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Research article

Close-to-convexity and partial sums for normalized Le Roy-type q-Mittag-Leffler functions

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Abstract: In recent years, researchers have explored the properties of close-to-convexity and partial sums for various Mittag-Leffler functions, including q-Mittag-Leffler, Bernas Mittag-Leffler, and Le Roy-type Mittag-Leffler functions. Building on previous research, this paper explores the Le Roy-type q-Mittag-Leffler function in the open unit disk, with a specific focus on its normalization. We also use the concept of q-close-to-convex functions, and examine whether the Le Roy-type q-Mittag-Leffler function possesses close-to-convexity. Moreover, we determine lower bounds for the normalized Le Roy-type q-Mittag-Leffler and its sequence of partial sums. We also present some lemmas, propositions, examples, and meaningful corollaries that highlight the importance of our findings. The results presented in this paper are new and demonstrate improvements over some existing findings in the literature.

Keywords: analytic functions; Mittag-Leffler functions; q-Mittag-Leffler function; q-difference operator; close-to-convexity; partial sums; Le Roy-type q-Mittag-Leffler function
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1. Introduction

Let $\mathcal{A}$ be the set of all analytic functions $h(\tau)$, in $\triangle = \{\tau : \tau \in \mathbb{C} \text{ and } |\tau| < 1\}$, which satisfy the normalized conditions as $h(0) = 0$ and $h^{'}(0) = 1$, and every $h \in \mathcal{A}$ has the Taylor-Maclaurin series representation

$$h(\tau) = \tau + \sum_{m=2}^{\infty} a_m \tau^m. \tag{1.1}$$

Also, let $\mathcal{S}$ denote the set of all univalent functions in $\triangle$. A function $h \in \mathcal{A}$ is said to be starlike in $\triangle$ with respect to 0 if its image, $h(\triangle)$, is a star-shaped domain. The set of such functions is denoted by $\mathcal{S}^*$ (see [1]), and these functions satisfy

$$\operatorname{Re}\left(\frac{\tau h'(\tau)}{h(\tau)}\right) > 0,\ \tau \in \triangle.$$

More generally, for $0 \leq \lambda < 1$, the starlike functions of order λ in $\triangle$ are denoted by $h \in \mathcal{S}^*(\lambda)$ and defined as:

$$\mathcal{S}^*(\lambda) = \left\{h \in \mathcal{A} : \operatorname{Re}\left(\frac{\tau h'(\tau)}{h(\tau)}\right) > \lambda,\ \tau \in \triangle\right\}.$$

It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$.

A function $h \in \mathcal{A}$ is convex if its image $h(\triangle)$, is a convex (see [1]). These functions denoted by C and can be defined as:

$$C = \left\{h \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{\tau h''(\tau)}{h'(\tau)}\right) > 0,\ \tau \in \triangle\right\}.$$

More generally, for $0 \leq \lambda < 1$, the convex functions of order λ in $\triangle$ are denoted by $h \in C(\lambda)$ and defined as:

$$C(\lambda) = \left\{h \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{\tau h''(\tau)}{h'(\tau)}\right) > \lambda,\ \tau \in \triangle\right\}.$$

It is well known that $C(0) = C$. The Alexander relations in [2] show that

$$\tau h' \in \mathcal{S}^* \Leftrightarrow h \in C.$$

A function h in $\mathcal{A}$ is considered close-to-convex in $\triangle$ if the image $h(\triangle)$ is close-to-convex, meaning its complement consists of non-overlapping half-lines. Alternatively, h in $\mathcal{A}$ is close-to-convex if there exists a starlike function g in $\triangle$ such that

$$\operatorname{Re}\left(\frac{\tau h'(\tau)}{g(\tau)}\right) > 0$$

for all τ in $\triangle$, denoted by $\mathcal{K}$.

Or, in another way, there exists a convex function g in $\triangle$ such that

$$\operatorname{Re}\left(\frac{h'(\tau)}{g'(\tau)}\right) > 0,$$

for all τ in $\triangle$. An important property of close-to-convex functions in $\triangle$ is that they are univalent, meaning they do not take on the same value twice. This implies that the class of close-to-convex functions in $\triangle$ is a subset of the larger class $\mathcal{S}$ of normalized univalent functions.

Geometric function theory (GFT) has extensive research on various subclasses of the normalized analytic function class $\mathcal{A}$, with diverse approaches and perspectives. The q-calculus and fractional q-calculus have emerged as powerful tools in this pursuit. A significant milestone was achieved by Ismail et al. [3], who introduced a q-analogue of the starlike function class $\mathcal{S}$ in $\triangle$ using the q-difference operator ($\mathcal{D}_q$). Building upon this foundation, Srivastava's work (see [4], pp. 347 et seq.) pioneered the application of q-calculus in GFT and explored the role of basic (or q-) hypergeometric functions.

To fully understand the concepts presented in this article, we will first outline the necessary notations and definitions.

First of all, for $q \in (0,1)$, the q-factorial $[\eta]_q!$ is given as

$$[\eta]_q! = 1, \text{ for } \eta = 0 \text{ and } [\eta]_q! = \prod_{k=1}^{n}[k]_q, \text{ for } n = \eta \in \mathbb{N}.$$

The q-generalized pochhammer symbol $[\eta]_{n,q}$, $\eta \in \mathbb{C}$, is defined as

$$[\eta]_{n,q} = 1, \text{ for } n = 0 \text{ and } \left([\eta]_q\right)_n = [\eta]_q[\eta+1]_q[\eta+2]_q...[\eta+n-1]_q, \text{ for } n \in \mathbb{N},$$

and

$$\left([\eta]_q\right)_n \leq \left([\eta]_q\right)^n.$$

For $q \in (0,1)$, $\alpha, \mu \in \mathbb{C}$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the q-shifted factorial $(\alpha; q)_\mu$ is defined by

$$(\alpha; q)_\mu = \prod_{k=0}^{\infty} \frac{1-\alpha q^k}{1-\alpha q^{\mu+k}},$$

so that

$$(\alpha; q)_n = \begin{cases} 1, & n = 0, \\ \prod\limits_{k=0}^{m-1}\left(1-\alpha q^k\right) & m \in \mathbb{N}, \end{cases}$$

and

$$(\alpha; q)_\infty = \prod_{k=0}^{\infty}\left(1-\alpha q^k\right), \ \alpha \in \mathbb{C}.$$

The definition of the q-Gamma function $\Gamma_q(\tau)$ is

$$\Gamma_q(\tau) = \frac{(q;q)_\infty}{(q^\tau;q)_\infty}(1-q)^{1-\tau}, \quad (0 < q < 1), \ \tau \in \mathbb{C},$$

and

$$\lim_{q \to 1^-} \Gamma_q(\tau) = \Gamma(\tau),$$

where Γ Gamma function. One of the defining features of the q-Gamma function $\Gamma_q(\tau)$ is that the following equation is satisfied

$$\Gamma_q(1+\tau) = [\tau]_q \Gamma_q(\tau), \quad (0 < q < 1), \ \tau \in \mathbb{C}, \tag{1.2}$$

where

$$[\tau]_q = \frac{1-q^\tau}{1-q}. \tag{1.3}$$

For a more comprehensive understanding of q-calculus and the aforementioned notations, readers are referred to the works of Gasper and Rahman [5] and Srivastava and Karlsson ([6], pp. 346–351).

