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Research article

Robust investment and reinsurance strategies under inflation risk and CEV model

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Abstract: This paper innovatively investigated the optimal investment and reinsurance problem with the objective of minimizing the probability of insolvency in a framework that simultaneously considers the ambiguity aversion perspective and inflation risk. Different from traditional research, this study assumed that insurance companies can not only purchase proportional reinsurance to minimize claims risk but also invest in the financial market, and examined the impact of inflation on the prices of risky assets through discounting. In particular, we incorporated the inflation effect into the pricing of risky assets and proposed a new investment model that takes inflation into account. The surplus process of an insurance company was modeled using a diffusion approximation model, and the financial market consisted of risk-free assets and risky assets, in which the price of risky assets was described by the constant elasticity of variance (CEV) model, which has an advantage in capturing market volatility compared to traditional models. Through the Radon-Nikodym derivative and Girsanov's theorem, we derived the wealth process of an insurance company from the perspective of ambiguity aversion. Based on the principle of dynamic programming, the HJB equation of the optimization problem was established, and the analytical solutions of the robust optimal investment and reinsurance strategies were obtained. Finally, the sensitivity of the model parameters to the theoretical results was analyzed through numerical simulation experiments, which provides quantitative guidance for insurance company's investment and reinsurance decisions in complex economic environments.

Keywords: probability of ruin; ambiguity aversion; inflation; HJB equation; robust investment strategy; robust reinsurance strategy; stochastic optimal control
Mathematics Subject Classification: 62P05, 93E20

1. Introduction

In today's complex and volatile global economic environment, financial institutions and insurance companies are facing unprecedented challenges. As market competition intensifies and financial market volatility increases, how to profitably ensure sound asset growth with an efficient portfolio in the financial market has become a core issue for the insurance industry to address. Reinsurance, as an important tool for risk management, provides an additional safety net for direct insurance companies by diversifying and transferring risks. This enhances their underwriting capacity and avoids the risk of huge losses and greatly reduces the probability of bankruptcy. Therefore, the development and implementation of a set of efficient and optimal investment and reinsurance strategies by insurance companies has become a key research issue.

In the actuarial field of insurance, one of the key objectives that insurance companies are committed to pursuing is to reduce the probability of insolvency, a concept initially proposed by Ferguson [1], who focused on how investors can formulate optimal investment strategies to minimize the probability of insolvency under the discrete time and state models. Subsequently, this perspective was first applied to insurance companies by Browne [2], who assumed that the financial flows of insurance companies follow Brownian motion with drift, allowing them to invest in risky assets and assuming that asset prices in risky markets follow the Black-Sholes model. Utilizing the dynamic programming methods of Fleming-Rishel [3] and Krylov [4], Browne [2] successfully derived an explicit expression for the minimum probability of bankruptcy and the corresponding optimal strategy, which marked a pioneering application of the dynamic programming principle and the theory of HJB equations to the probability of bankruptcy problem. Further, Hipp and Plum [5] investigated the investment strategy of how to minimize bankruptcy probability when an insurer invests in only one risky asset based on the composite Poisson risk model. On the other hand, Yang and Zhang [6] considered the cases in which an insurer can invest in risk-free and risky assets based on the jump-diffusion risk model and, for the different distributions of the claims, solved the problem of minimizing the bankruptcy probability using numerical methods. Promislow and Young [7], on the other hand, focused on the diffusion risk model and explored how to formulate an investment strategy to minimize the probability of bankruptcy of the insurer. Later, Chen et al. [8] introduced the dynamic VaR restriction and delved into the optimal investment and reinsurance strategies of the insurer under the objective of minimizing the probability of insolvency. Bai et al. [9], on the other hand, used a two-dimensional composite Poisson process to describe the dependent risk model and gave the optimal reinsurance strategy of the insurer under the objective of minimizing the probability of insolvency. Azcue and Muler [10] further considered the transaction costs and investigated the optimal investment strategy of the insurer with the objective of minimizing the probability of insolvency.

Predicting the amount of future claims that an insurer may face as well as the market price of risky assets is extremely complex and challenging, involving numerous uncertainties and variables that are difficult to quantify, which makes accurate prognostication extremely difficult. Because of this, insurance companies are usually ambiguity averse and would like to choose the optimal reinsurance and investment strategy, i.e., the robust optimal reinsurance and investment strategy, in order to achieve the goals of transferring claims risk and minimizing the probability of insolvency under the worst insurance and financial market environment. The above works are basically concerned with the case of model parameter determination, but model uncertainty has gradually attracted the attention of scholars in recent years. Maenhout [11] first proposed the concept of robust optimal investment and studied the portfolio problem with model uncertainty. Anderson et al. [12], for the first time, applied the robust optimal control method to study the fuzzy aversion problem. In recent years, many scholars have studied the investment-reinsurance problem with model uncertainty. For example, Chen and Yang

[13] considered the fuzzy aversion of the insurance company in the dependent case and obtained the analytical solutions of the robust optimal reinsurance and investment strategies with the objective of maximizing the expected exponential utility of the terminal wealth. Li et al. [14] considered the optimal investment problem under the influence of inflation with stochastic exit time and derived the analytical solutions. Wang et al. [15] studied the optimal investment reinsurance problem for a fuzzy averse insurer with the objective of minimizing the probability of bankruptcy in the case of proportional reinsurance. Yang [16] studied the robust optimal reinsurance and investment problem under simultaneous consideration of fuzzy aversion and inflation risk. Hu et al. [17] studied the problem of robust optimal reinsurance and investment strategy for insurance companies with time lag.

In portraying the price process of risky assets, most of the literature uses Geometric Brownian Motion (GBM) or its extended form, i.e., it is assumed that the price volatility of risky assets is deterministic, whereas according to Beckers [18], Campbell [19] and others, it was found that price volatility of risky assets is stochastic in financial markets. Cox and Ross [20] proposed the constant-variance elasticity (CEV) model, which is more advantageous in portraying extreme price volatility and asymmetry in the price process of risky assets, as well as the flexibility of application in option pricing and risk management. Wang and Rong [15], among others, studied the optimal investment reinsurance problem of a fuzzy averse insurer under the premise that the price process of a risky asset obeys the CEV model. Wang et al. [22] studied the optimal investment reinsurance strategy of an insurer based on the hopping-diffusion risk model with the objective of expected utility maximization under the CEV model. Wang et al. [22] assumed that the price process of a stock follows the CEV model, and considered the optimal investment reinsurance strategy of an insurer in the case of the jump diffusion risk model in the case of a stock that is not a risky asset under the objective of expected utility maximization. The CEV model, considers the reinsurance investment delay problem of insurers under the mean-variance criterion in a defaultable market. The time-consistent reinsurance investment strategies in the post-default and pre-default situations are explicitly derived through a game-theoretic framework. Deng et al. [23] investigated the robust optimal investment reinsurance problem for fuzzy avoidance insurers (AAI) involving defaulted securities with uncertain model parameters by describing the price process of stocks using the CEV model.

Based on the current state of research, the study of optimal reinsurance and investment strategies for insurance companies under model uncertainty has attracted increasing attention, particularly with the introduction of the Constant Elasticity of Variance (CEV) model to capture volatility in risky asset prices. For example, Hao et al. [26] investigated a fuzzy differential game for reinsurance and investment under the CEV framework, aiming to maximize a weighted expected utility of terminal wealth. Ning et al. [27] studied a delayed stochastic differential investment-reinsurance game between two competing insurers, with the objective of maximizing the utility of their relative terminal surplus. Moreover, Dong et al. [28] considered the correlation between risky asset prices and the claims process, analyzing a nonzero-sum game where each insurer maximizes the expected exponential utility of terminal wealth relative to its competitor.

Although the existing literature has proposed representative methodologies and frameworks, most studies primarily focus on the traditional objective of expected utility maximization. Few works consider the control of bankruptcy probability, which is a core concern in insurance regulation, nor do they systematically account for the impact of inflation on the asset–liability dynamics of insurance companies. Regarding model formulation, while some studies incorporate the Constant Elasticity of

Variance (CEV) model to describe the volatility of risky asset prices, most of them focus solely on the relationship between volatility and asset levels, without embedding inflation risks into both the asset and claims processes. Additionally, although some papers address either model ambiguity or inflation risk, few attempt to integrate both within a unified robust decision-making framework.

To address these gaps, this paper proposes the following innovations and extensions:

(1) From the perspective of minimizing the probability of insolvency, we develop a robust optimization model that aligns more closely with solvency regulation and risk control objectives in practice;

(2) We adopt the CEV model to better capture the volatility structure of risky asset prices, as opposed to the traditional geometric Brownian motion. At the same time, we introduce inflation-adjusted mechanisms into both asset and claims dynamics to reflect the systemic influence of macroeconomic factors on insurers' financial positions;

(3) We construct a novel decision-making framework that couples inflation uncertainty with insurers' ambiguity preferences, thus more accurately capturing behavioral traits in today's uncertain economic environment;

(4) In the numerical analysis, we compare the optimal strategies of both ambiguity-averse and ambiguity-neutral insurers, and conduct sensitivity analysis to illustrate how key factors—such as the degree of ambiguity and inflation volatility—affect investment and reinsurance decisions.

In summary, this study distinguishes itself from existing literature through its unique objective function, integrated inflation modeling, and robust control approach, thus offering new theoretical contributions and practical insights.

The paper is structured as follows: in Section 2, the investment and reinsurance models are established; in Section 3, the robust optimization problem is established by applying the dynamic programming principle under the fuzzy aversion perspective; in Section 4, the stochastic control theory is used to derive the analytical solutions of the robust optimal investment and reinsurance strategies; in Section 5, the influence of the model parameters on the robust optimal investment and reinsurance strategies is explored through numerical experiments; and Section 6. draws conclusions and summarizes the whole paper.

2. Model structure

In this paper, in order to make the described stochastic phenomena closer to reality and to consider the rigor of the model setup in the following sections, all random variables and stochastic processes involved in the paper are assumed to be defined on the complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$. T is the termination moment of the investment and reinsurance, $\{\mathcal{F}_t\}$ is all the information obtained by the insurance company up to moment t, and $\mathbb{P}$ is the probability measure of the reference model. Assume that all stochastic processes are adaptive on this probability space. Trading in the insurance and financial markets is continuous, short selling is allowed, and there are no transaction costs or taxes incurred.

2.1. Insurance market

Assume that the insurance company's claims process obeys Brownian motion with drift.

$$\mathrm{d}C(t) = a\mathrm{d}t - b\mathrm{d}W_1(t), \tag{2.1}$$

where a, b are constants denoting the average claim rate and claim volatility faced by the insurer, respectively, and $W_1(t)$ is a one-dimensional Standard Brownian Motion defined on the complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$. This embodies the stochastic nature of the claims. The insurance company calculates premiums according to the expected value premium principle, then $c_0 = (1+\omega)a$ denotes the premium rate, where $\omega > 0$ is the insurance company's safety loading factor. Without considering reinsurance and investment, according to equation (2.1), the surplus process of the insurance company satisfies the following stochastic differential equation

$$\mathrm{d}R(t) = c_0\mathrm{d}t - \mathrm{d}C(t) = (1+\omega)a\mathrm{d}t - \mathrm{d}C(t) = \omega a\mathrm{d}t + b\mathrm{d}W_1(t). \tag{2.2}$$

This paper assumes that an insurance company can purchase proportional reinsurance to mitigate claim risk. For any time $t \in [0, T]$ (where $T < \infty$), the insurance company retains a proportion $q(t) \in [0, 1]$ of each claim, meaning it is responsible for paying $q(t)$ of the claim amount. The remaining portion, $1 - q(t)$, is covered by the reinsurance company. The insurance company is required to pay a reinsurance premium to the reinsurer, calculated according to the expected value principle. That is, at any time t, the insurer purchases reinsurance at a premium rate of

$$c_1 = (1+\eta)a, \tag{2.3}$$

where η is the safety loading coefficient of the reinsurer. To avoid arbitrage behavior by the insurance company, we stipulate that $\eta > \omega > 0$. Otherwise, the insurance company could transfer all claims to the reinsurer, capturing the difference between the original premium and the reinsurance premium without bearing any risk.

