Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 14278–14287.
DOI: 10.3934/math.2025643
Received: 03 January 2025
Revised: 20 May 2025
Accepted: 06 June 2025
Published: 23 June 2025

Research article

On the minimum size of maximal IC-plane graphs

Rui Xu*

Key Laboratory of Computing and Stochastic Mathematics (Ministry of Education), School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, China

* **Correspondence:** Email: fireworks1990@126.com.

Abstract: A graph is IC-planar if it admits a drawing with at most one crossing per edge so that each vertex is incident to at most one crossing edge, and an IC-plane graph means such a drawing of an IC-planar graph. We show that any maximal IC-plane graph with n vertices has at least $2n-2$ edges.

Keywords: edge density; maximal IC-plane graph
Mathematics Subject Classification: 05C10

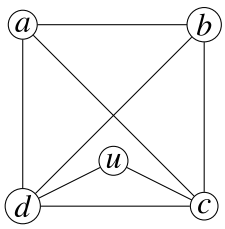
1. Introduction

All graphs considered here are simple and finite. A *drawing* of a graph $G=(V,E)$ is a mapping D that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc connecting $D(u)$ and $D(v)$. An edge is said to *cross* another edge if they intersect at a point that is not a vertex of a graph when it is drawn in the Euclidean plane. A drawing of a graph is 1-*planar* if each of its edges is crossed at most once. If a graph has a 1-planar drawing, then it is 1-*planar*. A 1-*plane* graph means such a drawing of a 1-planar graph. Notably, 1-planar graphs have been widely studied in the literature (see, e.g., [1–3]).

A graph G is said to be *maximal* in a graph class $\mathcal{G}$ if adding any edge to G would cause it to no longer belong to $\mathcal{G}$. It is well known that for every maximal planar graph G of order $n\geq 3$, all of its faces are triangles, and in addition, $|E(G)|=3n-6$. Nevertheless, with respect to the lower bound of the number of edges in maximal 1-plane graphs of a given order, there is still no single, unified formula. 1-planar graphs of order n have at most $4n-8$ edges, and the bound is tight for $n=8$ and all $n\geq 10$ [1]. But there are maximal 1-planar graphs with $\frac{45}{17}n-\frac{84}{17}$ edges [2]. Brandenburg et al. in [2] furthermore proved every maximal 1-plane graph G has at least $\frac{28}{13}n(G)-O(1)\approx 2.15n(G)-O(1)$ edges. Subsequently, Barát and Tóth [3] refined this lower bound. Very recently, Huang, Ouyang, Zhang and Dong [4] determined that for all $n\geq 5$, the minimum number of edges in a maximal 1 - plane graph is $\left\lceil\frac{7}{3}n\right\rceil-3$.

The concept of IC-planar graphs was first introduced by Albertson [5] in 2008. A graph is *IC-planar* (independent crossing planar) [5–7] if it admits a drawing with at most one crossing per edge so that each vertex is incident to at most one crossing edge, and an *IC-plane graph* means such a drawing of an IC-planar graph. In fact, as a subclass of 1-planar graphs, IC-planar graphs have been thoroughly investigated in a multitude of academic papers (see, e.g., [6–8]). A graph is NIC-*planar* if two pairs of crossing edges share at most one vertex. Brandenburg et al. showed that every IC-planar graph has a straight-line drawing in quadratic area. They also proved that testing whether a graph is IC-planar is NP-hard. Zhang and Liu [9] proved that every IC-planar graph with n vertices has at most $3.25n - 6$ edges, and this bound is tight. They also proved that NIC-planar graphs with n vertices have at most $\frac{18}{5}(n-2)$ edges [9, 10]. For the lower bound on the density of NIC-planar graphs, Bachmaier et al. [11] showed that this class of graphs has at least $\frac{16}{5}(n-2)$ edges with n vertices. Outer 1-planar graphs are another subclass of 1-planar graphs. They must admit a 1-planar embedding such that all vertices are in the outer face [12]. Auer et al. [13] proved that the lower bound on the density of outer 1-planar graphs red is $\frac{11}{5}n - \frac{18}{5}$ and the upper bound is $\frac{5}{2}n - 2$. In 2017, Bachmaier et al. [7] asserted that n-vertex maximal IC-planar graphs have at least $3n - 5$ edges, and the bound is tight.