As introduced by Jackson [7], the q-difference operator $\mathcal{D}_q$ applied to a function $h \in \mathcal{A}$ is defined as follows:

$$(\mathcal{D}_q h)(\tau) = \begin{cases} \frac{h(\tau)-h(q\tau)}{\tau(1-q)}, & \tau \in \triangle\backslash\{0\}; \ 0 < q < 1, \\ h'(0), & \tau = 0; \ 0 < q < 1. \end{cases} \tag{1.4}$$

One can clearly see from the definition (1.4) that

$$(\mathcal{D}_q h)(\tau) = 1 + \sum_{m=2}^{\infty} \frac{1-q^m}{1-q} a_m \tau^m = 1 + \sum_{m=2}^{\infty} [m]_q \, a_m \tau^m,$$

and

$$\lim_{q \to 1^-} (\mathcal{D}_q h)(\tau) = h'(\tau), \; \tau \in \triangle.$$

The following rules holds, as stated by [8]:

$$\begin{aligned} \mathcal{D}_q (h(\tau) \pm g(\tau)) &= \mathcal{D}_q h(\tau) \pm \mathcal{D}_q g(\tau), \\ \mathcal{D}_q (h(\tau) . g(\tau)) &= h(q\tau) \mathcal{D}_q g(\tau) + g(z) \mathcal{D}_q h(\tau), \\ \mathcal{D}_q \left(\frac{h(\tau)}{g(\tau)} \right) &= \frac{g(q\tau) \mathcal{D}_q h(\tau) - h(qz) \mathcal{D}_q g(\tau)}{g(q\tau) g(\tau)}. \end{aligned} \tag{1.5}$$

Definition 1.1. [9] The definition of the q-Jackson integrals from 0 to τ is provided by:

$$\int_0^{\tau} h(\zeta) \, d_q \zeta = \tau (1-q) \sum_{n=0}^{\infty} q^n h(\tau^n).$$

Definition 1.2. [3] We say that a function $h \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(q)$ if

$$\left| \frac{\tau \left(\mathcal{D}_q h \right)(\tau)}{h(\tau)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; \tau \in \triangle; \; 0 < q < 1. \tag{1.6}$$

Note that, when $q \to 1^-$, then $\mathcal{S}^*(q) = \mathcal{S}^*$.

Raghavendar and Swaminathan [10] extended the approach in (1.6) to introduce the class $\mathcal{K}(q)$ of q-close-to-convex functions in $\triangle$, providing the following definition in their 2012 work.

Definition 1.3. [10] A function $h \in \mathcal{A}$ belongs to the class $\mathcal{K}(q)$ if

$$\left| \frac{\tau \left(\mathcal{D}_q h \right)(\tau)}{g(\tau)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \; \tau \in \triangle; \; 0 < q < 1. \tag{1.7}$$

Remark 1. *When $q \to 1^-$, then*

$$\mathcal{K}(q) = \mathcal{K}$$

and it is easy to see that

$$S^*(q) \subset \mathcal{K}(q).$$

For a function h defined in $\triangle$, the q-Alexander transformation (q-Alexander operator) is given by the integral

$$\mathcal{I}[h(\tau)] = \int_0^{\tau} \frac{h(\tau)}{\varsigma} d_q \varsigma.$$

The operator transforms the power series as:

$$\mathcal{I}[h(\tau)] = \tau + \sum_{m=1}^{\infty} \frac{a_{m+1}}{[m+1]_q} \tau^{m+1}, \tag{1.8}$$

where

$$[m+1]_q = \frac{1-q^{m+1}}{1-q}.$$

The operator $\mathcal{I}[h]$ was previously examined as a particular case of the q-Srivastava-Attiya operator, as discussed in [11]. This operator satisfies the normalization conditions, specifically

$$\mathcal{I}[h(0)] = 0,\ \mathcal{I}\left[h^{'}(0)\right] = 1.$$

This transformation is known as the Alexander transformation when $q \to 1^{-}$ introduced and studied in [2].

1.1. Motivation

Geometric function theory (GFT), a key component of complex analysis, focuses on the geometric aspects of analytic functions. These functions play a vital role in various fields, including mathematics, physics, and statistics. Researchers have been drawn to problems related to the geometric properties of families of analytic functions, especially those involving special functions [12–14]. Recent studies have examined the geometric properties of various special functions. Mehrez et al. [15] investigated the products of modified Bessel functions of the first kind, while Mehrez [16] explored a class of functions related to Fox-Wright functions. Moreover, Mehrez and Raza [17] studied the geometric properties of Mittag-Leffler-Prabhakar functions of Le Roy-type in the open unit disk. Recent studies have further explored its geometric properties, with notable research from Aktas [18], Aktas and Orhan [19], and Aktas and Baricz [20]. Special functions play a vital role in mathematical analysis, particularly in fractional calculus and its generalizations. q-calculus extends classical calculus and introduces the q-Mittag-Leffler function, a powerful tool for solving q-fractional differential equations and modeling complex systems in mathematical physics and engineering [8, 21–23]. The Le Roy-type q-Mittag-Leffler function is a notable subclass that enables better modeling of anomalous processes [24]. Recent studies have examined geometric properties like convexity [25] and partial sums [26, 27], essential for univalence, stability, and numerical approximation. This paper builds upon these foundations, offering new results on the convexity and partial sums of normalized Le Roy-type q-Mittag-Leffler functions, and contributing to GFT and fractional analysis.

The Swedish mathematician Gösta Magnus Mittag-Leffler introduced the one-parameter [28] version of the Mittag-Leffler function as follows:

$$E_c(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(cm+1)} \tau^m,\ c, \tau \in \mathbb{C},\ \operatorname{Re}(c) > 0.$$

The two-parameter Mittag-Leffler function [29], denoted as $E_{c,d}(\tau)$, is a mathematical function that is defined by the series

$$E_{c,d}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(cm+d)} \tau^m,\ c, d, \tau \in \mathbb{C},\ \operatorname{Re}(c) > 0.$$

In a recent development, Gerhold [30] and Garra and Polito [31] each proposed the Le Roy-type Mittag-Leffler function, which is characterized by the following definition:

$$E_{c,d}^{\sigma}(\tau) = \sum_{m=0}^{\infty} \frac{1}{(\Gamma(cm+d))^{\sigma}} \tau^m,\ c, d, \sigma > 0, \tau \in \mathbb{C}.$$

When $c = d = 1$, in $E^{\sigma}_{c,d}(\tau)$, we have the Le Roy-type functions (LRFs) investigated in [32] and further studied by Mehrez and Das in [35]. This function, denoted as $E^{\sigma}(\tau)$, is defined as

$$E^{\sigma}(\tau) = \sum_{m=0}^{\infty} \frac{1}{(m!)^{\sigma}} \tau^{m},$$

where σ is a positive real number and τ is a complex number.

Now, we consider the Le Roy-type q-Mittag Laffler function, which is defined by the series

$$E^{\sigma,q}_{c,d}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\left(\Gamma_q(cm+d)\right)^{\sigma}} \tau^{m}, \ c, d, \sigma > 0, \tau \in \mathbb{C}, \ 0 < q < 1. \tag{1.9}$$

The normalized Le Roy type q-Mittag-Leffler function, denoted as $M^{\sigma,q}_{c,d}(\tau)$, and defined as:

$$\begin{aligned} M^{\sigma,q}_{c,d}(\tau) &= \tau \left(\Gamma_q(d)\right)^{\sigma} E^{\sigma,q}_{c,d}(\tau) \\ &= \sum_{m=0}^{\infty} \left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma} \tau^{m+1}, \end{aligned} \tag{1.10}$$

where $c, d, \sigma > 0$, $\tau \in \mathbb{C}$, and $0 < q < 1$.

The functions given in (1.9) and (1.10) provide an extended version of many well-known special functions.

Special Cases:

(i) When $q \to 1^{-}$, then $M^{\sigma,q}_{c,d}(\tau)$ reduces to the Le Roy type Mittag-Leffler function defined in [30], (see also [31]).