After considering reinsurance strategies, the surplus process $R^q(t)$ of the insurance company satisfies the following equation

$$\mathrm{d}R^q(t) = [c_0 - (1-q(t))c_1]\mathrm{d}t - q(t)\mathrm{d}C(t) = [\omega - \eta(1-q(t))]a\mathrm{d}t + bq(t)\mathrm{d}W_1(t). \tag{2.4}$$

2.2. Financial market

Apart from reinsurance, insurance companies need to invest in financial markets to increase their wealth. Suppose insurance companies can allocate their funds into risk-free assets with price $\{M(t), t > 0\}$ and risky assets with price $\{S(t), t > 0\}$. The price of risk-free assets, $M(t)$, satisfies the ordinary differential equation

$$\mathrm{d}M(t) = \lambda M(t)\,\mathrm{d}t, \tag{2.5}$$

where $\lambda > 0$ represents the interest rate of risk-free assets. To more accurately capture the stochastic changes in risky asset prices, we assume that the price of risky assets $S(t)$ follows the Constant Elasticity of Variance (CEV) model, satisfying the following stochastic differential equation

$$\mathrm{d}S(t) = S(t)\left[\mu\,\mathrm{d}t + \sigma S^{\beta}(t)\,\mathrm{d}W_2(t)\right], \tag{2.6}$$

where $\mu > 0$ represents the return rate of risky assets, σ denotes the volatility of risky asset prices, β is known as the elasticity coefficient, and $\sigma S^{\beta}(t)$ is referred to as the instantaneous volatility. $W_2(t)$ is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. In reality, there may be a certain correlation and mutual influence between the insurance market and the financial market. Therefore, we assume that the Brownian motions $W_1(t)$ and $W_2(t)$ are correlated, with a correlation coefficient of ρ_{12}.

2.3. Inflationary impact

As the economy progresses, the emergence of inflation becomes an inevitable factor in financial markets. This paper, from a theoretical perspective, investigates the impact of inflation on investment, assuming that the inflation rate is stochastic and that the price $L(t)$ at time t satisfies the following stochastic differential equation

$$\mathrm{d}L(t) = L(t)[p\mathrm{d}t + h\mathrm{d}W_3(t)], \tag{2.7}$$

where p and h denote the expected growth rate and expected volatility of the inflation rate, respectively, while $W_3(t)$ represents a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Given that inflation exerts certain influences on both the insurance and financial markets, we assume that $W_1(t)$ and $W_3(t)$, as well as $W_2(t)$ and $W_3(t)$, are correlated, with correlation coefficients ρ_{13} and ρ_{23}, respectively.

Remark 1 (On the correlation between claim risk and inflation)**.** In the literature on optimal investment and reinsurance strategies, the correlations between insurance claim risk and risky assets, as well as between risky assets and inflation risk, have been widely acknowledged and extensively studied. This paper further investigates the correlation between insurance claim risk and inflation risk. Empirical and theoretical findings from previous studies, including Cummins and Outreville [29], Han and Hung [30], and Yang [16], provide strong support for the existence of a significant relationship between the two. Accordingly, the specification of the correlation coefficient ρ_{13} in this study is well grounded in both theoretical rationale and empirical evidence. □

To more comprehensively analyze the impact of inflation risk on the insurer's investment decisions, we incorporate an inflation-adjustment mechanism into the modeling of risky asset prices, similar to the approach adopted in Yang [16]. Specifically, we employ a real price adjustment approach, wherein the nominal price of the risky asset is deflated by a price index to represent its value in real terms, thus reflecting changes in purchasing power.

We define the inflation-adjusted risky asset price as:

$$S_1(t) = \frac{S(t)}{L(t)}, \tag{2.8}$$

where $S(t)$ denotes the nominal price of the risky asset at time t, and $L(t)$ is a price index representing the level of inflation. This formulation implies that $S_1(t)$ represents the real value of the risky asset, adjusted for inflation, and is therefore better suited to capture the true trend of asset prices under inflationary conditions.

By modeling the asset dynamics from a real-price perspective, this adjustment enables us to more accurately assess the effect of inflation on investment outcomes. This is particularly important in high-inflation or inflation-volatile environments, where nominal prices may fail to reflect actual investment value. Using real prices not only enhances the economic interpretability of the model but also more effectively reveals the erosion of returns due to inflation risk.

Accordingly, the introduction of the inflation-adjusted asset price $S_1(t)$ lays a more solid and realistic foundation for our subsequent analysis of optimal reinsurance and investment strategies.

Based on the stochastic differential equations satisfied by $S(t)$ and $L(t)$, by applying Itô's formula to $S_1(t)$, we obtain the following stochastic differential equation satisfied by $S_1(t)$

$$\mathrm{d}S_1(t) = S_1(t)\left[(\mu - p + h^2 - \rho_{23}\sigma S_1^{\beta} h)\mathrm{d}t + \sigma S_1^{\beta}\mathrm{d}W_2(t) - h\mathrm{d}W_3(t)\right]. \tag{2.9}$$

2.4. Wealth dynamics

Next, we consider an insurance company investing in a financial market affected by inflation. Suppose that at time t, the insurance company invests an amount $\pi(t)$ in risky assets, while the remaining funds $R^q(t) - \pi(t)$ are invested in risk-free assets. Assuming the insurance company's investment-reinsurance strategy is denoted as $\alpha = (\pi(t), q(t))$, then after considering reinsurance and investment, the insurer's wealth process satisfies the following stochastic differential equation

$$\begin{aligned} \mathrm{d}R^{\alpha}(t) &= \mathrm{d}R^{q}(t) + \pi(t)\frac{\mathrm{d}S_1(t)}{S_1(t)} + [R^{\alpha}(t) - \pi(t)]\frac{\mathrm{d}M(t)}{M(t)} \\ &= \Big[\lambda R^{\alpha}(t) + \Big(\omega - \eta(1 - q(t))a + \pi(t)(\mu - p + h^2 - \lambda)\Big)\Big]\,\mathrm{d}t \\ &\quad - \pi(t)\rho_{23}\sigma S_1^{\beta} h\mathrm{d}t + bq(t)\mathrm{d}W_1(t) + \pi(t)\Big[\sigma S_1^{\beta}\mathrm{d}W_2(t) - h\mathrm{d}W_3(t)\Big]. \end{aligned} \tag{2.10}$$

2.5. Fuzzy aversion situation

The aforementioned framework represents the traditional investment reinsurance model, wherein the insurer is ambiguity-neutral. Currently, most articles in actuarial science related to minimizing the probability of ruin focus on the optimal investment-reinsurance strategies for ambiguity-neutral insurers (ANIs), implying complete trust in the existing reference measure $\mathbb{P}$ by the insurer. However, in reality, many investors do not trust models described by the reference measure. To better capture this scenario, this paper considers the optimal investment-reinsurance strategies for ambiguity-averse insurers (AAIs). Due to model uncertainty, AAIs seek robust optimal investment-reinsurance strategies across a range of alternative models. The discrepancies between alternative models and the reference model are reflected through transformations between different probability measures. Assuming the current reference probability measure is $\mathbb{P}$, and a particular alternative probability measure is $\mathbb{Q}$, which is equivalent to $\mathbb{P}$, the set of alternative probability measures is denoted as $\boldsymbol{Q} = \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}$.

Based on the aforementioned assumptions, we utilize the Girsanov Theorem to transform probability measures such that the alternative models differ from the reference model solely in terms of their drift functions. According to the Girsanov Theorem, for any $\mathbb{Q} \in \boldsymbol{Q}$, there exists a distortion process.

$$\{\epsilon(t) = (\theta_1(t), \theta_2(t), \theta_3(t)) \,|\, t \in [0, T]\}, \tag{2.11}$$

such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \Lambda^{\epsilon}(T), \tag{2.12}$$

where

$$\Lambda^{\epsilon}(T) = \exp\left\{\int_0^t \sum_{i=1}^{3} \theta_i(s)\,\mathrm{d}W_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^{3} [\theta_i(s)]^2\ \mathrm{d}s\right\}, \tag{2.13}$$

is a P-martingale. If $\mathbb{P} \in \boldsymbol{Q}$, then $\theta_1(t) = \theta_2(t) = \theta_3(t) = 0$.

Definition 1. A stochastic process $\{\epsilon(t) = (\theta_1(t), \theta_2(t), \theta_3(t))|t \in [0, T]\}$ satisfies the following conditions:

1. For any $t \in [0, T]$, $\theta_1(t)$, $\theta_2(t)$, and $\theta_3(t)$ are progressively measurable.
2. For any $t \in [0, T]$, $\theta_1(t) > 0$, $\theta_2(t) > 0$, and $\theta_3(t) > 0$.

3.
$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T\sum_{i=1}^{3}(\theta_i(t))^2\,\mathrm{d}t\right)\right]<\infty.$$

Condition 3 is also known as the Novikov condition.

According to Girsanov's theorem, we present the form of the drift term in the aforementioned model under the equivalent measure $\mathbb{Q}$

$$\begin{cases}\mathrm{d}W_1^{\mathbb{Q}}(t)=\mathrm{d}W_1(t)-\theta_1(t)\,\mathrm{d}t,\\ \mathrm{d}W_2^{\mathbb{Q}}(t)=\mathrm{d}W_2(t)-\theta_2(t)\,\mathrm{d}t,\\ \mathrm{d}W_3^{\mathbb{Q}}(t)=\mathrm{d}W_3(t)-\theta_3(t)\,\mathrm{d}t.\end{cases} \tag{2.14}$$

Under the probability distribution $\mathbb{Q}$, the stochastic processes $W_i^{\mathbb{Q}}(t)$ are standard Brownian motions. Since the correlation coefficient between $W_1(t)$ and $W_2(t)$ is ρ_{12}, the correlation coefficient between $W_2(t)$ and $W_3(t)$ is ρ_{23}, and the correlation coefficient between $W_1(t)$ and $W_3(t)$ is ρ_{13}, it follows that the correlation coefficients between $W_1^{\mathbb{Q}}(t)$ and $W_2^{\mathbb{Q}}(t)$, $W_2^{\mathbb{Q}}(t)$ and $W_3^{\mathbb{Q}}(t)$, and $W_1^{\mathbb{Q}}(t)$ and $W_3^{\mathbb{Q}}(t)$ remain ρ_{12}, ρ_{23}, and ρ_{13}, respectively.

Under the probability distribution $\mathbb{Q}$, the price process of risky assets is given by

$$\begin{aligned}\mathrm{d}S^{\mathbb{Q}}(t)&=S(t)[(\mu\,\mathrm{d}t+\sigma S^{\beta}(t))(\mathrm{d}W_2^{\mathbb{Q}}(t)+\theta_2(t)\,\mathrm{d}t)]\\&=S(t)[(\mu+\sigma S^{\beta}(t)\theta_2(t))\,\mathrm{d}t+\sigma S^{\beta}(t)\,\mathrm{d}W_2^{\mathbb{Q}}(t)].\end{aligned} \tag{2.15}$$

After considering inflation, the price $L^{\mathbb{Q}}(t)$ at time t satisfies the following stochastic differential equation

$$\begin{aligned}\mathrm{d}L^{\mathbb{Q}}(t)&=L(t)\left[(p\,\mathrm{d}t+h)\left(\mathrm{d}W_3^{\mathbb{Q}}(t)+\theta_3(t)\,\mathrm{d}t\right)\right]\\&=L(t)\left[(p+h\theta_3(t))\,\mathrm{d}t+h\mathrm{d}W_3^{\mathbb{Q}}(t)\right].\end{aligned} \tag{2.16}$$

The discounted form of the risky asset price is as follows

$$\begin{aligned}\mathrm{d}S_1^{\mathbb{Q}}(t)&=S_1(t)\left[(\mu-p+h^2-\rho_{23}\sigma S_1^{\beta}h)\mathrm{d}t+\sigma S_1^{\beta}\mathrm{d}W_2(t)-h\mathrm{d}W_3(t)\right]\\&=S_1(t)\left[(\mu-p+h^2-\rho_{23}\sigma S_1^{\beta}h)\mathrm{d}t+\sigma S_1^{\beta}\left(\mathrm{d}W_2^{\mathbb{Q}}(t)+\theta_2(t)\mathrm{d}t\right)-h\left(\mathrm{d}W_3^{\mathbb{Q}}(t)+\theta_3(t)\mathrm{d}t\right)\right]\\&=S_1(t)\left[(\mu-p+h^2-\rho_{23}\sigma S_1^{\beta}h+\sigma S_1^{\beta}\theta_2(t)-h\theta_3(t))\mathrm{d}t+\sigma S_1^{\beta}\mathrm{d}W_2^{\mathbb{Q}}(t)-h\mathrm{d}W_3^{\mathbb{Q}}(t)\right].\end{aligned} \tag{2.17}$$

