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Research article

New oscillation results for noncanonical quasilinear differential equations of neutral type

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Abstract: This article explored oscillation conditions for quasi-linear neutral differential equations of noncanonical form. By establishing new iterative monotonic properties of solutions, we derived novel oscillation criteria that extend and refine existing results in the literature. To illustrate the significance of our findings, we provided three concrete examples demonstrating the applicability of the proposed conditions.

Keywords: oscillatory; nonoscillatory; neutral differential equations; second-order; noncanonical
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1. Introduction

Differential equations (DEs) are essential for modeling and comprehending a wide range of real-world phenomena, offering significant insights into the dynamics and behaviors of these systems. These equations often describe intricate processes in disciplines such as physics, biology, engineering, and economics. However, many differential equations encountered in practical problems lack closed-form solutions, meaning they cannot be explicitly resolved using standard mathematical functions. Consequently, numerical and approximate techniques are employed to analyze and solve these equations. Over recent decades, the formulation of various differential equation models has spurred extensive research into their qualitative properties, including existence, oscillation, periodicity, finiteness, and stability. Understanding these properties allows researchers and practitioners to grasp the long-term behavior and stability of solutions, despite the absence of explicit analytical solutions, see [1–3].

Neutral differential equations (NDEs) have become a critical tool for modeling diverse phenomena in science and engineering. These equations are particularly useful in analyzing systems such as electric networks with lossless transmission lines and the dynamics of vibrating masses connected to elastic bars. These applications highlight the importance of understanding the qualitative behavior of NDEs, including their existence, stability, asymptotic behavior, and oscillatory properties. Despite the significant interest, the oscillatory and nonoscillatory nature of solutions to NDEs remains complex and unresolved. Since the foundational work of Sturm in 1836, oscillation theory has been extensively developed, particularly for functional differential equations. Researchers have expanded the scope of NDEs through various methods, such as the Riccati transformation, integral averaging, and comparison techniques, to enhance oscillation conditions, see [4–7].

In this work, we consider the second-order noncanonical neutral differential equation (NDE):

$$(\mathfrak{a}(s)\,(z'(s))^{\alpha})' + \mathfrak{q}(s)u^{\beta}(\mathfrak{h}(s)) = 0,\ s \geq s_0, \tag{1.1}$$

where $z(s) = u(s) + h(s)\,u(\mathfrak{z}(s))$. Throughout the paper we assume that:

(H_1) $0 < \alpha \leq 1$, $\alpha \geq \beta$ are ratios of odd positive integers;
(H_2) $h, \mathfrak{q} \in C([s_0, \infty), \mathbb{R}^+)$, $0 \leq h(s) < 1$ and $\mathfrak{q}(s) > 0$;
(H_3) $\mathfrak{a} \in C([s_0, \infty), \mathbb{R}^+)$ satisfies $\pi(s_0) < \infty$, where

$$\pi(s) := \int_s^{\infty} \frac{1}{\mathfrak{a}^{1/\alpha}(\varrho)} d\varrho;$$

(H_4) $\mathfrak{z}, \mathfrak{h} \in C^1([s_0, \infty), \mathbb{R})$ satisfies $\mathfrak{h}(s) \leq s$, $\mathfrak{z}(s) \leq s$, $\mathfrak{h}'(s) > 0$ and $\lim_{s\to\infty} \mathfrak{z}(s) = \lim_{s\to\infty} \mathfrak{h}(s) = \infty$;
(H_5) $h(s) < \frac{\pi(s)}{\pi(\mathfrak{z}(s))}$.

A function $u(s) \in C([s_u, \infty), \mathbb{R})$, $s_u \geqslant s_0$, is said to be a solution of (1.1) which has the property $\mathfrak{a}(s)(z'(s))^{\alpha} \in C^1[s_u, \infty)$, and it satisfies Eq (1.1) for all $s \in [s_u, \infty)$. We consider only those solutions $u(s)$ of (1.1) which exist on some half-line $[s_u, \infty)$ and satisfy the condition

$$\sup\{|u(s)| : s \geqslant S\} > 0,\ \text{for all } S \geq s_u.$$

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Oscillation theory aims to establish conditions ensuring that all solutions of a given differential equation exhibit oscillatory behavior. Ladde et al. [8] were pioneers in this field, extensively discussing oscillation theory up to 1984. However, their focus primarily lay on how varying arguments influence solution oscillations, without addressing neutral delay equations. Gyori and Ladas [9], on the other hand, made substantial contributions to linearized oscillation theory. Their work elucidated the relationship between the root distribution of characteristic equations and the oscillatory nature of all solutions, marking a significant advancement in oscillation theory. Oscillation theory has seen significant advancements in recent years, particularly in understanding the conditions necessary for the existence of solutions with specific asymptotic properties and calculating the separation between the zeros of oscillatory solutions. This progress is well-documented in several monographs that summarize the literature from the past decade, offering new results, and methods, see [10–12]. Recent publications have introduced improved criteria for assessing the oscillation of delay equations [13–16], and neutral differential equations [17–20]. These studies have developed oscillation criteria for both canonical and non-canonical situations by employing an extended Riccati substitution and comparisons with first-order differential equations. The collective efforts in these areas highlight the dynamic evolution of oscillation theory and its expanding scope of applications.

Grammatikopoulos et al. [21] established conditions under which the NDE

$$(u(\mathrm{s}) + \mathrm{h}(\mathrm{s})\, u(\mathrm{s} - \mathfrak{z}))'' + \mathfrak{q}(\mathrm{s})\, u(\mathrm{s} - \mathfrak{b}) = 0$$

is oscillatory, provided

$$0 < \mathrm{h}(\mathrm{s}) \leq 1,\ \mathfrak{q}(\mathrm{s}) \geq 0,$$

and

$$\int_{\mathrm{s}_0}^{\infty} \mathfrak{q}(\varrho)\,(1 - \mathrm{h}(\varrho - \mathfrak{b}))\, \mathrm{d}\varrho = \infty.$$

Subsequent studies by Xu and Xia [22] generalized this result to include

$$(u(\mathrm{s}) + \mathrm{h}(\mathrm{s})\, u(\mathrm{s} - \mathfrak{z}))'' + \mathfrak{q}(\mathrm{s})\, f(u(\mathrm{s} - \mathfrak{b})) = 0,$$

under conditions

$$0 \leq \mathrm{h}(\mathrm{s}) < \infty,\ \mathfrak{q}(\mathrm{s}) \geq M > 0.$$

Grace and Lalli [23] extended the analysis to the equation

$$(\mathfrak{a}(\mathrm{s})\,(u(\mathrm{s}) + \mathrm{h}(\mathrm{s})\, u(\mathrm{s} - \mathfrak{z}))')' + \mathfrak{q}(\mathrm{s})\, f(u(\mathrm{s} - \mathfrak{b})) = 0,$$

susubject to

$$\frac{f(u)}{u} \geq k > 0,\ \int_{\mathrm{s}_0}^{\infty} \frac{1}{\mathfrak{a}(\varrho)} \mathrm{d}\varrho = \infty,$$

and

$$\int_{\mathrm{s}_0}^{\infty} \left(\rho(\varrho)\, \mathfrak{q}(\varrho)\,(1 - \mathrm{h}(\varrho - \mathfrak{b})) - \frac{(\rho'(\varrho))^2\, \mathfrak{a}(\varrho - \mathfrak{b})}{4k\rho(\varrho)} \right) \mathrm{d}\varrho = \infty,$$

where $\rho(\mathrm{s})$ is an optional function.

Grace et al. [24] studied the oscillatory behavior of second-order nonlinear noncanonical NDEs. They established sufficient conditions for oscillation, analyzing equations of the form:

$$(\mathfrak{a}(\mathrm{s})\,(u(\mathrm{s}) + \mathrm{h}(\mathrm{s})\, u(\mathrm{s} - \mathfrak{z}))')' + \mathfrak{q}(\mathrm{s})\, f(u(\mathrm{s} - \mathfrak{b})) = 0.$$

Bohner et al. [25], Jadlovska et al. [26], Nabih et al. [27], and Al-Jaser and Moaaz [28] investigated the oscillation of the second-order half-linear NDE

$$(\mathfrak{a}(\mathrm{s})\,[(u(\mathrm{s}) + \mathrm{h}(\mathrm{s})\, u(\mathfrak{z}(\mathrm{s})))']^{\alpha})' + \mathfrak{q}(\mathrm{s})\, u^{\alpha}(\mathfrak{b}(\mathrm{s})) = 0 \tag{1.2}$$

in both canonical and noncanonical cases. The asymptotic and oscillatory behavior of neutral DEs with distributed deviating arguments was studied by Moaaz et al. [29] and Al Themairi et al. [30], with a damping term by Althobati et al. [31] and Moaaz et al. [32].