(ii) Moreover, when $q \to 1^{-}$, and $\sigma = 1$, the function $M^{\sigma,q}_{c,d}(\tau)$ reduces to the two-parameter Mittag-Leffler function $E_{c,d}(\tau)$ defined in [29] and extensively studied in 2016 and 2021, (see [33, 34]).

(iii) When $q \to 1^{-}$ and $c = d = 1$, the function $M^{\sigma,q}_{c,d}(\tau)$ becomes the Le Roy-type function (LRFs) $E^{\sigma}(\tau)$ defined in [32] and further investigated by Mehrez and Das in [35].

(iv) Finally, when $q \to 1^{-}$ and $c = \sigma = 1$, the function $M^{\sigma,q}_{c,d}(\tau)$ becomes the classical Mittag-Leffler function $E_d(\tau)$, defined and studied by Mittag-Leffler in 1903.

Some others special cases of the normalized Le Roy type q-Mittag-Leffler function $M^{\sigma,q}_{c,d}(\tau)$ are also listed below:

$$\begin{aligned} M^{1,q}_{0,1}(\tau) &= \frac{\tau}{1-\tau}, \\ M^{1,q}_{1,1}(\tau) &= \tau e^{\tau}_{q}, \\ M^{1,q}_{1,2}(\tau) &= e^{\tau}_{q} - 1, \end{aligned} \tag{1.11}$$

where e^{τ}_{q} is one of the q-analogues of the exponential function e^{τ}, which is given by (see ([36], p. 488, Eq 6.3 (7))

$$e^{\tau}_{q} = \sum_{m=0}^{\infty} \frac{\tau^{m}}{\Gamma_q(m+1)}.$$

2. Main results

This section focuses on analyzing the close-to-convexity, partial sums, and establishing lower bounds for the normalized Le Roy type q-Mittag-Leffler function (LR-q-MLFs).

2.1. Close-to-convexity of normalized Le Roy-type q-Mittag-Leffler functions

This section presents the following key results and methods: First, we show that the normalized LR-q-MLFs, as defined in (1.10), are q-close-to-convex in $\triangle$ with respect to $g(\tau) = \tau\backslash 1 - \tau$, with the proof facilitated by Lemma 2.

Lemma 1. *([10], Lemma 1.1(1)). Let* $h(\tau) = \tau + \sum_{m=2}^{\infty} A_m \tau^m$*, and*

$$\sum_{m=1}^{\infty} |D_{m+1} - D_m| \leq 1$$

with

$$D_n = \frac{A_m (1 - q^m)}{1 - q}.$$

Then, $h(\tau) \in \mathcal{K}(q)$ with $g(\tau) = \tau\backslash 1 - \tau$.

Lemma 1 leads to the following lemma, which was proved by Sahoo and Sharma in their work [37].

Lemma 2. *[37]. Let* $\{A_m\}_{m\in\mathbb{N}}$ *be a sequence of real numbers, and define another sequence* $\{D_m\}_{m\in\mathbb{N}}$,

$$D_n = \frac{A_m (1 - q^m)}{1 - q}$$

for all $n \geq 1$. Suppose that

$$1 \geq D_2 \geq D_3 \geq D_4 \geq \cdots \geq D_n \geq \cdots \geq 0,$$

or

$$1 \leq D_2 \leq D_3 \leq D_4 \leq \cdots \leq D_n \leq \cdots \leq 2.$$

Then, $h(\tau) = \tau + \sum_{m=2}^{\infty} A_m \tau^m \in \mathcal{K}(q)$ with respect to $g(z) = z\backslash 1 - z$.

Theorem 3. *For each* $c, d, \sigma \geq 1$*, satisfying* $c^2\sigma \geq d$*,* $c\sigma \geq 1$*, and* $c + d \geq 3$*, subject to the condition*

$$\left(\Gamma_q (c + d)\right)^{\sigma} \geq (1 + q) \left(\Gamma_q (d)\right)^{\sigma},$$

the normalized LR-q-MLFs $M_{c,d}^{\sigma,q}(\tau) \in \mathcal{K}(q)$ in $\triangle$ with respect to $g(\tau) = \tau\backslash 1 - \tau$.

Proof. To show that $M_{c,d}^{\sigma,q}(\tau) \in \mathcal{K}(q)$ in $\triangle$ with respect to $g(\tau) = \tau\backslash 1 - \tau$, we recall (1.10), and thus we have

$$\begin{aligned} M_{c,d}^{\sigma,q}(\tau) &= \tau + \sum_{m=2}^{\infty} \left(\frac{\Gamma_q (d)}{\Gamma_q (c (m - 1) + d)} \right)^{\sigma} \tau^m \\ &= \tau + \sum_{m=2}^{\infty} A_m \tau^m, \end{aligned}$$

where

$$A_m = \left(\frac{\Gamma_q (d)}{\Gamma_q (c (m - 1) + d)} \right)^{\sigma}, \ m \in \mathbb{N},$$

and

$$D_m = \frac{1-q^m}{1-q}A_m = \frac{1-q^m}{1-q}\left(\frac{\Gamma_q(d)}{\Gamma_q(c(m-1)+d)}\right)^{\sigma}.$$

Note that $D_1 = 1$ and $D_m \geq 0$, for all $m \in \mathbb{N}$. If $c^2\sigma \geq d$ and $c\sigma \geq 1$, then

$$D_2 = (1+q)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma} \leq 1 = D_1.$$

Next, we show that

$$D_{m+1} \leq D_m, \ \ (m \in \mathbb{N}\backslash\{1\}),$$

that is,

$$\frac{1-q^{m+1}}{1-q}\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma} \leq \frac{1-q^m}{1-q}\left(\frac{\Gamma_q(d)}{\Gamma_q(c(m-1)+d)}\right)^{\sigma}, \ \ (m \in \mathbb{N}\backslash\{1\}),$$

which is equivalent to

$$\left(1-q^{m+1}\right)\left(\Gamma_q(c(m-1)+d)\right)^{\sigma} \leq (1-q^m)\left(\Gamma_q(cm+d)\right)^{\sigma}, \ \ (m \in \mathbb{N}\backslash\{1\}). \tag{2.1}$$

Inequality (2.1) is verified by the fact that

$$\begin{aligned}(1-q^m)\left(\Gamma_q(cm+d)\right)^{\sigma} &= (1-q^m)\left(\Gamma_q(c(m-1)+c+d)\right)^{\sigma} \\ &\geq (1-q^m)\left(\Gamma_q(c(m-1)+d+1)\right)^{\sigma}.\end{aligned} \tag{2.2}$$

Using (1.2), then (2.2) becomes

$$(1-q^m)\left(\Gamma_q(cm+d)\right)^{\sigma} \geq (1-q^m)\left([c(m-1)+d]_q\,\Gamma_q(c(m-1)+d)\right)^{\sigma}. \tag{2.3}$$

Again using (1.3), then (2.3) becomes

$$(1-q^m)\left(\Gamma_q(cm+d)\right)^{\sigma} \geq (1-q^m)\left(\frac{1-q^{c(m-1)+d}}{1-q}\right)^{\sigma}\left(\Gamma_q(c(m-1)+d)\right)^{\sigma}. \tag{2.4}$$