The wealth process of an insurance company under the $\mathbb{Q}$ measure can be expressed as follows

$$\begin{aligned}\mathrm{d}R^{\mathbb{Q}}(t)=&\left[\lambda R^{\alpha}(t)+(\omega-\eta(1-q(t))a+\pi(t)(\mu-p+h^2-\lambda))\right]\mathrm{d}t\\&-\pi(t)\rho_{23}\sigma S_1^{\beta}h\mathrm{d}t+bq(t)\left(\mathrm{d}W_1^{\mathbb{Q}}(t)+\theta_1(t)\mathrm{d}t\right)\\&+\pi(t)\left[\sigma S_1^{\beta}\left(\mathrm{d}W_2^{\mathbb{Q}}(t)+\theta_2(t)\mathrm{d}t\right)-h\left(\mathrm{d}W_3^{\mathbb{Q}}(t)+\theta_3(t)\mathrm{d}t\right)\right]\\=&\left[\lambda R^{\alpha}(t)+(\omega-\eta(1-q(t))a+\pi(t)(\mu-p+h^2-\lambda))\right.\\&\left.+\theta_1(t)bq(t)+\theta_2(t)\pi(t)\sigma S_1^{\beta}-\theta_3(t)\pi(t)h\right]\mathrm{d}t+bq(t)\mathrm{d}W_1^{\mathbb{Q}}(t)\\&+\pi(t)\sigma S_1^{\beta}\mathrm{d}W_2^{\mathbb{Q}}(t)-\pi(t)h\mathrm{d}W_3^{\mathbb{Q}}(t)-\pi(t)\rho_{23}\sigma S_1^{\beta}h\mathrm{d}t.\end{aligned} \tag{2.18}$$

In order to address model uncertainty, insurance companies adopt alternative models. However, it is crucial that these alternative models do not deviate too far from the reference model. To this end, we introduce the notion of relative entropy, which serves as a penalty for deviations of alternative models from the reference model, thereby allowing us to measure the differences between them.

$$m(\mathbb{Q}_t \mid \mathbb{P}_t) = \mathbb{E}^{\mathbb{Q}}\left[\ln\left(\frac{\mathrm{d}\mathbb{Q}_t}{\mathrm{d}\mathbb{P}_t}\right)\right], \tag{2.19}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under the $\mathbb{Q}$ measure. Substitute (2.11) (2.13) into (2.18), we have

$$\begin{aligned} m(\mathbb{Q}_t \mid \mathbb{P}_t) &= \mathbb{E}^{\mathbb{Q}}\left[\ln\left(\frac{\mathrm{d}\mathbb{Q}_t}{\mathrm{d}\mathbb{P}_t}\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left\{\int_0^t \sum_{i=1}^{3} \theta_i(s)\,\mathrm{d}W_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^{3} [\theta_i(s)]^2\,\mathrm{d}s\right\} \\ &= \mathbb{E}^{\mathbb{Q}}\left\{\int_0^t \sum_{i=1}^{3} \theta_i(s)\,\mathrm{d}W_i^{\mathbb{Q}}(s) + \frac{1}{2}\int_0^t \sum_{i=1}^{3} [\theta_i(s)]^2\,\mathrm{d}s\right\} \\ &= \mathbb{E}^{\mathbb{Q}}\left\{\frac{1}{2}\int_0^t \sum_{i=1}^{3} [\theta_i(s)]^2\,\mathrm{d}s\right\}. \end{aligned} \tag{2.20}$$

In order to obtain the optimal reinsurance and investment strategies in the following context, we present the definition of admissible strategies.

Definition 2 (Admissible Strategy). A strategy $\alpha = (\pi(t), q(t))$ is deemed feasible, or admissible, if:

1. For any $t > 0$, $(\pi(t), q(t))$ is $\mathcal{F}_t$-measurable.
2. For any $t > 0$, $q(t) > 0$; and

$$E^{\mathbb{Q}}\left[\int_0^{\infty} \pi^2(t)S^2(t)\beta\,\mathrm{d}t\right] < \infty, \quad E^{\mathbb{Q}}\left[\int_0^{\infty} q^2(t)\,\mathrm{d}t\right] < \infty.$$

3. $(\pi(t), q(t))$ ensures that Eq (2.9) has a unique strong solution.

The set of all admissible strategies is denoted by $\mathcal{A}$.

3. The problem of robust optimization

In this paper, we investigate the robust optimization problem of minimizing the probability of insolvency under model uncertainty. To begin with, we define the moment of insolvency for the insurance company as

$$\tau_\alpha = \inf\{t : R(t) < c\}, \tag{3.1}$$

which represents the first time the insurer's surplus $R(t)$ falls below a critical threshold c under control strategy α.

Given an initial level of wealth r, the probability of bankruptcy is denoted by

$$v_\alpha(r, s) = \mathbb{P}(\tau_\alpha < \infty \mid R(0) = r, S_1(0) = s). \tag{3.2}$$

The insurer's optimization objective is to minimize this probability over all admissible strategies

$$v(r,s) = \inf_{\alpha\in\mathcal{A}} v_\alpha(r,s). \tag{3.3}$$

Under model uncertainty, the time of insolvency under an alternative probability measure $\mathbb{Q} \in \mathcal{Q}$ is given by

$$\tau_\alpha^{\mathbb{Q}} = \inf\left\{t : R^{\mathbb{Q}}(t) < c\right\}. \tag{3.4}$$

Then, the corresponding probability of insolvency becomes

$$v_\alpha^{\mathbb{Q}}(r,s) = \mathbb{Q}\left(\tau_\alpha^{\mathbb{Q}} < \infty \mid R^{\mathbb{Q}}(0) = r, S_1{}^{\mathbb{Q}}(0) = s\right). \tag{3.5}$$

The form of the robust optimal value function is shown below

$$v^{\mathbb{Q}}(r,s) = \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ v_\alpha^{\mathbb{Q}}(r,s) - E_{t,r,s}^{\mathbb{Q}}\left[\int_t^{\tau_\alpha^{\mathbb{Q}}} \Phi\left(v^{\mathbb{Q}}(R^{\mathbb{Q}}(u),s)\right) \times \frac{1}{2}\sum_{i=1}^{3}\xi_i\left[\theta_i(u)\right]^2 \mathrm{d}u\right]\right\}, \tag{3.6}$$

where

$$\Phi\left(v^{\mathbb{Q}}\left(R^{\mathbb{Q}}(t),s\right)\right) = v^{\mathbb{Q}}\left(R^{\mathbb{Q}}(t),s\right),$$

$$E_{t,r,s}^{\mathbb{Q}}[\cdot] = E^{\mathbb{Q}}\left[\cdot \mid R^{\mathbb{Q}}(t) = r, S_1{}^{\mathbb{Q}}(t) = s\right], r \geq c.$$

Following the approach in Liu and Zhou et al. [25], we introduce a normalized penalty function $\Phi(\cdot) > 0$, which ensures the penalty term is of the same order of magnitude as the robust value function $v^{\mathbb{Q}}(r)$. This normalization enhances interpretability and avoids scaling issues in the optimization process.

The integral interval $\left[t, \tau_\alpha^{\mathbb{Q}}\right]$ reflects the idea that model uncertainty is considered only before insolvency occurs. The parameters $\xi_i > 0\,(i = 1,2,3)$ capture the insurer's level of ambiguity aversion toward different sources of uncertainty, including insurance risk, financial market fluctuations, and inflation risk.

4. Model solution

The purpose of this section is to derive a general framework for the optimization problem (3.6) and then to find the optimal reinsurance and investment strategies, where we define the generating element of equation (2.17) as follows

$$\begin{aligned}\mathcal{G}_\alpha f(r,t) = &\left[\lambda R^\alpha(t) + (\omega - \eta(1-q(t))a + \pi(t)(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^\beta h)\right.\\ &+ \theta_1(t)bq(t) + \theta_2(t)\pi(t)\sigma S_1^\beta - \theta_3(t)\pi(t)h)\left.\right] f_r(r,t) + f_t(r,t)\\ &+ \frac{1}{2}\left[b^2q^2(t) + \pi^2(t)\sigma^2 S_1^{2\beta} + \pi^2(t)h^2 + 2\rho_{12}b\sigma S_1^\beta q(t)\pi(t)\right.\\ &\left.-2\rho_{13}bhq(t)\pi(t) - 2\rho_{23}\sigma S_1^\beta h\pi^2(t)\right] f_{rr}(r,t).\end{aligned} \tag{4.1}$$

Define $C^{2,2}(\mathrm{R}\times\mathrm{R}) := \{f(r,t) \mid f(r,\cdot)$ as a second-order continuously derivable on R with respect to r, and $f(\cdot,t)$ as a second-order continuously derivable on R with respect to $t\}$.

The verification theorem is given as follows based on dynamic programming principles.

Theorem 1 (Verification Theorem): If there exists a function $V(r,t) \in C^{2,2}(\mathrm{R} \times \mathrm{R})$ satisfying

$$\inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathcal{G}_\alpha V(r,t) - \frac{1}{2}\xi_i \theta_i^2(t) V(r,t) \right\} = 0, \tag{4.2}$$

and satisfying the boundary conditions

$$V(c,t) = 1, V(\infty,t) = 0. \tag{4.3}$$

Then the value function $v^{\mathbb{Q}}(r,t)$ defined by equation (3.6) satisfies

$$v^{\mathbb{Q}}(r,t) = V(r,t). \tag{4.4}$$

The optimal investment reinsurance strategy and the worst-case measure (α^*, ϵ^*) satisfy the

$$(\alpha^*, \epsilon^*) = \arg \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathcal{G}_\alpha V(r,t) - \frac{1}{2}\xi_i \theta_i^2(t) V(r,t) \right\} = 0. \tag{4.5}$$

Proof. If there exists a function $V(r,t) \in C^{2,2}(\mathrm{R} \times \mathrm{R})$ that satisfies the HJB equation (4.2) and the boundary conditions $v(c,T) = 1, v(\infty,T) = 0$, and further satisfies the concavity condition $\frac{\partial^2 v(r,t)}{\partial x^2} < 0$, then we can conclude that $v(r,t) = V(r,t)$.

Assume that $\left(\theta_1^*(t), \theta_2^*(t), \theta_3^*(t)\right)$ attains the sup term in the HJB equation (4.2). Then, there exists a corresponding probability measure $\mathbb{Q}^*$ associated with this choice of $\theta_i^*(t)$. Under this measure, we apply Itô's formula to the process $v(R(t),t)^\alpha_{\mathbb{Q}^*}$.

$$\begin{aligned} v(R(T),T)^\alpha_{\mathbb{Q}^*} = v(R(t),t)^\alpha_{\mathbb{Q}^*} + \int_t^T \mathcal{G}_\alpha v(R(z),z)^\alpha_{\mathbb{Q}^*} \mathrm{d}z + \int_t^T bq(z)\,\mathrm{d}W_1^{\mathbb{Q}^*}(z) \\ + \int_t^T \pi(z)\sigma S_1^\beta \mathrm{d}W_2^{\mathbb{Q}^*}(z) + \int_t^T \pi(z) h \mathrm{d}W_3^{\mathbb{Q}^*}(z). \end{aligned} \tag{4.6}$$

Taking both sides of Eq (4.6) together with the conditional expectation $E_{r,t}^{\mathbb{Q}^*}[\cdot]$ yields

$$E_{r,t}^{\mathbb{Q}^*}\left[v(R(T),T)^\alpha_{\mathbb{Q}^*}\right] = v(r,t) + E_{r,t}^{\mathbb{Q}^*}\left[\int_t^T \mathcal{G}_\alpha v(R(z),z)^\alpha_{\mathbb{Q}^*} \mathrm{d}z\right]. \tag{4.7}$$

From the fact that $v(r,t)$ satisfies the HJB equation (4.2), there are

$$\begin{aligned} 0 &= \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathcal{G}_\alpha v(R(z),z)^\alpha_{\mathbb{Q}} - \frac{1}{2}\xi_i \theta_i^2(t) v(R(z),z)^\alpha_{\mathbb{Q}} \right\} \\ &\le \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathcal{G}_\alpha v(R(z),z)^\alpha_{\mathbb{Q}} - \frac{1}{2}\xi_i \theta_i^2(t) v(R(z),z)^\alpha_{\mathbb{Q}} \right\} \\ &= \mathcal{G}_\alpha v(R(z),z)^\alpha_{\mathbb{Q}^*} - \frac{1}{2}\xi_i [\theta_i^*(t)]^2 v(R(z),z)^\alpha_{\mathbb{Q}^*}. \end{aligned} \tag{4.8}$$

