian Physical Society*, **118** (2003), 475–492. https://doi.org/10.1393/ncb/i2003-10012-9

27. H. A. Diao, X. X. Fei, H. Y. Liu, K. Yang, Visibility, invisibility, and unique recovery of inverse electromagnetic problems with conical singularities, *Inverse Probl. Imag.*, **18** (2024), 541–570. https://doi.org/10.3934/ipi.2023043

28. H. A. Diao, H. Y. Liu, L. Y. Tao, Stable determination of an impedance obstacle by a single far-field measurement, *Inverse Probl.*, **40** (2024), 055005. https://doi.org/10.1088/1361-6420/ad3087

29. H. A. Diao, R. X. Tang, H. Y. Liu, J. X. Tang, Unique determination by a single far-field measurement for an inverse elastic problem, *Inverse Probl. Imag.*, **18** (2024), 1405–1430. https://doi.org/10.3934/ipi.2024020

30. B. O'neill, *Semi-Riemannian geometry with applications to relativity*, New York: Academic Press, 1983.

31. D. G. Zill, P. D. Shanahan, *Complex analysis: A first course in complex analysis with applications*, Burlington: Jones & Bartlett Learning, 2009.

32. H. Simsek, M. Özdemir, Generating hyperbolical rotation matrix for a given hyperboloid, *Linear Algebra Appl.*, **496** (2016), 221–245. https://doi.org/10.1016/j.laa.2016.01.038

33. P. S. Modenov, A. S. Parkhomenko, *Euclidean and affine transformations: geometric transformations*, New York: Academic Press, 2014.

34. J. L. Ramírez, Inversions in an ellipse, *Forum Geometricorum*, **14** (2014), 107–115.