In light of the inequality

$$\left(\Gamma_q(c+d)\right)^{\sigma} \geq (1+q)\left(\Gamma_q(d)\right)^{\sigma},$$

and $c+d \geq 3$, we have that

$$c(m-1)+d \geq m+1, \ \ m \in \mathbb{N}\backslash\{1\}.$$

Thus, (2.4) becomes

$$\begin{aligned}(1-q^m)\left(\Gamma_q(cm+d)\right)^{\sigma} &\geq (1-q^m)\left(\frac{1-q^{m+1}}{1-q}\right)^{\sigma}\left(\Gamma_q(c(m-1)+d)\right)^{\sigma} \\ &\geq (1-q^m)\left(\frac{1-q^{m+1}}{1-q}\right)\left(\Gamma_q(c(m-1)+d)\right)^{\sigma} \\ &= [m]_q\left(1-q^{m+1}\right)\left(\Gamma_q(c(m-1)+d)\right)^{\sigma} \\ &\geq \left(1-q^{m+1}\right)\left(\Gamma_q(c(m-1)+d)\right)^{\sigma}, \ \ m \in \mathbb{N}\backslash\{1\},\end{aligned}$$

which proves our required inequality (2.1). Now, utilizing Lemma 2, the normalized LR-q-MLFs $M_{c,d}^{\sigma,q}(\tau) \in \mathcal{K}(q)$ with respect to $g(\tau) = \tau\backslash 1-\tau$. □

Setting $c = 1$ in Theorem 3 yields the following proposition.

Proposition 1. *For each $\sigma \geq 1$ and $d \geq 2$ satisfying the inequality*

$$\left(\Gamma_q (1 + d)\right)^{\sigma} \geq (1 + q) \left(\Gamma_q (d)\right)^{\sigma},$$

then,

$$M_{1,d}^{\sigma,q} (\tau) = \sum_{m=0}^{\infty} \left(\frac{\Gamma_q (d)}{\Gamma_q (m + d)}\right)^{\sigma} \tau^{m+1} \in \mathcal{K} (q).$$

Proof. When $c = 1$ in Theorem 3, and the inequality

$$\left(\Gamma_q (1 + d)\right)^{\sigma} \geq (1 + q) \left(\Gamma_q (d)\right)^{\sigma}.$$

Upon simplification, we have

$$\left([d]_q\right)^{\sigma} \geq (1 + q),$$

or

$$\left(\frac{1 - q^d}{1 - q}\right)^{\sigma} \geq (1 + q),$$

and it is holds true only $\sigma \geq 1$ and $d \geq 2$. Thus, the normalized $M_{1,d}^{\sigma,q} (\tau) \in \mathcal{K} (q)$ for $\sigma \geq 1$ and $d \geq 2$. □

Setting $c = 2$ in Theorem 3 yields the following proposition.

Proposition 2. *For each $\sigma \geq 1$ and $d \geq 1$, the inequality*

$$\left(\Gamma_q (2 + d)\right)^{\sigma} \geq (1 + q) \left(\Gamma_q (d)\right)^{\sigma},$$

then

$$M_{2,d}^{\sigma,q} (\tau) = \sum_{m=0}^{\infty} \left(\frac{\Gamma_q (d)}{\Gamma_q (2m + d)}\right)^{\sigma} \tau^{m+1} \in \mathcal{K} (q).$$

Proof. When $c = 2$ in Theorem 3 and the inequality

$$\left(\Gamma_q (2 + d)\right)^{\sigma} \geq (1 + q) \left(\Gamma_q (d)\right)^{\sigma},$$

Upon simplification, we have

$$\left([1 + d]_q\right)^{\sigma} \geq (1 + q),$$

or

$$\left(\frac{1 - q^{d+1}}{1 - q}\right)^{\sigma} \geq (1 + q),$$

and it is satisfied only for $\sigma \geq 1$ and $d \geq 1$. Thus, the normalized $M_{2,d}^{\sigma,q} (\tau) \in \mathcal{K} (q)$ for $\sigma \geq 1$ and $d \geq 1$. □

Remark 2. *If we set $c = 1$, $\sigma = 1$, and $d \geq 2$ in (1.10), then*

$$M_{1,d}^{1,q} (\tau) = \sum_{m=0}^{\infty} \frac{\Gamma_q (d)}{\Gamma_q (m + d)} \tau^{m+1} \in \mathcal{K} (q).$$

Remark 3. *If we set $c = 2$, $\sigma = 1$, and $d \geq 1$ in (1.10), then*

$$M_{2,d}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{\Gamma_q(d)}{\Gamma_q(2m+d)} \tau^{m+1} \in \mathcal{K}(q).$$

Example 1. *From Remark 2, we readily deduce that*

$$M_{1,1}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma_q(m+1)} \tau^{m+1} = \tau e_q^{\tau} \in \mathcal{K}(q), \tag{2.5}$$

$$M_{1,2}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma_q(m+2)} \tau^{m+1} = e_q^{\tau} - 1 \in \mathcal{K}(q), \tag{2.6}$$

$$M_{1,3}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{[2]_q}{\Gamma_q(m+3)} \tau^{m+1} = \frac{(1+q)\left(e_q^{\tau} - \tau - 1\right)}{\tau} \in \mathcal{K}(q). \tag{2.7}$$

Example 2. *From Remark 3, we readily deduce that*

$$M_{2,1}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma_q(2m+1)} \tau^{m+1} = z \cosh_q \sqrt{z} = z\left(\frac{e_q^{\sqrt{z}} + e_q^{-\sqrt{z}}}{2}\right) \in \mathcal{K}(q),$$

$$M_{2,2}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma_q(2m+2)} \tau^{m+1} = \sinh_q \sqrt{z} = \left(\frac{e_q^{\sqrt{z}} - e_q^{-\sqrt{z}}}{2}\right) \in \mathcal{K}(q),$$

$$\begin{aligned} M_{2,3}^{1,q}(\tau) &= \sum_{m=0}^{\infty} \frac{[2]_q}{\Gamma_q(2m+3)} \tau^{m+1} = (1+q)\left(\cosh_q \sqrt{z} - 1\right) \\ &= (1+q)\left(\frac{e_q^{\sqrt{z}} + e_q^{-\sqrt{z}}}{2} - 1\right) \in \mathcal{K}(q). \end{aligned}$$

If $\sigma = 1$ in Theorem 3, we obtain the following known result proved in [38].

Corollary 1. *[38]. For each $c \geq 1$ and $d \geq 1$, satisfying the inequality*

$$\Gamma_q(c+d) \geq (1+q)\Gamma_q(d),$$

the normalized q-Mittag-leffler functions $\mathcal{R}_{c,d}^{q}(\tau) \in \mathcal{K}(q)$ in $\triangle$ with respect to $g(\tau) = \tau\backslash 1 - \tau$.

2.2. Partial sums for normalized Le Roy-type q-Mittag-Leffler functions

For any function h in $\mathcal{A}$, we can form a partial sum, denoted as $h_\delta(\tau)$, by truncating its Taylor series expansion to the δ-th term:

$$h_\delta(\tau) = \tau + \sum_{m=2}^{\delta} a_m \tau^m. \tag{2.8}$$

Finding partial sums of analytic functions in the largest possible disk, $\triangle_r = \{\tau \in \mathbb{C} : |\tau| < r\}$, where the partial sum $h_\delta(\tau)$ is one-to-one is a challenging task. In 1928, Szegö [39] made a significant breakthrough by proving that for functions in the class $\mathcal{S}$, each partial sum is one-to-one within the disk $\triangle_{1\backslash 4}$. However, this result does not guarantee that partial sums of functions in $\mathcal{S}$ are always one-to-one throughout the entire unit disk $\triangle$. A counter example is the convex univalent function $h(\tau) = \tau/(1-\tau)$, which demonstrates that this is not always the case. Moreover, the Koebe function

$$Q(\tau) = \frac{\tau}{(1-\tau)^2}$$

has a second partial sum $h_2(\tau) = \tau + 2\tau^2$ that is one-to-one within $\triangle_{1\backslash 4}$, and remarkably, the radius $1\backslash 4$ is the largest possible.