This paper aims to address a significant gap in the literature by extending the investigation of oscillatory behavior to second-order NDEs. Our research is inspired by Baculíková's work [33], which explored the asymptotic properties and oscillations of second-order delay differential equations (DDEs):

$$\left(\mathfrak{a}(\mathrm{s})x'(\mathrm{s})\right)' + \mathfrak{q}(\mathrm{s})x(\mathfrak{h}(\mathrm{s})) = 0. \tag{1.3}$$

This means that (1.3) is a special case of (1.1) where $\alpha = \beta = 1$, and $\mathrm{h}(\mathrm{s}) = 0$. Inspired by their findings, we have extended the equation and used the same technique, thereby contributing to the broader understanding of NDEs.

2. Preliminary results

Let us define

$$K^{[0]}(\mathrm{s}) := K(\mathrm{s}) \text{ and } K^{[j]}(\mathrm{s}) := K\left(K^{[j-1]}(\mathrm{s})\right), \text{ for } j = 1, 2, ..., m,$$

$$\mathrm{H}_m(\mathrm{s}) := \sum_{i=0}^{m}\left(\prod_{j=0}^{2i} \mathrm{h}\left(\mathfrak{z}^{[j]}(\mathrm{s})\right)\right)\left(\frac{1}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - \frac{\pi(\mathfrak{z}^{[2i+1]}(\mathrm{s}))}{\pi(\mathfrak{z}^{[2i]}(\mathrm{s}))}\right),$$

and

$$\widetilde{\mathfrak{q}}(\mathrm{s}) := \mathfrak{q}(\mathrm{s})\mathrm{H}_m^{\beta}(\mathfrak{h}(\mathrm{s})),$$

for $\mathrm{s}_1 \in [\mathrm{s}_0, \infty)$.

According to a generalization of Kiguradze's lemma in [34], the set of positive solutions of (1.1) possesses the following structure.

Lemma 2.1. *[34] Suppose that $u(\mathrm{s})$ is an eventually positive solution of (1.1). Then the corresponding function $z(\mathrm{s})$ satisfies one of two cases eventually:*

$$\begin{aligned}(N_1) \quad &: \quad z(\mathrm{s}) > 0,\ z'(\mathrm{s}) > 0,\ \left(\mathfrak{a}(\mathrm{s})(z'(\mathrm{s}))^{\alpha}\right)' < 0,\\ (N_2) \quad &: \quad z(\mathrm{s}) > 0,\ z'(\mathrm{s}) < 0,\ \left(\mathfrak{a}(\mathrm{s})(z'(\mathrm{s}))^{\alpha}\right)' < 0,\end{aligned}$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

Lemma 2.2. *[35] Suppose that u is an eventually positive solution of (1.1). Then, eventually,*

$$u(t) > \sum_{i=0}^{m}\left(\prod_{j=0}^{2i} \mathrm{h}\left(\mathfrak{z}^{[j]}(\mathrm{s})\right)\right)\left(\frac{z\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - z\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)\right), \tag{2.1}$$

where $m > 0,\ m \in \mathbb{Z}$.

Lemma 2.3. *[36] Suppose that u is an eventually positive solution of (1.1) satisfying (N_2). Then*

$$z^{\beta/\alpha}(\mathrm{s}) \geq v(\mathrm{s})\, z(\mathrm{s}),$$

where

$$v(\mathrm{s}) := \begin{cases} 1 & \text{if } \alpha = \beta, \\ v_1 & \text{if } \alpha > \beta, \\ v_2 \pi^{(\beta-\alpha)/\alpha}(\mathrm{s}) & \text{if } \alpha < \beta. \end{cases}$$

Here, v_1 and v_2 are constants.

The following considerations are intended to show that the class (N_2) is the essential one.

Lemma 2.4. *If*

$$\int_{s_0}^{\infty}\left(\frac{1}{\mathfrak{a}(\varsigma)}\int_{s_0}^{\varsigma}\widetilde{\mathfrak{q}}(\varrho)\,d\varrho\right)^{1/\alpha}d\varsigma=\infty, \tag{2.2}$$

then, the positive solution $u(s)$ *of (1.1) satisfies* (N_2) *in Lemma 2.1 and, moreover,*
$(\Upsilon_{1,1})$ $\mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s)+z(s)\geqslant 0$;
$(\Upsilon_{1,2})$ $z(s)/\pi(s)$ *is increasing*;
$(\Upsilon_{1,3})$ $(\mathfrak{a}(s)(z'(s))^{\alpha})'\leq-\widetilde{\mathfrak{q}}(s)\,z^{\beta}(\mathfrak{h}(s))$;
$(\Upsilon_{1,4})$ $\lim_{s\to\infty}z(s)=0$.

Proof. Suppose on the contrary that u is an eventually positive solution of (1.1) satisfying condition (N_1) in Lemma 2.1 for $s\geq s_1\geq s_0$. Then there exists a constant $c_0>0$ such that $z(s)\geq c_0$ and $z(\mathfrak{h}(s))\geq c_0$ eventually. In light of Lemma 2.2, we conclude that

$$\begin{aligned}u(t) &> \sum_{i=0}^{m}\left(\prod_{j=0}^{2i}\mathrm{h}\left(\mathfrak{z}^{[j]}(s)\right)\right)\left(\frac{z\left(\mathfrak{z}^{[2i]}(s)\right)}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(s)\right)}-z\left(\mathfrak{z}^{[2i+1]}(s)\right)\right)\\ &\geq \sum_{i=0}^{m}\left(\prod_{j=0}^{2i}\mathrm{h}\left(\mathfrak{z}^{[j]}(s)\right)\right)\left(\frac{1}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(s)\right)}-1\right)z\left(\mathfrak{z}^{[2i+1]}(s)\right).\end{aligned} \tag{2.3}$$

Since

$$\frac{\pi(\mathfrak{z}^{[2i+1]}(s))}{\pi(\mathfrak{z}^{[2i]}(s))}\geq 1,$$

then

$$\frac{1}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(s)\right)}-1\geq\frac{1}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(s)\right)}-\frac{\pi(\mathfrak{z}^{[2i+1]}(s))}{\pi(\mathfrak{z}^{[2i]}(s))}. \tag{2.4}$$

Combining (2.3), (2.4), we get

$$u(t)\geq\sum_{i=0}^{m}\left(\prod_{j=0}^{2i}\mathrm{h}\left(\mathfrak{z}^{[j]}(s)\right)\right)\left(\frac{1}{\mathrm{h}\left(\mathfrak{z}^{[2i]}(s)\right)}-\frac{\pi(\mathfrak{z}^{[2i+1]}(s))}{\pi(\mathfrak{z}^{[2i]}(s))}\right)z\left(\mathfrak{z}^{[2i+1]}(s)\right)\geq c_0\mathrm{H}_m(s). \tag{2.5}$$

Substituting (2.5) into (1.1), we deduce that

$$(\mathfrak{a}(s)(z'(s))^{\alpha})'=-\mathfrak{q}(s)u^{\beta}(\mathfrak{h}(s))\leq-c_0^{\beta}\mathfrak{q}(s)\mathrm{H}_m^{\beta}(\mathfrak{h}(s))=-c_0^{\beta}\widetilde{\mathfrak{q}}(s).$$

Integrating the resulting inequality from s_1 to s, we have

$$\mathfrak{a}(s_2)(z'(s_2))^{\alpha}\geq c_0^{\beta}\int_{s_1}^{s}\widetilde{\mathfrak{q}}(\varrho)\,d\varrho. \tag{2.6}$$

It follows from (2.2) and (H_3) that $\int_{s_1}^{s}\widetilde{\mathfrak{q}}(\varrho)\,d\varrho$ must be unbounded. Further, since $\pi'(s)<0$, it is easy to see that

$$\int_{s_1}^{s}\widetilde{\mathfrak{q}}(\varrho)\,d\varrho\to\infty\text{ as }s\to\infty, \tag{2.7}$$

which with (2.6) gives a contradiction.