Substituting Eq (4.8) into Eq (4.7) yields

$$E_{r,t}^{\mathbb{Q}^*}\left[v(R(T),T)^\alpha_{\mathbb{Q}^*}\right] \ge v(r,t) + E_{r,t}^{\mathbb{Q}^*}\left[\int_t^T \frac{1}{2}\xi_i [\theta_i^*(t)]^2 v(R(z),z)^\alpha_{\mathbb{Q}^*} \mathrm{d}z\right]. \tag{4.9}$$

Thus it can be concluded that

$$\begin{aligned} v\left(r,t\right) &\leq E_{r,t}^{\mathbb{Q}^*}\left[v(R\left(T\right),T)_{\mathbb{Q}^*}^{\alpha}\right] - E_{r,t}^{\mathbb{Q}^*}\left[\int_t^T \frac{1}{2}\xi_i[\theta_i^*\left(t\right)]^2 v(R\left(z\right),z)_{\mathbb{Q}^*}^{\alpha}\mathrm{d}z\right] \\ &\leq \sup_{\mathbb{Q}\in Q} E_{r,t}^{\mathbb{Q}}\left[v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha} - \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\mathrm{d}z\right]. \end{aligned} \tag{4.10}$$

Since Eq (4.10) holds for all $\alpha = (\pi\left(t\right), q\left(t\right))$, then

$$v\left(r,t\right) \leq \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in Q} E_{r,t}^{\mathbb{Q}}\left[v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha} - \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\mathrm{d}z\right]. \tag{4.11}$$

From $v\left(r,T\right) = 1$, the following can be obtained

$$v\left(r,t\right) \leq \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in Q} E_{r,t}^{\mathbb{Q}}\left[1 - \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\mathrm{d}z\right] = V\left(r,t\right). \tag{4.12}$$

On the other hand, assuming that $(\pi^*\left(t\right), q^*\left(t\right))$ satisfies the inf term of the HJB equation (4.2), for $v(R\left(t\right),t)_{\mathbb{Q}^*}^{\alpha}$, and using Itô's formula, we get

$$\begin{aligned} v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha^*} &= v(R\left(t\right),t)_{\mathbb{Q}}^{\alpha^*} + \int_t^T \mathcal{G}_{\alpha^*}v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha^*}\mathrm{d}z + \int_t^T bq^*\left(z\right)\mathrm{d}W_1^{\mathbb{Q}}\left(z\right) \\ &\quad + \int_t^T \pi^*\left(z\right)\sigma S_1^{\beta}\mathrm{d}W_2^{\mathbb{Q}}\left(z\right) + \int_t^T \pi^*\left(z\right)h\mathrm{d}W_3^{\mathbb{Q}}\left(z\right). \end{aligned} \tag{4.13}$$

From the HJB equation in Eq (4.2), we get

$$\begin{aligned} 0 &= \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in Q}\left\{\mathcal{G}_{\alpha}v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha} - \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\right\} \\ &\geq \inf_{\alpha\in\mathcal{A}}\left\{\mathcal{G}_{\alpha}v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha} - \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\right\} \\ &= \mathcal{G}_{\alpha^*}v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha^*} - \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha^*}. \end{aligned} \tag{4.14}$$

The association of Eqs (4.13) and (4.14) yields

$$\begin{aligned} v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha^*} \leq{}& v(R\left(t\right),t)_{\mathbb{Q}}^{\alpha^*} + \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha^*}\mathrm{d}z + \int_t^T bq^*\left(z\right)\mathrm{d}W_1^{\mathbb{Q}}\left(z\right) \\ &+ \int_t^T \pi^*\left(z\right)\sigma S_1^{\beta}\mathrm{d}W_2^{\mathbb{Q}}\left(z\right) + \int_t^T \pi^*\left(z\right)h\mathrm{d}W_3^{\mathbb{Q}}\left(z\right). \end{aligned} \tag{4.15}$$

Taking both sides of Eq (4.15) simultaneously with the conditional expectation $E_{r,t}^{\mathbb{Q}^*}\left[\cdot\right]$ yields

$$\begin{aligned} v\left(r,t\right) &\geq E_{r,t}^{\mathbb{Q}}\left[v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha^*} - \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha^*}\mathrm{d}z\right] \\ &= \inf_{\alpha\in\mathcal{A}} E_{r,t}^{\mathbb{Q}}\left[v(R\left(T\right),T)_{\mathbb{Q}}^{\alpha} - \int_t^T \frac{1}{2}\xi_i\theta_i^2\left(t\right)v(R\left(z\right),z)_{\mathbb{Q}}^{\alpha}\mathrm{d}z\right]. \end{aligned} \tag{4.16}$$

By the fact that Eq (4.16) is satisfied for all $\mathbb{Q} \in Q$, it follows that

$$v(r,t) \geq \inf_{\alpha\in\mathcal{A}} \sup_{\mathbb{Q}\in Q} E_{r,t}^{\mathbb{Q}} \left[v(R(T),T)_{\mathbb{Q}}^{\alpha} - \int_t^T \frac{1}{2}\xi_i\theta_i^2(t)\, v(R(z),z)_{\mathbb{Q}}^{\alpha} \mathrm{d}z \right] = V(r,t). \tag{4.17}$$

The theorem is successfully proved. □

The main objective of this section is to obtain a robust optimal reinsurance and investment strategy for problem (3.6) by simply solving the HJB equation (4.2).

Before giving the solution of problem (3.6), the following notation is first defined:

$$\begin{cases} \gamma_1(t) = \xi_1\eta a e^{-\lambda(T-t)} \\ \gamma_2(t) = \xi_2\xi_3\left(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta} h\right) e^{-\lambda(T-t)} \\ \Gamma = \xi_3 l\sigma^2 S_1^{2\beta}(1+\xi_2) + \xi_2 l h^2(1+\xi_3) - 2\xi_2\xi_3 l\rho_{23}\sigma S_1^{\beta} h \end{cases} \tag{4.18}$$

Theorem 2. In practical economic settings, the parameters $\xi_1, \xi_2, \xi_3, l, b, \eta, a$ are inherently positive due to their underlying economic interpretations—such as risk aversion coefficients, premium loading factors, and return ratios. Beyond these intrinsic properties, we impose the following economically reasonable assumptions: the net return of the risky asset exceeds the inflation rate and associated risk penalties,

$$\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta} h > 0,$$

and the correlation-weighted volatility spread between the asset and the inflation index remains positive,

$$\rho_{12}\sigma S_1^{\beta} - \rho_{13} h > 0.$$

Additionally, the auxiliary function Γ is assumed to satisfy the following inequality:

$$\Gamma > \frac{\xi_1 l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2\xi_2\xi_3}{b(1+\xi_1)}.$$

Under the above economic rationale and technical assumptions, if the value function in Problem (3.6) admits the form given in Eq (4.6), then the solution to Problem (3.6) is explicitly characterized. The optimal investment–reinsurance strategy associated with the AAI under the objective of minimizing bankruptcy probability is given by:

$$\alpha^* = (\pi^*(t), q^*(t)), \tag{4.19}$$

where the optimal investment strategy $\pi^*(t)$ satisfies

$$\pi^*(t) = \frac{\xi_1 lb\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)\gamma_1(t) + lb^2(1+\xi_1)\gamma_2(t)}{bl\left\{b(1+\xi_1)\Gamma - \xi_1 l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\xi_2\xi_3\right\}}, \tag{4.20}$$

the optimal reinsurance strategy $q^*(t)$ satisfies

$$q^*(t) = \frac{\Gamma\gamma_1(t) + \xi_2\xi_3 lb\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)\gamma_2(t)}{bl\left\{b(1+\xi_1)\Gamma - \xi_1 l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\xi_2\xi_3\right\}}. \tag{4.21}$$

The form of the corresponding optimal value function satisfies

$$v^{\mathbb{Q}}(r,s) = -\frac{1}{l}e^{-l\left[Re^{\lambda(T-t)}+f(t)\right]}, \tag{4.22}$$

where

$$f(t) = \int_t^T \left[\omega a e^{\lambda(T-s)} + G\left(\pi^*(s), q^*(s)\right)\right] \mathrm{d}s. \tag{4.23}$$

The worst case measure is

$$\epsilon^*(t) = \left(\theta_1^*(t), \theta_2^*(t), \theta_3^*(t)\right), \tag{4.24}$$

where

$$\begin{cases} \theta_1^*(t) = -\frac{bq(t)le^{\lambda(T-t)}}{\xi_1} \\ \theta_2^*(t) = -\frac{\sigma S_1^{\beta}\pi(t)le^{\lambda(T-t)}}{\xi_2} \\ \theta_3^*(t) = \frac{h\pi(t)le^{\lambda(T-t)}}{\xi_3} \end{cases}. \tag{4.25}$$

Remark 2 (Admissibility of the Optimal Reinsurance Strategy $q^*(t)$)**.** Under appropriate economic and parameter conditions, the optimal reinsurance proportion $q^*(t)$, given by

$$q^*(t) = \frac{\Gamma\gamma_1(t) + \xi_2\xi_3 lb\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)\gamma_2(t)}{bl\left\{b(1+\xi_1)\Gamma - \xi_1 l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\xi_2\xi_3\right\}},$$

is strictly positive for all $t \in [0, T]$, i.e., $q^*(t) > 0$, and hence admissible as required by Definition 2.1.

Consider the auxiliary functions:

$$\begin{cases} \gamma_1(t) = \xi_1\eta a e^{-\lambda(T-t)} \\ \gamma_2(t) = \xi_2\xi_3\left(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h\right)e^{-\lambda(T-t)} \\ \Gamma = \xi_3 l\sigma^2 S_1^{2\beta}(1+\xi_2) + \xi_2 lh^2(1+\xi_3) - 2\xi_2\xi_3 l\rho_{23}\sigma S_1^{\beta}h \end{cases}$$

Let $E = e^{-\lambda(T-t)} > 0$. Then the numerator of $q^*(t)$ becomes

$$\begin{aligned} \text{Numerator} &= \Gamma\gamma_1(t) + \xi_2\xi_3 lb(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)\gamma_2(t) \\ &= E\left[\Gamma\xi_1\eta a + (\xi_2\xi_3)^2 lb(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h)\right]. \end{aligned}$$

Since $E > 0$, it suffices to show the expression inside the brackets is positive. Assume the following economically reasonable conditions:

$$\xi_1, \xi_2, \xi_3, l, b, \eta, a > 0,$$

$$\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h > 0, \quad \rho_{12}\sigma S_1^{\beta} - \rho_{13}h > 0.$$

Then, each term in the brackets is strictly positive, ensuring that the numerator is positive.

Next, consider the denominator:

$$\text{Denominator} = bl\left\{b(1+\xi_1)\Gamma - \xi_1 l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\xi_2\xi_3\right\}.$$

Since $b > 0$, it suffices that

$$b\,(1+\xi_1)\,\Gamma > \xi_1 l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2\xi_2\xi_3,$$

or equivalently,

$$\Gamma > \frac{\xi_1 l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2\xi_2\xi_3}{b\,(1+\xi_1)}.$$

Substituting the expression of Γ:

$$\Gamma = l\left[\xi_3\sigma^2 S_1^{2\beta}(1+\xi_2) + \xi_2 h^2(1+\xi_3) - 2\xi_2\xi_3\rho_{23}\sigma S_1^{\beta}h\right],$$

we observe that as long as the negative third term is not dominant (e.g., when ρ_{23} is moderate), $\Gamma > 0$ holds, and thus the denominator is positive.