Researchers have extensively studied partial sums of analytic functions, particularly those in the class $\mathcal{S}^*$. A notable contribution was made by Robertson [40], who established the radius of starlikeness for partial sums of functions in [40]. Additionally, Silvia [41] introduced the concept of lower bounds for the real parts of partial sum ratios, which was later built upon by Silverman [42]. Silverman developed techniques for finding partial sums of starlike and convex functions, laying the groundwork for further research. Subsequent studies have expanded these results to various subclasses of analytic functions [43–45] and special functions, such as the normalized Struve functions, Dini functions, and Wright functions [46–48].

Taking motivation from [17, 48, 49], in this section, we study the lower bounds of the sequence of partial sums $\left(M_{c,d}^{\sigma,q}(\tau)\right)_\delta(\tau)$ and $M_{c,d}^{\sigma,q}(\tau)$.

The sequence of partial sums $\left(M_{c,d}^{\sigma,q}(\tau)\right)_\delta(\tau)$ of the normalized LR-q-MLFs is defined by

$$\left(M_{c,d}^{\sigma,q}(\tau)\right)_\delta(\tau) = \tau + \sum_{m=1}^{\delta}\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^\sigma \tau^{m+1}, \quad \delta \in \mathbb{N}. \tag{2.9}$$

For $\delta = 0$, then the sequence of partial sums is

$$\left(M_{c,d}^{\sigma,q}(\tau)\right)_0(\tau) = \tau.$$

The main goal of this section is to find the lower bounds for

$$\operatorname{Re}\left(\frac{M_{c,d}^{\sigma,q}(\tau)}{\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}\right), \quad \operatorname{Re}\left(\frac{\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}{M_{c,d}^{\sigma,q}(\tau)(\tau)}\right), \tag{2.10}$$

$$\operatorname{Re}\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}\right), \quad \operatorname{Re}\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}\right), \tag{2.11}$$

$$\operatorname{Re}\left(\frac{\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)(\tau)}{\left(\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)\right)_\delta(\tau)}\right), \quad \operatorname{Re}\left(\frac{\left(\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)\right)_\delta(\tau)}{\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)(\tau)}\right), \tag{2.12}$$

where $\mathcal{I}\left(M_{c,d}^{\sigma,q}(\tau)\right)$ is the Alexander transform of $M_{c,d}^{\sigma,q}(\tau)$.

To solve our problems, we first derive the following two lemmas.

Lemma 4. *Let $0 < q < 1$ and c, d, and σ be arbitrary positive real numbers.*

(i) If $cd \geq 1$, $c^2\sigma \geq d$, then the sequence $(A_m)_{m\geq 1}$ defined by

$$A_m = \Gamma_q(m+1)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^\sigma$$

is decreasing.
(ii) Also, if $cd \geq 1$ and $2c^2\sigma \geq d$, then the sequence $(C_m)_{m\geq 1}$ defined by

$$C_m = \Gamma_q(m+2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}$$

is also decreasing.

Proof. To show that $A_m = \Gamma_q(m+1)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}$ is decreasing, we have to show that $\frac{A_{m+1}}{A_m} < 1$. For this,

$$\begin{aligned}\frac{A_{m+1}}{A_m} &= [m+1]_q\left(\frac{\Gamma_q(cm+d)}{\Gamma_q(c(m+1)+d)}\right)^{\sigma}\\ &= \frac{1-q^{m+1}}{1-q}\left(\frac{\Gamma_q(cm+d)}{\Gamma_q(c(m+1)+d)}\right)^{\sigma}.\end{aligned} \tag{2.13}$$

We observe that when $m \to \infty$, then

$$0 < \frac{1-q^{m+1}}{1-q} < 1. \tag{2.14}$$

Applying the definition of the q-Gamma function on $\frac{\Gamma_q(cm+d)}{\Gamma_q(c(m+1)+d)}$, we have

$$\frac{\Gamma_q(cm+d)}{\Gamma_q(c(m+1)+d)} = \prod_{k=0}^{\infty}\frac{1-q^{cm+d+k}}{1-q^{c(m+1)+d+k}}. \tag{2.15}$$

Since $0 < q < 1$, both the terms in the numerator and denominator will be positive and less than 1. Hence, by (2.14), and (2.15), we have

$$A_{m+1} < A_m.$$

Thus, for $\sigma \geq 0$, the sequence $(A_m)_{m\geq 1}$

$$A_m = \Gamma_q(m+1)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}$$

is decreasing.

Similarly, the sequence

$$C_m = \Gamma_q(m+2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}$$

is also decreasing for $0 < q < 1$, $cd \geq 1$ and $2c^2\sigma \geq d$. □

Lemma 5. *Assume that c, d, and σ are arbitrary positive real numbers. Then, the function $M_{c,d}^{\sigma,q}(\tau) : \triangle \to \mathbb{C}$ defined by (1.10) satisfies the following inequalities:*

(i) If $cd \geq 1$ and $c^2\sigma \geq d$, then

$$\left|M_{c,d}^{\sigma,q}(\tau)\right| \leq 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\left(e_q - 1\right), \ \tau \in \triangle.$$

(ii) If $cd \geq 1$ and $2c^2\sigma \geq d$, then

$$\left|\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)\right| \leq 1 + (1+q)\,\Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\left(e_q - 1\right),\ \tau \in \triangle.$$

(iii) If $cd \geq 1$ and $c^2\sigma \geq d$, then

$$\left|\left[\mathcal{I}\left(M_{c,d}^{\sigma,q}(\tau)\right)\right](\tau)\right| \leq 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\left(e_q - [2]_q\right),\ \tau \in \triangle,$$

where $[2]_q = 1 + q$.

Proof. (i) If $cd \geq 1$ and $c^2\sigma \geq d$, then according to Lemma 4, we have

$$\begin{aligned}
\left|M_{c,d}^{\sigma,q}(\tau)\right| &= \left|\tau + \sum_{m=1}^{\infty}\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\tau^{m+1}\right| \\
&= \left|\tau + \sum_{m=1}^{\infty}\frac{\Gamma_q(m+1)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}}{[m]_q!}\tau^{m+1}\right| \\
&\leq 1 + \sum_{m=1}^{\infty}\frac{A_1(c,d,q,\sigma)}{[m]_q!} \\
&\leq 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\sum_{m=1}^{\infty}\frac{1}{[m]_q!} \\
&= 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\left(e_q - 1\right).
\end{aligned}$$

(ii) If $cd \geq 1$ and $2c^2\sigma \geq d$, then according to Lemma 4, we have

$$\begin{aligned}
\left|\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)\right| &= \left|1 + \sum_{m=1}^{\infty}([m+1]_q\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\tau^{m}\right| \\
&= \left|1 + \sum_{m=1}^{\infty}\frac{[m+1]_q\,\Gamma_q(m+1)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}}{[m]_q!}\tau^{m}\right| \\
&= \left|1 + \sum_{m=1}^{\infty}\frac{\Gamma_q(m+2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}}{[m]_q!}\tau^{m}\right| \\
&\leq 1 + \sum_{m=1}^{\infty}\frac{C_1(c,d,q,\sigma)}{[m]_q!} \\
&\leq 1 + \Gamma_q(3)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\sum_{m=1}^{\infty}\frac{1}{[m]_q!}
\end{aligned}$$

$$= 1 + \Gamma_q(3)\left(\frac{\Gamma_q(d)}{\Gamma_q(c+d)}\right)^{\sigma}\left(e_q - 1\right).$$