($\Upsilon_{1,1}$) From case (N_2) of Lemma 2.1, we note that $z(\mathrm{s})$ is positive and decreasing for all $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$. By the definition of $z(\mathrm{s})$, we have $z(\mathrm{s}) \geq u(\mathrm{s})$ and

$$u(\mathrm{s}) \geq z(\mathrm{s}) - \mathrm{h}(\mathrm{s})\, z(\mathfrak{z}(\mathrm{s})),\ \ \mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0. \tag{2.8}$$

Since $\mathfrak{a}(\mathrm{s})(z'(\mathrm{s}))^{\alpha}$ is decreasing, we get

$$\mathfrak{a}^{1/\alpha}(\mathrm{s})\, z'(\mathrm{s}) \geq \mathfrak{a}^{1/\alpha}(\varrho)\, z'(\varrho)\ \text{ for } \varrho \geq \mathrm{s}.$$

Dividing the last inequality by $\mathfrak{a}^{1/\alpha}(\varrho)$ and integrating the resulting inequality from s to ∞, we have

$$\mathfrak{a}^{1/\alpha}(\mathrm{s}) z'(\mathrm{s}) \pi(\mathrm{s}) + z(\mathrm{s}) \geqslant 0. \tag{2.9}$$

($\Upsilon_{1,2}$) From (2.9), we obtain

$$\left(\frac{z(\mathrm{s})}{\pi(\mathrm{s})}\right)' = \frac{\mathfrak{a}^{1/\alpha}(\mathrm{s}) z'(\mathrm{s}) \pi(\mathrm{s}) + z(\mathrm{s})}{\mathfrak{a}^{1/\alpha}(\mathrm{s}) \pi^2(\mathrm{s})} \geqslant 0.$$

($\Upsilon_{1,3}$) Since $z(\mathrm{s})/\pi(\mathrm{s})$ is increasing and $\mathfrak{z}^{[2i]}(\mathrm{s}) \geq \mathfrak{z}^{[2i+1]}(\mathrm{s})$, we derive that

$$z(\mathfrak{z}^{[2i+1]}(\mathrm{s})) \leq \frac{\pi(\mathfrak{z}^{[2i+1]}(\mathrm{s}))}{\pi(\mathfrak{z}^{[2i]}(\mathrm{s}))} z(\mathfrak{z}^{[2i]}(\mathrm{s})).$$

In light of Lemma 2.2, we conclude that

$$\begin{aligned} x(t) &> \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \mathrm{h}\left(\mathfrak{z}^{[j]}(\mathrm{s})\right) \right) \left(\frac{z\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}{\mathrm{h}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - z\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right) \right) \\ &\geq \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \mathrm{h}\left(\mathfrak{z}^{[j]}(\mathrm{s})\right) \right) \left(\frac{1}{\mathrm{h}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - \frac{\pi(\mathfrak{z}^{[2i+1]}(\mathrm{s}))}{\pi(\mathfrak{z}^{[2i]}(\mathrm{s}))} \right) z\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right). \end{aligned} \tag{2.10}$$

Since $z' < 0$, inequality (2.10) simplifies to

$$x(t) > \sum_{i=0}^{m} \left(\prod_{j=0}^{2i} \mathrm{h}\left(\mathfrak{z}^{[j]}(\mathrm{s})\right) \right) \left(\frac{1}{\mathrm{h}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - \frac{\pi(\mathfrak{z}^{[2i+1]}(\mathrm{s}))}{\pi(\mathfrak{z}^{[2i]}(\mathrm{s}))} \right) z(\mathrm{s}) = \mathrm{H}(\mathrm{s})\, z(\mathrm{s}). \tag{2.11}$$

By substituting (2.11) into (1.1), we have

$$\begin{aligned} (\mathfrak{a}(\mathrm{s})(z'(\mathrm{s}))^{\alpha})' &= -\mathfrak{q}(\mathrm{s}) u^{\beta}(\mathfrak{h}(\mathrm{s})) \\ &\leq -\mathfrak{q}(\mathrm{s}) \mathrm{H}_m^{\beta}(\mathfrak{h}(\mathrm{s}))\, z^{\beta}(\mathfrak{h}(\mathrm{s})) \\ &\leq -\widetilde{\mathfrak{q}}(\mathrm{s})\, z^{\beta}(\mathfrak{h}(\mathrm{s})), \end{aligned}$$

which leads to

$$(\mathfrak{a}(\mathrm{s})(z'(\mathrm{s}))^{\alpha})' \leq -\widetilde{\mathfrak{q}}(\mathrm{s})\, z^{\beta}(\mathfrak{h}(\mathrm{s})). \tag{2.12}$$

$(\Upsilon_{1,4})$ Since $z(\mathrm{s})>0,$ *and* $z'(\mathrm{s})<0$, then $\lim_{\mathrm{s}\to\infty} z(\mathrm{s})=c\geq 0$. We claim that $c=0$. If not, then $z(\mathrm{s})\geq c>0$ for $\mathrm{s}\geq \mathrm{s}_2\geq \mathrm{s}_1$. Then, integrating $(\Upsilon_{1,3})$ from s_1 to s, we arrive at

$$\begin{aligned}\mathfrak{a}(\mathrm{s})\left(z'(\mathrm{s})\right)^{\alpha} &\leq \mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha}-\int_{\mathrm{s}1}^{\mathrm{s}}\widetilde{\mathfrak{q}}(\varrho)\,z^{\beta}(\mathfrak{h}(\varrho))\,\mathrm{d}\varrho\\ &\leq -c^{\beta}\int_{\mathrm{s}_1}^{\mathrm{s}}\widetilde{\mathfrak{q}}(\varrho)\,\mathrm{d}\varrho.\end{aligned}$$

By integrating this inequality from s_1 to ∞, we get

$$z(\mathrm{s}_1)\geq c^{\beta/\alpha}\int_{\mathrm{s}_1}^{\infty}\left(\frac{1}{\mathfrak{a}(\varsigma)}\int_{\mathrm{s}1}^{\varsigma}\widetilde{\mathfrak{q}}(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha}\mathrm{d}\varsigma\to\infty \text{ as } \mathrm{s}\to\infty,$$

which leads to a contradiction with (2.2). Consequently, $c=0$.

This completes the proof of the lemma. □

3. Main results

In this part, we introduce novel monotonicity characteristics for the solutions of (1.1).

Lemma 3.1. *Assume that* $u(\mathrm{s})$ *is a positive solution of (1.1). If* $\delta_0\in(0,1)$ *with*

$$\frac{1}{\alpha}\mathfrak{a}^{1/\alpha}(\mathrm{s})\,\widetilde{\mathfrak{q}}(\mathrm{s})\,\pi^{\alpha+1}(\mathrm{s})\geqslant\delta_0^{\alpha},\ \ \rho_0=\nu_1\delta_0, \tag{3.1}$$

then

$(\Upsilon_{2,1})$ $z(\mathrm{s})/\pi^{\rho_0}(\mathrm{s})$ *is decreasing;*

$(\Upsilon_{2,2})$ $\lim_{\mathrm{s}\to\infty} z(\mathrm{s})/\pi^{\rho_0}(\mathrm{s})=0$;

$(\Upsilon_{2,3})$ $z(\mathrm{s})/\pi^{1-\rho_0}(\mathrm{s})$ *is increasing.*

Moreover, for the special case when $\alpha=\beta=1$, *we have* $\nu_1=1$.

Proof. Assume that $u(\mathrm{s})$ is an eventually positive solution of (1.1). From (3.1) we find that

$$\begin{aligned}\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\mathfrak{a}(\varsigma)}\int_{\mathrm{s}_1}^{\varsigma}\widetilde{\mathfrak{q}}(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha}\mathrm{d}\varsigma &\geq \alpha^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\mathfrak{a}(\varsigma)}\int_{\mathrm{s}_1}^{\varsigma}\frac{1}{\mathfrak{a}^{1/\alpha}(\varrho)\,\pi^{\alpha+1}(\varrho)}\mathrm{d}\varrho\right)^{1/\alpha}\mathrm{d}\varsigma\\ &= \alpha^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{\mathfrak{a}^{1/\alpha}(\varsigma)}\left(\int_{\mathrm{s}_1}^{\varsigma}\frac{1}{\mathfrak{a}^{1/\alpha}(\varrho)\,\pi^{\alpha+1}(\varrho)}\mathrm{d}\varrho\right)^{1/\alpha}\mathrm{d}\varsigma\\ &= \delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{\mathfrak{a}^{1/\alpha}(\varsigma)}\left(\pi^{-\alpha}(\varsigma)-\pi^{-\alpha}(\mathrm{s}_1)\right)^{1/\alpha}\mathrm{d}\varsigma.\end{aligned}$$

From the fact that $\lim_{\mathrm{s}\to\infty} z(\mathrm{s})=0$, there exists a $\mathrm{s}_1\geq \mathrm{s}_0$ such that $\pi^{-\alpha}(\mathrm{s})-\pi^{-\alpha}(\mathrm{s}_1)\geq\epsilon\pi^{-\alpha}(\mathrm{s})$ for $\epsilon\in(0,1)$. Thus

$$\begin{aligned}\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{\mathfrak{a}(\varsigma)}\int_{\mathrm{s}_1}^{\varsigma}\widetilde{\mathfrak{q}}(\varrho)\,\mathrm{d}\varrho\right)^{1/\alpha}\mathrm{d}\varsigma &\geq \epsilon^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{\mathfrak{a}^{1/\alpha}(\varsigma)\pi(\varsigma)}\mathrm{d}\varsigma\\ &= \epsilon^{1/\alpha}\delta_0\lim_{\mathrm{s}\to\infty}\ln\frac{\pi(\mathrm{s}_0)}{\pi(\mathrm{s})}\to\infty.\end{aligned}$$

Hence, from Lemma 2.4, we have that $(\Upsilon_{1,1})$–$(\Upsilon_{1,4})$ hold.