Thus it can be determined that the optimal reinsurance policy $q^*(t)$ is strictly positive for all $t \in [0, T]$. □

Proof. First, the fuzzy parameters of the inner layer are determined, and the first-order optimality condition is applied to Eq (4.2) $\theta_i\,(t)\,(i = 1, 2, 3)$, respectively

$$\begin{cases} \theta_1^*\,(t) = \frac{bq(t)V_r(r,t)}{\xi_1 V(r,t)} \\ \theta_2^*\,(t) = \frac{\sigma S_1^{\beta}\pi(t)V_r(r,t)}{\xi_2 V(r,t)} \\ \theta_3^*\,(t) = -\frac{h\pi(t)V_r(r,t)}{\xi_3 V(r,t)} \end{cases}. \tag{4.26}$$

Substituting Eq (4.26) into Eq (4.2) yields

$$\begin{aligned} &\inf_{\alpha\in\mathcal{A}}\Big\{V_t\,(r,t) + \left[\lambda R^{\alpha}\,(t) + (\omega - \eta\,(1-q\,(t))\,a + \pi\,(t)\left(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h\right)\right] V_r\,(r,t) \\ &+ \frac{1}{2}\left[b^2q^2\,(t) + \pi^2\,(t)\,\sigma^2 S_1^{2\beta} + \pi^2\,(t)\,h^2 + 2\rho_{12}b\sigma S_1^{\beta}q\,(t)\,\pi\,(t) - 2\rho_{13}bhq\,(t)\,\pi\,(t) - 2\rho_{23}\sigma S_1^{\beta}h\pi^2\,(t)\right] V_{rr}\,(r,t) \\ &+ \frac{1}{2}\left(\frac{b^2q^2\,(t)}{\xi_1} + \frac{\pi^2\,(t)\,\sigma^2 S_1^{2\beta}}{\xi_2} + \frac{\pi^2\,(t)\,h^2}{\xi_3}\right)\frac{V_r^2\,(r,t)}{V\,(r,t)}\Big\} = 0. \end{aligned} \tag{4.27}$$

To ensure analytical tractability and capture the diminishing marginal effect of wealth on insolvency risk, we adopt an exponential specification for the candidate value function. This functional form is well-supported by prior studies in robust control and insurance optimization, such as Liu and Zhou et al. [25] and Yang [16], which have employed similar structures in the context of ruin probability minimization. The exponential form not only reflects the insurer's risk preferences but also facilitates the derivation of explicit solutions to the associated HJB equation.

$$V\,(r,t) = -\frac{1}{l}e^{-l\left[Re^{\lambda(T-t)} + f(t)\right]}, \tag{4.28}$$

$f\,(t)$ is a function, whose exact form will be determined later, which satisfies the boundary conditions $f\,(T) = 0$. The partial derivatives of $V\,(r,t)$ with respect to each variable are

$$\begin{cases} V_t\,(r,t) = \left[\lambda Re^{\lambda(T-t)} - f'\,(t)\right] lV\,(r,t) \\ V_r\,(r,t) = -le^{\lambda(T-t)}V\,(r,t) \\ V_{rr}\,(r,t) = l^2e^{2\lambda(T-t)}V\,(r,t) \end{cases}. \tag{4.29}$$

Substituting Eq (4.29) into Eq (4.27) yields

$$\inf_{\alpha\in\mathcal{A}}\Big\{\Big[\lambda Re^{\lambda(T-t)}-f'(t)\Big]lV(r,t)-\Big[\lambda R^{\alpha}(t)+(\omega-\eta(1-q(t))a+\pi(t)\Big(\mu-p+h^2-\lambda-\rho_{23}\sigma S_1^{\beta}h\Big)\Big]$$
$$le^{\lambda(T-t)}V(r,t)+\frac{1}{2}\Big[b^2q^2(t)+\pi^2(t)\sigma^2S_1^{2\beta}+\pi^2(t)h^2+2\rho_{12}b\sigma S_1^{\beta}q(t)\pi(t)-2\rho_{13}bhq(t)\pi(t)-2\rho_{23}\sigma S_1^{\beta}h\pi^2(t)\Big]$$
$$l^2e^{2\lambda(T-t)}V(r,t)+\frac{1}{2}\left(\frac{b^2q^2(t)}{\xi_1}+\frac{\pi^2(t)\sigma^2S_1^{2\beta}}{\xi_2}+\frac{\pi^2(t)h^2}{\xi_3}\right)\frac{V_r^2(r,t)}{V(r,t)}\Big\}=0, \tag{4.30}$$

thus

$$\begin{aligned}\inf_{\alpha\in\mathcal{A}}\Big\{&-f'(t)-\Big[\Big(\omega-\eta(1-q(t))a+\pi(t)(\mu-p+h^2-\lambda-\rho_{23}\sigma S_1^{\beta}h)\Big)\Big]e^{\lambda(T-t)}\\&+\frac{l}{2}\Big[b^2q^2(t)+\pi^2(t)\sigma^2S_1^{2\beta}+\pi^2(t)h^2+2\rho_{12}b\sigma S_1^{\beta}q(t)\pi(t)-2\rho_{13}bhq(t)\pi(t)-2\rho_{23}\sigma S_1^{\beta}h\pi^2(t)\Big]e^{2\lambda(T-t)}\\&+\frac{l}{2}\Big(\frac{b^2q^2(t)}{\xi_1}+\frac{\pi^2(t)\sigma^2S_1^{2\beta}}{\xi_2}+\frac{\pi^2(t)h^2}{\xi_3}\Big)e^{2\lambda(T-t)}\Big\}=0.\end{aligned} \tag{4.31}$$

The collation leads to

$$\inf_{\alpha\in\mathcal{A}}\Big\{-f'(t)+(\eta-\omega)ae^{\lambda(T-t)}+G(\pi(t),q(t))\Big\}, \tag{4.32}$$

where

$$\begin{aligned}G(\pi(t),q(t))&=-\eta aq(t)e^{\lambda(T-t)}-\pi(t)\Big(\mu-p+h^2-\lambda-\rho_{23}\sigma S_1^{\beta}h\Big)e^{\lambda(T-t)}\\&+\frac{l}{2}\Big(b^2q^2(t)+\pi^2(t)\sigma^2S_1^{2\beta}+\pi^2(t)h^2\Big)e^{2\lambda(T-t)}\\&+\frac{l}{2}\left(\frac{b^2q^2(t)}{\xi_1}+\frac{\pi^2(t)\sigma^2S_1^{2\beta}}{\xi_2}+\frac{\pi^2(t)h^2}{\xi_3}\right)e^{2\lambda(T-t)}\\&+l\pi(t)\Big(\rho_{12}b\sigma S_1^{\beta}q(t)-\rho_{13}bhq(t)-\rho_{23}\sigma S_1^{\beta}h\pi(t)\Big)e^{2\lambda(T-t)}.\end{aligned} \tag{4.33}$$

The partial derivatives of $G(\pi(t),q(t))$ with respect to $\pi(t)$ and $q(t)$, respectively, are given by

$$\begin{cases}\frac{\partial G(\pi(t),q(t))}{\partial q(t)}=-\eta ae^{\lambda(T-t)}+\frac{lb^2q(t)(1+\xi_1)e^{2\lambda(T-t)}}{\xi_1}+lb\pi(t)\Big(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\Big)e^{2\lambda(T-t)}\\ \frac{\partial G(\pi(t),q(t))}{\partial\pi(t)}=-\Big(\mu-p+h^2-\lambda-\rho_{23}\sigma S_1^{\beta}h\Big)e^{\lambda(T-t)}+\frac{l\sigma^2S_1^{2\beta}\pi(t)(1+\xi_2)e^{2\lambda(T-t)}}{\xi_2}\\ +\frac{lh^2\pi(t)(1+\xi_3)e^{2\lambda(T-t)}}{\xi_3}+l\Big(\rho_{12}b\sigma S_1^{\beta}q(t)-\rho_{13}bhq(t)-2\rho_{23}\sigma S_1^{\beta}h\pi(t)\Big)e^{2\lambda(T-t)}\end{cases}. \tag{4.34}$$

Letting the above two partial derivatives be 0, we obtain the following system of linear equations for $q(t)$ and $\pi(t)$

$$\begin{cases}lb^2(1+\xi_1)q(t)+\xi_1lb\pi(t)\Big(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\Big)=\gamma_1(t)\\ \xi_2\xi_3lb\Big(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\Big)q(t)+\Gamma\pi(t)=\gamma_2(t)\end{cases}, \tag{4.35}$$

where $\gamma_1(t),\gamma_2(t)$ and Γ are defined by Eq (4.18).

The determinant of the coefficient matrix of the system of Eq (4.27) is given by

$$\begin{aligned}
&\begin{vmatrix} lb^2(1+\xi_1) & \xi_1 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right) \\ \xi_2\xi_3 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right) & \Gamma \end{vmatrix} \\
&= \begin{vmatrix} lb^2(1+\xi_1) & \xi_1 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right) \\ \xi_2\xi_3 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right) & \xi_3 l\sigma^2 S_1^{2\beta}(1+\xi_2)+\xi_2 lh^2(1+\xi_3)-2\xi_2\xi_3 l\rho_{23}\sigma S_1^{\beta}h \end{vmatrix} \\
&= -\xi_1 l^2 b\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3+\xi_3 l^2\sigma^2 S_1^{2\beta}b^2(1+\xi_1)(1+\xi_2) \\
&-2b^2(1+\xi_1)h\xi_2\xi_3 S_1^{\beta}\rho_{23}l^2\sigma+b^2l^2h^2\xi_2(1+\xi_1)(1+\xi_3) \\
&= -\xi_1 l^2 b\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3+b^2 l(1+\xi_1)\Gamma \\
&= bl\left\{b(1+\xi_1)\Gamma-\xi_1 l\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3\right\},
\end{aligned} \tag{4.36}$$

moreover,

$$b(1+\xi_1)\Gamma \neq \xi_1 l\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3. \tag{4.37}$$

Then the system of Eqs (4.27) has a unique root $(\widehat{\pi}(t),\widehat{q}(t))$, and $\widehat{\pi}(t)$ and $\widehat{q}(t)$ satisfy the following equation, respectively:

$$\widehat{\pi}(t)=\frac{\xi_1 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)\gamma_1(t)+lb^2(1+\xi_1)\gamma_2(t)}{bl\left\{b(1+\xi_1)\Gamma-\xi_1 l\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3\right\}}, \tag{4.38}$$

and

$$\widehat{q}(t)=\frac{\Gamma\gamma_1(t)+\xi_2\xi_3 lb\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)\gamma_2(t)}{bl\left\{b(1+\xi_1)\Gamma-\xi_1 l\left(\rho_{12}\sigma S_1^{\beta}-\rho_{13}h\right)^2\xi_2\xi_3\right\}}. \tag{4.39}$$

Based on the convexity of $G(\pi(t),q(t))$ with respect to $\pi(t)$ and $q(t)$, and the definition of the permissive strategy, the candidate robust optimal reinsurance strategy $q^*(t)$ and the investment strategy $\pi^*(t)$ are given by Eqs (4.20) and (4.21), respectively, when $\widehat{q}(t)\in(0,1)$. □

When model uncertainty is not taken into account, the problem degenerates into a nonrobust optimal control problem under a reference measure $\mathbb{P}$. In this case, the wealth process of ANI satisfies Eq (2.9), the optimal objective satisfies Eq (3.3), and the corresponding HJB equation for the optimal control problem simplifies to

$$\inf_{\alpha\in\mathcal{A}}\{\mathcal{G}_\alpha V(r,t)\}=0, \tag{4.40}$$

and satisfies the boundary conditions

$$V(c,t)=1,\ V(\infty,t)=0. \tag{4.41}$$

The optimal investment-reinsurance strategy for ANI under the CEV model is given below by Theorem 3.

Theorem 3. In realistic economic settings, the model parameters l,b,η,a are strictly positive due to their economic interpretations, such as liability scale, premium rate, reinsurance cost coefficient, and profitability factor. Suppose that the net return of the risky asset exceeds the inflation rate and the associated risk penalties, that is,

$$\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta} h > 0,$$

and assume that the correlation-weighted volatility spread between the risky asset and the inflation index remains positive, that is,

$$\rho_{12}\sigma S_1^{\beta} - \rho_{13}h > 0.$$

In addition, let the auxiliary function Γ^0 satisfy the inequality

$$\Gamma^0 > l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2.$$

Then the baseline reinsurance strategy $q^0(t)$ is strictly positive for all $t \in [0, T]$.