Following the approach in [7], we can show that every face in an IC - drawing of a maximal IC-planar graph is triangular. However, for maximal IC-plane graphs G, not every face is necessarily a triangle, and there may exist vertices of degree two in G. Thus, maximal IC-plane graphs may not be maximal IC-planar graphs. For example, we can draw a maximal IC-plane graph G as in Figure 1. But G cannot be a maximal IC-planar graph since moving the vertex u to the exterior of the region bounded by *abcda* and adding the edges *ua* and *ub* results in a new graph G' that is 1-planar, contradicting the maximality of G. Based on these observations, we establish a smaller lower bound on the size among maximal IC-plane graphs with given orders $n \geq 4$.



G

Figure 1. The drawing of G.

Theorem 1. *Let G be a maximal IC-plane graph with $n \geq 4$ vertices. Then G has at least $2n - 2$ edges.*

2. Preliminaries

In this section, we introduce some basic concepts and terminology. Let G be a graph and v be a vertex of G. A *component* of G is a connected subgraph H such that no subgraph of G that properly contains H is connected. Let $G - v$ or $G - S$ be the subgraph obtained from G by deleting a vertex v or a set of vertices S. A vertex $v \in V(G)$ is called a cut-vertex of a connected graph G if $G - v$ is disconnected. We write $G[T]$ for the *induced subgraph* by the vertex set T in G. K_n is a complete graph with n vertices, where every pair of distinct vertices is connected by an edge. A *spanning subgraph* for G is a subgraph of G that contains every vertex of G.

For any drawing D, the associated plane graph $D^\times$ of D is the plane graph that is obtained from D by turning all crossings of D into new vertices of degree four. An edge e of G is called *non-crossing* if it crosses no other edges under the drawing ϕ, otherwise, it is called a *crossing* edge.

Let G be a 1-plane graph. The edges of G divide the plane into *faces*. A face is bounded by edges and edge segments. The endpoints of these edges and edge segments are vertices or crossings. The *boundary* of a face is the ordered set of edges and edge segments that enclose the face. $\partial(f)$ denotes the boundary of the face f. The *degree* of a face is defined as the total number of edges and edge segments that form the closed walk of its boundary. A face is called a *k-face* if its degree is k. A 1-plane graph G is a *triangulation* if every face of G is of degree 3. On the other hand, a connected component D of G whose boundary contains a crossing point is a *fake face*. The degree of a fake face D is the length of the boundary of $D^\times$.

For other terminology and notations not defined here, we refer to [14].

3. Proof of Theorem 1

Below, we give some auxiliary lemmas.

Lemma 1 ([3]). *Let G be a 1-plane graph with at least two vertices. Then the boundary of each face of G has at least two vertices.*

The following lemma follows directly from the definition of IC-plane graphs.

Lemma 2. *Let G be an IC-plane graph, and let G' be the drawing obtained from G by adding an edge uv from the complement of G. If uv is non-crossing in G', then G' is still an IC-plane.*

Lemma 3. *Let G be a maximal IC-plane graph. If edges uv and xy are crossed with each other, then $G[\{u, v, x, y\}]$ is isomorphic to K_4, where ux,xv,vy,yu are uncrossed edges in G.*

Proof. Assume that uv crosses xy at a point c. If there are two vertices among u,v,x, and y that are not adjacent, namely x and u, then we can obtain G' from G by drawing an uncrossed edge xu, which is infinitely close to the segments xc and uc. By Lemma 2, G' is IC-plane. This contradicts the maximality of G. Similarly, $\{xv, vy, yu\} \subseteq E(G)$, and thus $G[\{u, v, x, y\}] \cong K_4$. Since edges uv and xy cross each other in G, if any of the edges ux, xv, vy, or yu is crossed in G, then G violates IC-planarity, leading to a contradiction. Therefore, ux, xv, vy, yu are uncrossed in G. □

Lemma 4. *Let G be a maximal IC-plane graph. If f is a k-face in G, then* $3 \leq k \leq 4$.