$(\Upsilon_{2,1})$ Integrating (1.1) from s_1 to s, we obtain

$$\begin{aligned} -\mathfrak{a}(s)\left(z'(s)\right)^{\alpha} &= -\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} + \int_{s_1}^{s} \mathfrak{q}(\varrho)u^{\beta}(\mathfrak{h}(\varrho))d\varrho \\ &\geqslant -\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} + \int_{s_1}^{s} \widetilde{\mathfrak{q}}(\varrho)\, z^{\beta}(\mathfrak{h}(\varrho))\, d\varrho \\ &\geqslant -\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} + z^{\beta}(s) \int_{s_1}^{s} \widetilde{\mathfrak{q}}(\varrho)\, d\varrho. \end{aligned}$$

By using (3.1), we get

$$\begin{aligned} -\mathfrak{a}(s)\left(z'(s)\right)^{\alpha} &\geqslant -\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} + z^{\beta}(s) \int_{s_1}^{s} \frac{\alpha\delta_0^{\alpha}}{\mathfrak{a}^{1/\alpha}(\varrho)\, \pi^{\alpha+1}(\varrho)} d\varrho \\ &= -\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} + \delta_0^{\alpha}\frac{z^{\beta}(s)}{\pi^{\alpha}(s)} - \delta_0^{\alpha}\frac{z^{\beta}(s)}{\pi^{\alpha}(s_1)}. \end{aligned} \tag{3.2}$$

Since $\lim_{s\to\infty} z(s) \to \infty$, there is a $s_2 \in (s_1, \infty)$ such that

$$-\mathfrak{a}(s_1)\left(z'(s_1)\right)^{\alpha} - \delta_0^{\alpha}\frac{z^{\beta}(s)}{\pi^{\alpha}(s_1)} \geq 0,$$

and so, (3.2) becomes

$$-\mathfrak{a}^{1/\alpha}(s)z'(s) \geqslant \delta_0 \frac{z^{\beta/\alpha}(s)}{\pi(s)}, \;\; s \geq s_2,$$

and so,

$$\mathfrak{a}^{1/\alpha}(s)\pi(s)z'(s) + \delta_0 z^{\beta/\alpha}(s) \leq 0. \tag{3.3}$$

Furthermore, from Lemma 2.3 we see that

$$\mathfrak{a}^{1/\alpha}(s)\pi(s)z'(s) + v_1\delta_0 z(s) \leq \mathfrak{a}^{1/\alpha}(s)\pi(s)z'(s) + \delta_0 z^{\beta/\alpha}(s) \leq 0,$$

which leads to

$$\mathfrak{a}^{1/\alpha}(s)\pi(s)z'(s) + \rho_0 z(s) \leq 0. \tag{3.4}$$

Consequently,

$$\left(\frac{z(s)}{\pi^{\rho_0}(s)}\right)' = \frac{\mathfrak{a}^{1/\alpha}(s)\pi(s)z'(s) + \rho_0 z(s)}{\mathfrak{a}^{1/\alpha}(s)\pi^{1+\rho_0}(s)} \leq 0,$$

so $z(s)/\pi^{\rho_0}(s)$ is decreasing.

$(\Upsilon_{2,2})$ Since $z(s)/\pi^{\rho_0}(s)$ is positive and decreasing, $\lim_{s\to\infty} z(s)/\pi^{\rho_0}(s) = c_1 \geqslant 0$. We claim that $c_1 = 0$. If not, then $z(s)/\pi^{\rho_0}(s) \geqslant c_1 > 0$ eventually. Now, we introduce the function

$$w(s) = \left(\mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s) + z(s)\right)\pi^{-\rho_0}(s).$$

In view of $(\Upsilon_{1,1})$ in Lemma 2.4, we note that $w(s) > 0$ and

$$w'(u) \;=\; \left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)'\pi^{1-\rho_0}(s) - (1-\rho_0)\, z'(s)\pi^{-\rho_0}(s) + z'(s)\pi^{-\rho_0}(s) + \rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\left(\mathfrak{a}(s)\left(z'(s)\right)^{\alpha}\right)'\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi^{1-\rho_0}(s)+\rho_0 z'(s)\pi^{-\rho_0}(s)+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&= -\frac{1}{\alpha}\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi^{1-\rho_0}(s)\mathfrak{q}(s)u^{\beta}(\mathfrak{h}(s))+\rho_0 z'(s)\pi^{-\rho}(s)+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&\le -\frac{1}{\alpha}\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi^{1-\rho_0}(s)\widetilde{\mathfrak{q}}(s)\,z^{\beta}(\mathfrak{h}(s))+\rho_0 z'(s)\pi^{-\rho_0}+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}.
\end{aligned}
$$

By using (3.1), (3.3), and (3.4), we find

$$
\begin{aligned}
w'(s) &\le -\left(\frac{\delta_0 z^{\beta/\alpha}(s)}{\pi(s)}\right)^{1-\alpha}\pi^{1-\rho_0}(s)\frac{\delta_0^{\alpha}}{\mathfrak{a}^{1/\alpha}(s)\pi^{\alpha+1}(s)}z^{\beta}(s)+\rho_0 z'(s)\pi^{-\rho_0}+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&\le -\delta_0 z^{\beta/\alpha}(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}+\rho_0 z'(s)\pi^{-\rho_0}(s)+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&\le -\nu_1\delta_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}+\rho_0 z'(s)\pi^{-\rho_0}(s)+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&\le -\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}+\rho_0 z'(s)\pi^{-\rho_0}(s)+\rho_0 z(s)\frac{\pi^{-1-\rho_0}(s)}{\mathfrak{a}^{1/\alpha}(s)}\\
&\le \rho_0 z'(s)\pi^{-\rho_0}(s)\\
&\le -\rho_0\pi^{-\rho_0}(s)\,\frac{\rho_0 z(s)}{\mathfrak{a}^{1/\alpha}(s)\pi(s)}\\
&\le -\frac{\rho_0^2}{\mathfrak{a}^{1/\alpha}(s)\pi(s)}\frac{z(s)}{\pi^{\rho_0}(s)}.
\end{aligned}
$$

Using the fact that $z(s)/\pi^{\rho_0}(s) \geqslant c_1$, we get

$$
w'(s) \leqslant -\frac{\rho_0^2 c_1}{\mathfrak{a}^{1/\alpha}(s)\pi(s)} < 0.
$$

Applying the integration from s_1 to s, on the previous inequality, we obtain

$$
w(s_1) \ge \rho_0^2 c_1 \ln\frac{\pi(s_1)}{\pi(s)} \to \infty \text{ as } s \to \infty,
$$

which is a contradiction. Thus, $c_1 = 0$.

($\Upsilon_{2,3}$) Finally, we have

$$
\begin{aligned}
\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s)+z(s)\right)' &= \left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)'\pi(s)-z'(s)+z'(s)\\
&= \left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)'\pi(s)\\
&= \frac{1}{\alpha}\left(\mathfrak{a}(s)\left(z'(s)\right)^{\alpha}\right)'\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi(s)\\
&\le -\frac{1}{\alpha}\widetilde{\mathfrak{q}}(s)z^{\beta}(s)\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi(s)\\
&\le -\delta_0^{\alpha}\frac{1}{\mathfrak{a}^{1/\alpha}(s)\pi^{1+\alpha}(s)}z^{\beta}(s)\left(-\delta_0\frac{z^{\beta/\alpha}(s)}{\pi(s)}\right)^{1-\alpha}\pi(s)
\end{aligned}
$$

$$\begin{aligned}
&\leq -\delta_0^{\alpha}\frac{1}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\,\pi^{\alpha}(\mathrm{s})}z^{\beta}(\mathrm{s})\left(\delta_0\frac{z^{\beta/\alpha}(\mathrm{s})}{\pi(\mathrm{s})}\right)^{1-\alpha}\\
&\leq \frac{-\delta_0}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\,\pi(\mathrm{s})}z^{\beta/\alpha}(\mathrm{s})\\
&\leq \frac{-\nu_1\delta_0}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\,\pi(\mathrm{s})}z(\mathrm{s})\\
&\leq \frac{-\rho_0}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\,\pi(\mathrm{s})}z(\mathrm{s}).
\end{aligned}$$

Applying the integration from s to ∞, on the previous inequality, we obtain

$$\begin{aligned}
\mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+z(\mathrm{s}) &\geq \rho_0\int_{\mathrm{s}}^{\infty}\frac{1}{\mathfrak{a}^{1/\alpha}(\varrho)}\frac{z(\varrho)}{\pi(\varrho)}\mathrm{d}\varrho\\
&\geq \rho_0\frac{z(\mathrm{s})}{\pi(\mathrm{s})}\int_{\mathrm{s}}^{\infty}\frac{1}{\mathfrak{a}^{1/\alpha}(\varrho)}\mathrm{d}\varrho\\
&\geq \rho_0 z(\mathrm{s}).
\end{aligned}$$

Thus

$$\mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+(1-\rho_0)\,z(\mathrm{s})\geq 0,$$

and hence

$$\left(\frac{z(\mathrm{s})}{\pi^{1-\rho_0}(\mathrm{s})}\right)'=\frac{\mathfrak{a}^{1/\alpha}(\mathrm{s})\pi(\mathrm{s})z'(\mathrm{s})+(1-\rho_0)z(\mathrm{s})}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\pi^{2-\rho_0}(\mathrm{s})}\geqslant 0.$$

Therefore, the proof is concluded. □

Theorem 3.1. *Suppose there exists a* $\rho_0\in(0,1)$ *such that (3.1) holds. If*

$$\rho_0>\frac{1}{2}, \tag{3.5}$$

then, Eq (1.1) is oscillatory.