If the wealth process of the agent with no ambiguity (ANI) satisfies Eq (2.10), and the value function satisfies the form given in Eq (3.3), then the corresponding optimal investment–reinsurance strategy is explicitly characterized.

$$\alpha^0 = \left(\pi^0(t), q^0(t)\right), \tag{4.42}$$

where the optimal investment strategy $\pi^0(t)$ satisfies

$$\pi^0(t) = \frac{lb\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)\gamma_1^0(t) + lb^2\gamma_2^0(t)}{b^2 l\left\{\Gamma^0 - l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\right\}}, \tag{4.43}$$

the optimal reinsurance strategy $q^0(t)$ satisfies

$$q^0(t) = \frac{\Gamma^0\gamma_1^0(t) + lb\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)\gamma_2^0(t)}{b^2 l\left\{\Gamma^0 - l\left(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h\right)^2\right\}}, \tag{4.44}$$

where

$$\begin{cases} \gamma_1^0(t) = \eta a e^{-\lambda(T-t)}, \\ \gamma_2^0(t) = (\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta} h)e^{-\lambda(T-t)}, \\ \Gamma^0 = l\left(\sigma^2 S_1^{2\beta} + h^2 - 2\rho_{23}\sigma S_1^{\beta} h\right). \end{cases} \tag{4.45}$$

Remark 3 (Verification of $q^0(t) > 0$)**.** The baseline optimal reinsurance proportion is given by:

$$q^0(t) = \frac{\Gamma^0\gamma_1^0(t) + lb(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)\gamma_2^0(t)}{b^2 l\left\{\Gamma^0 - l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2\right\}},$$

where the auxiliary functions are defined as:

$$\begin{cases} \gamma_1^0(t) = \eta a e^{-\lambda(T-t)}, \\ \gamma_2^0(t) = (\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta} h)e^{-\lambda(T-t)}, \\ \Gamma^0 = l\left(\sigma^2 S_1^{2\beta} + h^2 - 2\rho_{23}\sigma S_1^{\beta} h\right). \end{cases}$$

Let $E = e^{-\lambda(T-t)} > 0$. The numerator becomes:

$$\begin{aligned}\text{Numerator} &= \Gamma^0 \gamma_1^0(t) + lb(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)\gamma_2^0(t)\\ &= E\left[\Gamma^0 \eta a + lb(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)(\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h)\right].\end{aligned}$$

Each term inside the brackets is strictly positive under the assumptions:

$$\mu - p + h^2 - \lambda - \rho_{23}\sigma S_1^{\beta}h > 0, \quad \rho_{12}\sigma S_1^{\beta} - \rho_{13}h > 0, \quad l, b, \eta, a > 0.$$

Therefore, the numerator is positive.
Now consider the denominator:

$$\text{Denominator} = b^2 l \left\{\Gamma^0 - l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2\right\}.$$

Since $b, l > 0$, the denominator is positive if:

$$\Gamma^0 > l(\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2.$$

Substituting the expression of Γ^0:

$$\Gamma^0 = l\left(\sigma^2 S_1^{2\beta} + h^2 - 2\rho_{23}\sigma S_1^{\beta}h\right),$$

the inequality becomes:

$$\sigma^2 S_1^{2\beta} + h^2 - 2\rho_{23}\sigma S_1^{\beta}h > (\rho_{12}\sigma S_1^{\beta} - \rho_{13}h)^2.$$

This condition is satisfied if ρ_{23} is moderate and the volatility penalty term does not dominate. Therefore, the denominator is also positive.

Thus it can be determined that the optimal reinsurance policy $q^0(t)$ is strictly positive for all $t \in [0, T]$. □

The procedure for proving Theorem 3 is similar to that of Theorem 2 and will not be repeated here.

5. Sensitivity analysis and economic interpretation

In Section 4, we derived the analytical solutions for the optimal reinsurance and investment strategies of both ambiguity-averse and ambiguity-neutral insurers based on theoretical analysis.

This section aims to further investigate the impact of ambiguity preferences by conducting numerical simulations that compare the optimal strategies under the two types of insurers. The model parameters used in the simulations are summarized in Table 1.

Table 1. Model parameter settings.

ξ_1	ξ_2	ξ_3	l	b	ρ_{12}	ρ_{13}	ρ_{23}	μ	t
0.4	0.5	0.9	1.0	0.7	0.5	0.6	0.3	0.06	3
σ	S_1	β	p	h	λ	η	a	T	
0.2	3.5	0.5	0.5	0.09	0.02	0.2	1.0	12	

5.1. *The effect of model parameters on robust optimal reinsurance strategies*

In this section, we analyze the effect of model parameters on the optimal reinsurance strategy $q^*(t)$, which is computed by Eq (4.21).

Figure 1 illustrates that the optimal retention level $q^*(t)$ decreases monotonically with claim volatility b, implying that insurers tend to cede a larger proportion of claims to reinsurers as uncertainty in claim sizes increases. This behavior reflects a rational response to heightened risk exposure: greater volatility raises the likelihood of extreme losses, prompting ambiguity-averse insurers to transfer more risk to maintain financial stability. From a managerial standpoint, this finding highlights the importance of incorporating claim volatility into reinsurance decisions, suggesting that insurers operating in high-risk environments should adopt more conservative retention strategies to enhance solvency and mitigate risk.

Figure 2 illustrates the effect of inflation volatility h on the optimal risk retention level $q^*(t)$, under various values of the correlation coefficient p between the insurance claims process and the inflation process. It is important to emphasize that in our model, $q^*(t)$ represents the *retention ratio*, i.e., the proportion of risk that the insurer chooses to retain, rather than the proportion ceded to the reinsurer.

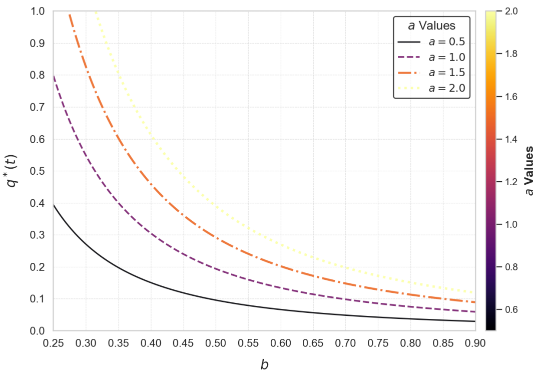

Figure 1. Effect of b on $q^*(t)$.

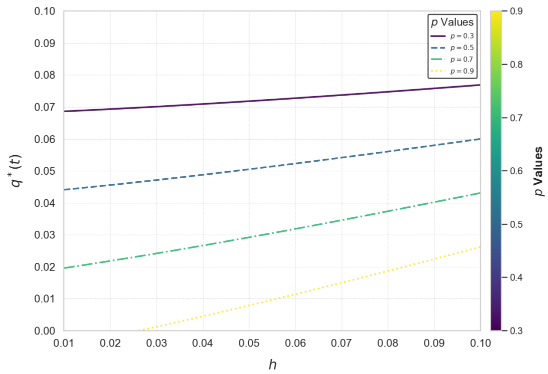

Figure 2. Effect of h on $q^*(t)$.

We observe that $q^*(t)$ is monotonically increasing in h for all values of p. Economically, this phenomenon can be understood as follows: when the volatility of inflation increases, the insurer faces greater uncertainty not only in investment returns but also in claim amounts, especially when the two are positively correlated. In such an environment, fuzzy-averse insurers—characterized by a heightened sensitivity to model ambiguity—tend to reduce their reliance on external hedging mechanisms such as reinsurance, which may become less effective or more costly under ambiguity. Instead, they prefer to retain a larger share of the risk, thereby exerting tighter control over their wealth dynamics and minimizing the subjective probability of ruin.

Furthermore, inflation may drive up reinsurance premiums, making risk transfer less attractive from a cost-benefit perspective. As a result, increasing h leads the insurer to increase $q^*(t)$, indicating a more conservative and self-reliant strategy. This observation highlights the unique behavior of fuzzy-averse insurers under model uncertainty, which may differ from conventional intuition based solely on risk transfer incentives.

Figure 3 discusses the effect of the ambiguity aversion coefficients ξ_1, ξ_2, ξ_3 on $q^*(t)$. ξ_1, ξ_2, ξ_3 represent ambiguities due to claims, risky assets, and inflation, respectively. The larger ξ_1, ξ_2, ξ_3 are, the greater the ambiguity, i.e., the greater the risk. As can be seen from Figure 3, $q^*(t)$ is increasing with respect to ξ_1 and decreasing with respect to ξ_2 and ξ_3. This implies that when the risk of claims increases, the insurer increases the proportion of self-retained risk to reduce the probability of an insolvency event. When the risk from risky assets and inflation increases, insurance companies want to purchase reinsurance to reduce the proportion of retained risk. Compared to ξ_2 and ξ_3, ξ_1 has the most significant effect on $q^*(t)$. This is because the risk in the insurance market has a more direct effect on the reinsurance strategy than the risk in the financial market.

Figure 4 visualizes the impact on $q^*(t)$ of the correlation coefficient ρ_{12} between the claim process and the risky asset price process, ρ_{13} between the claim process and the inflation process, and ρ_{23} between the risky asset price process and the inflation process. It can be seen that $q^*(t)$ is increasing with respect to ρ_{12} and decreasing with respect to ρ_{13} and ρ_{23}. This means that the larger ρ_{12} is, the better the insurance company is able to grasp the risks in the insurance and financial markets, and therefore the willingness to reinsure decreases. Larger ρ_{13} and ρ_{23} can be seen as an increase in the impact brought about by the risk of inflation, in which the insurance company realizes that the more severe the inflation is, the higher risk it faces, so the insurance company is more likely to want to purchase reinsurance to transfer the risk.

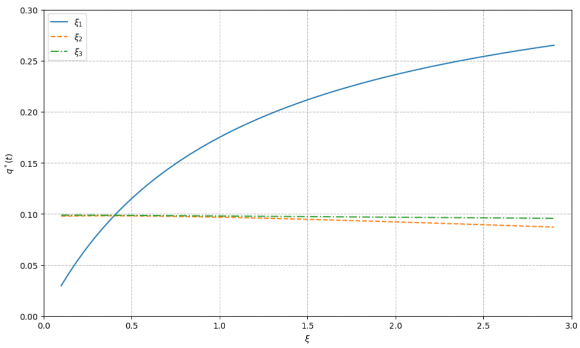

Figure 3. Effect of ξ on $q^*(t)$.

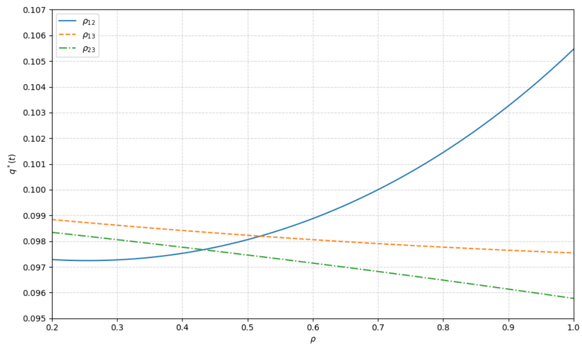

Figure 4. Effect of ρ on $q^*(t)$.

Figure 5 analyzes the joint effect of the absolute risk aversion coefficient l and the risky asset ambiguity coefficient ξ_2 of the insurance company on $q^*(t)$. It can be seen that $q^*(t)$ is decreasing with respect to both l and ξ_2. The larger l and ξ_2 are, the more risk averse the insurer is and the more it wants to reinsure. Further observation of Figure 5 also reveals that there is little difference in the degree of influence of l and ξ_2 on $q^*(t)$. This suggests that insurance companies do not differentiate between financial markets and their own risk aversion.

The combined influence of ρ_{12} and ρ_{13} on $q^*(t)$ is further examined in Figure 6. It is evident that ρ_{13} has a more pronounced impact on $q^*(t)$ compared to ρ_{12}, indicating that insurers are more sensitive to the dependence between the claims process and the inflation process. This highlights the importance insurers place on the interaction between insurance claims and inflation when designing reinsurance strategies, as inflation risk directly affects the real value of liabilities and can amplify claim costs, thereby reinforcing the necessity for effective risk transfer.

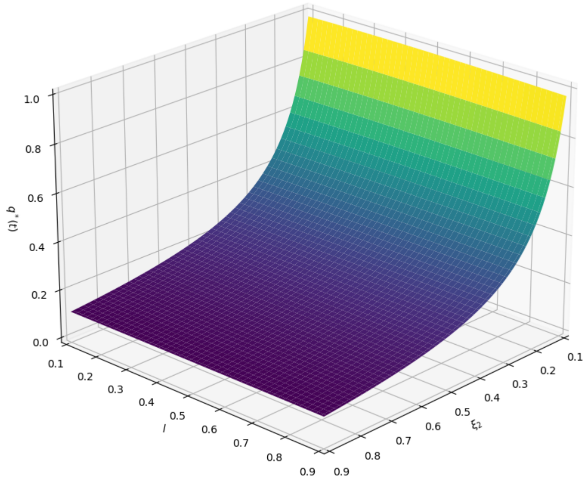

Figure 5. The joint effect of l and ξ_2 on $q^*(t)$.