Proof. Suppose there exists a face of degree 2 in G. This would necessarily result in either self-loops or parallel edges, which is impossible. Thus, $k \geq 3$. If $k \geq 5$ and $\partial(f)$ contains at least 5 vertices (denoted sequentially as u_1, u_2, u_3, u_4, u_5), the maximality of G requires that: (1) Any two vertices on $\partial(f)$ must be connected. (2) The edges u_1u_3 and u_2u_4 must cross each other outside f. (3) The edges u_1u_3 and u_2u_5 must cross each other outside f. This configuration would force at least one edge to cross twice, violating the IC-planarity of G. For the remaining cases, if $\partial(f)$ contains at most 3 vertices, it contradicts the IC-planarity condition. If $\partial(f)$ has exactly 4 vertices, they induce a complete graph K_4 in the exterior region, creating exactly one crossing. By IC-planarity, no edges on $\partial(f)$ can be crossed elsewhere. Thus, f must be a true 4-face. Then we conclude that $k \leq 4$.

□

Lemma 5. *Let G be a maximal IC-plane graph with n vertices, c crossings, and f_4 4-faces, then the number of edges in G satisfies $|E(G)| = 3n - 6 + c - f_4$.*

Proof. Let T be the set of faces of G. Let f_3 denote the number of 3-faces in G. We have $\sum_{f \in T} deg(f) = 2|E(G^{\times})|$, and $|V(G^{\times})| - |E(G^{\times})| + f_4 + f_3 = 2$. Substituting $|V(G^{\times})| = n + c$, $|E(G^{\times})| = |E(G)| + 2c$, and when combining with Lemma 4, we derive:

$$3f_3 + 4f_4 = 2|E(G)| + 4c \quad \text{and} \quad n - |E(G)| + f_3 + f_4 = 2 + c.$$

Eliminating f_3, we obtain $|E(G)| = 3n - 6 + c - f_4$. □

Proof of Theorem 1:

Proof. Let G be a maximal IC-plane graph on n vertices. We first give the following claim.

Claim 1. *If $n \geq 5$, then G has at least one crossing point.*

Proof. Suppose that G has no crossing; then, by Lemma 2 and the condition of maximality of G, we may assume that G is a maximal plane graph. Note that every maximal plane graph is triangulated. It is known the minimum degree of every maximal plane graph is at most 5 and at least 3. Thus, G can be embedded in the sphere, with three subdrawings based on the degree of a vertex u_0, denoted as H_1, H_2, and H_3, as shown in Figure 2.

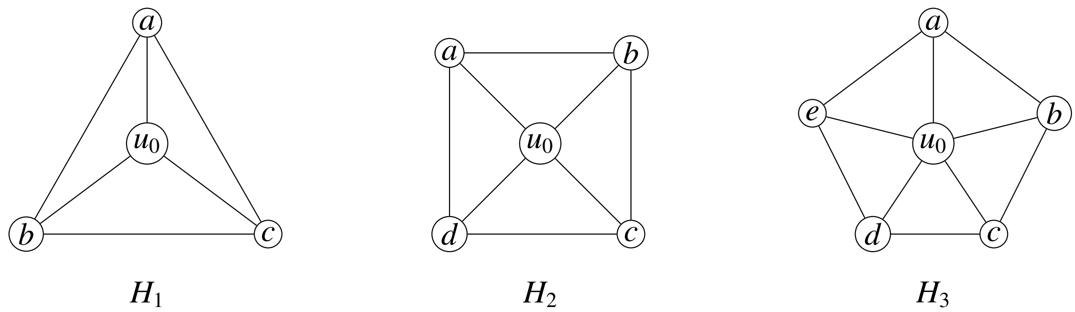


Figure 2. Three subdrwings H_1, H_2 and H_3.

Case 1. *G contains H_1.*

Since $n \geq 5$, we can find a vertex v_1 in H_1 that is not the vertex c such that it, together with a and b, forms a triangular face outside of abc. Now we can draw the edge u_0v_1 in G, and the resulting graph remains an IC graph, a contradiction.

Case 2. *G contains H_2.*

For H_2, the edge ac must exist; otherwise, we could draw the edge ac within the region enclosed by $abcda$ without violating the IC property. Similarly, the edge bd must also exist. Since G does not contain crossings, both ac and bd must connect outside the region enclosed by $abcda$. This leads to a contradiction because they would definitely cross each other, contradicting the assumption that G contains no crossings.

Case 3. *G contains H_3.*

Similar to Case 2, for H_3, we can still add one of the edges ac or bd. This contradicts the maximality of G.

The three aforementioned cases all lead to contradictions. Hence, G has at least one crossing. □

Basis step: We shall apply induction on the number of vertices of G. First, it is not difficult to prove that when $|V(G)| = 4, G$ is isomorphic to K_4, which has 6 edges and satisfies the bound in the theorem. Thus, the base case of induction holds.