Proof. Suppose u is an eventually positive solution of (1.1). According to Lemma 3.1, the functions $z(\mathrm{s})/\pi^{\rho_0}(\mathrm{s})$ and $z(\mathrm{s})/\pi^{1-\rho_0}(\mathrm{s})$ are shown to be decreasing and increasing, respectively, for $\mathrm{s}\geq\mathrm{s}_1$. Thus

$$\mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+\rho_0 z(\mathrm{s})\leq 0, \tag{3.6}$$

and

$$\mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+(1-\rho_0)\,z(\mathrm{s})\geq 0. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\begin{aligned}
0 &\leq \mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+(1-\rho_0)\,z(\mathrm{s})\\
&= \mathfrak{a}^{1/\alpha}(\mathrm{s})z'(\mathrm{s})\,\pi(\mathrm{s})+\rho_0 z(\mathrm{s})+(1-2\rho_0)\,z(\mathrm{s})\\
&\leq (1-2\rho_0)\,z(\mathrm{s}).
\end{aligned}$$

Since $z(\mathrm{s})>0$, then $1-2\rho_0\geq 0$, which means that

$$\rho_0\leq 1/2,$$

which is a contradiction. The proof is complete. □

If $\rho_0 \leq \frac{1}{2}$, we can enhance the findings stated in Lemma 3.1. Given that $\pi(\mathrm{s})$ is decreasing, there exists a constant $\lambda \geq 1$ such that

$$\frac{\pi(\mathfrak{h}(\mathrm{s}))}{\pi(\mathrm{s})} \geq \lambda. \tag{3.8}$$

We define the constant $\rho_1 > \rho_0$ by

$$\rho_1 = \rho_0 \sqrt[\alpha]{\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}}. \tag{3.9}$$

Lemma 3.2. *Assume $u(\mathrm{s})$ is a positive solution of (1.1) and (3.1) holds. If (3.8) holds, then*

$(\Upsilon_{3,1})$ $z(\mathrm{s})/\pi^{\rho_1}(\mathrm{s})$ *is decreasing;*

$(\Upsilon_{3,2})$ $\lim_{\mathrm{s}\to\infty} z(\mathrm{s})/\pi^{\rho_1}(\mathrm{s}) = 0$;

$(\Upsilon_{3,3})$ $z(\mathrm{s})/\pi^{1-\rho_1}(\mathrm{s})$ *is increasing.*

Proof. Suppose that u is an eventually positive solution of (1.1) satisfying condition (N_2) in Lemma 2.1 for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$. According to Lemma 2.4, conditions $(\Upsilon_{1,1})$–$(\Upsilon_{1,4})$ are satisfied. Moreover, Lemma 3.1 ensures that $(\Upsilon_{2,1})$–$(\Upsilon_{2,3})$ hold.

$(\Upsilon_{3,1})$ Integrating $(\Upsilon_{1,3})$ from s_1 to s, we get

$$-\mathfrak{a}(\mathrm{s})\left(z'(\mathrm{s})\right)^{\alpha} \geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \int_{\mathrm{s}_1}^{\mathrm{s}} \widetilde{\mathfrak{q}}(\varrho)\, z^{\beta}\left(\mathfrak{h}(\varrho)\right) \mathrm{d}\varrho.$$

By using the fact that $z(\mathrm{s})/\pi^{\rho_0}(\mathrm{s})$ is decreasing, we have

$$\begin{aligned}
-\mathfrak{a}(\mathrm{s})\left(z'(\mathrm{s})\right)^{\alpha} &\geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\frac{z(\varrho)}{\pi^{\rho_0}(\varrho)}\right)^{\beta} \pi^{\beta\rho_0}(\mathfrak{h}(\varrho))\widetilde{\mathfrak{q}}(\varrho)\mathrm{d}\varrho \\
&\geq \geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \pi^{\beta\rho_0}(\mathfrak{h}(\varrho))\widetilde{\mathfrak{q}}(\varrho)\mathrm{d}\varrho.
\end{aligned}$$

By using (3.1) and (3.8), we get

$$\begin{aligned}
-\mathfrak{a}(\mathrm{s})\left(z'(\mathrm{s})\right)^{\alpha} &\geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}}{\mathfrak{a}^{1/\alpha}(\varrho)\pi^{\alpha+1}(\varrho)}\pi^{\beta\rho_0}(\varrho)\mathrm{d}\varrho \\
&\geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}\left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta} \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\pi^{-1-\alpha+\beta\rho_0}(\varrho)}{\mathfrak{a}^{1/\alpha}(\varrho)}\mathrm{d}\varrho \\
&\geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta}\left[\pi^{\beta\rho_0-\alpha}(\mathrm{s}) - \pi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\right] \\
&\geqslant -\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\pi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta} \\
&\quad + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{z^{\beta}(\mathrm{s})}{\pi^{\alpha}(\mathrm{s})}.
\end{aligned}$$

Using $(\Upsilon_{2,2})$, there is $\mathrm{s}_2 \in [\mathrm{s}_1, \infty)$, such that

$$-\mathfrak{a}(\mathrm{s}_1)\left(z'(\mathrm{s}_1)\right)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\pi^{\beta\rho_0-\alpha}(\mathrm{s}_1)\left(\frac{z(\mathrm{s})}{\pi^{\rho_0}(\mathrm{s})}\right)^{\beta} \geq 0,$$

for $\mathrm{s} \geq \mathrm{s}_2$, and so

$$-\mathfrak{a}(\mathrm{s})\left(z'(\mathrm{s})\right)^{\alpha} \geqslant \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{z^{\beta}(\mathrm{s})}{\pi^{\alpha}(\mathrm{s})},$$

and so

$$\begin{aligned} z'(\mathrm{s}) &\geq \delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\pi(\mathrm{s})\,\mathfrak{a}^{1/\alpha}(\mathrm{s})}z^{\beta/\alpha}(\mathrm{s}) \\ &\geq v(\mathrm{s})\,\delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\pi(\mathrm{s})\,\mathfrak{a}^{1/\alpha}(\mathrm{s})}z(\mathrm{s}) \\ &= \rho_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\pi(\mathrm{s})\,\mathfrak{a}^{1/\alpha}(\mathrm{s})}z(\mathrm{s}) \\ &= \rho_1\frac{1}{\pi(\mathrm{s})\,\mathfrak{a}^{1/\alpha}(\mathrm{s})}z(\mathrm{s}), \end{aligned}$$

or equivalently,

$$\mathfrak{a}^{1/\alpha}(\mathrm{s})\pi(\mathrm{s})z'(\mathrm{s}) + \rho_1 z(\mathrm{s}) \leq 0. \tag{3.10}$$

Consequently,

$$\left(\frac{z(\mathrm{s})}{\pi^{\rho_1}(\mathrm{s})}\right)' = \frac{\mathfrak{a}^{1/\alpha}(\mathrm{s})\pi(\mathrm{s})z'(\mathrm{s}) + \rho_1 z(\mathrm{s})}{\mathfrak{a}^{1/\alpha}(\mathrm{s})\pi^{1+\rho_1}(\mathrm{s})} \leqslant 0,$$

so $z(\mathrm{s})/\pi^{\rho_1}(\mathrm{s})$ is decreasing.

Following the same steps outlined in the proof of Lemma 3.1, we confirm the validity of $(\Upsilon_{3,2})$ and $(\Upsilon_{3,3})$. □

Let

$$\rho_n = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_{n-1}}}{1-\frac{\beta}{\alpha}\rho_{n-1}}}. \tag{3.11}$$

Furthermore, following the proof structure of Lemma 3.2, we confirm the following:
$(\Upsilon_{n,1})$ $z(\mathrm{s})/\pi^{\rho_n}(\mathrm{s})$ is decreasing;
$(\Upsilon_{n,2})$ $\lim_{\mathrm{s}\to\infty} z(\mathrm{s})/\pi^{\rho_n}(\mathrm{s}) = 0$;
If $\rho_1 < 1/2$, iterating the above process yields $\delta_2 > \delta_1$,

$$\rho_2 = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_1}}{1-\frac{\beta}{\alpha}\rho_1}}.$$

Generally, if $\rho_i < 1/2$ for $i = 1, 2, ..., n-1$, we define
$(\Upsilon_{n,3})$ $z(\mathrm{s})/\pi^{1-\rho_n}(\mathrm{s})$ as increasing.