(iii) If $cd \geq 1$ and $c^2\sigma \geq d$, then according to Lemma 4, we have

$$\begin{aligned}
\left|\mathcal{I}\left[\left(M_{c,d}^{\sigma,q}(\tau)\right)(\tau)\right]\right| &= \left|\tau + \sum_{m=1}^{\infty}\frac{1}{[m+1]_q}\left(\frac{\Gamma(d)}{\Gamma(cm+d)}\right)^{\sigma}\tau^{m+1}\right| \\
&= \left|\tau + \sum_{m=1}^{\infty}\frac{\Gamma_q(m+1)}{[m+1]_q\,[m]_q!}\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\tau^{m+1}\right| \\
&= \left|\tau + \sum_{m=1}^{\infty}\frac{\Gamma_q(m+1)}{[m+1]_q!}\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\tau^{m+1}\right| \\
&\leq 1 + A_1(c,d,q,\sigma)\sum_{m=1}^{\infty}\frac{1}{[m+1]_q!} \\
&\leq 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\sum_{m=1}^{\infty}\frac{1}{[m+1]_q!} \\
&= 1 + \Gamma_q(2)\left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^{\sigma}\left(e_q - [2]_q\right).
\end{aligned}$$

□

Remark 4. *The numerical table of Lemma 5 for specific q, c, d, σ to illustrate tightness is as follows:*

q	c	d	σ	R.H.S (i)	R.H.S (ii)	R.H.S (iii)
0.9	1.5	1.0	0.7	∼ 1.059	∼ 1.097	∼ 1.121
0.8	2.0	1.0	1.0	∼ 1.034	∼ 1.063	∼ 1.080
0.95	1.2	1.1	0.6	∼ 1.072	∼ 1.110	∼ 1.135
0.99	1.1	0.9	0.5	∼ 1.084	∼ 1.120	∼ 1.149

Example 3. *If $c = d = \sigma = 1$, in Lemma 5, then,*

$$\left|M_{1,1}^{1,q}(\tau)\right| \leq 1 + \Gamma_q(2)\left(\frac{\Gamma_q(1)}{\Gamma_q(2)}\right)\left(e_q - 1\right). \tag{2.16}$$

Using the definition of Γ_q and (1.11), in (2.16), we obtain

$$\left|\tau e_q^{\tau}\right| \leq e_q,\ \tau \in \triangle.$$

Similarly,

$$\begin{aligned}
\left|\mathcal{D}_q\left(M_{1,1}^{1,q}(\tau)\right)\right| &\leq 1 + (1+q)\,\Gamma_q(2)\left(\frac{\Gamma_q(1)}{\Gamma_q(2)}\right)\left(e_q - 1\right) \\
\left|\mathcal{D}_q\left(\tau e_q^{\tau}\right)\right| &\leq 1 + (1+q)\left(e_q - 1\right).
\end{aligned} \tag{2.17}$$

Using (1.5), in (2.17), we obtain

$$\left|(q\tau + 1)\,e_q^{\tau}\right| \leq 1 + (1+q)\left(e_q - 1\right),\ \tau \in \triangle,$$

and

$$\left|\mathcal{I}\left(\tau e^{\tau}\right)\right| \leq e_q - q,\ \tau \in \triangle.$$

Example 4. *If $c = \sigma = 1$, and $d = 2$, in the Lemma 5, then*

$$\left|e_q^\tau - 1\right| \le 1 + \frac{1}{1+q}\left(e_q - 1\right), \ \left|e_q^\tau\right| \le e_q, \ \tau \in \triangle,$$

and

$$\left|\mathcal{I}\left(e_q^\tau - 1\right)\right| \le \frac{1}{1+q} e_q, \ \tau \in \triangle.$$

Consider an analytic function $u(\tau)$ defined in the unit disk $\triangle$, (see [51]).

$$\mathrm{Re}\left(\frac{1+u(\tau)}{1-u(\tau)}\right) > 0, \ \tau \in \triangle \Leftrightarrow |u(\tau)| < 1, \ \tau \in \triangle.$$

Theorem 6. *If $cd \ge 1$, $c^2\sigma \ge d$, and $\left(\Gamma_q(c+d)\right)^\sigma \ge \Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)$, then*

$$\mathrm{Re}\left(\frac{M_{c,d}^{\sigma,q}(\tau)}{\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}\right) \ge \frac{\left(\Gamma_q(c+d)\right)^\sigma - \Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)}{\left(\Gamma_q(c+d)\right)^\sigma}, \tag{2.18}$$

and

$$\mathrm{Re}\left(\frac{\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)}{M_{c,d}^{\sigma,q}(\tau)}\right) \ge \frac{\left(\Gamma_q(c+d)\right)^\sigma}{\left(\Gamma_q(c+d)\right)^\sigma + \Gamma_q(2)\left(\Gamma(d)\right)^\sigma\left(e_q - 1\right)}. \tag{2.19}$$

Proof. Using inequality (i) from Lemma 5, we derive

$$1 + \sum_{m=1}^{\infty} |A_m| \le \frac{\left(\Gamma_q(c+d)\right)^\sigma + \Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)}{\left(\Gamma_q(c+d)\right)^\sigma}.$$

Alternatively,

$$\frac{\left(\Gamma_q(c+d)\right)^\sigma}{\Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)} \sum_{m=1}^{\infty} |A_m| \le 1,$$

where

$$A_m = \left(\frac{\Gamma_q(d)}{\Gamma_q(cm+d)}\right)^\sigma. \tag{2.20}$$

To establish the inequality (2.18) let $u(\tau)$ in $\triangle$ be given by

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)} &= \frac{\left(\Gamma_q(c+d)\right)^\sigma}{\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)\Gamma_q(2)} \\ &\quad \times \left(\frac{M_{c,d}^{\sigma,q}(\tau)}{\left(M_{c,d}^{\sigma,q}\right)_\delta(\tau)} - \frac{\left(\Gamma_q(c+d)\right)^\sigma - \Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)}{\left(\Gamma_q(c+d)\right)^\sigma}\right) \\ &= \frac{1}{1 + \sum_{m=1}^{\delta} A_m \tau^m}\left(1 + \sum_{m=1}^{\delta} A_m \tau^m + \frac{\left(\Gamma_q(c+d)\right)^\sigma}{\Gamma_q(2)\left(\Gamma_q(d)\right)^\sigma\left(e_q - 1\right)} \sum_{m=\delta+1}^{\infty} A_m \tau^m\right).\end{aligned} \tag{2.21}$$

Using Eq (2.21), we can express this as

$$u(\tau)=\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}A_m\tau^m}{2+2\sum_{m=1}^{\delta}A_m\tau^m+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}A_m\tau^m},$$

and we have

$$|u(\tau)|\leq\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|}{2-2\sum_{m=1}^{\delta}|A_m|-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|}.$$

This means that $|u(\tau)|\leq 1$ if and only if

$$\frac{2\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|\leq 2-2\sum_{m=1}^{\delta}|A_m|.$$

This further implies that

$$\sum_{m=1}^{\delta}|A_m|+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|\leq 1. \tag{2.22}$$

Our goal is to prove that the L.H.S (left-hand-side) of Eq (2.22) is bounded above by

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=1}^{\infty}|A_m|,$$

which can be rewritten as

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=1}^{\delta}|A_m|\geq 0. \tag{2.23}$$

Thus, by virtue of (2.23), the proof of the inequality in (2.18) is now complete.