Inductive step: Assume that the theorem holds for maximal IC-plane graphs with n' vertices such that $5 \leq n' < n$. By Claim 1 and Lemma 3, G has a K_4-subgraph F that contains a crossing in its drawing, shown in Figure 3. We set v_1, v_2, v_3, v_4 for the four vertices of F, where v_1v_3 crosses v_2v_4. Let S_1 and S_2 be all the vertices within the interior and exterior of the region bounded by $v_1v_2v_3v_4v_1$, respectively.

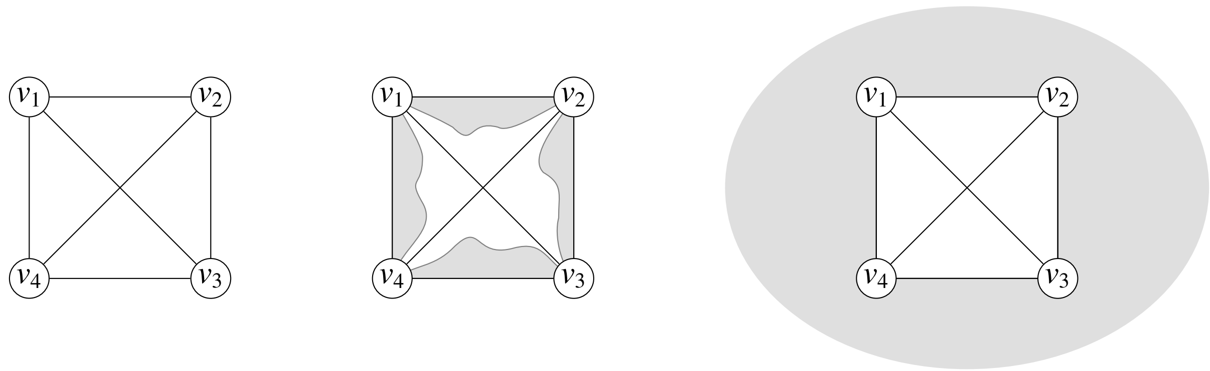


Figure 3. The drawings F, G_1, and G_2.

We next consider three separate cases.

Case 1. S_1 and S_2 are not empty.

In this case, let $G_1 = G[S_1 \cup V(F)]$. Let $G_2 = G[S_2 \cup V(F)]$. Then we have

$$|V(G)| = |V(G_1)| + |V(G_2)| - 4 \quad \text{and} \quad |E(G)| = |E(G_1)| + |E(G_2)| - 6,$$

due to the overlap of 4 vertices and 6 edges between G_1 and G_2 in the subgraph F.

Claim 2. *Both G_1 and G_2 are maximal IC-plane graphs.*

Proof. First, G_1 is an IC-plane graph since it is a subgraph of G. Suppose G_1 is not a maximal IC-plane graph. Then, we can add at least one edge xy to G_1 such that the new graph G_1' remains IC-plane. We assert that xy must lie within the interior of the fake face of F. Otherwise, adding xy would either create a multiple edge or a crossing on the boundary of F, contradicting the IC-planarity of G_1.

Next, we add vertices and edges within the interior of the true face of F (as in G) to obtain a new graph G'. If xy is non-crossing in G_1, then G' is an IC-plane graph by Lemma 2. If xy crosses an edge uv in G_1, then none of the vertices x, y, u, v belong to F (otherwise, G_1 would not be an IC-plane graph). Thus, xy does not cross any edges outside F. If G' were not an IC-plane graph, this would imply G_1 is not an IC-plane graph—a contradiction. Hence, G' is an IC-plane graph.

However, since G is a maximal IC-plane graph, this is impossible. Therefore, G_1 must be a maximal IC-plane graph. Similarly, G_2 is also maximal IC-plane. □

By Claim 2, if both S_1 and S_2 are non-empty, the vertex number of G_1 or G_2 is less than G. By the induction hypothesis, we have $|E(G_1)| \geq 2|V(G_1)| - 2$ and $|E(G_2)| \geq 2|V(G_2)| - 2$. Then

$$\begin{aligned}|E(G)| &= |E(G_1)| + |E(G_2)| - 6 \\ &\geq 2|V(G_1)| - 2 + 2|V(G_2)| - 2 - 6 \\ &= 2(|V(G_1)| + |V(G_2)|) - 10 \\ &= 2(n+4) - 10 \\ &= 2n - 2.\end{aligned}$$

Case 2. S_2 is empty.