Theorem 3.2. *Assume that there exists a* $\rho_0 \in (0,1)$ *for which (3.1) is satisfied. If there also exists an* $n \in \mathbb{N}$ *such that*

$$\rho_n > \frac{1}{2}, \tag{3.12}$$

then Eq (1.1) is oscillatory.

Theorem 3.3. *Assume that (2.2), (3.1), and (3.8) hold. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{s\to\infty} \int_{\mathfrak{h}(s)}^{s} \frac{\pi(\varrho)\widetilde{\mathfrak{q}}(\varrho)}{\pi^{1-\alpha}(\mathfrak{h}(\varrho))} d\varrho > \frac{\alpha v_1^{-\alpha}\rho_n^{\alpha-1}(1-\rho_n)}{e}, \tag{3.13}$$

then, Eq (1.1) is oscillatory.

Proof. Assume on the contrary that (1.1) possesses an eventually positive solution $u(s)$. Condition (2.2) guarantees that $u(s)$ satisfies (N_2). From Lemma 2.4, we have that $(\Upsilon_{1,1})$–$(\Upsilon_{1,4})$ hold. We construct sequence $\{\rho_n\}$ by (3.11).

Now, we define the function

$$w(s) = \mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s) + z(s).$$

In view of $(\Upsilon_{1,1})$ in Lemma 2.4, we note that $w(s) > 0$ and from $(\Upsilon_{n,1})$ we can obtain

$$\mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s) + \rho_n z(s) \leq 0.$$

Then, from the definition of $w(s)$, we have

$$\begin{aligned} w(s) &= \mathfrak{a}^{1/\alpha}(s)z'(s)\pi(s) + \rho_n z(s) - \rho_n z(s) + z(s) \\ &\leq (1-\rho_n)z(s). \end{aligned} \tag{3.14}$$

From $(\Upsilon_{1,3})$ and (3.4), we obtain

$$\begin{aligned} w'(s) &= \left(\mathfrak{a}^{1/\alpha}(s)z(s)\right)'\pi(s) \\ &\leq \frac{1}{\alpha}\left(\mathfrak{a}(s)(z'(s))^{\alpha}\right)'\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi(s) \\ &\leq -\frac{1}{\alpha}\widetilde{\mathfrak{q}}(s)z^{\beta}(\mathfrak{h}(s))\left(\mathfrak{a}^{1/\alpha}(s)z'(s)\right)^{1-\alpha}\pi(s) \\ &\leq -\frac{1}{\alpha}\widetilde{\mathfrak{q}}(s)z^{\beta}(\mathfrak{h}(s))\left(\rho_n\frac{z(s)}{\pi(s)}\right)^{1-\alpha}\pi(s) \\ &\leq -\frac{1}{\alpha}\rho_n^{1-\alpha}\widetilde{\mathfrak{q}}(s)\pi(s)z^{\beta}(\mathfrak{h}(s))\left(\frac{z(s)}{\pi(s)}\right)^{1-\alpha}. \end{aligned} \tag{3.15}$$

From $(\Upsilon_{1,2})$ in Lemma 2.4, we note that $z(s)/\pi(s)$ is increasing, and then

$$\frac{z(\mathfrak{h}(s))}{\pi(\mathfrak{h}(s))} \leq \frac{z(s)}{\pi(s)},$$

and

$$\left(\frac{z(\mathfrak{h}(s))}{\pi(\mathfrak{h}(s))}\right)^{1-\alpha} \leq \left(\frac{z(s)}{\pi(s)}\right)^{1-\alpha}.$$

From this (3.15) becomes

$$w'(s) \leq -\frac{1}{\alpha}\rho_n^{1-\alpha}\widetilde{\mathfrak{q}}(s)\pi(s)z^{\beta}(\mathfrak{h}(s))\left(\frac{z(\mathfrak{h}(s))}{\pi(\mathfrak{h}(s))}\right)^{1-\alpha}$$

$$\leq \quad -\frac{1}{\alpha}\rho_n^{1-\alpha}\widetilde{\mathfrak{q}}(\mathrm{s})\frac{\pi(\mathrm{s})}{\pi^{1-\alpha}(\mathfrak{h}(\mathrm{s}))}z^{\beta-\alpha}(\mathfrak{h}(\mathrm{s}))\,z(\mathfrak{h}(\mathrm{s}))\,.$$

From Lemma 2.3 we know that $z^{\beta-\alpha}(\mathfrak{h}(\mathrm{s})) \geq v_1^{\alpha}$. Therefore the above inequality leads to

$$w'(\mathrm{s}) \leq -\frac{v_1^{\alpha}}{\alpha}\rho_n^{1-\alpha}\widetilde{\mathfrak{q}}(\mathrm{s})\frac{\pi(\mathrm{s})}{\pi^{1-\alpha}(\mathfrak{h}(\mathrm{s}))}z(\mathfrak{h}(\mathrm{s}))\,.$$

By using (3.14) we see that $w(\mathrm{s})$ is a positive solution of

$$w'(\mathrm{s}) + \frac{v_1^{\alpha}}{\alpha}\frac{\rho_n^{1-\alpha}}{(1-\rho_n)}\frac{\pi(\mathrm{s})\,\widetilde{\mathfrak{q}}(\mathrm{s})}{\pi^{1-\alpha}(\mathfrak{h}(\mathrm{s}))}w(\mathfrak{h}(\mathrm{s})) \leq 0. \tag{3.16}$$

This is a contradiction since by Theorem 2.1.1 in [8] condition (3.13) guarantees that (3.16) has no positive solution. This contradiction completes the proof of the theorem. □

We use illustrative examples to show the importance of the obtained results.

Example 3.1. *Consider the NDE*

$$\left(\mathrm{s}^{2\alpha}\left((u(\mathrm{s}) + \mathrm{h}_0 u(\mathfrak{z}_0\mathrm{s}))'\right)^{\alpha}\right)' + \mathfrak{q}_0\mathrm{s}^{\alpha-1}u^{\beta}(\mathfrak{h}_0\mathrm{s}) = 0,\ \mathrm{s} \geq 1, \tag{3.17}$$

where $\alpha > \beta$, $0 \leq \mathrm{h}_0 < 1$, $\mathfrak{z}_0, \mathfrak{h}_0 \in (0,1)$, *and* $\mathfrak{q}_0 > 0$. *By comparing (1.1) and (3.17) we note that* $\mathfrak{a}(\mathrm{s}) = \mathrm{s}^{2\alpha}$, $\mathfrak{q}(\mathrm{s}) = \mathfrak{q}_0\mathrm{s}^{\alpha-1}$, $\mathrm{h}(\mathrm{s}) = \mathrm{h}_0$, $\mathfrak{h}(\mathrm{s}) = \mathfrak{h}_0\mathrm{s}$, *and* $\mathfrak{z}(\mathrm{s}) = \mathfrak{z}_0\mathrm{s}$. *It is easy to find that*

$$\pi(\mathrm{s}) = \frac{1}{\mathrm{s}},\ \frac{\pi(\mathfrak{z}(\mathfrak{h}(\mathrm{s})))}{\pi(\mathfrak{h}(\mathrm{s}))} = \frac{1}{\mathfrak{z}_0},$$

$$\mathrm{H}_m(\mathrm{s}) = \mathrm{H}_m = \left(\frac{1}{\mathrm{h}_0} - \frac{1}{\mathfrak{z}_0}\right)\sum_{i=0}^{m}\mathrm{h}_0^{2i+1},$$

and

$$\widetilde{\mathfrak{q}}(\mathrm{s}) = \mathfrak{q}_0\mathrm{s}^{\alpha-1}\mathrm{H}_m^{\beta}.$$

For (3.1), we set

$$\delta_0 = \frac{1}{\alpha^{1/\alpha}}\mathfrak{q}_0^{1/\alpha}\mathrm{H}_m^{\beta/\alpha}.$$

From (3.8), we have $\lambda = \frac{1}{\mathfrak{h}_0}$. *Now, we define the sequence* $\{\beta_{\mathrm{i}}\}_{\mathrm{i}=1}^{m}$ *as*

$$\rho_n = \rho_0\sqrt[\alpha]{\frac{1}{1-\frac{\beta}{\alpha}\rho_{n-1}}\left(\frac{1}{\mathfrak{h}_0}\right)^{\beta\rho_{n-1}}},$$

with

$$\rho_0 = v_1\frac{1}{\alpha^{1/\alpha}}\mathfrak{q}_0^{1/\alpha}\mathrm{H}_m^{\beta/\alpha}.$$