To prove (2.19), we write

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)}&=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\\&\quad\times\left(\frac{\left(M_{c,d}^{\sigma,q}\right)_{\delta}(\tau)}{M_{c,d}^{\sigma,q}(\tau)}-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\right)\\&=\frac{1+\sum_{m=1}^{\delta}A_m\tau^m+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}A_m\tau^m}{1+\sum_{m=1}^{\delta}A_m\tau^m}.\end{aligned}$$

Therefore,

$$|u(\tau)| \leq \frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)(\Gamma(d))^{\sigma}\left(e_q-1\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|}{2-2\sum_{m=1}^{\delta}|A_m|-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|} \leq 1.$$

This is equivalent to the inequality

$$\sum_{m=1}^{\delta}|A_m| + \frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m| \leq 1. \tag{2.24}$$

Thus, the L.H.S of (2.24) is bounded above by

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}|A_m|.$$

Thus, the proof of the theorem is complete. □

When $q \to 1^-$, the above result reduces to the following known result proved in [49].

Corollary 2. *[49]. If $cd \geq 1$, $c^2\sigma \geq d$, and $\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma} \geq (e-1)$, then*

$$\operatorname{Re}\left(\frac{M_{c,d}^{\sigma}(\tau)}{\left(M_{c,d}^{\sigma}\right)_{\delta}(\tau)}\right) \geq 1-\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma}(e-1),$$

and

$$\operatorname{Re}\left(\frac{\left(M_{c,d}^{\sigma}\right)_{\delta}(\tau)}{M_{c,d}^{\sigma}(\tau)}\right) \geq \frac{1}{1+\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma}(e-1)}.$$

Theorem 7. *If $cd \geq 1$, $2c^2\sigma \geq d$, and $\left(\Gamma_q(c+d)\right)^{\sigma} \geq [2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)$, then*

$$\operatorname{Re}\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_{\delta}(\tau)}\right) \geq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}-[2]_q\Gamma_q(2)(\Gamma(d))^{\sigma}\left(e_q-1\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}}, \quad \tau \in \triangle, \tag{2.25}$$

and

$$\operatorname{Re}\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_{\delta}(\tau)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}\right) \geq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)(\Gamma(d))^{\sigma}\left(e_q-1\right)}, \quad \tau \in \triangle, \tag{2.26}$$

where $[2]_q = 1+q$.

Proof. Using inequality (ii) from Lemma 5, we derive

$$1+\sum_{m=1}^{\infty}[m+1]_q|A_m| \leq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}}.$$

Alternatively,

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}(e-1)}\sum_{m=1}^{\infty}[m+1]_q\,|A_m|\leq 1,$$

where A_m is given by (2.20). To establish the inequality (2.25), let us consider the function $u(\tau)$ in $\triangle$ given by:

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)}&=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\\&\quad\times\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_{\delta}(\tau)}-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}}\right)\\&=\frac{1}{1+\sum_{m=1}^{\delta}[m+1]_q\,A_m\tau^m}\left(1+\sum_{m=1}^{\delta}[m+1]_q\,A_m\tau^m+\right.\\&\quad\left.\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,A_m\tau^m\right).\end{aligned}\tag{2.27}$$

Now, from (2.27), we can write

$$u(\tau)=\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,A_m\tau^m}{2+2\sum_{m=1}^{\delta}[m+1]_q\,A_m\tau^m+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,A_m\tau^m},$$

and we have

$$|u(\tau)|\leq\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,|A_m|}{2-2\sum_{m=1}^{\delta}(m+1)\,|A_m|-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,|A_m|}.$$

This means that $|u(\tau)|\leq 1$ if and only if

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,|A_m|\leq 2-2\sum_{m=1}^{\delta}[m+1]_q\,|A_m|.$$

As a result, we can also conclude that

$$\sum_{m=1}^{\delta}[m+1]_q\,|A_m|+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q\,|A_m|\leq 1.\tag{2.28}$$

To complete the proof, we need to demonstrate that the inequality (2.28) is bounded above by

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\,\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=1}^{\infty}[m+1]_q\,|A_m|.$$

This is equivalent to the inequality

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-2\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=1}^{\delta}[m+1]_q|A_m|\geq 0. \tag{2.29}$$

Thus, by virtue of (2.29), the proof of the inequality in (2.26) is now complete.

To establish equation (2.26), consider the expression

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)}&=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\\&\times\left(\frac{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}\right)_{\delta}(\tau)}{\mathcal{D}_q\left(M_{c,d}^{\sigma,q}(\tau)\right)}-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\right)\\&=\frac{1}{1+\sum_{m=1}^{\delta}[m+1]_qA_m\tau^m}\left(1+\sum_{m=1}^{\delta}A_m[m+1]_q\tau^m\right.\\&+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\\&\left.\times\sum_{m=\delta+1}^{\infty}[m+1]_qA_m\tau^m\right).\end{aligned}$$

Therefore,

$$\begin{aligned}|u(\tau)|\leq&\frac{T_1(q,c,d)\sum_{m=\delta+1}^{\infty}(m+1)|A_m|}{2-2\sum_{m=1}^{\delta}[m+1]_q|A_m|-T_2(q,c,d)\sum_{m=\delta+1}^{\infty}[m+1]_q|A_m|},\\&\leq 1.\end{aligned}$$

where

$$\begin{aligned}T_1(q,c,d)&=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)},\\T_2(q,c,d)&=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}.\end{aligned}$$

This is equivalent to the inequality

$$\sum_{m=1}^{\delta}[m+1]_q|A_m|+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q|A_m|\leq 1. \tag{2.30}$$

Therefore, we can see that the L.H.S of (2.30) is bounded above by

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{[2]_q\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-1\right)}\sum_{m=\delta+1}^{\infty}[m+1]_q|A_m|.$$

Thus, we have completed the proof. □

When $q \to 1^-$, the above result reduces to the following known result proved in [49].

Corollary 3. *[49]. If $cd \geq 1$, $c^2\sigma \geq d$, and $\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma} \geq 2(e-1)$, then,*

$$\operatorname{Re}\left(\frac{M^{\sigma}_{c,d}(\tau)}{\left(M^{\sigma}_{c,d}\right)_{\delta}(\tau)}\right) \geq 1-2\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma}(e-1),$$

and

$$\operatorname{Re}\left(\frac{\left(M^{\sigma}_{c,d}\right)_{\delta}(\tau)}{M^{\sigma}_{c,d}(\tau)}\right) \geq \frac{1}{1+2\left(\frac{\Gamma(d)}{\Gamma(c+d)}\right)^{\sigma}(e-1)}.$$

Theorem 8. *If $cd \geq 1$, $c^2\sigma \geq d$, and $\left(\Gamma_q(c+d)\right)^{\sigma} \geq \Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)$, then*

$$\operatorname{Re}\left(\frac{\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)(\tau)}{\left(\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)\right)_{\delta}(\tau)}\right) \geq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}-\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}},\ \tau \in \triangle, \tag{2.31}$$

and

$$\operatorname{Re}\left(\frac{\left(\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)\right)_{\delta}(\tau)}{\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)(\tau)}\right) \geq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)},\ \tau \in \triangle. \tag{2.32}$$

Proof. To establish equation (2.31), we recall from part (iii) of Lemma 5 that

$$1+\sum_{m=1}^{\infty}\frac{A_m}{[m+1]_q} \leq \frac{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}}.$$

Alternatively,

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=1}^{\infty}\frac{A_m}{[m+1]_q} \leq 1,$$

where A_m is given by (2.20). Now, we write

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)} &= \frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\\ &\quad\times\left(\frac{\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)(\tau)}{\left(\mathcal{I}\left(M^{\sigma,q}_{c,d}\right)\right)_{\delta}(\tau)}-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}{\left(\Gamma_q(c+d)\right)^{\sigma}}\right)\\ &= \frac{1}{1+\sum_{m=1}^{\delta}\frac{A_m}{[m+1]_q}\tau^m}\left(1+\sum_{m=1}^{\delta}\frac{A_m}{[m+1]_q}\tau^m+\right.\\ &\quad\left.\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{A_m}{[m+1]_q}\tau^m\right).\end{aligned} \tag{2.33}$$