In this case, there is at least one of the four regions within the interior of F that has vertices and edges. Let S'_1 denote the set of vertices lying inside the interior of one of the four regions. Let us assume that this region contains vertices v_3 and v_4. Let S'_2 be the vertices within the interior of the other three regions. Let $G'_1 = G[S'_1 \cup V(F)]$. Let $G'_2 = G[S'_2 \cup V(F)]$, shown in Figure 4.

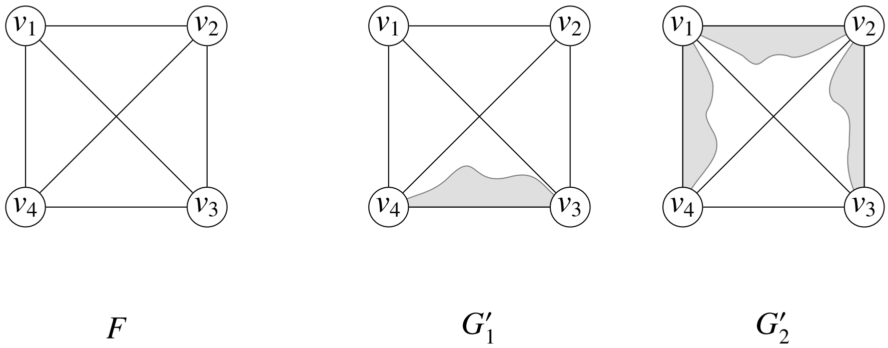


Figure 4. The drawings F, G'_1 and G'_2.

Subcase 2.1. S'_2 is non-empty.

In this case, similar to Claim 2, we have both G'_1 and G'_2 are maximal IC-plane graphs. Both S'_1 and S'_2 are non-empty. Hence, the vertex number of G'_1 or G'_2 is less than G. By the induction hypothesis, we have $|E(G'_1)| \geq 2|V(G'_1)| - 2$ and $|E(G'_2)| \geq 2|V(G'_2)| - 2$.

Since

$$|V(G)| = \left|V\left(G'_1\right)\right| + \left|V\left(G'_2\right)\right| - 4 \quad \text{and} \quad |E(G)| = \left|E\left(G'_1\right)\right| + \left|E\left(G'_2\right)\right| - 6,$$

due to the overlap of 4 vertices and 6 edges between G'_1 and G'_2 in the subgraph F, we have

$$\begin{aligned}|E(G)| &= |E(G'_1)| + |E(G'_2)| - 6 \\ &\geq 2|V(G'_1)| - 2 + 2|V(G'_2)| - 2 - 6 \\ &= 2(|V(G'_1)| + |V(G'_2)|) - 10 \\ &= 2(n+4) - 10 \\ &= 2n - 2.\end{aligned}$$

Subcase 2.2. S'_2 is empty.

Then $G = G'_1$. Let c denote the number of crossings of G. If $c = 1$, $G[S'_1 \cup \{v_3, v_4\}]$ will be a maximal plane graph. Then we have

$$\begin{cases} |E(G[S'_1 \cup \{v_3, v_4\}])| = 3|V(G[S'_1 \cup \{v_3, v_4\}])| - 6, \\ |V(G[S'_1 \cup \{v_3, v_4\}])| = n - 2, \\ |E(G[S'_1 \cup \{v_3, v_4\}])| = |E(G)| - 5. \end{cases} \tag{3.1}$$

Then we have $|E(G)| = 3n - 7 \geq 2n - 2$ with $n \geq 5$.

If $c \geq 2$, then there exists at least a K_4 within the interior of the fake face of F. Any such K_4 must satisfy one of the two configurations illustrated in Figure 5. Moreover, the region R_5 necessarily contains vertices and edges strictly in its interior.

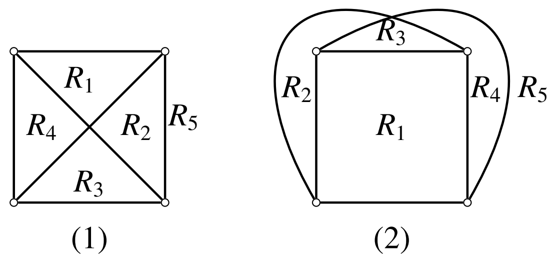


Figure 5. Crossing patterns of K_4 subgraphs in G.