Then, condition (3.5) reduces to

$$\mathfrak{q}_0 > \frac{\alpha}{(2v_1)^{\alpha}\,\mathrm{H}_m^{\beta}}, \tag{3.18}$$

and condition (3.13) becomes

$$\begin{aligned}\liminf_{\mathrm{s}\to\infty}\int_{\mathfrak{h}(\mathrm{s})}^{\mathrm{s}}\frac{\pi(\varrho)\widetilde{\mathfrak{q}}(\varrho)}{\pi^{1-\alpha}(\mathfrak{h}(\varrho))}\mathrm{d}\varrho &= \liminf_{\mathrm{s}\to\infty}\int_{\mathfrak{h}_0\mathrm{s}}^{\mathrm{s}}\frac{1}{\varrho}\mathfrak{h}_0^{1-\alpha}\varrho^{1-\alpha}\mathfrak{q}_0\varrho^{\alpha-1}\mathrm{H}_m^{\beta}\mathrm{d}\varrho\\ &= \liminf_{\mathrm{s}\to\infty}\int_{\mathfrak{h}_0\mathrm{s}}^{\mathrm{s}}\frac{1}{\varrho}\mathfrak{h}_0^{1-\alpha}\mathfrak{q}_0\mathrm{H}_m^{\beta}\mathrm{d}\varrho\\ &= \mathfrak{h}_0^{1-\alpha}\mathfrak{q}_0\mathrm{H}_m^{\beta}\liminf_{\mathrm{s}\to\infty}\int_{\mathfrak{h}_0\mathrm{s}}^{\mathrm{s}}\frac{1}{\varrho}\mathrm{d}\varrho\\ &= \mathfrak{h}_0^{1-\alpha}\mathfrak{q}_0\mathrm{H}_m^{\beta}\ln\frac{1}{\mathfrak{h}_0},\end{aligned}$$

which leads to

$$\mathfrak{q}_0 > \frac{\alpha v_1^{-\alpha}\rho_n^{\alpha-1}(1-\rho_n)}{\mathfrak{h}_0^{1-\alpha}\mathrm{H}_m^{\beta}\ln\frac{1}{\mathfrak{h}_0}}\frac{1}{\mathrm{e}}. \tag{3.19}$$

Using Theorem 3.1 and Theorem (3.3), we note that Eq (3.17) is oscillatory if either (3.18) or (3.19) holds, respectively.

Example 3.2. *Consider the NDE*

$$\left(\mathrm{s}^{2/3}\left((u(\mathrm{s})+0.25u(0.75\mathrm{s}))'\right)^{1/3}\right)' + \frac{\mathfrak{q}_0}{\mathrm{s}^{2/3}}u^{1/5}(0.3\mathrm{s}) = 0,\ \mathrm{s}\geq 1. \tag{3.20}$$

Clearly:

$\alpha = 1/3, \beta = 1/5$, $\mathfrak{a}(\mathrm{s}) = \mathrm{s}^{2/3}$, $\mathfrak{q}(\mathrm{s}) = \frac{\mathfrak{q}_0}{\mathrm{s}^{2/3}}$, $\mathrm{h}(\mathrm{s}) = 0.25$, $\mathfrak{h}(\mathrm{s}) = 0.3\mathrm{s}$, *and* $\mathfrak{z}(\mathrm{s}) = 0.75\mathrm{s}$. *It is easy to find that*

$$\pi(\mathrm{s}) = \frac{1}{\mathrm{s}},\ \frac{\pi(\mathfrak{z}(\mathfrak{h}(\mathrm{s})))}{\pi(\mathfrak{h}(\mathrm{s}))} = 1.3,$$

$$\mathrm{H}_{10} = \left(\frac{1}{0.25}-\frac{1}{0.75}\right)\sum_{i=0}^{10}(0.25)^{2i+1} \approx 0.711,$$

and

$$\widetilde{\mathfrak{q}}(\mathrm{s}) = 0.934\,06\mathfrak{q}_0\mathrm{s}^{-2/3}.$$

For (3.1), we set

$$\delta_0 = 22.003\mathfrak{q}_0^3.$$

From (3.8), we have $\lambda = 3.3$. *Now, we define the sequence* $\{\beta_{\mathfrak{a}}\}_{\mathfrak{a}=1}^{m}$ *as*

$$\rho_n = \rho_0\frac{1}{\left(1-\frac{3}{5}\rho_{n-1}\right)^3}(3.3)^{\frac{3\rho_{n-1}}{5}},$$

with

$$\rho_0 = 22.003\mathfrak{q}_0^3 v_1,\ v_1 > 0.$$

Then, condition (3.18) reduces to

$$\mathfrak{q}_0 > \frac{0.283\,25}{\sqrt[3]{v_1}}, \tag{3.21}$$

and condition (3.19) leads to

$$_0 > 0.13283v_1^{-1/3}\rho_n^{-2/3}(1-\rho_n)\frac{1}{e}, \ \ v_1 > 0. \tag{3.22}$$

Using Theorem 3.1 and Theorem (3.3), we note that Eq (3.20) is oscillatory if either (3.21) or (3.22) holds, respectively.

Notation 3.1. *From the previous example we get*

$$\rho_0 = 22.003\mathfrak{q}_0^3 v_1, \ v_1 > 0.$$

Then condition (3.5) in Theorem 3.1 is satisfied if

$$22.003\mathfrak{q}_0^3 v_1 > \frac{1}{2},$$

or

$$\mathfrak{q}_0^3 v_1 > 0.02724.$$

If we choose $v_1 = 1.5$, *then*

$$\mathfrak{q}_0 > 0.26285.$$

Example 3.3. *Consider the NDE*

$$\left(s^2\left(u(s) + \frac{1}{4}u\left(\frac{1}{3}s\right)\right)'\right)' + \mathfrak{q}_0 u\left(\frac{1}{2}s\right) = 0, \ s \geq 1, \tag{3.23}$$

with $\mathfrak{q}_0 > 0$. *Clearly:*

$$\alpha = \beta = 1, \ v_1 = 1, \ h(s) = \frac{1}{4}, \ \pi(s) = \frac{1}{s}, \ \frac{\pi(\mathfrak{z}(\mathfrak{h}(s)))}{\pi(\mathfrak{h}(s))} = 3, \ \lambda = 4,$$

$$H_{10} = (4-3)\sum_{i=0}^{10}\left(\frac{1}{4}\right)^{2i+1} \approx 0.266\,67, \ \widetilde{\mathfrak{q}}(s) = 0.26667\mathfrak{q}_0,$$

and

$$\rho_0 = \delta_0 = 0.26667\mathfrak{q}_0.$$

By setting $\mathfrak{q}_0 = 1.9$, *we obtain*

$$\rho_0 = 0.50667,$$

and it is verified that condition (3.5) holds for $\mathfrak{q}_0 \geq 1.9$*, which ensures that Eq (3.23) oscillates.*

For $\mathfrak{q}_0 = 1.3$, *we find the following values for* ρ_n :

n	0	1
ρ_n	0.346 67	0.85802

Condition (3.12) holds for $n = 1$, *ensuring the oscillatory behavior of Eq (3.23).*

Next, for $\mathfrak{q}_0 = 0.7$, *we get the following values for* ρ_n :

n	0	1	2	3
ρ_n	0.18667	0.29730	0.401 14	0.543 58

Condition (3.12) is satisfied for $n = 3$, *again guaranteeing the oscillation of Eq (3.23).*
Finally, for $\mathfrak{q}_0 = 0.6$, *we find the following values for* ρ_n :

n	0	1	2	3	4	5	6	7
ρ_n	0.16	0.23778	0.29188	0.33864	0.543 58	0.386 87	0.44616	0.53623

Condition (3.12) holds for $n = 7$, *once again ensuring the oscillation of Eq (3.23).*

4. Conclusions

In this paper, we have addressed the issue of finding sufficient conditions to ensure oscillatory behavior for all solutions of a class of nonlinear NDEs. Our investigation particularly focused on the non-canonical case, where we explored new monotonic properties of positive solutions and derived novel oscillation criteria. Building upon the seminal work of Baculíková [33] on second-order DDEs, our research extends the analysis to second-order NDEs. By adopting and further developing their techniques, we have contributed to a broader understanding of oscillatory phenomena in differential equations.

Moving forward, an exciting direction for future research would be to apply similar methodologies to study even-order nonlinear NDEs of the form

$$\left(\mathfrak{a}(s)\left(z^{(n-1)}(s)\right)^{\alpha}\right)' + \mathfrak{q}(s)u^{\beta}(\mathfrak{h}(s)) = 0,\ n \geq 4.$$

This approach promises to deepen our insights into the intricate dynamics of these systems and potentially uncover new oscillation patterns.