Now, from (2.33), we can write

$$u(\tau)=\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{1}{[m+1]_q}A_m\tau^m}{2+2\sum_{m=1}^{\delta}\frac{1}{[m+1]_q}A_m\tau^m+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{1}{[m+1]_q}A_m\tau^m},$$

and we have

$$|u(\tau)|\leq\frac{\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{1}{[m+1]_q}|A_m|}{2-2\sum_{m=1}^{\delta}\frac{1}{[m+1]_q}|A_m|-\left(\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\right)\sum_{m=\delta+1}^{\infty}\frac{1}{[m+1]_q}|A_m|}.$$

This implies that $|u(\tau)|\leq 1$ if and only if

$$\frac{2\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}\leq 2-2\sum_{m=1}^{\delta}\frac{|A_m|}{[m+1]_q}.$$

This further implies that

$$\sum_{m=1}^{\delta}\frac{|A_m|}{[m+1]_q}+\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}\leq 1. \tag{2.34}$$

It is enough to demonstrate that equation (2.28) is bounded above by

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=1}^{\infty}\frac{|A_m|}{[m+1]_q}.$$

Alternatively,

$$\frac{\left(\Gamma_q(c+d)\right)^{\sigma}-\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=1}^{\delta}\frac{1}{[m+1]_q}|A_m|\geq 0. \tag{2.35}$$

Thus, by virtue of (2.35), the proof of the inequality in (2.31) is now complete.

To prove (2.32), we write

$$\begin{aligned}\frac{1+u(\tau)}{1-u(\tau)}&=T_3(q,c,d)\left(\frac{\left(\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)\right)_{\delta}(\tau)}{\mathcal{I}\left(M_{c,d}^{\sigma,q}\right)(\tau)}-\frac{\left(\Gamma_q(c+d)\right)^{\sigma}}{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}\right)\\&=\frac{1+\sum_{m=1}^{\delta}\frac{A_m}{[m+1]_q}\tau^m+T_3(q,c,d)\sum_{m=\delta+1}^{\infty}\frac{A_m}{[m+1]_q}\tau^m}{1+\sum_{m=1}^{\delta}\frac{A_m}{[m+1]_q}\tau^m},\end{aligned}$$

where

$$T_3(q,c,d)=\frac{\left(\Gamma_q(c+d)\right)^{\sigma}+\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}{\Gamma_q(2)\left(\Gamma_q(d)\right)^{\sigma}\left(e_q-2\right)}.$$

Therefore,

$$|u(\tau)| \leq \frac{T_3\left(q,c,d\right)\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}}{2-2\sum_{m=1}^{\delta}\frac{|A_m|}{[m+1]_q}-T_4\left(q,c,d\right)\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}} \leq 1, \tag{2.36}$$

where

$$T_4\left(q,c,d\right)=\frac{\left(\Gamma_q\left(c+d\right)\right)^{\sigma}-\Gamma_q\left(2\right)\left(\Gamma_q\left(d\right)\right)^{\sigma}\left(e_q-2\right)}{\Gamma_q\left(2\right)\left(\Gamma_q\left(d\right)\right)^{\sigma}\left(e_q-2\right)}.$$

Inequality (2.36) is equivalent to

$$\sum_{m=1}^{\delta}\frac{|A_m|}{[m+1]_q}+\frac{\left(\Gamma_q\left(c+d\right)\right)^{\sigma}}{\Gamma_q\left(2\right)\left(\Gamma_q\left(d\right)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}\leq 1. \tag{2.37}$$

Thus, the L.H.S of (2.37) is bounded above by

$$\frac{\left(\Gamma_q\left(c+d\right)\right)^{\sigma}}{\Gamma_q\left(2\right)\left(\Gamma_q\left(d\right)\right)^{\sigma}\left(e_q-2\right)}\sum_{m=\delta+1}^{\infty}\frac{|A_m|}{[m+1]_q}.$$

With this, the proof is complete. □

Example 5. *By substituting the values* $\delta=0$, $\sigma=1$, $c=1$, *and* $d=1$, *we have the following special functions satisfying Theorems 6 and 7:*

$$\operatorname{Re}\left(e_q^{\tau}\right)\geq e_q \text{ and } \operatorname{Re}\left(\frac{1}{e_q^{\tau}}\right)\geq\frac{1}{e_q},\ \tau\in\triangle,$$

and

$$\operatorname{Re}\left(\frac{\left(q\tau+1\right)e_q^{\tau}}{\tau}\right)\geq 1-\left(1+q\right)\left(e_q-1\right),\ \tau\in\triangle,$$
$$\operatorname{Re}\left(\frac{\tau}{\left(q\tau+1\right)e_q^{\tau}}\right)\geq\frac{1}{1+\left(1+q\right)\left(e_q-1\right)},\tau\in\triangle,$$

where

$$M_{1,1}^{1,q}\left(\tau\right)=\sum_{m=0}^{\infty}\frac{1}{\Gamma_q\left(m+1\right)}\tau^{m+1}=\tau e_q^{\tau}.$$

Example 6. *By substituting the values* $\delta=0,\sigma=1,c=1$, *and* $d=2$, *then we have the following special functions satisfying Theorems 6 and 7:*

$$\operatorname{Re}\left(\frac{e_q^{\tau}-1}{\tau}\right)\geq\frac{2+q-e_q}{1+q} \text{ and } \operatorname{Re}\left(\frac{\tau}{e_q^{\tau}-1}\right)\geq\frac{1+q}{q+e_q}.$$

Further,

$$\operatorname{Re}\left(\frac{\mathcal{D}_q\left(e_q^{\tau}-1\right)}{\tau}\right)\geq 2+e_q,\ \tau\in\triangle,$$

and

$$\mathrm{Re}\left(\frac{\tau}{\mathcal{D}_q\left(e_q^{\tau}-1\right)}\right) \geq \frac{1}{e_q},\ \tau \in \triangle,$$

where

$$M_{1,2}^{1,q}(\tau) = \sum_{m=0}^{\infty} \frac{1}{\Gamma_q(m+2)} \tau^{m+1} = e_q^{\tau} - 1.$$

3. Conclusions

In this paper, Section 1 discussed analytic functions and the background of some of their subclasses. We then provided fundamentals of q-calculus and defined the class $K(q)$ of q-close-to-convex functions. The motivation section provided background on Mittag-Leffler functions, performed normalization of the Le Roy-type q-Mittag-Leffler function, and discussed some special cases. Section 2, "Main results" is divided into two subsections. The first subsection presented known lemmas to prove the q-close-to-convexity of the Le Roy-type q-Mittag-Leffler function in Theorem 3, along with supporting propositions, corollaries, and examples. The next subsection proved essential lemmas and established a lower bound for the sequence of partial sums. Supporting examples demonstrate the practicality of our findings. Future research directions include investigating geometric properties like starlikeness, convexity, and coefficient inequalities, potentially combining the normalized function with harmonic or p-valent functions.

Author contributions

Khaled Matarneh: Conceptualization, formal analysis, investigation, methodology, validation, software, writing-original draft, writing-review and editing; Suha B. Al-Shaikh: Data curation, formal analysis, methodology, software, visualization; Mohammad Faisal Khan: Data curation, methodology, software, visualization; Ahmad A. Abubakar: Formal analysis, investigation, project administration, resources, supervision, validation, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no competing interest.

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