Claim 3. *There are no vertices and edges within the interior of R_i ($i \in 1, 2, 3, 4$).*

Proof. Assuming there exist vertices and edges within the interior of R_i ($i \in 1, 2, 3, 4$), we can: (1) designate the K_4 as F, and (2) invoke the inductive argument of Case 1 or Subcase 2.1 to derive the edge bound $|E(G)| \geq 2n - 2$. □

Let f_4 be the number of the 4-face in G.

Claim 4. *Every boundary of a 4-face contains at most one crossing.*

Proof. Clearly, if the boundary of the 4-face contains at least two crossings, it contradicts the IC-property. □

By Claim 4, every 4-face in G follows two types, as shown in Figure 6.

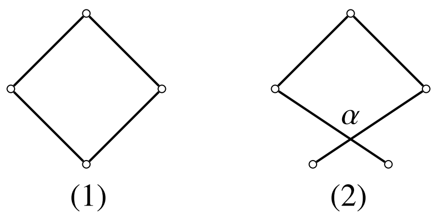


Figure 6. Possible configurations of 4-faces in G.

Claim 5. *Each crossing within the interior of $G[S'_1 \cup \{v_3, v_4\}]$ belongs to the boundary of at most one 4-face.*

Proof. By Claim 4, any fake 4-face in G must follow the configuration illustrated in Figure 6(2). For a K_4 subgraph spanned by crossing α within $G[S_1' \cup \{v_3, v_4\}]$, if it follows Figure 5(1), then no 4-face containing α on its boundary can exist. This is true unless at least one region R_i ($i \in 1, 2, 3, 4$) contains interior vertices or edges; otherwise, Claim 3 would be violated. Conversely, if the K_4 follows Figure 5(2), exactly one such 4-face with α on its boundary may exist within R_5, provided no vertices or edges are present in R_i ($i \in 2, 3, 4$), which is again consistent with Claim 3. □

Claim 6. $f_4 \leq 2c$.

Proof. By Claim 5, if every K_4 follows Figure 5(1) spanned by the new crossings, then f_4 will be exactly 2, one of which comes from the face bounded by v_1, v_2, v_3, v_4, the other face contains the crossing between edges v_1v_3, v_2v_4. In this case, $f_4 = 2 \leq 2c$ when $c \geq 2$. If every K_4 follows Figure 5(2) spanned by the new crossings, then each K_4 generates both a new fake 4-face and a new true 4-face, since R_1 has no vertices and edges in this configuration. In this case, f_4 reaches its maximum value, where $f_4 = 2c$. Then we have $f_4 \leq 2c$. □

By Lemma 5, we have $|E(G)| \geq 3n - 6 + c - 2c = 3n - 6 - c$. Moreover, since G is an IC-plane graph, it holds that $c \leq \frac{n}{4}$. Then $|E(G)| \geq 3n - 6 - \frac{n}{4} = \frac{11}{4}n - 6$. Because of $c \geq 2$, then $n \geq 8$. We can ensure that $|E(G)| \geq \frac{11}{4}n - 6 \geq 2n - 2$, for any $n \geq 8$.

Case 3. S_1 is empty.

If $G[S_2]$ contains no crossings, then G is obtained by adding a single crossing edge to a maximal planar graph, yielding $|E(G)| = 3n - 6 + 1 = 3n - 5 \geq 2n - 2$, for any $n \geq 5$. Let c denote the number of crossings of G. If $c \geq 2$ and each crossing corresponds to a topological K_4 formed by its four incident vertices. These copies of K_4 are vertex-disjoint. If each K_4 contains no other vertices and edges within its own fake face, then $|E(G)| = 3n - 6 + c \geq 2n - 2$ with $n \geq 5$. Therefore, there must exist at least one K_4 whose fake face contains other vertices and edges. If some vertices and edges of G lie in the true face of this K_4, then we can select the corresponding K_4 for the inductive steps, reducing the problem to Case 1 (as S_1 and S_2 become non-empty when this K_4 is designated as F). Alternatively, if all the residual vertices and edges of G reside strictly within the interior of the fake face of this K_4, the scenario becomes identical to Case 2.

Therefore, the theorem holds. □

In this paper, we establish a lower bound on the number of edges for maximal IC-plane graphs. However, determining the tight lower bound on their density remains an open and intriguing question, as the current bound seems to be far from optimal.