Author contributions

Hail S. Alrashdi: Writing-original draft, Methodology, Investigation; Fahd Masood: Writing – review & editing, Writing-original draft, Methodology, Investigation; Ahmad M. Alshamrani: Supervision, Methodology, Investigation; Sameh S. Askar: Supervision, Methodology, Investigation; Monica Botros: Writing – review & editing, Writing-original draft, Methodology, Investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no competing interests.

References

1. J. K. Hale, *Theory of Functional Differential Equations*, 2004, New York: Springer, 1977. https://doi.org/10.1007/978-1-4612-9892-2
2. M. Braun, *Qualitative Theory of Differential Equations: Differential Equations and Their Applications*, New York: Springer, 1993.
3. A. Zafer, *Oscillatory and nonoscillatory properties of solutions of functional differential equations and difference equations*, Iowa State University, PhD Thesis, 1992. https://doi.org/10.31274/rtd-180813-12264
4. Y. S. Yılmaz, A. Zafer, Bounded oscillation of nonlinear neutral differential equations of arbitrary order, *Czechoslovak Math. J.*, **51** (2001), 185–195. https://doi.org/10.1023/A:1013763409361
5. J. R. Graef, M. K. Grammatikopoulos, P.W. Spikes, On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equation, *J. Math. Anal. Appl.*, **156** (1991), 23–39. https://doi.org/10.1016/0022-247x(91)90379-e
6. B. Qaraad, O. Bazighifan, O. T. A. Nofal, A. H. Ali, Neutral differential equations with distribution deviating arguments: Oscillation conditions, *J. Ocean Eng. Sci.,* 2022, In press. https://doi.org/10.1016/j.joes.2022.06.032
7. F. Masood, O. Moaaz, S.S. Santra, U. Fernandez-Gamiz, H.A. El-Metwally, Oscillation theorems for fourth-order quasi-linear delay differential equations, *AIMS Mathematics,* **8** (2023), 16291–16307. https://doi.org/10.3934/math.2023834
8. G.S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, New York: Marcel Dekker, 1987.
9. I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford: Clarendon Press, 1991. https://doi.org/10.1093/oso/9780198535829.001.0001
10. R. P. Agarwal, M. Bohner, W.-T. Li, *Nonoscillation and oscillation: Theory for functional differential equations*, Boca Raton: CRC Press, 2004. https://doi.org/10.1201/9780203025741
11. S. Saker, *Oscillation Theory of Delay Differential and Difference Equations: Second and Third Orders*, Saarbrücken: Lambert Academic Publishing, 2010.
12. P. Drábek, A. Kufner, K. Kuliev, Oscillation and nonoscillation results for solutions of half-linear equations with deviated argument, *J. Math. Anal. Appl.*, **447** (2016), 371–382. https://doi.org/10.1016/j.jmaa.2016.10.019
13. J. Džurina, I. Jadlovská, I. P. Stavroulakis, Oscillatory results for second-order noncanonical delay differential equations, *Opusc. Math.*, **39** (2019), 483–495. https://doi.org/10.7494/opmath.2019.39.4.483
14. M. Alatwi, O. Moaaz, W. Albalawi, F. Masood, H. El-Metwally, Asymptotic and Oscillatory Analysis of Fourth-Order Nonlinear Differential Equations with p-Laplacian-like Operators and Neutral Delay Arguments, *Mathematics*, **12** (2024), 470. https://doi.org/10.3390/math12030470

15. G. E. Chatzarakis, S. R. Grace, I. Jadlovská, On the sharp oscillation criteria for half-linear second-order differential equations with several delay arguments, *Appl. Math. Comput.*, **397** (2021), 125915. https://doi.org/10.1016/j.amc.2020.125915

16. B. Almarri, F. Masood, A. Muhib, O. Moaaz, New Comparison Results for Oscillation of Even-Order Delay Differential Equations, *Symmetry*, **14** (2022), 946. https://doi.org/10.3390/sym14050946

17. I. Jadlovská, New Criteria for Sharp Oscillation of Second-Order Neutral Delay Differential Equations, *Mathematics*, **9** (2021), 2089. https://doi.org/10.3390/math9172089

18. F. Masood, C. Cesarano, O. Moaaz, S. S. Askar, A. M. Alshamrani, H. El-Metwally, Kneser-Type Oscillation Criteria for Half-Linear Delay Differential Equations of Third Order, *Symmetry*, **15** (2023), 1994. https://doi.org/10.3390/sym15111994

19. B. Qaraad B, O. Bazighifan, A. H. Ali, A. A. Al-Moneef, A. J. Alqarni, K. Nonlaopon, Oscillation Results of Third-Order Differential Equations with Symmetrical Distributed Arguments, *Symmetry,* **14** (2022), 2038. https://doi.org/10.3390/sym14102038

20. O. Moaaz, F. Masood, C. Cesarano, S. A. M. Alsallami, E. M. Khalil, M. L. Bouazizi, Neutral Differential Equations of Second-Order: Iterative Monotonic Properties, *Mathematics*, **10** (2022), 1356. https://doi.org/10.3390/math10091356

21. M. K. Grammatikopoulos, G. Ladas, A. Meimaridou, Oscillation of second order neutral delay differential equation, *Rad. Mat.*, **1** (1985), 267–274.

22. R. Xu, Y. Xia, A Note on the Oscillation of Second-Order Nonlinear Neutral Functional Differential Equations 1, *Int. J. Contemp. Math. Sci.*, **3** (2008), 1441–1450.

23. S. R. Grace, B.S. Lalli, Oscillation of nonlinear second order neutral delay differential equations, *Rad. Math.*, **3** (1987), 77–84.

24. S. R. Grace, J. R. Graef, T. Li, E. Tunç, Oscillatory behavior of second-order nonlinear noncanonical neutral differential equations, *Acta Univ. Sapientiae Math.*, **15** (2023), 259–271. https://doi.org/10.2478/ausm-2023-0014

25. M. Bohner, S. Grace, I. Jadlovská, Oscillation criteria for second-order neutral delay differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2017** (2017), 1–12. https://doi.org/10.14232/ejqtde.2017.1.60

26. I. Jadlovska, G. E. Chatzarakis, E. Tunç, Kneser-type oscillation theorems for second-order functional differential equations with unbounded neutral coefficients, *Math. Slovaca*, **74** (2024), 637–664. https://doi.org/10.1515/ms-2024-0049

27. A. Nabih, A. Al-Jaser, O. Moaaz, Neutral Emden-Fowler Differential equation of second order: Oscillation Criteria of Coles type, *Symmetry*, **16** (2024), 931. https://doi.org/10.3390/sym16070931

28. A. Al-Jaser, O. Moaaz, Second-order general Emden-Fowler differential equations of neutral type: Improved Kamenev-type oscillation criteria, *Electron. Res. Arch.*, **32** (2024), 5231–5248. https://doi.org/10.3934/era.2024241

29. O. Moaaz, C. Park, E.M. Elabbasy, W. Muhsin, New oscillation criteria for second-order neutral differential equations with distributed deviating arguments, *Bound. Value Probl.*, **2021** (2021), 35. https://doi.org/10.1186/s13661-021-01512-x

30. A. Al Themairi, B. Qaraad, O. Bazighifan, K. Nonlaopon, New conditions for testing the oscillation of third-order differential equations with distributed arguments, *Symmetry*, **14** (2022), 2416. https://doi.org/10.3390/sym14112416

31. S. Althobati, J. Alzabut, O. Bazighifan, Non-Linear Neutral Differential Equations with Damping: Oscillation of Solutions, *Symmetry*, **13** (2021), 285. https://doi.org/10.3390/sym13020285

32. O. Moaaz, G.E. Chatzarakis, T. Abdeljawad, C. Cesarano, A. Nabih, Amended oscillation criteria for second-order neutral differential equations with damping term, *Adv. Differ. Equ.*, **2020** (2020), 553. https://doi.org/10.1186/s13662-020-03013-0

33. B. Baculíková, Oscillatory behavior of the second order noncanonical differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **2019** (2019), 1–11. https://doi.org/10.14232/ejqtde.2019.1.89

34. I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Dordrecht: Kluwer Academic Publishers Group, 1993. https://doi.org/10.1007/978-94-011-1808-8

35. O. Moaaz, C. Cesarano, B. Almarri, An Improved Relationship between the Solution and Its Corresponding Function in Fourth-Order Neutral Differential Equations and Its Applications, *Mathematics*, **11** (2023), 1708. https://doi.org/10.3390/math11071708

36. B. Batiha, N. Alshammari, F. Aldosari, F. Masood, O. Bazighifan, Nonlinear neutral delay differential equations: Novel criteria for oscillation and asymptotic behavior, *Mathematics*, **13** (2025), 147. https://doi.org/10.3390/math13010147

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