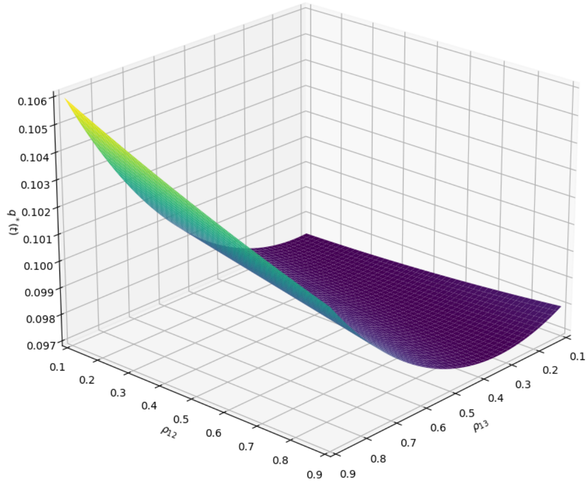

Figure 6. The joint effect of ρ_{12} and ρ_{13} on $q^*(t)$.

5.2. The effect of model parameters on robust optimal investment strategies

In this section, we analyze the effect of the model parameters on the optimal reinsurance strategy $\pi^*(t)$, which is computed by Eq (4.20).

Figure 7 visualizes the effect of the fuzzy aversion coefficients ξ_1, ξ_2, ξ_3 on $\pi^*(t)$. As can be seen from Figure 7, $\pi^*(t)$ is increasing with respect to ξ_1 and decreasing with respect to ξ_2 and ξ_3. This implies that insurers increase the proportion of investment in risky assets when the risk of claims increases. When the risk from risky assets and inflation increases, the insurance company decreases the proportion of investment in risky assets. Compared to ξ_2 and ξ_3, ξ_1 has the most significant effect on $\pi^*(t)$. This is because insurers are more concerned with claim-induced risk than investment risk and inflation risk.

Figure 8 illustrates how the optimal investment strategy $\pi^*(t)$ responds to changes in the correlation coefficients ρ_{12}, ρ_{13}, and ρ_{23}. The results show that $\pi^*(t)$ increases with ρ_{12} and ρ_{13}, but decreases with ρ_{23}, with the effect of ρ_{23} being the most pronounced. This suggests that the inflation risk embedded in the financial market plays a critical role in shaping investment behavior. A stronger positive correlation between asset returns and inflation increases uncertainty in real investment returns, prompting insurers to reduce their exposure to risky assets.

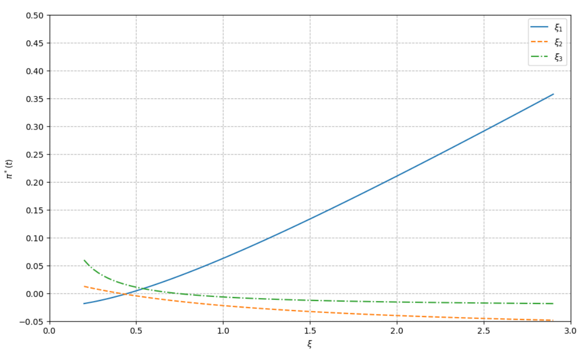

Figure 7. Effect of ξ on $\pi^*(t)$.

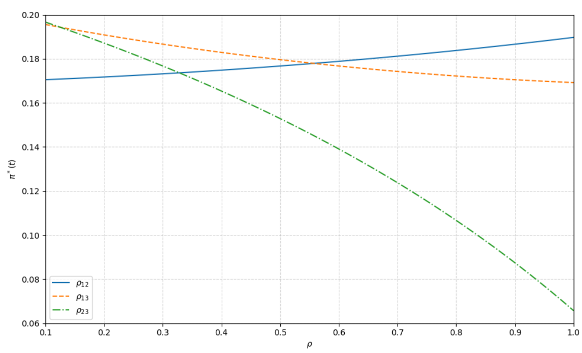

Figure 8. Effect of ρ on $\pi^*(t)$.

Figure 9 illustrates the impact of the volatility σ of the risky asset price on the optimal investment strategy $\pi^*(t)$ under different values of the elasticity coefficient β and the expected return μ. It is observed that $\pi^*(t)$ decreases monotonically as σ increases, and the influence of β on $\pi^*(t)$ is more pronounced compared to that of μ. This reflects the economic mechanism that, as the volatility of the risky asset rises, the associated investment risk intensifies, leading insurers to reduce their allocation to such assets to mitigate potential losses. Moreover, β, which measures the sensitivity of the asset price to external shocks, amplifies this effect; the higher the β, the more sensitive the asset price is to market fluctuations, making the investment strategy more responsive to changes in β. These findings highlight the importance for insurers to closely monitor the elasticity of risky asset prices when dynamically adjusting their portfolio allocations.

Figure 10 examines the joint influence of the insurer's absolute risk aversion coefficient l and inflation-related ambiguity aversion coefficient ξ_3 on the optimal investment strategy $\pi^*(t)$. The analysis shows that $\pi^*(t)$ declines with increases in both l and ξ_3. This indicates that insurers with stronger aversion to financial risk and inflation uncertainty tend to reduce their exposure to risky assets. From an economic perspective, higher values of l reflect a more conservative attitude toward investment risk, while a greater ξ_3 implies heightened concern over ambiguity in the inflation process. The combined effect leads to a more cautious asset allocation strategy.

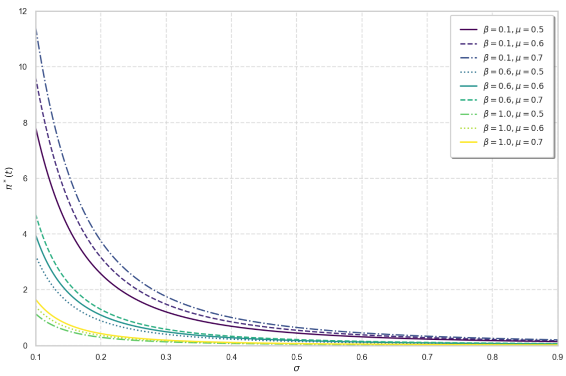

Figure 9. Effect of σ on $\pi^*(t)$.

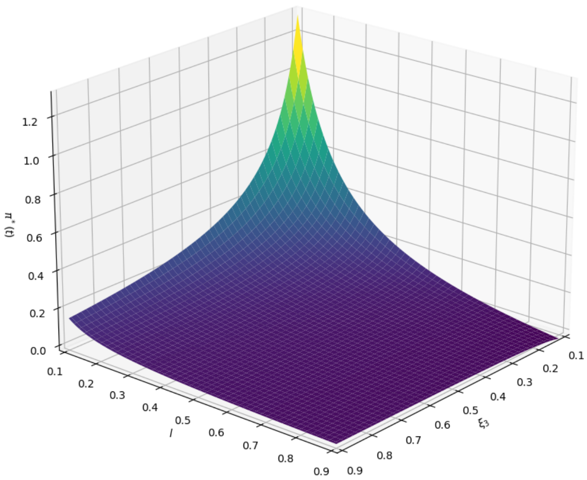

Figure 10. The joint effect of l and ξ_3 on $\pi^*(t)$.

Figure 11 provides an insight into the specific impact of the expected return μ on risky assets and the asset price volatility σS_1^β on $\pi^*(t)$. The results of the analysis show that when the expected return of risky assets μ shows an increasing trend, insurers tend to adopt a more conservative investment strategy and therefore reduce the amount of investment in risky assets. This suggests that insurers' rational consideration of the risk-return trade-off makes reducing risk exposure to protect capital security their primary choice when the return rate increases but the potential risk is not clearly reduced. On the contrary, when the volatility of asset prices, σS_1^β, increases, i.e., market volatility increases, signaling the possibility of higher return opportunities, the insurer's investment strategy demonstrates a certain risk appetite. In this scenario, insurers tend to increase the amount invested in risky assets with a view to obtaining potentially higher returns by taking on higher risks. Such adjustments not only reflect insurers' keenness to capture changes in market dynamics, but also demonstrate their ability to flexibly adjust their portfolios to maximize returns in a complex market environment.

Figure 12 demonstrates the joint effect of the correlation coefficients ρ_{12} and ρ_{13} on the optimal investment strategy $\pi^*(t)$. The results reveal that $\pi^*(t)$ increases as both ρ_{12} and ρ_{13} rise, indicating that stronger positive correlations between the insurance claims process and the risky asset price process, as well as between the claims process and the inflation process, lead to higher investment in the risky asset.

This behavior reflects the insurer's strategic response to perceived hedging effects. When the claims process moves more in tandem with financial and inflationary variables, the insurer may view the risky asset as a partial hedge against liabilities or inflation-induced losses, thereby justifying a greater investment allocation. Such correlations reduce the relative uncertainty in the joint dynamics of assets and liabilities, prompting the insurer to pursue more aggressive investment strategies.

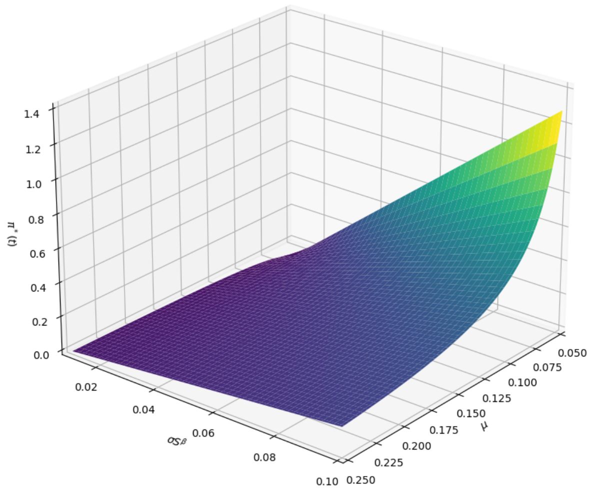

Figure 11. The joint effect of μ and σS_1^{β} on $\pi^*(t)$.

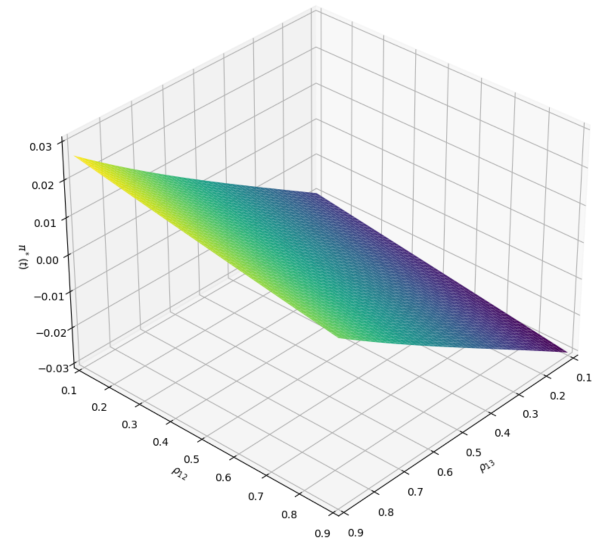

Figure 12. The joint effect of ρ_{12} and ρ_{13} on $\pi^*(t)$.

5.3. *The effect of model parameters on optimal reinsurance strategies in the case of model determination*

In this section, we focus on analyzing the effect of model parameters on the optimal reinsurance strategy $q^0(t)$ in the model determination case. $q^0(t)$ is calculated by Eq (4.44).

Figure 13 explores the effect of the correlation coefficients ρ on the optimal reinsurance strategy $q^0(t)$ under the assumption of model certainty. The results show that $q^0(t)$ increases with ρ_{12} but decreases with ρ_{13} and ρ_{23}, with the influence of ρ_{12} being the most prominent. This pattern suggests that when the dependence between the claims process and the risky asset price strengthens, insurers are inclined to retain a larger proportion of the claims risk themselves, possibly due to improved predictability or hedging opportunities between financial assets and liabilities. Conversely, a stronger correlation between the claims process and either inflation or asset prices (as captured by ρ_{13} and ρ_{23}) may increase uncertainty in the insurer's overall exposure, prompting more conservative reinsurance decisions.

Figure 14 examines the joint impact of the correlation coefficients ρ_{12} and ρ_{23} on the optimal reinsurance strategy $q^0(t)$ in the case of model certainty. The numerical results indicate that $q^0(t)$ increases as both ρ_{12} and ρ_{23} increase. This outcome suggests that, when the financial and macroeconomic environments become more tightly linked to the insurer's liabilities, the insurer tends to retain a higher proportion of the risk. This result highlights the strategic value of exploiting inter-market dependencies in the design of reinsurance programs. Insurers that can effectively monitor and utilize such correlations may reduce reliance on external risk transfer mechanisms, enhancing cost efficiency and capital utilization.