4. Conclusions

This study establishes a lower bound on edge density for maximal IC-plane graphs, demonstrating that any such graph with n vertices contains at least $2n - 2$ edges. We rigorously differentiate the edge density properties between maximal IC-plane graphs and maximal IC-planar graphs, highlighting structural constraints in the former that necessitate higher minimal edge counts. Furthermore, we prove key properties of these graphs through combinatorial analysis and mathematical induction. This work provides a foundational benchmark for future research toward identifying the exact tight lower bound on edge density for maximal IC-plane graphs.

Use of Generative-AI tools declaration

We declare that generative AI tools were used solely for the purpose of checking spelling, grammar, and basic syntax errors in this document. These tools were applied as automated proofreading aids to correct surface-level language mechanics. No generative AI was used for any other purpose, including but not limited to: (1) generating or modifying core ideas, arguments, or content; (2) paraphrasing or rewriting substantive text; (3) conducting research or data analysis; (4) creating structural outlines or conceptual frameworks.

The intellectual content, reasoning, and conclusions remain entirely our original work.

Acknowledgments

This paper is supported by the program for excellent talents Hunan Normal University (ET13101).

Conflict of interest

The author declares that there is no conflict of interest.

References

1. Y. Suzuki, Re-embeddings of maximum 1-planar graphs, *SIAM J. Discrete Math.*, **24** (2010), 1527–1540. https://doi.org/10.1137/090746835
2. F. J. Brandenburg, D. Eppstein, A. Gleißner, M. T. Goodrich, K. Hanauer, J. Reislhuber, On the density of maximal 1-planar graphs, In: W. Didimo, M. Patrignani, *Graph drawing*, Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, **7704** (2013), 327–338. https://doi.org/10.1007/978-3-642-36763-2_29
3. J. Barát, G. Tóth, Improvements on the density of maximal 1-planar graphs, *J. Graph Theory*, **88** (2018), 101–109. https://doi.org/10.1002/jgt.22187
4. Y. Huang, Z. Ouyang, L. Zhang, F. Dong, Determining the minimum size of maximal 1-plane graphs, *arXiv Preprint*, 2025. https://doi.org/10.48550/arXiv.2502.11696
5. M. O. Albertson, Chromatic number, independence ratio, and crossing number, *Ars Math. Contemp.*, **1** (2008), 1–6. https://doi.org/10.26493/1855-3974.10.2d0
6. F. J. Brandenburg, W. Didimo, W. S. Evans, P. Kindermann, G. Liotta, F. Montecchiani, Recognizing and drawing IC-planar graphs, *Theor. Comput. Sci.*, **636** (2016), 1–16. https://doi.org/10.1016/j.tcs.2016.04.026
7. C. Bachmaier, F. J. Brandenburg, K. Hanauer, A note on IC-planar graphs, *arXiv Preprint*, 2017. https://doi.org/10.48550/arXiv.1707.08652
8. Y. Wang, Y. Wang, W. Wang, L. Zheng, A note on the list vertex-arboricity of IC-planar graphs, *Discrete Math.*, **347** (2024), 113998. https://doi.org/10.1016/j.disc.2024.113998
9. X. Zhang, G. Liu, The structure of plane graphs with independent crossingsand its application to coloring problems, *Cent. Eur. J. Math.*, **11** (2013), 308–321. https://doi.org/10.2478/s11533-012-0094-7

10. X. Zhang, Drawing complete multipartite graphs on the plane with restrictions on crossings, *Acta Math. Sin.-English Ser.*, **30** (2014), 2045–2053. https://doi.org/10.1007/s10114-014-3763-6

11. C. Bachmaier, F. J. Brandenburg, K. Hanauer, D. Neuwirth, J. Reislhuber, NIC-planar graphs, *Discrete Appl. Math.*, **232** (2017), 23–40. https://doi.org/10.1016/j.dam.2017.08.015

12. S. H. Hong, P. Eades, N. Katoh, G. Liotta, P. Schweitzer, Y. Suzuki, A linear-time algorithm for testing outer-1-planarity, *Algorithmica*, **72** (2015), 1033–1054. https://doi.org/10.1007/s00453-014-9890-8

13. C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, et al., Outer 1-planar graphs, *Algorithmica*, **74** (2015), 1293–1320. https://doi.org/10.1007/s00453-015-0002-1

14. D. B. West, *Introduction to graph theory*, 2 Eds., Prentice Gall, 2001.