Figure 15 analyzes the impact of the absolute risk aversion coefficient l of insurance companies and the correlation coefficients ρ_{12} of the claims process with the risky asset price process, ρ_{13} of the claims process with the inflation process, and ρ_{23} of the risky asset price process with the inflation process on the joint production of $q^*(t)$, $q^0(t)$. The results reveal that the effect of risk aversion on reinsurance decisions is significantly modulated by the underlying correlation structures. Specifically, when ρ_{12} increases, the insurer tends to retain more risk, but this tendency weakens as l increases, reflecting a trade-off between hedging benefits and risk tolerance. Conversely, higher values of ρ_{13} and ρ_{23} amplify the impact of risk aversion, leading to a sharper decline in retention levels as l increases, due to the compounding uncertainty from inflation-related risks.

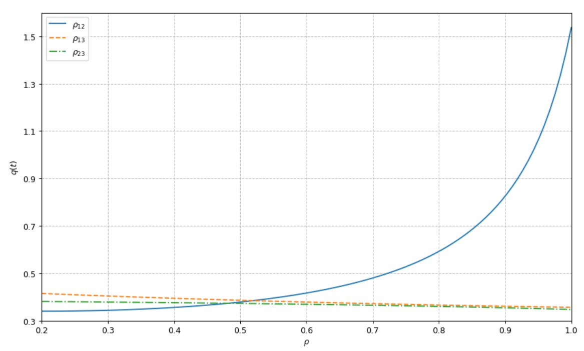

Figure 13. Effect of ρ on $q^0(t)$.

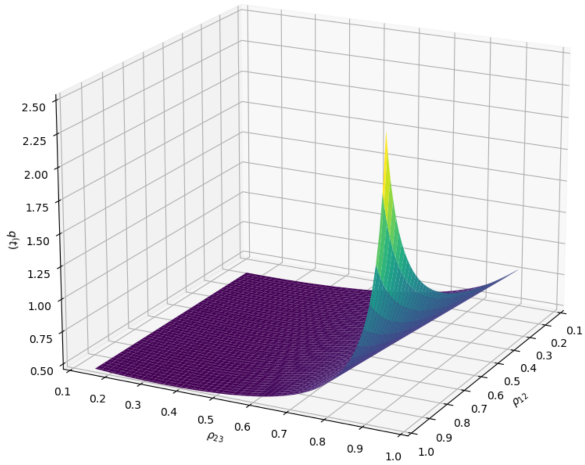

Figure 14. The joint effect of ρ_{12} and ρ_{23} on $q^0(t)$.

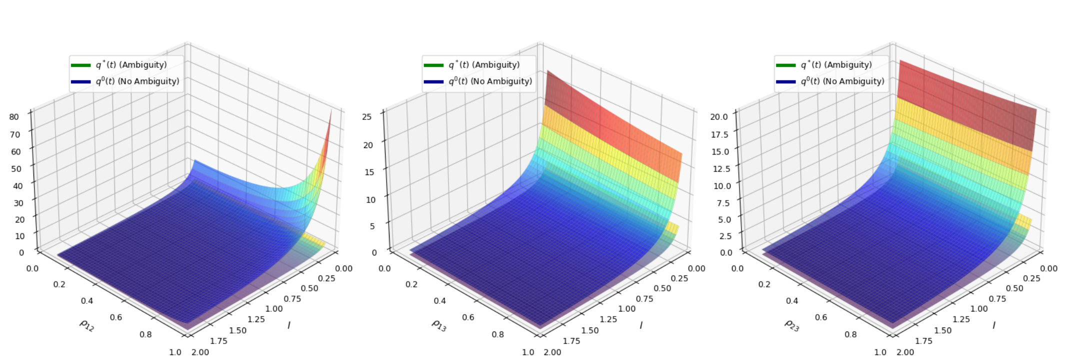

Figure 15. The joint effect of ρ and l on $q(t)$.

These findings suggest that reinsurance decisions are not only shaped by risk preferences or economic dependencies alone but by their interaction. Ignoring such joint effects could lead to suboptimal risk management, particularly under ambiguity.

5.4. *The effect of model parameters on optimal investment strategies in the case of model determination*

In this section, we focus on analyzing the effect of model parameters on the optimal investment strategy $\pi^0(t)$ under model determination, which is calculated by Eq (4.43).

Figure 16 visually discusses the effect of the correlation coefficient ρ on $\pi^0(t)$ for the model determination case. $\pi^0(t)$ is increasing with respect to ρ_{12} and decreasing with respect to ρ_{13} and ρ_{23}, with ρ_{12} having the most significant effect on $\pi^0(t)$. This pattern highlights that, when there is a stronger positive correlation between the claims process and the risky asset price process, insurers are more inclined to increase investment in the risky asset. Conversely, stronger correlations between claims and inflation, or between asset prices and inflation, reduce the attractiveness of risky investments, due to increased macroeconomic uncertainty that amplifies the volatility of returns and diminishes the effectiveness of traditional hedging strategies.

Figure 17 explores the joint impact of the correlation coefficients ρ_{13} and ρ_{23} on the optimal investment strategy $\pi^0(t)$ under the model determination framework. The results indicate that $\pi^0(t)$ declines as both ρ_{13} and ρ_{23} increase. This pattern implies that when the risks associated with insurance claims and asset prices are more strongly correlated with inflation uncertainty, insurers tend to adopt a more conservative investment strategy. The simultaneous increase in these correlations signals heightened exposure of both liabilities and assets to inflation shocks, thereby elevating the insurer's overall risk profile. To mitigate this compounded uncertainty and maintain financial resilience, insurers reduce their investment in risky assets. Such behavior reflects a rational and prudent response to systemic inflation risk, consistent with robust risk management practices in volatile economic conditions.

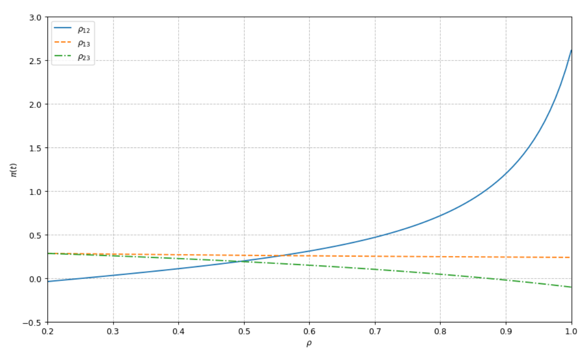

Figure 16. Effect of ρ on $\pi^0(t)$.

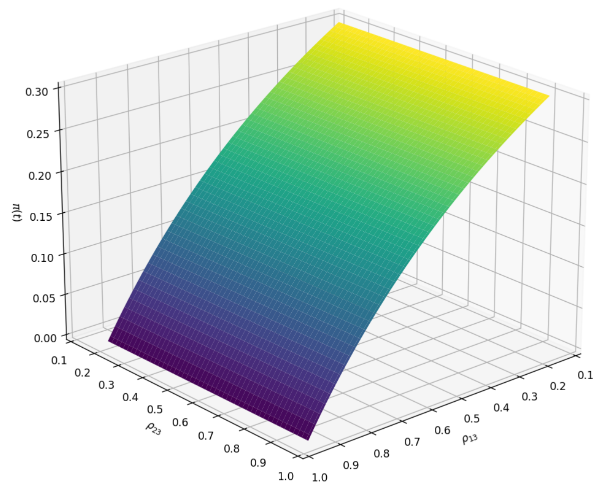

Figure 17. The joint effect of ρ_{13} and ρ_{23} on $\pi^0(t)$.

Figure 18 investigates the joint effects of the insurer's absolute risk aversion coefficient l and the correlation coefficients ρ_{12}, ρ_{13}, and ρ_{23} on the optimal investment strategies $\pi^*(t)$ under ambiguity and $\pi^0(t)$ under no ambiguity. The three panels respectively present the joint impact of l with each correlation coefficient.

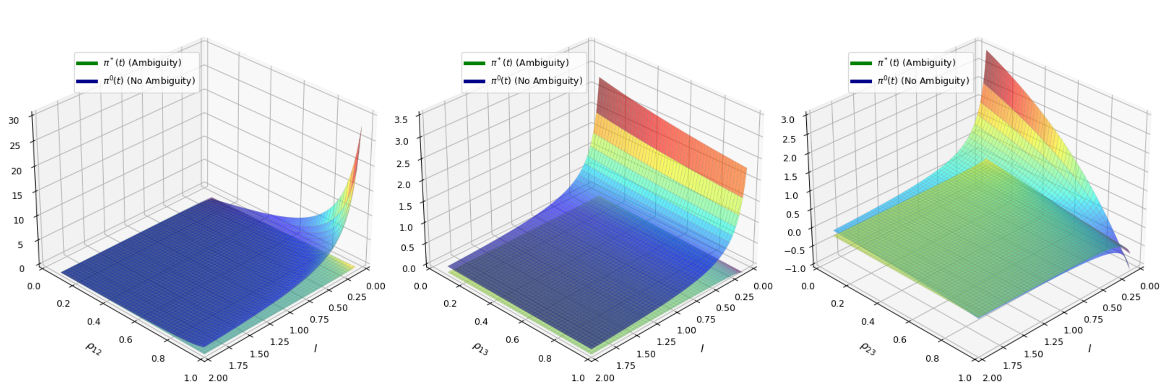

Figure 18. The joint effect of ρ and l on $\pi(t)$.

In all three subplots, $\pi^*(t)$ and $\pi^0(t)$ exhibit a decreasing trend as l increases, reflecting a more conservative investment approach with rising risk aversion. Additionally, the correlation coefficients introduce significant heterogeneity into the investment behavior. Specifically, as ρ_{12} increases, both $\pi^*(t)$ and $\pi^0(t)$ rise, especially at low levels of risk aversion, indicating insurers' sensitivity to claim-asset dependence when ambiguity is absent or moderate. Conversely, increasing ρ_{13} and ρ_{23} lead to a decline in both $\pi^*(t)$ and $\pi^0(t)$, suggesting that inflation-related risks on both liability and asset sides reduce insurers' willingness to invest in risky assets. Moreover, the gap between $\pi^*(t)$ and $\pi^0(t)$ becomes more pronounced with lower l and higher correlation values, highlighting that ambiguity aversion amplifies the sensitivity of optimal strategies to underlying stochastic dependencies.

6. Conclusions

In the real-world financial environment, insurance companies face constant and unpredictable challenges and must be ready to respond to a variety of potential claims demands. To effectively manage these financial risks, insurance companies employ a variety of strategies, including transferring claims risk by purchasing proportional reinsurance and increasing wealth by investing in both risky and risk-free assets in financial markets as a means of achieving the goal of reducing the probability of insolvency. However, as the global economy continues to grow and becomes more complex, the problem of inflation becomes more serious, exerting a profound impact on the investment strategies of insurance companies and increasing the complexity of investment decisions.

This paper investigates the optimal investment and reinsurance problems of ambiguity-averse insurance companies with the objective of minimizing the probability of bankruptcy. The article adopts the diffusion risk model to describe the surplus process of insurance companies and characterizes the price dynamics of risky assets by combining the CEV model with the effects of inflation. Based on the dynamic programming principle, the corresponding HJB equations are established, and the analytical solutions of the robust optimal investment and reinsurance strategies for insurance companies are derived. Finally, the effects of model parameters on the optimal strategies are verified through numerical analysis, which further confirms the validity and practicality of the model. The results provide an important basis for insurance companies to make investment and reinsurance decisions in practice.

The research findings are summarized as follows:

(1) Ambiguity-averse insurance companies, due to their heightened sensitivity to model uncertainty, tend to adopt more conservative investment and reinsurance strategies. Compared with ambiguity-neutral insurance companies, they exhibit a lower level of risk tolerance and thus maintain relatively lower exposure in financial and insurance markets.

(2) Insurance market risks have a more direct impact on investment and reinsurance strategies than financial market and inflation risks.

(3) Ambiguity-averse insurers adopt more conservative investment and reinsurance strategies under the combined influence of the absolute risk aversion coefficient and the fuzzy coefficient.

(4) Through these findings, this paper not only deepens the understanding of risk management strategies of insurance companies under the ambiguity aversion perspective but also provides new ideas and directions for related research in similar settings. In future research, the model can be extended and improved by considering the introduction of other more complex portfolio influences or discussing optimal investment and reinsurance strategies under different risk asset price models.

Author contributions

Chen Wang: Conceptualization, Methodology, Validation, Writing – Original Draft, Visualization; Hongmin Xiao: Formal analysis, Investigation, Writing-review and editing.All authors have read and agreed to the published version of the manuscript.

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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